https://doi.org/10.15388/vu.thesis.614
https://orcid.org/0000-0003-4796-2760

VILNIUS UNIVERSITY
CENTER FOR PHYSICAL SCIENCES AND TECHNOLOGY

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Renormalization of the Grimus-Neufeld model

DOCTORAL DISSERTATION

Natural Sciences,
Physics (N 002)

VILNIUS 2024

The dissertation was prepared between 2019 and 2023 at Vilnius University.

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This doctoral dissertation will be defended in a public meeting of the Dissertation Defence Panel:

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The dissertation shall be defended at a public meeting of the Dissertation Defence Panel at 3 pm on the 19th of June 2024 in the Room A101 of the National Center for Physical Sciences and Technology.
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The text of this dissertation can be accessed at the libraries of Vilnius University, as well as on the website of Vilnius University:
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VILNIAUS UNIVERSITETAS
FIZINIŲ IR TECHNOLOGIJOS MOKSLŲ CENTRAS

Simonas Draukšas

Grimus-Neufeld modelio renormalizacija

DAKTARO DISERTACIJA

Gamtos mokslai,
Fizika (N 002)

VILNIUS 2024

Disertacija rengta 2019–2023 metais Vilniaus universitete.

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Disertacija ginama viešame Gynimo tarybos posėdyje 2024 m. birželio mėn. 19 d. 15 val. Nacionalinio fizinių ir technologijos mokslų centro A101 auditorijoje. Adresas: (Saulėtekio al. 3, Vilnius, Lietuva).

Disertaciją galima peržiūrėti Vilniaus universiteto, Fizinių ir technologijos mokslų centro bibliotekose ir interneto svetainėje adresu:
`https://www.vu.lt/naujienos/ivykiu-kalendorius`

ABBREVIATIONS

BBM	beyond base model
BSM	beyond Standard Model
BRST	Becchi-Rouet-Stora-Tyutin
BM	base model
CDF	collider detector at Fermilab
CKM	Cabbibo-Kobayashi-Maskawa
CP	charge-parity
EW	electroweak
FJ	Fleischer-Jegerlehner
GNM	Grimus-Neufeld model
H.C.	Hermitian conjugation
LEP	Large Electron Collider
LSZ	Lehmann-Symanzik-Zimmerman
MSSM	minimal supersymmetric standard model
NPM	new physics model
OS	on-shell
PV	Passarino-Veltmann
QED	quantum electrodynamics
QFT	quantum field theory
RG	renormalization group
SM	Standard Model
THDM	two Higgs doublet model
UV	ultraviolet
VEV	vacuum expectation value
1PI	1 particle irreducible

CONTENTS

INTRODUCTION

Early days of the quantum field theory (QFT) encountered a variant of the ultraviolet (UV) catastrophe — in the 1930s-1940s perturbative calculations of observables in quantum electrodynamics (QED) seemed to give infinite predictions. This obstacle was overcome by Tomonaga, Schwinger, Feynman, and Dyson, who formulated the *renormalization* procedure, which yielded finite results [1–10]. Even with this success the physics community regarded the renormalization procedure as a trick, which "swept the UV divergences under the rug" [11], and even produced somewhat desperate jokes [12]. Nowadays, the procedure has a much more solid reputation, especially due to the ideas of the renormalization group (RG) of the 1970s [12–19], and is part of quantum field theory textbooks and courses.

The nature of UV divergences can be explained by the roughness of the quantum fields at short distances or, alternatively, by the naivety that our theories hold up to arbitrary energy (distance) scales. In any case, one needs to *regulate* the theory with a cut-off [20] or some other method to have control over the divergences. However, the observables should not depend on the arbitrary regulator. Interestingly, the ideas of the renormalization group tell us that the different scales decouple — one does not need to know about the quarks to predict the hydrogen spectrum, and one does not need to know how the streets in the Vilnius old town are arranged to predict the orbit of Jupiter, *etc.* In turn, the regulator, corresponding to UV scales, simply cancels out when comparing two observables — this is the renormalization procedure. In other words, predictions expressed in terms of a set of earlier measurements are regulator-independent and UV-finite.

Even though the general concepts of renormalization seem to be well understood, the devil is in the detail — there are cases where it is not entirely clear how exactly the renormalization procedure should be carried out. In turn, the literature seems to suggest many renormalization schemes, with their up- and downsides, but no unified recipe is present. Of particular interest are theories with particle mixing and the corresponding mixing matrices, since that includes the Standard Model (SM) as well as many of its extensions, both supersymmetric and not. In the SM mixing is present in the quark sector, where the mixing is conveniently parameterized by the Cabbibo-Kobayashi-Maskawa (CKM) [21, 22] matrix. Probably the first attempt to perform the renormalization procedure in the presence of mixing was in [23] for the CKM matrix. However, it was later found that this particular approach made the CKM counterterm gauge-dependent [24, 25] — an unwanted feature since mixing is a physical effect. In

the last 30 years, there have been quite a few attempts at the renormalization in the presence of fermion mixing [24, 26–31] aiming to improve the initial approach in [23]. In addition, the discussion is not limited to fermions but is also relevant for extended scalar sectors where mixing is almost inevitable and, in turn, there are many renormalization approaches [32–44] and the list is likely not extensive.

The main result of this thesis is exactly this — the renormalization procedure in the presence of particle mixing. In particular, we have come up with a renormalization scheme that is rather conceptual in its nature. We do use the by-now standard On-Shell renormalization conditions [45], but they are implemented in a rather non-standard fashion in the Fleischer-Jegerlehner (FJ) tadpole scheme [46, 47]. Non-standartness of the scheme presents itself in the definitions of the off-diagonal field renormalization constants, which are selected as coefficients of the $m_i^2 - m_j^2$ mass structure, where i and j are flavor indices. Interestingly, this allows for a trivial mixing matrix counterterm, *i.e.* it is set to 0, which immediately satisfies all the requirements of mixing renormalization formulated in [27, 36, 43]. Importantly, we argue that the scheme is valid to all orders in perturbation theory and that mixing matrices (angles), being basis-dependent quantities, should not receive counterterms for consistency.

Apart from general and model-independent considerations, the main driver in presenting the new scheme is the Standard Model extended with an additional Higgs doublet as well as a Majorana singlet — we call this the Grimus-Neufeld model (GNM) [48–52]. The model is interesting both technically and physically. On the technical side, the model combines the seesaw mechanism as well as radiative corrections to generate neutrino masses. The model contains four Majorana neutrinos in the mass eigenstate basis; two of them are massless at tree level while at 1-loop level one of them receives a radiative mass. These mechanisms and especially the massless states, are interesting to consider for a scheme, that relies on selecting mass structures. On the physical side, neutrinos are one of the very few and elusive hints of beyond Standard Model (BSM) physics and the GNM, with the seesaw mechanism and radiative mass generation mechanisms, provides a possible explanation for the smallness of neutrino masses. In addition, the heavy Majorana singlet (decoupled at low energies) may be a suitable candidate for dark matter. Therefore, in order to make predictions and check the validity of the model it is important to renormalize the model.

It is important to not only make predictions but to also easily compare them with predictions of other models as well as with experimental data. A

common way to do so is via the so-called oblique parameters S, T, U introduced in [53, 54] and later extend with X, V, W in [55], although there are other related parameterizations [56–60]. The oblique parameters are a useful tool in approximating the electroweak (EW) sector predictions as long as only loop corrections to the gauge-boson propagators are sufficient and other corrections such as box and vertex diagrams can be neglected. It is important to note that the oblique parameters are UV-finite and gauge-independent so that they can directly parameterize the observables. Traditionally these parameters take as input (renormalize) three parameters in the EW sector and are used to compare the SM with any other model that has the same gauge group as well as the custodial $SU(2)$ symmetry, which keeps the well-known tree-level relation between the W and Z boson masses via the Weinberg angle, namely, $m_W = \cos\theta_W m_Z$. In turn, the Grimus-Neufeld model may be compared with the SM, or the GNM may be taken as a base model with which some other model could be compared. However, in light of the new measurement of the W boson mass by the CDF collaboration [61], which predicts a larger mass than the SM, models with broken custodial symmetry become interesting since $m_W = \cos\theta_W m_Z$ no longer holds. In such models, many authors have found the T parameter to be UV-divergent [60, 62–70] and so it cannot be used to parameterize observables. One can come around this by renormalizing an additional EW parameter, this eliminates the T parameter, however, the reduced set of oblique parameters can be used only for models where $m_W \neq \cos\theta_W m_Z$ at tree-level. In turn, the oblique parameters are not suitable for comparing the SM or GNM with a model, where the custodial symmetry is broken — such a comparison requires full computations including the neglected corrections.

The thesis is organized as follows: below we briefly list the main goal and tasks of the thesis, followed by the remarks on novelty and relevance, we also list the statements to be defended after which the four main parts of the thesis follow. We begin the first part in Section 1, where we give a basic introduction to renormalization, in Section 2 we present the ideas behind particle mixing, we conclude this part of the thesis in Section 3, where we show that mixing angle counterterms should be trivial on general grounds. In the following part, we develop a new renormalization scheme: in Section 4 we set up the needed notation, in Section 5 we present the on-shell renormalization conditions and discuss their implementation in our scheme both at 1-loop and to all orders, in Section 6 we more explicitly discuss the properties (divergences, gauge-dependence) of our scheme and also perform comparisons with other schemes at 1-loop in Section 6.3, the part is concluded in Section 7 by showing that our scheme works to all orders in perturbation theory. The next part consists of

Section 8, where we explicitly apply our renormalization scheme to the Grimus-Neufeld model at 1-loop. The final part of the thesis considers the oblique corrections in the electroweak sector and consists of two sections: in Section 9 we give the needed introduction and definitions, while in Section 10 we discuss the corrections to selected observables both when $m_W = \cos\theta_W m_Z$ does and does not hold at tree-level. We finalize the thesis by giving our conclusions, the bibliography, a summary (Lithuanian), and copies of publications contributing to the thesis.

Main goal and tasks

The main goal of the thesis is to arrive as close as possible to the full renormalization of the Grimus-Neufeld model. The main obstacle in the renormalization of the model is the presence of particle mixing both in the fermion and scalar sectors, which requires coming up with a consistent renormalization scheme. To do so, we have completed the following tasks:

1. We have shown that the mixing angles should not receive counterterms.
2. We have devised a 1-loop renormalization scheme with explicitly trivial mixing angle counterterm. The scheme is based on the usual On-shell renormalization conditions [45].
3. At 1-loop we have renormalized the fermion fields, masses, and mixing matrices in the Grimus-Neufeld model.
4. We have extended the scheme to all orders in perturbation theory.

Another significant goal is to parameterize the observables such that they can be quickly compared to the experiment. One of the most popular approaches is that of the oblique parameters, however, this approach is not immediately valid for all models. In particular, for models with $m_W \neq \cos\theta_W m_Z$, the T parameter is ultraviolet divergent. To solve the problem we have completed the following tasks:

1. In the electroweak (EW) sector we have in addition renormalized the mass of the W-boson.
2. We have derived expressions for the EW observables in terms of the reduced set of oblique parameters, where the T parameter is no longer present.

Novelty and relevance

The renormalization procedure is in some regards well-understood, however, there are a few issues, most notable of which is the renormalization in the presence of particle mixing. For the past thirty years, there have been many attempts to renormalize the quark mixing matrix in the Standard model and, more recently, mixing matrices in extended scalar sectors. Up to this day, there is no agreed-upon recipe and so there are many schemes with properties of varying convenience and consistency. In this work, we provide a consistent conceptual approach towards the renormalization of mixing matrices (angles)

and give a new renormalization scheme that explicitly implements the conceptual approach.

Another novelty in terms of renormalization is that of the consistent application of the oblique parameters to models where the well-known relation $m_W = \cos\theta_W m_Z$ no longer holds at tree level. It turns out that the T parameter, which was found to be divergent in such models by many authors, can be removed from the usual set of oblique parameters by an additional renormalization condition in the electroweak sector, namely, the renormalization of the mass of the W-boson. While this allows to parameterize the observables in terms of the reduced set of oblique parameters, this also implies that one cannot compare the Standard Model predictions with predictions of a model, where $m_W \neq \cos\theta_W m_Z$ at tree level, by using the oblique parameters.

Statements of the thesis

1. Counterterms of rotation (mixing) angles are inconsistent in the sense that they obstruct the commutativity of basis rotations and the renormalization procedure, *i.e.* there is a difference depending on the basis in which the renormalization procedure is performed. In turn, there should be no rotation (mixing) angle counterterms.

2. The proposed renormalization scheme for fermions is an explicit example of an On-Shell scheme, where the mixing angles do not receive counterterms. Moreover, the scheme is valid for arbitrary orders in perturbation theory.

3. It is impossible to compare the observables between the Standard Model (or the Grimus-Neufeld model) and a model, where $m_W = \cos\theta_W m_Z$ does not hold at tree level, by using the oblique parameters. The available options are as follows:

    - if both models *preserve* $m_W = \cos\theta_W m_Z$ at tree level, the standard S, T, U (sometimes extended with X, V, W) oblique parameters may be used,
    - if both models *violate* $m_W = \cos\theta_W m_Z$ at tree level, the T parameter can be removed from the usual set via a simple replacement rule.
    - in-between cases — one model preserves and the other violates $m_W = \cos\theta_W m_Z$ at tree level — cannot be captured by oblique parameters and full computations must be performed in each respective model before comparison.

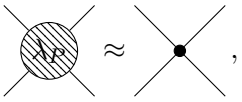
1. Basic ideas behind renormalization

In the following, we review the basics of renormalization in some detail and also discuss the current status of the renormalization procedure in the presence of particle mixing. This section is rather loosely based on [11, 20, 71, 72] as well as personal insight unless explicitly specified otherwise.

Let us consider the ϕ^4 theory to present the basic ideas behind renormalization. The Lagrangian is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4\,, \tag{1.1}$$

where ∂_μ is a shorthand for $\frac{\partial}{\partial x^\mu}$, m is the mass, λ is a quartic coupling, and ϕ is a real scalar field. The Lagrangian and especially its parameters, should be connected to the real world. For example, one may measure the $2 \to 2$ scattering in some experiment and also compute the amplitude (or cross-section) for the very same process using the above Lagrangian and the usual Feynman rules. After the measurement the experimentalist would assign the *physical* value to the coupling, namely λ_P, while the theorist would begin with the 1st order of the coupling, *i.e.* the tree-level, in perturbation theory and would use the coupling λ from the Lagrangian. Diagrammatically one has



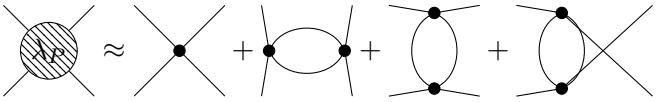
, (1.2)

such that the couplings are easily equated at this order

$$-i\lambda_P = -i\lambda\,. \tag{1.3}$$

The important thing is that the experimentalist measures *everything* at once without picking any specific order, hence the blob on the l.h.s., while the theorist began with the first order in the coupling λ, hence the single diagram on the r.h.s. with the interaction vertex marked as a dot.

At the next order in perturbation theory, loop diagrams may be considered such that one gets



(1.4)

and using the momentum space Feynman rules

$$\begin{aligned} -i\lambda_P = &-i\lambda + \frac{1}{2}(-i\lambda)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2} \frac{i}{(p-k)^2-m^2} \\ &+ \frac{1}{2}(-i\lambda)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2} \frac{i}{(q-k)^2-m^2} \\ &+ \frac{1}{2}(-i\lambda)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2} \frac{i}{(l-k)^2-m^2} \end{aligned} \quad , \tag{1.5}$$

where k is the 4-momentum running in the loop, while $p = p_1 + p_2$, $q = p_1 - p_4$, $l = p_1 - p_3$ with p_i being the momenta of external particles. Even without performing the integration, it is obvious that the relation between the measured coupling λ_P and the Lagrangian parameter λ is no longer straightforward as in the tree-level relation in Eq. (1.3) and going beyond 1-loop only increases the complexity. Another thing that may be noticed is that the integrals depend on the momenta of external particles, such that λ_P must also depend on the momenta, *i.e.* λ_P depends on the scale at which the measurement is performed, while there is no such scale dependence in the parameter λ. Combining these two ideas we may write the relation between λ_P and λ as a power series

$$\lambda_P(\mu) = \sum_{k=1}^{\infty} \lambda^k A_k(\mu)\,, \tag{1.6}$$

where μ is the scale at which the experiment is performed and $A_k(\mu)$ are scale-dependent coefficients coming from the integrals. The coefficients $A_k(\mu)$ may be computed and the coupling $\lambda_P(\mu)$ can be measured, however, the parameter λ is *a priori* unknown — at this point, the Lagrangian cannot be used to make predictions. A way around this is to invert the relation such that λ is expressed in terms of measured and computed quantities

$$\lambda = \sum_{k=1}^{\infty} \left[\lambda_P(\mu)\right]^k B_k(\mu)\,. \tag{1.7}$$

This is at the heart of renormalization — one expresses the so-called "bare" parameters in the Lagrangian in terms of measured (and computed) quantities. Of course, this implies some initial measurements must be made to fix the bare parameters and make the theory predictive. For example, say that one performs a measurement at some scale μ_0 and fixes λ in Eq. (1.7), then this fixed value may be inserted in Eq. (1.6) to get

$$\lambda_P(\mu) = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} \left[\lambda_P(\mu_0)\right]^k B_k(\mu_0) \right)^n A_n(\mu)\,. \tag{1.8}$$

The important point is that now everything on the r.h.s. is either known or can be computed such that the l.h.s. can be both predicted and measured.

It would be rather compelling to state that renormalization is needed because of the complexity in Eqs. (1.5) and (1.6), which arises because of perturbation theory. However, even in the simple tree-level case of Eq. (1.3) the parameter λ is unknown and must be fixed by measurement. In a crude sense, renormalization — expressing the bare parameters in terms of physical ones — is nothing more but the choice of initial conditions that one encounters already in classical mechanics. Lagrangians can and do incorporate many of the physics principles (*i.e.* Lorentz invariance), but the same Lagrangian can describe different systems such that it is needed to first fit the parameters to the data in order to specify the exact realization and make the theory predictive.

What earns "renormalization" a separate name from that of "specifying the initial conditions" are the additional problems that it solves. In the Introduction we have mentioned that the renormalization procedure takes care of ultraviolet divergences appearing in the perturbation theory. For example, by taking the $k \gg p, m$ it can be seen that the integrals in Eq. (1.5) are divergent in the UV region

$$-\int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2} \frac{i}{(p-k)^2-m^2} \xrightarrow{k\gg p,m} \int \frac{d^4k}{k^4} \sim i\int_0^\infty \frac{dk}{k}\,, \tag{1.9}$$

which gives a divergent logarithm upon completing the integration and the factor of i is from the Wick rotation. To better control the divergences it is needed to regulate the integrals, *i.e.* express the divergences in a consistent manner. For clarity and simplicity, we limit the integral bounds to go only up to the cut-off Λ, this makes the integral finite, while the divergences can be retrieved by taking $\Lambda \to \infty$. With the cut-off the integrals become

$$-\int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2} \frac{i}{(p-k)^2-m^2} = a\log\frac{\Lambda^2}{p^2} + b(p^2)\,, \tag{1.10}$$

where a, b are finite factors, but b may also depend on p^2. Plugging this into Eq. (1.5) and choosing a specific scale p_0^2 we get

$$\begin{aligned} -i\lambda_P(p_0^2) = &-i\lambda + \frac{i}{2}\lambda^2 a\log\frac{\Lambda^2}{p_0^2} + \frac{i}{2}\lambda^2 b(p_0^2) \\ &+ (p_0 \to q_0, p_0 \to l_0)\,. \end{aligned} \tag{1.11}$$

Here all three momenta combinations corresponding to the usual Mandelstam variables are manifest, however, the momenta of incoming (outgoing) particles still sum up to p_0^2 in all diagrams/integrals.

For simplicity let us keep only tree-level and p_0^2 terms. Then we proceed with renormalization by inserting Eq. (1.7) into Eq. (1.11) at second order in λ_P

$$\begin{aligned} -i\lambda_P(p_0^2) = &- i\left(\lambda_P(p_0^2)B_1(p_0^2) + \left[\lambda_P(p_0^2)\right]^2 B_2(p_0^2)\right) \\ &+ \frac{i}{2}\left[\lambda_P(p_0^2)B_1(p_0^2)\right]^2 a\log\frac{\Lambda^2}{p_0^2} + \frac{i}{2}\left[\lambda_P(p_0^2)B_1(p_0^2)\right]^2 b(p_0^2)\,. \end{aligned} \tag{1.12}$$

By collecting the powers of λ_P it is then not hard to determine that $B_1(p_0^2) = 1$ and

$$B_2(p_0^2) = \frac{1}{2}a\log\frac{\Lambda^2}{p_0^2} + \frac{1}{2}b(p_0^2)\,, \tag{1.13}$$

where analogous terms with q_0, l_0 should be included for completeness, but these do not influence the discussion.

Having the coefficients B_i or, equivalently, the expansion of the bare parameters λ in terms of the physical one λ_P we may make a prediction for λ_P at some scale p_1^2. Inserting the expansion in Eq. (1.8) (or Eq. (1.11) with $p_0 \to p_1$) up to second order in λ_P we get

$$\begin{aligned} -i\lambda_P(p_1^2) = &- i\lambda_P(p_0^2) + \frac{i}{2}\left[\lambda_P(p_0^2)\right]^2 a\log\frac{p_1^2}{p_0^2} \\ &+ \frac{i}{2}\left[\lambda_P(p_0^2)\right]^2\left(b(p_1^2) - b(p_0^2)\right)\,. \end{aligned} \tag{1.14}$$

As expected, here on the r.h.s. every quantity is either known(measured) or can be found by explicit computation, but the amazing achievement of the renormalization procedure is that the UV cut-off Λ canceled and can be freely taken to infinity without producing any UV divergences in the prediction for $\lambda_P(p_1^2)$! Even more so, one may encounter divergences that are not logarithmic, but renormalization still takes care of them, even to all orders in perturbation theory [73–75]. It is important to note that for *renormalizable* theories it is enough to perform a finite number of measurements, while the *non-renormalizable* theories require an infinite amount of measurements. Nonetheless, both types of theories have predictive power.

The fact that the UV divergences, coming from the high-momentum region, cancel out is rooted in the fact that different scales decouple. In other words, the precise implementation of the UV theory has little influence on low-energy physics. For example, after taking $k \gg p, m$ the integral in Eq. (1.9) depends only on the loop momentum k — there are no other scales present so the integral gives the same divergence irrespective of the low-energy scale at which the

experiment is performed. This allows for the cancellation of divergences in the renormalization procedure. In essence, we do not know if the physics at scales such as the Planck scale are even represented by QFT, but this is conveniently hidden under the regulator such as the UV cut-off Λ.

More precisely, the UV divergences do cancel out, but the cut-off can be present in effective (non-renormalizable) theories in the so-called irrelevant operators (of mass dimension > 4), which it suppresses. The higher the cut-off, the more suppressed the irrelevant operators are at low energies. This almost complete independence of UV physics at low energies is both a blessing and a curse. On the one hand, we can make low-energy theories and test them without knowing the precise microscopic theory, on the other hand, it is really hard to find out anything about the UV theories from low-energy experiments. There seem to be two ways to study UV physics — either increase the energy of the experiments or gather enough data to measure the small effects of the irrelevant operators. In any case, effective theories are not the focus of this thesis and we will not delve further into them.

Perhaps the last thing that can be said regarding the UV divergences is that since the coefficients B_i are divergent, the bare parameter λ must also be divergent — this is quite the threat to perturbation theory. By expressing the bare parameter in terms of the measured parameters one gets around the problem, since the series is then in terms of a UV-finite coupling λ_P and this justifies the perturbative expansion.

It is customary to make the renormalization procedure somewhat more organized and efficient. Taking the same quartic coupling as an example we write Eq. (1.7) as

$$\lambda = Z_\lambda \lambda_R = (1 + \delta Z_\lambda)\, \lambda_R = \lambda_R + \delta\lambda\,, \tag{1.15}$$

where λ is the bare parameter, Z_λ is the renormalization constant, λ_R is the renormalized coupling, δZ_λ and $\delta\lambda$ are the counterterms. The counterterms are taken to be polynomials in powers of the renormalized coupling and are easily related between themselves

$$\delta\lambda = \delta Z_\lambda \lambda_R\,, \tag{1.16}$$

the point being that the renormalization may be performed both multiplicatively and additively. However, note that $\delta\lambda$ starts at the second order in λ_R, while δZ_λ starts at first order.

It is important to note that we have used the renormalized coupling λ_R instead of the measured λ_P. Renormalization does have parallels with setting

the initial condition, however, the main focus is on the UV divergences, such that Eq. (1.15) is necessarily required to hold only for UV parts, but the fine parts can be distributed at will. The renormalized coupling is UV-finite, but it can be different from the measured one. More precisely,

$$\begin{aligned} \lambda =& Z_\lambda \lambda_R = Z'_\lambda \lambda_P \\ \to& \lambda_P - \lambda_R = \delta\lambda - \delta\lambda' \,, \end{aligned} \tag{1.17}$$

where $\delta\lambda$ and $\delta\lambda'$ must have identical UV parts. The freedom of choosing the finite parts gives rise to renormalization schemes such as Minimal Subtraction (MS), Modified Minimal Subtraction ($\overline{\text{MS}}$), On-Shell (OS), etc. Importantly, different schemes may be more or less convenient for particular tasks, but the physical result is necessarily scheme-independent, given that the procedure is carried out correctly. Different schemes only specify how the physical (scheme-independent) renormalization *conditions* are fulfilled.

For example, Eq. (1.11) is the renormalization condition at some scale p_0, but we may choose to fix the counterterm $\delta\lambda$ such that only the divergent term is canceled (this is the MS scheme)

$$\delta\lambda = \frac{1}{2}\left[\lambda_R(p_0^2)\right]^2 a \log\frac{\Lambda^2}{p_0^2} \,. \tag{1.18}$$

With the counterterm fixed in this way Eq. (1.11) further gives (with q, l terms again dropped for simplicity)

$$\lambda_P(p_0^2) - \lambda_R(p_0^2) = -\frac{1}{2}\left[\lambda_R(p_0^2)\right]^2 b(p_0^2) = -\frac{1}{2}\left[\lambda_P(p_0^2)\right]^2 b(p_0^2) \,, \tag{1.19}$$

where the final equality holds since the measured and renormalized couplings differ at least at second order. Alternatively, we may express the renormalized coupling in terms of the measured one.

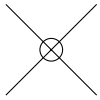
$$\lambda_R(p_0^2) = \lambda_P(p_0^2) + \frac{1}{2}\left[\lambda_P(p_0^2)\right]^2 b(p_0^2) \,. \tag{1.20}$$

This simply reiterates the point that the final predictions are scheme-independent — finite terms can be distributed between the renormalized coupling and the counterterm without changing physical results. However, as we will see in the course of this thesis, some ways of distributing are better than others.

Further, renormalizing the bare coupling conveniently rewrites the Lagrangian, for instance, the quartic term becomes

$$\mathcal{L}_4 = -\frac{\lambda}{4!}\phi^4 = -\frac{\lambda_R}{4!}\phi^4 - \frac{\delta\lambda}{4!}\phi^4 \,. \tag{1.21}$$

This automatically gives a Feynman rule for the counterterm



$$= -i\delta\lambda\,, \tag{1.22}$$

which can be used in computing amplitudes. In turn, at a given order it is enough to fix the counterterms once and then use them to make predictions.

So far we have not discussed the two-point functions, which are extremely important for the results of this thesis. However, the Lagrangian Eq. (1.1) is insufficient for the 1-loop discussion (example), since some UV divergences are seen only starting at two loops. Hence, we consider a very general and schematic case but still deal with real scalar fields.

The truth is that it is not enough to renormalize only the parameters in the Lagrangian, but it is also needed to renormalize the fields themselves. For instance, if one in general considers corrections to the 2-point function (inverse-propagator) $\Pi(p^2)$ one may get

$$\Pi(p^2) = p^2 - m^2 + \left(p^2 - m^2\right)\text{div.} + m^2\text{div.}' + \text{finite}\,, \tag{1.23}$$

where div. and div.$'$ stand for UV divergences and also include numerical factors and any other couplings. Importantly, the divergences are proportional to the mass as well as the momentum and in general are different in the two terms. By renormalizing the mass as $m^2 = m_R^2 + \delta m^2$ we get

$$\Pi(p^2) = p^2 - m_R^2 - \delta m^2 + \left(p^2 - m_R^2\right)\text{div.} + m_R^2\text{div.}' + \text{finite} \tag{1.24}$$

up to 1-loop order. We can choose some scale p_0^2 and fix the mass counterterm, however, it is easy to see that a prediction for Π at some other scale will remain UV-divergent, for example,

$$\Pi(p_1^2) - \Pi(p_0^2) = p_1^2 - p_0^2 + \left(p_1^2 - p_0^2\right)\text{div.} + \Delta\text{finite}\,. \tag{1.25}$$

Here Δfinite indicates the finite parts, which did not cancel because of the different momenta. Importantly, the different momenta also obstruct the cancellation of divergences. These momentum-proportional divergences can be remedied in two almost identical ways: by the field renormalization or by the Lehmann-Symanzik-Zimmermann (LSZ) factors [76]. Let us briefly discuss the former and leave the LSZ factors for Section 5.

The divergences proportional to momentum come from the derivative terms $\partial_\mu\phi\partial^\mu\phi$, but there are no parameters — one can only renormalize the fields

themselves. Field renormalization is done in complete analogy to the parameter case[1]

$$\phi = Z_\phi \phi_R = (1 + \delta Z_\phi)\, \phi_R. \tag{1.26}$$

With the field renormalization, the two-point function becomes

$$\begin{aligned} \Pi(p^2) = &\left(p^2 - m_R^2\right) + 2\delta Z_\phi \left(p^2 - m_R^2\right) + \left(p^2 - m_R^2\right) \text{div.} \\ &- \delta m^2 + m_R^2 \text{div.}' + \text{finite}\,. \end{aligned} \tag{1.27}$$

Then it is not hard to see that the divergences proportional to $p^2 - m_R^2$ can be easily canceled with the field counterterm δZ_ϕ at some scale p_0^2 such that the two-point function remains finite at some other scale p_1^2. Importantly, correctly fixing the field renormalization is essential since field renormalization constants enter every term in the Lagrangian — renormalization of the 2-point functions influences the renormalization of any other n-point function.

While we will discuss the 2-point functions in much more detail, this concludes the basic introduction to the renormalization procedure.

2. Renormalization of mixing angles

We have presented the basic ideas of renormalization, however, models with particle mixing pose a challenge. Next, we discuss this challenge in more detail and also propose a new renormalization scheme that is problem-free.

2.1. The usual approach

The idea behind particle mixing is rather simple: if one cannot simultaneously diagonalize all the interactions concerning some particle families, there is said to be particle mixing. For example, consider updating the ϕ^4 Lagrangian in Eq. (1.1) with fermions $\boldsymbol{\psi}$

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi_0 \partial^\mu \phi_0 - \frac{m_0^2}{2}\phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4 + \overline{\boldsymbol{\psi}}_0 \left(i\not{\partial} - \boldsymbol{M}_0\right) \boldsymbol{\psi}_0 - \overline{\boldsymbol{\psi}}_0 \boldsymbol{Y}_0 \boldsymbol{\psi}_0 \phi_0\,. \tag{2.1}$$

Here we have added "0"s to indicate bare quantities, $\not{\partial} = \gamma_\mu \partial^\mu$ with γ_μ being the Dirac gamma matrices, $\boldsymbol{\psi}_0$ is to be understood as a vector containing n fermion flavors

$$\boldsymbol{\psi}_0 = \begin{pmatrix} \psi_1^0 \\ \psi_2^0 \\ \vdots \\ \psi_n^0 \end{pmatrix}, \tag{2.2}$$

[1] It is conventional to write $\phi = \sqrt{Z_\phi}\phi_R$, however, this only amounts to rather annoying numerical factors and has no influence on the physics.

likewise, $\boldsymbol{M}_0$ is the bare mass matrix

$$\boldsymbol{M}_0 = \begin{pmatrix} M_{11}^0 & \dots & M_{1n}^0 \\ \vdots & \vdots & \vdots \\ M_{n1}^0 & \dots & M_{nn}^0 \end{pmatrix} \tag{2.3}$$

and $\boldsymbol{Y}_0$ is the bare Yukawa coupling with the same form as the mass matrix. Note that the fermion fields have a chiral structure, hence, the mass matrix and the Yukawa coupling can be composed of left- and right-handed parts to account for this structure in the fields, for example,

$$\boldsymbol{M}_0 = \boldsymbol{M}_{L\,0} P_L + \boldsymbol{M}_{R\,0} P_R \,, \tag{2.4}$$

where $P_L = \frac{1}{2}(1-\gamma^5)$ and $P_R = \frac{1}{2}(1+\gamma^5)$ are the usual chiral projectors. In the initial basis, neither $\boldsymbol{M}_0$ nor $\boldsymbol{Y}_0$ are diagonal. However, one may perform some (unitary) basis rotation $\boldsymbol{U}$ (also composed of left- and right-handed parts) on the fields

$$\boldsymbol{\psi}_0 = \boldsymbol{U}\boldsymbol{\psi}_0' \tag{2.5}$$

such that the rotation diagonalizes $\boldsymbol{M}_0$

$$\mathcal{L}_{\boldsymbol{\psi}} = \overline{\boldsymbol{\psi}}_0' \left(i\partial\!\!\!/ - \boldsymbol{M}_0'\right) \boldsymbol{\psi}_0' - \overline{\boldsymbol{\psi}}_0' \gamma^0 \boldsymbol{U}^\dagger \gamma^0 \boldsymbol{Y}_0 \boldsymbol{U} \boldsymbol{\psi}_0' \phi_0 \,. \tag{2.6}$$

Here we took just the fermionic terms and $\boldsymbol{M}_0' = \gamma^0 \boldsymbol{U}^\dagger \gamma^0 \boldsymbol{M}_0 \boldsymbol{U} = \text{diag.}\,(m_1', \dots, m_n')$. Conversely, the matrix $\gamma^0 \boldsymbol{U}^\dagger \gamma^0 \boldsymbol{Y}_0 \boldsymbol{U}$ is not necessarily diagonal, hence, even if the tree level 2-point function is diagonal in the flavor space, it will develop off-diagonal contributions through diagrams such as this

. (2.7)

Here the dashed line denotes the propagator of the scalar field ϕ_0, and solid lines with arrows denote fermion propagators with i, j, k being the family indices. Importantly, the incoming fermion flavor is not necessarily the same as that of the outgoing one, hence, the mixing. There is no mixing only if $\boldsymbol{M}_0$ and $\boldsymbol{Y}_0$ are simultaneously diagonalizable or, equivalently, if the two matrices commute as, for example, noted in [77]. Although this is an example of particle mixing and contains the important point of diagonalizing the mass matrix, it is not entirely relevant for the renormalization procedure since one can simply rename $\gamma^0 \boldsymbol{U}^\dagger \gamma^0 \boldsymbol{Y}_0 \boldsymbol{U}$ to $\boldsymbol{Y}_0'$, such that the rotation matrices disappear from the Lagrangian.

The more usual mixing case is when the basis rotations do not cancel out and cannot be absorbed into other parameters — one then has to think of

renormalizing these matrices. A good and physical example of this is the SM quark sector, where in the flavor basis one has

$$\begin{aligned}\mathcal{L}_q =&\overline{\boldsymbol{u}}_0 \left(i\partial\!\!\!/ - \boldsymbol{M}_{u\,0}\right) \boldsymbol{u}_0 + \overline{\boldsymbol{d}}_0 \left(i\partial\!\!\!/ - \boldsymbol{M}_{d\,0}\right) \boldsymbol{d}_0 \\ &- \left\{ \frac{g_0}{\sqrt{2}} \overline{\boldsymbol{u}}_0 W\!\!\!/_0 P_L \boldsymbol{d}_0 + H.C. \right\} .\end{aligned} \tag{2.8}$$

Here $\boldsymbol{u}_0$ and $\boldsymbol{d}_0$ are the vectors of bare up- and down-type bare quark fields, $\boldsymbol{M}_{u,d\,0}$ are the corresponding non-diagonal mass matrices, W_0 is the bare field of the W-boson, g_0 is the bare weak coupling constant, and $H.C.$ stands for hermitian conjugation. As is usual, we can now rotate into the mass eigenstate basis of up-type quarks with the matrix $\boldsymbol{U}$ and of down-type quarks with the matrix $\boldsymbol{D}$

$$\begin{aligned}\mathcal{L}_q =&\overline{\boldsymbol{u}}'_0 \left(i\partial\!\!\!/ - \boldsymbol{M}'_{u\,0}\right) \boldsymbol{u}'_0 + \overline{\boldsymbol{d}}'_0 \left(i\partial\!\!\!/ - \boldsymbol{M}'_{d\,0}\right) \boldsymbol{d}'_0 \\ &- \left\{ \frac{g_0}{\sqrt{2}} \overline{\boldsymbol{u}}'_0 W\!\!\!/_0 P_L \boldsymbol{U}^\dagger \boldsymbol{D} \boldsymbol{d}'_0 + H.C. \right\} .\end{aligned} \tag{2.9}$$

Importantly, even though the matrices $\boldsymbol{U}$ and $\boldsymbol{D}$ are unitary, the product $\boldsymbol{U}^\dagger \boldsymbol{D} \neq \mathbb{1}$. Since the weak interaction is chiral, only the left-handed part of $\boldsymbol{U}^\dagger \boldsymbol{D}$ contributes, which is the usual CKM matrix [21, 22] parameterizing the quark mixing

$$\boldsymbol{V}_0^{\text{CKM}} = \boldsymbol{U}_L^\dagger \boldsymbol{D}_L \,. \tag{2.10}$$

Unlike the previous example with Yukawa couplings, this mixing matrix cannot be absorbed into any other parameter and is usually renormalized, hence, the added "0" subscript to indicate this upcoming step.

Renormalization constants in the quark sector are introduced just like in Section 1

$$\boldsymbol{M}'_{u,d\,0} = \boldsymbol{M}_{u,d} + \delta \boldsymbol{M}_{u,d} \,, \tag{2.11}$$

$$\boldsymbol{u}'_0 = \boldsymbol{Z}_u \boldsymbol{u} \qquad \text{and} \qquad \boldsymbol{d}'_0 = \boldsymbol{Z}_d \boldsymbol{d} \,, \tag{2.12}$$

$$\boldsymbol{V}_0^{\text{CKM}} = \boldsymbol{V}^{\text{CKM}} + \delta \boldsymbol{V}^{\text{CKM}} \,, \tag{2.13}$$

where we do not bother with the explicit renormalization of g_0 and the W-boson since that does not influence the discussion. Most importantly, in the usual approach, the mass counterterms are considered to be *diagonal*, while the field counterterms are *non-diagonal* and account for the non-diagonal and UV-divergent loop corrections to the propagator. Fermion field counterterms also account for the chiral structure of the fields and have the corresponding left- and right-handed parts. With these constants, we may write the 1-loop

counterterm Lagrangian in the quark sector

$$\begin{aligned}\mathcal{L}_{q\text{ct.}} =& \overline{\boldsymbol{u}}\left(\delta\boldsymbol{Z}_u^\dagger\left(i\not{\partial}-\boldsymbol{M}_u\right)+\left(i\not{\partial}-\boldsymbol{M}_u\right)\delta\boldsymbol{Z}_u-\delta\boldsymbol{M}_u\right)\boldsymbol{u}\\ &+\overline{\boldsymbol{d}}\left(\delta\boldsymbol{Z}_d^\dagger\left(i\not{\partial}-\boldsymbol{M}_d\right)+\left(i\not{\partial}-\boldsymbol{M}_d\right)\delta\boldsymbol{Z}_d-\delta\boldsymbol{M}_d\right)\boldsymbol{d}\\ &-\left\{\frac{g}{\sqrt{2}}\overline{\boldsymbol{u}}\not{W}P_L\left(\delta\boldsymbol{V}^{\text{CKM}}+\delta\boldsymbol{Z}_u^\dagger\boldsymbol{V}^{\text{CKM}}+\boldsymbol{V}^{\text{CKM}}\delta\boldsymbol{Z}_d\right)\boldsymbol{d}+H.C.\right\}.\end{aligned} \tag{2.14}$$

The fermion field counterterms, as well as mass matrix counterterms, can be fixed from the two-point function. The weak coupling constant and the W-boson are not included here but can be renormalized from the SM gauge sector (*e.g.* [78]). Therefore, at this point, the only non-fixed counterterm is that of the CKM mixing matrix, which may be fixed from the W-boson decay amplitude.

In the initial approach to renormalize the CKM matrix [23], it has been found that in the $W\overline{u}d$ 1-loop amplitude all the divergences cancel except for those coming from the anti-hermitian part of the field renormalization

$$\langle W\overline{u}d\rangle\big|_{\text{UV}} = \delta\boldsymbol{V}^{\text{CKM}} - \delta\boldsymbol{Z}_u^A\boldsymbol{V}^{\text{CKM}} + \boldsymbol{V}^{\text{CKM}}\delta\boldsymbol{Z}_d^A\,\big|_{\text{UV}} \neq 0\,, \tag{2.15}$$

where $\delta\boldsymbol{Z}_{u,d}^A$ is the anti-hermitian part of the field renormalization obeying

$$\left(\delta\boldsymbol{Z}_{u,d}^A\right)^\dagger = -\delta\boldsymbol{Z}_{u,d}^A\,. \tag{2.16}$$

In turn, the only counterterm that can be used to cancel the left-over UV divergences is the CKM counterterm, so that a definition follows naturally

$$\delta\boldsymbol{V}^{\text{CKM}} = \delta\boldsymbol{Z}_u^A\boldsymbol{V}^{\text{CKM}} - \boldsymbol{V}^{\text{CKM}}\delta\boldsymbol{Z}_d^A\,. \tag{2.17}$$

Also, note that this definition obeys the unitarity properties of the CKM counterterm

$$\delta\left(\boldsymbol{V}^{\text{CKM}\,\dagger}\boldsymbol{V}^{\text{CKM}}\right) = \mathbb{0} \quad\Longrightarrow\quad \delta\boldsymbol{V}^{\text{CKM}\,\dagger}\boldsymbol{V}^{\text{CKM}} = -\boldsymbol{V}^{\text{CKM}\,\dagger}\delta\boldsymbol{V}^{\text{CKM}}\,, \tag{2.18}$$

such that, for example, the hermitian part of the field renormalization could not be accounted for by the CKM counterterm without violating unitarity.

At this point, the CKM matrix counterterm in Eq. (2.17) seems natural, however, there are quite a few problems with this counterterm. For example, in the On-Shell scheme, this makes the CKM counterterm gauge-dependent [25], which also makes the $W\overline{u}d$ amplitude gauge-dependent even though the process is physical [24, 79, 80]. This and other problems, which we discuss in the

following section, sparked a series of papers attempting to arrive at a suitable counterterm [24, 25, 27, 29, 31, 81–83].

The takeaway from this section is that the "usual approach" consists of a few key steps

i. One goes from the initial (flavor) basis into the mass eigenstate basis in the bare Lagrangian.

ii. The Lagrangian is then renormalized such that the mass counterterms are diagonal and the field counterterms are not.

iii. The mixing matrix counterterms are defined to cancel the UV divergences appearing in amplitudes such as $W\overline{u}d$.

Unfortunately, the mixing matrix counterterm is then expressed in terms of the field renormalization and is gauge-dependent, this alone already calls for a different definition.

2.2. Requirements for mixing renormalization

Gauge independence is not the only requirement or, at the very least, wanted feature for mixing renormalization. Over a span of roughly 20 years there accumulated a list of 5 requirements for mixing renormalization, which is not exclusive to the CKM matrix [27, 36, 43]:

1. The mixing counterterm should cancel the UV divergences that are left after the renormalization of other quantities such as masses, couplings, and fields.

2. The mixing matrix counterterm should be gauge-independent.

3. Renormalization should preserve the basic structure of the theory, in particular, the mixing matrix counterterm should not spoil unitarity (orthogonality).

4. The mixing renormalization should be symmetric w.r.t. the mixing degrees of freedom and also independent of a specific physical process.

5. The mixing angle renormalization should not spoil the numerical stability of the perturbative expansion.

   (a) In the limit of degenerate masses of the mixing particles, no singularities should be introduced into physical observables

(b) There should be no "dead corners" in the parameter space, where a renormalized input parameter goes to infinity.

Let us consider this list in more detail. The very first point is almost obvious — UV-finiteness is one of the main focuses of the renormalization procedure and renormalization of the mixing is not an exception. However, the important point is that the mixing renormalization should take care of the "leftover" divergences after the renormalization of other quantities. In particular, once we discuss our scheme in Section 6, we will see that it is possible to define the counterterms in a way that leaves no such UV divergences to be renormalized by the mixing.

The second point we already mentioned right before the start of this section. It has been found that in the OS scheme, the $W\overline{u}d$ amplitude is UV-divergent, but gauge independent if the CKM counterterm is not included [24, 83, 84]. In turn, if the definition of the mixing matrix counterterm is gauge-dependent it will necessarily make such amplitudes gauge-dependent as well. Alternatively, the mixing matrix parameterizes a physical process so it is usually considered to be physical quantity. Gauge dependence is not a welcome feature in counterterms of physical quantities.

The third point is already evident in Eq. (2.18), where we require the mixing matrix counterterm to not violate the unitarity of the bare mixing matrix. This is required by many considerations as noted in [27]: Becchi-Rouet-Stora-Tyutin (BRST) [85, 86] symmetry and Ward-Takahashi identities [87, 88] would be broken, the unitarity triangle would lose its meaning beyond tree-level for a unitarity non-preserving mixing matrix counterterm.

The fourth point is both conceptual and practical. Mixing matrices on their own do not prefer a certain flavor, so it is preferable for the counterterms to also be flavor-symmetric. This is often spoiled in schemes, where the mixing matrix counterterms are based on a specific physical process. On the practical side, such non-universal counterterms can be hard to compute at, and especially beyond, 1-loop. It may also be cumbersome to apply a scheme from one model to another since the particle content and processes can differ.

Finally, the fifth point considers the numerical stability, which is important both for keeping the perturbative expansion meaningful and for practical considerations. There can be a few sources for this instability. For example, non-universal process-dependent schemes can bring unnatural numerical corrections(*e.g.* [41]), although, this is not necessarily the case (*e.g.* [36]). Another extremely common source of numerical instability is the factor $1/(m_i^2 - m_j^2)$, which appears in the mixing renormalization and is singular in the limit, when both masses m_i and m_j become equal, *i.e.* in the $m_j \to m_i$ limit. Naturally,

parameter scans in the vicinity of this singularity are cumbersome.

In the following section, we show that points 2, 3, and 5 can be trivially satisfied by simply setting the mixing matrix counterterm to 0, while points 1 and 4 require the discussion of a specific renormalization scheme, which we do in Section 6.

3. Triviality of the mixing matrix counterterm

This section is based on our publication [A1], where we discuss the interplay between basis rotations and the renormalization procedure and make the case for trivial mixing matrix counterterms. The discussion there, as well as here, is based on considering two approaches present in the literature: the basis-independent one and the basis-dependent one. The basis-independent approach seems to be sought-after (mostly) at tree-level, where the idea is to form basis-invariants that are to be used in expressing the observables [89–97]. Interestingly, it is also stated that the mixing angles are basis-dependent quantities and cannot be physically meaningful (except if a basis is preferred by some additional requirements) [94, 95, 98]. On the other hand, renormalization (attribution of a counterterm) of these non-physical quantities seems to be the standard basis-dependent approach. For example, the list of requirements for mixing renormalization we listed was also to be understood in the context of non-trivial mixing matrix counterterms. Below we give arguments for choosing the basis-independent approach beyond tree-level both on conceptual and practical grounds.

3.1. Basis rotations and renormalization

In our work [A1] we have considered a system of scalars, however, here let us consider fermions both to show the universality of the discussion and to make a better connection with the renormalization scheme for fermions we present in Section 6. To do so, let us consider just the kinetic term from Eq. 2.1

$$\mathcal{K} = \overline{\psi}_0 \left(i\partial\!\!\!/ - \boldsymbol{M}_0\right) \psi_0 \,. \tag{3.1}$$

Let us further consider an arbitrary unitary basis rotation $\boldsymbol{\mathcal{U}}_0$ that rotates the fields ψ_0 into the fields $\boldsymbol{\chi}_0$ and does *not* diagonalize the mass matrix

$$\psi_0 = \boldsymbol{\mathcal{U}}_0 \boldsymbol{\chi}_0 \,. \tag{3.2}$$

With this rotation may write the kinetic term in a few different ways

$$\mathcal{K} = \overline{\psi}_0 \left(i\partial\!\!\!/ - \boldsymbol{M}_0\right) \psi_0 \tag{3.3a}$$

$$= \overline{\boldsymbol{\chi}}_0 \left(i\not{\partial} - \gamma^0 \boldsymbol{\mathcal{U}}_0^\dagger \gamma^0 \boldsymbol{M}_0 \boldsymbol{\mathcal{U}}_0 \right) \boldsymbol{\chi}_0 \tag{3.3b}$$

$$= \overline{\boldsymbol{\chi}}_0 \left(i\not{\partial} - \widetilde{\boldsymbol{M}}_0 \right) \boldsymbol{\chi}_0 \,. \tag{3.3c}$$

Here we have used the unitarity of $\boldsymbol{\mathcal{U}}_0$ and defined

$$\widetilde{\boldsymbol{M}}_0 = \gamma^0 \boldsymbol{\mathcal{U}}_0^\dagger \gamma^0 \boldsymbol{M}_0 \boldsymbol{\mathcal{U}}_0 \,. \tag{3.4}$$

Next we perform the renormalization procedure for Eqs. (3.3a) and (3.3c)

$$\begin{aligned} \mathcal{K} = & \overline{\boldsymbol{\psi}} \{ i\not{\partial} - \boldsymbol{M} \\ & + \gamma^0 \delta \boldsymbol{Z}_\psi^\dagger \gamma^0 \left(i\not{\partial} - \boldsymbol{M} \right) + \left(i\not{\partial} - \boldsymbol{M} \right) \boldsymbol{Z}_\psi - \delta \boldsymbol{M} \Big\} \boldsymbol{\psi} \,, \end{aligned} \tag{3.5a}$$

$$\begin{aligned} \mathcal{K} = & \overline{\boldsymbol{\chi}} \Big\{ i\not{\partial} - \widetilde{\boldsymbol{M}} \\ & + \gamma^0 \delta \boldsymbol{Z}_\chi^\dagger \gamma^0 \left(i\not{\partial} - \widetilde{\boldsymbol{M}} \right) + \left(i\not{\partial} - \widetilde{\boldsymbol{M}} \right) \boldsymbol{Z}_\chi - \delta \widetilde{\boldsymbol{M}} \Big\} \boldsymbol{\chi} \,. \end{aligned} \tag{3.5b}$$

Here the counterterms follow the same logic as in Section 1 and we have kept only terms that are linear in the counterterms, however, neither $\boldsymbol{M}_0$ nor $\widetilde{\boldsymbol{M}}_0$ are diagonal, their respective counterterms $\delta \boldsymbol{M}$ and $\delta \widetilde{\boldsymbol{M}}$ are also non-diagonal.

Analogously, we may renormalize the kinetic term in Eq. (3.3b) which includes the rotation matrix $\mathcal{U}$

$$\begin{aligned} \mathcal{K} = & \overline{\boldsymbol{\chi}} \Big\{ i\not{\partial} - \widetilde{\boldsymbol{M}} + \gamma^0 \delta \boldsymbol{Z}_\chi^\dagger \gamma^0 \left(i\not{\partial} - \widetilde{\boldsymbol{M}} \right) + \left(i\not{\partial} - \widetilde{\boldsymbol{M}} \right) \delta \boldsymbol{Z}_\chi \\ & - \gamma^0 \delta \boldsymbol{\mathcal{U}}^\dagger \boldsymbol{\mathcal{U}} \gamma^0 \widetilde{\boldsymbol{M}} - \delta \widetilde{\boldsymbol{M}} \boldsymbol{\mathcal{U}}^\dagger \delta \boldsymbol{\mathcal{U}} - \gamma^0 \boldsymbol{\mathcal{U}}^\dagger \gamma^0 \delta \boldsymbol{M} \boldsymbol{\mathcal{U}} \Big\} \boldsymbol{\chi} \,, \end{aligned} \tag{3.6}$$

where we have

$$\widetilde{\boldsymbol{M}} = \gamma^0 \boldsymbol{\mathcal{U}}^\dagger \gamma^0 \boldsymbol{M} \boldsymbol{\mathcal{U}} \,. \tag{3.7}$$

Next, we split the field renormalization into its hermitian and anti-hermitian parts

$$\delta \boldsymbol{Z}_\chi = \delta \boldsymbol{Z}_\chi^H + \delta \boldsymbol{Z}_\chi^A \,, \tag{3.8}$$

with

$$\left(\delta \boldsymbol{Z}_\chi^H \right)^\dagger = \delta \boldsymbol{Z}_\chi^H \qquad \text{and} \qquad \left(\delta \boldsymbol{Z}_\chi^A \right)^\dagger = -\delta \boldsymbol{Z}_\chi^A \,. \tag{3.9}$$

By using this and the unitarity of the matrix $\boldsymbol{\mathcal{U}}$ (analogous to Eq. (2.18) but with Dirac structure present)

$$\delta \left(\boldsymbol{\mathcal{U}}^\dagger \boldsymbol{\mathcal{U}} \right) = \mathbb{0} \quad \Rightarrow \quad \delta \boldsymbol{\mathcal{U}}^\dagger \boldsymbol{\mathcal{U}} = -\boldsymbol{\mathcal{U}}^\dagger \delta \boldsymbol{\mathcal{U}} \tag{3.10}$$

we may rewrite the kinetic term as follows

$$\begin{aligned} \mathcal{K} = & \overline{\boldsymbol{\chi}} \Big\{ i\not{\partial} - \widetilde{\boldsymbol{M}} + \gamma^0 \delta \boldsymbol{Z}_\chi^H \gamma^0 \left(i\not{\partial} - \widetilde{\boldsymbol{M}} \right) + \left(i\not{\partial} - \widetilde{\boldsymbol{M}} \right) \delta \boldsymbol{Z}_\chi^H \\ & - \gamma^0 \left[\gamma^0 \widetilde{\boldsymbol{M}}, \boldsymbol{\mathcal{U}}^\dagger \delta \boldsymbol{\mathcal{U}} + \delta \boldsymbol{Z}_\chi^A \right] - \gamma^0 \boldsymbol{\mathcal{U}}^\dagger \gamma^0 \delta \boldsymbol{M} \boldsymbol{\mathcal{U}} \Big\} \boldsymbol{\chi} \,, \end{aligned} \tag{3.11}$$

where $[\ldots,\ldots]$ is the commutator. Comparing either this form of the kinetic term or the one in Eq. (3.6) with Eq. (3.5b) it is evident that even though all equations are in the $\boldsymbol{\chi}$ basis, the counterterms are different — there seems to be a difference depending on when the renormalization procedure is performed. Another thing that we see in Eq. (3.11) is that the anti-hermitian part of the field renormalization is degenerate with the mixing angle counterterms as they both enter the commutator. — this is a generalized (and fermionic) version of the comment made in [42]. This degeneracy implies that the mixing in the 2-point function may be renormalized through the field renormalization and mass counterterms by simply setting $\delta\boldsymbol{\mathcal{U}} = 0$. For example, this is what enables the scheme for squarks in [39]. However, we intend to make the statement stronger — the mixing angle counterterms *should* always be set to 0, instead of being an alternative choice. We do so by comparing the kinetic term in Eqs. (3.5a), (3.6), and (3.5b) and discussing basis independence, gauge dependence and the degenerate mass limit.

3.2. Basis independence

As we have mentioned, basis-independent features are very welcome at tree-level and we believe they should be present in the renormalization procedure as well. For example, in Eqs. (3.5a) and (3.5b) the form of the kinetic term is the same, although the bases are different. On the other hand, the form of the kinetic term in Eq. (3.6) does not match either of the bases in Eq. (3.5). Of course, this is due to the presence of the mixing matrix counterterm $\delta\boldsymbol{\mathcal{U}}$.

It is simple to see that by identifying the bare mixing matrix with the renormalized one, $\boldsymbol{\mathcal{U}}_0 = \boldsymbol{\mathcal{U}} \Leftrightarrow \delta\boldsymbol{\mathcal{U}} = 0$, or by absorbing the mixing matrix counterterm into the anti-hermitian part of the field renormalization, the forms of the kinetic term in Eqs. (3.6) and (3.5) matches. Then a relation between mass counterterms easily follows from equating the two forms of the χ basis

$$\delta\widetilde{\boldsymbol{M}} = \gamma^0\boldsymbol{\mathcal{U}}^\dagger\gamma^0\delta\boldsymbol{M}\boldsymbol{\mathcal{U}}\,. \tag{3.12}$$

Further, if

$$\delta\boldsymbol{Z}_\chi = \boldsymbol{\mathcal{U}}^\dagger\delta\boldsymbol{Z}_\psi\boldsymbol{\mathcal{U}} \tag{3.13}$$

and

$$\boldsymbol{\chi} = \boldsymbol{\mathcal{U}}\boldsymbol{\psi}\,, \tag{3.14}$$

then Eqs. (3.5a) and (3.5b) correspond to the same bare kinetic term. While these relations are easy to find once the bare mixing matrix is identified with the renormalized one, this does not fully convey the necessary point and a more detailed analysis is needed.

Let us consider the case with $\delta\boldsymbol{\mathcal{U}} \neq 0$. Even though a specific basis may be chosen to fix the counterterms, it should still be possible to perform basis rotations. Let us rotate the fields in Eq. (3.11) by $\boldsymbol{\mathcal{U}}^\dagger$ (*i.e.* the inverse of $\boldsymbol{\mathcal{U}}$)

$$\begin{aligned}\mathcal{K} =&\overline{\boldsymbol{\chi}'}\left\{i\partial\!\!\!/ - \boldsymbol{M} + \gamma^0\delta\boldsymbol{Z}_\psi^H\gamma^0\left(i\partial\!\!\!/ - \boldsymbol{M}\right) + \left(i\partial\!\!\!/ - \boldsymbol{M}\right)\delta\boldsymbol{Z}_\psi^H\right.\\ &\left.-\gamma^0\left[\gamma^0\boldsymbol{M}, \delta\boldsymbol{\mathcal{U}}\boldsymbol{\mathcal{U}}^\dagger + \boldsymbol{\mathcal{U}}\delta\boldsymbol{Z}_\chi^A\boldsymbol{\mathcal{U}}^\dagger\right] - \delta\boldsymbol{M}\right\}\boldsymbol{\chi}'\,.\end{aligned} \tag{3.15}$$

Here $\boldsymbol{\chi}' = \boldsymbol{\mathcal{U}}^\dagger\boldsymbol{\chi}$ [2], we have used Eq. (3.12) for the mass counterterm and Eq. (3.13) for the hermitian part of the field renormalization, since only the anti-hermitian part is degenerate with the mixing matrix counterterm. All the renormalized quantities in the χ' basis are identical to those of the ψ basis, but the counterterms differ due to the presence of $\delta\boldsymbol{\mathcal{U}}$. In turn, one computes identical amplitudes in both bases, but they are renormalized with different sets of counterterms.

One simple problem caused by the mixing matrix counterterm is the change of the form of basis transformations, *i.e.* Eq. (3.13) no longer holds for the anti-hermitian part of the field renormalization. Instead, the anti-hermitian parts in the χ' and ψ bases are related by

$$\delta\boldsymbol{Z}_\psi^A \overset{!}{=} \delta\boldsymbol{\mathcal{U}}\boldsymbol{\mathcal{U}}^\dagger + \boldsymbol{\mathcal{U}}\delta\boldsymbol{Z}_\chi^A\boldsymbol{\mathcal{U}}^\dagger\,. \tag{3.16}$$

Obviously, one gets back the relation in Eq. (3.13) by setting $\delta\boldsymbol{\mathcal{U}}$ to 0.

A stronger problem following from the non-triviality of the mixing matrix counterterm is that in the χ' basis the counterterm $\delta\boldsymbol{\mathcal{U}}$ does not have an associated parameter. In turn, this obstructs the re-construction of the bare Lagrangian and we have

$$\mathcal{K}' = \overline{\boldsymbol{\chi}'}_0\left\{i\partial\!\!\!/ - \boldsymbol{M}_0 - \gamma^0\left[\gamma^0\boldsymbol{M}, \delta\boldsymbol{\mathcal{U}}\boldsymbol{\mathcal{U}}^\dagger + \boldsymbol{\mathcal{U}}\delta\boldsymbol{Z}_\chi^A\boldsymbol{\mathcal{U}}^\dagger - \delta\boldsymbol{Z}_\psi^A\right]\right\}\boldsymbol{\chi}'_0 \neq \mathcal{K}\,. \tag{3.17}$$

Here we have "un-renormalized" the fields by inverting $\boldsymbol{\chi}'_0 = \delta\boldsymbol{Z}_\psi\chi'$, so that we get back the Lagrangian in the ψ_0 basis up to the commutator term. Of course, the commutator term can be made to vanish if Eq. (3.16) holds, but a much more simple approach is to set $\delta\boldsymbol{\mathcal{U}}$ to 0. In that case, both the form of the kinetic term and the form of the basis transformations in Eq. (3.13) are preserved.

Perhaps an even stronger statement is that in Eq. (3.17) we have $\mathcal{K}' \neq \mathcal{K}$. Let us consider the *path* which leads to the mismatch of bare Lagrangians:

1. We start with the kinetic term in Eq. (3.3a) and rotate it by $\boldsymbol{\mathcal{U}}_0$ to Eq. (3.3b).

[2] One has $\boldsymbol{\chi}' = \psi$ for $\delta\boldsymbol{\mathcal{U}} = 0$.

2. We renormalize the theory in the χ_0 basis and include the counterterms for the mixing (basis transformation) matrix $\boldsymbol{\mathcal{U}}$.

3. We *rotate back* the renormalized fields by $\boldsymbol{\mathcal{U}}^\dagger$.

4. We "un-renormalize" the fields and arrive at $\mathcal{K}'$,

Alternatively, a different path may be taken:

1. We start with the kinetic term in Eq. (3.3a) and immediately renormalize the theory to get Eq. (3.5a).

2. We rotate the theory by $\boldsymbol{\mathcal{U}}$ and by using Eqs. (3.12) and (3.13) we arrive at Eq. (3.6).

3. We "un-renormalize" the theory and easily arrive at Eq. (3.5b)

4. A final rotation by $\boldsymbol{\mathcal{U}}_0^\dagger$ brings one back to the kinetic term $\mathcal{K}$.

The two paths differ in whether the renormalization is performed *before* or *after* basis rotations. It seems that renormalization after basis rotations (1st path) makes any further basis rotations impossible. More precisely, it is the attribution of counterterms to basis-dependent quantities such as rotation (mixing) matrices that selects a preferred basis and obstructs any further basis rotations. On the other hand, this is a rather unwelcome feature, since there is nothing special about basis rotations or the renormalization procedure and it should always be possible to return to the same bare and theory-defining Lagrangian irrespective of the chosen path. In other words, *basis rotations should commute with the renormalization procedure* [A1–A3]. This is rather simply achievable if the bare rotation matrix is identified with the renormalized one, *i.e.* $\boldsymbol{\mathcal{U}}_0 = \boldsymbol{\mathcal{U}}$ and $\delta\boldsymbol{\mathcal{U}} = 0$.

Probably the main outcome of identifying the bare mixing matrix with the renormalized one allows to freely change the basis irrespective of whether the renormalization procedure is already performed or not. This may be formulated as having a basis-independent *set* of counterterms:

$$\{\delta\boldsymbol{Z}_\psi, \delta\boldsymbol{M}, \delta\lambda\} \xrightarrow{\text{basis change}} \left\{\delta\boldsymbol{Z}_\chi, \delta\widetilde{\boldsymbol{M}}, \delta\widetilde{\lambda}\right\} \tag{3.18}$$

but not

$$\{\delta\boldsymbol{Z}_\psi, \delta\boldsymbol{M}, \delta\lambda\} \xrightarrow{\text{basis change}} \left\{\delta\boldsymbol{Z}_\chi, \delta\boldsymbol{\mathcal{U}}, \delta\widetilde{\boldsymbol{M}}, \delta\widetilde{\lambda}\right\}\,, \tag{3.19}$$

where $\delta\lambda$ and $\delta\widetilde{\lambda}$ stand for the counterterms of other parameters in the theory in the $\boldsymbol{\psi}$ and $\boldsymbol{\chi}$ bases, respectively.

A more philosophical or foundational formulation is also possible. The renormalization procedure is concerned with implementing a selected reference measurement into the theory to make it predictive. The standard device for doing so is the counterterms. As the observables are basis-independent it seems convenient for the renormalization procedure to also replicate this feature. In turn, having a set of basis-independent counterterms or, equivalently, trivial mixing matrix counterterms, $\delta\boldsymbol{\mathcal{U}} = 0$, is a welcome feature and a step towards basis independence. A possible counter-argument is that mixing matrices such as CKM are being measured and, hence, are physical quantities. However, the mixing itself is physical, but mixing matrices such as CKM are only a convenient parameterization. For example, a mixing matrix such as the CKM is derived by diagonalizing the mass matrix and can be expressed in terms of the elements of the "renormalized" mass matrix. It is the renormalization of these mass matrices that provides the basis-independent set of counterterms, but not the attribution of counterterms to mixing matrices. Even more so, one may work in completely arbitrary bases, *i.e.* one can select *numbers* for the matrix $\boldsymbol{\mathcal{U}}$ as seen fit and only a very specific selection of these numbers diagonalizes the mass matrix. This shows that mixing matrices are not theory-defining parameters and, hence, should not be renormalized.

While the above is important for conceptual consistency, it is also important to discuss the practical benefits of trivial mixing matrix counterterms. For example, from this conceptual discussion it already follows that the second mixing renormalization requirement from Section 2.2 is trivially satisfied. Both the bare and the renormalized mixing matrices have the same unitarity properties as they are identified and the counterterm vanishes.

Next, we turn to discuss the two other remaining requirements of gauge dependence and numerical stability.

3.3. Gauge dependence

To consider the gauge dependence let us keep considering the $\delta\boldsymbol{\mathcal{U}} \neq 0$ case. While gauge independence of the mixing matrix counterterm is required, achieving it is rather difficult. Conceptually the mixing matrices are non-physical and are degenerate with the field renormalization, Eq. (3.11). In turn, it is natural that these counterterms turn out to be gauge-dependent — a clear sign of non-physicality. On the practical side, we may use the Nielsen identities [79, 99], which are a type of Slavnov-Taylor identities [100, 101], to make the statement more precise.

Let us use the $\boldsymbol{\chi}$ basis (*e.g.*, Eq. 3.3b) and consider the gauge-dependence

of the bare fermion self-energy $\boldsymbol{\Sigma}^0(\not{p})$ at 1-loop in momentum space. The gauge-derivative of $\boldsymbol{\Sigma}^0(\not{p})$ w.r.t. the gauge parameter ξ is [79]

$$\partial_\xi \boldsymbol{\Sigma}^0(\not{p}) = \gamma^0 \left(\boldsymbol{\Lambda}(\not{p})\right)^\dagger \gamma^0 \left(\not{p} - \widetilde{\boldsymbol{M}}\right) + \left(\not{p} - \widetilde{\boldsymbol{M}}\right) \boldsymbol{\Lambda}(\not{p}) \,. \tag{3.20}$$

Here $\boldsymbol{\Lambda}$ is a correlation function involving BRST sources, moreover, it has Dirac structure and is a matrix in flavor space. Here we also imposed pseudo-hermiticity for simplicity, which is evident from the first term[3]. In addition, in the above expression, we have the renormalized masses instead of the bare ones since $\boldsymbol{\Lambda}$'s begin at 1-loop order and the mass counterterms contribute to higher orders. We may split $\boldsymbol{\Lambda}(\not{p})$ into its hermitian and anti-hermitian parts as we did for the field renormalization, such that the derivative of the self-energy becomes

$$\begin{aligned} \partial_\xi \boldsymbol{\Sigma}^0(\not{p}) =& \gamma^0 \left(\boldsymbol{\Lambda}(\not{p})\right)^H \gamma^0 \left(\not{p} - \widetilde{\boldsymbol{M}}\right) + \left(\not{p} - \widetilde{\boldsymbol{M}}\right) \left(\boldsymbol{\Lambda}(\not{p})\right)^H \\ &+ \gamma^0 \left[\gamma^0 \left(\not{p} - \widetilde{\boldsymbol{M}}\right), \left(\boldsymbol{\Lambda}(\not{p})\right)^A\right] . \end{aligned} \tag{3.21}$$

Further, we also consider the renormalized self-energy, which is just the bare self-energy plus the counterterms from Eq. (3.11)

$$\begin{aligned} \boldsymbol{\Sigma}^R(\not{p}) =& \boldsymbol{\Sigma}^0(\not{p}) + \gamma^0 \delta \boldsymbol{Z}_\chi^H \gamma^0 \left(\not{p} - \widetilde{\boldsymbol{M}}\right) + \left(\not{p} - \widetilde{\boldsymbol{M}}\right) \delta \boldsymbol{Z}_\chi^H \\ &- \gamma^0 \left[\gamma^0 \widetilde{\boldsymbol{M}}, \boldsymbol{\mathcal{U}}^\dagger \delta \boldsymbol{\mathcal{U}} + \delta \boldsymbol{Z}_\chi^A\right] - \gamma^0 \boldsymbol{\mathcal{U}}^\dagger \gamma^0 \delta \boldsymbol{M} \boldsymbol{\mathcal{U}} \,, \end{aligned} \tag{3.22}$$

where the momentum $\not{p}$ is from the Fourier transform of the derivatives $i\not{\partial}$. It is now simple to take the gauge derivative of the renormalized self-energy with the help of Eq. (3.21)

$$\begin{aligned} \partial_\xi \boldsymbol{\Sigma}^R(\not{p}) =& \gamma^0 \left(\partial_\xi \delta \boldsymbol{Z}_\chi^H + \left(\boldsymbol{\Lambda}(\not{p})\right)^H\right) \gamma^0 \left(\not{p} - \widetilde{\boldsymbol{M}}\right) \\ &+ \left(\not{p} - \widetilde{\boldsymbol{M}}\right) \left(\partial_\xi \delta \boldsymbol{Z}_\chi^H + \left(\boldsymbol{\Lambda}(\not{p})\right)^H\right) \\ &+ \gamma^0 \left[\gamma^0 \left(\not{p} - \widetilde{\boldsymbol{M}}\right), \boldsymbol{\mathcal{U}}^\dagger \partial_\xi \delta \boldsymbol{\mathcal{U}} + \partial_\xi \delta \boldsymbol{Z}_\chi^A + \left(\boldsymbol{\Lambda}(\not{p})\right)^A\right] \\ &- \gamma^0 \boldsymbol{\mathcal{U}}^\dagger \gamma^0 \partial_\xi \delta \boldsymbol{M} \boldsymbol{\mathcal{U}} \,. \end{aligned} \tag{3.23}$$

Here we have assumed that only counterterms can be gauge-dependent. Also note that in the third line we have added a vanishing term

$$\gamma^0 \left[\gamma^0 \not{p}, \boldsymbol{\mathcal{U}}^\dagger \partial_\xi \delta \boldsymbol{\mathcal{U}} + \partial_\xi \delta \boldsymbol{Z}_\chi^A\right] = 0 \tag{3.24}$$

to keep the structure, since

$$\gamma^0 \left[\gamma^0 \not{p}, \boldsymbol{\mathcal{U}} \left(\boldsymbol{\Lambda}(\not{p})\right)^A\right] \neq= 0 \tag{3.25}$$

[3]Note that the hermitian conjugation on $\boldsymbol{\Lambda}(\not{p})$ is both in flavor and spinor space.

unlike in the scalar case in [A1].

In Eq. (3.23) one can notice that the field counterterms, as well as the mixing matrix counterterms, are in a commutator together with $\boldsymbol{\mathcal{U}}\left(\boldsymbol{\Lambda}\left(\not{p}\right)\right)^{A}$. In other words, the mixing matrix and the field counterterms are naturally associated with gauge-dependent structures. Such natural association is what makes it hard to find a gauge-independent definition of the mixing matrix counterterm. Of course, this is easy to overcome if one identifies the bare and renormalized mixing matrices, such that $\delta\boldsymbol{\mathcal{U}} = 0$. This identification then trivially satisfies the 2nd requirement for mixing renormalization from Section 2.2.

On the other hand, the mass counterterm is free of gauge-dependent structures as can be seen by the non-presence of $\boldsymbol{\Lambda}$ on the last line of Eq. (3.23). In turn, only non-physical renormalization conditions can induce gauge dependence in the mass counterterm.

3.4. The degenerate mass limit

Next, we discuss the 5th requirement from Section 2.2 or, at the very least, point 5a on the degenerate mass limit. For the discussion let us again consider the case with $\delta\boldsymbol{\mathcal{U}} \neq 0$ and for simplicity choose a basis where the mass is diagonal

$$\widetilde{\boldsymbol{M}} = \text{diag.}\left(m_1, m_2, \ldots, m_n\right) . \tag{3.26}$$

Strictly speaking, it is not necessary to choose a mass eigenstate basis, but then the discussion is more cumbersome, because of the need to invert the commutator terms in more complicated ways.

Let us take Eq. (3.22) in the diagonal mass basis

$$\begin{aligned}\Sigma^{R}_{ji}\left(\not{p}\right) =& \Sigma^{0}_{ji}\left(\not{p}\right) + \gamma^{0}\delta Z^{H}_{\chi\, ji}\gamma^{0}\left(\not{p} - m_i\right) + \left(\not{p} - m_j\right)\delta Z^{H}_{\chi\, ji} \\ & - \gamma^{0}\left[\gamma^{0}\widetilde{\boldsymbol{M}}, \boldsymbol{\mathcal{U}}^{\dagger}\delta\boldsymbol{\mathcal{U}} + \delta\boldsymbol{Z}^{A}_{\chi}\right]_{ji} - \left(\gamma^{0}\boldsymbol{\mathcal{U}}^{\dagger}\gamma^{0}\delta\boldsymbol{M}\boldsymbol{\mathcal{U}}\right)_{ji} .\end{aligned} \tag{3.27}$$

Here i, j are flavor indices, the bold notation was removed (where appropriate) to indicate elements in the flavor space and the mass counterterm $\gamma^{0}\boldsymbol{\mathcal{U}}^{\dagger}\gamma^{0}\delta\boldsymbol{M}\boldsymbol{\mathcal{U}}$ is in general non-diagonal even if $\widetilde{\boldsymbol{M}}$ is.

As the counterterms must cancel the UV divergences irrespective of the chosen renormalization scheme let us keep only UV-divergent terms in Eq. (3.27). In addition, let us drop all the UV-divergent terms proportional to $\not{p} - m_i$ or $\not{p} - m_j$, then the cancellation of UV divergences in the remaining parts of the renormalized self-energy is ensured by the following equality

$$\Sigma^{0}_{ji}\left(\not{p}\right)\big|_{\text{UV},\cancel{p = m_{i,j}}} = \gamma^{0}\left[\gamma^{0}\widetilde{\boldsymbol{M}}, \boldsymbol{\mathcal{U}}^{\dagger}\delta\boldsymbol{\mathcal{U}} + \delta\boldsymbol{Z}^{A}_{\chi}\right]_{ji} + \left(\gamma^{0}\boldsymbol{\mathcal{U}}^{\dagger}\gamma^{0}\delta\boldsymbol{M}\boldsymbol{\mathcal{U}}\right)_{ji} . \tag{3.28}$$

To make this more transparent, let us write the commutator term explicitly

$$\begin{aligned}\gamma^0\left[\gamma^0\widetilde{\boldsymbol{M}},\boldsymbol{\mathcal{U}}^\dagger\delta\boldsymbol{\mathcal{U}}+\delta\boldsymbol{Z}_\chi^A\right]_{ji}=&m_j\left(\boldsymbol{\mathcal{U}}^\dagger\delta\boldsymbol{\mathcal{U}}+\delta\boldsymbol{Z}_\chi^A\right)_{ji}\\&-\gamma^0\left(\boldsymbol{\mathcal{U}}^\dagger\delta\boldsymbol{\mathcal{U}}+\delta\boldsymbol{Z}_\chi^A\right)_{ji}\gamma^0 m_i\\=&P_L\left\{m_j\left(\boldsymbol{\mathcal{U}}_L^\dagger\delta\boldsymbol{\mathcal{U}}_L+\delta\boldsymbol{Z}_{\chi L}^A\right)_{ji}\right.\\&\left.-\left(\boldsymbol{\mathcal{U}}_R^\dagger\delta\boldsymbol{\mathcal{U}}_R+\delta\boldsymbol{Z}_{\chi R}^A\right)_{ji}m_i\right\}\\&+P_R\left\{m_j\left(\boldsymbol{\mathcal{U}}_R^\dagger\delta\boldsymbol{\mathcal{U}}_R+\delta\boldsymbol{Z}_{\chi R}^A\right)_{ji}\right.\\&\left.-\left(\boldsymbol{\mathcal{U}}_L^\dagger\delta\boldsymbol{\mathcal{U}}_L+\delta\boldsymbol{Z}_{\chi L}^A\right)_{ji}m_i\right\}.\end{aligned}\tag{3.29}$$

Here we have decomposed the commutator into its left- and right-handed parts. Inconveniently, the coefficients of projectors $P_{L,R}$ have contributions from both the left- and right-handed components of the field renormalization and mixing matrices so that it is needed to solve for these components. In contrast, the scalar case in [A1] is much simpler. In any case, for simplicity, we may decompose the bare self-energy in Eq. (3.28)

$$\Sigma_{ji}^0\left(\not{p}\right)\big|_{\mathrm{UV},\cancel{p=m_{i,j}}}=P_L a_{ji}+P_R b_{ji}\tag{3.30}$$

with some scalar and UV-divergent coefficients a_{ji} and b_{ji}. Then taking into account this decomposition as well as the explicit form of the commutator, we may write the following solutions

$$\begin{aligned}-\left(m_i^2-m_j^2\right)\left(\boldsymbol{\mathcal{U}}_L^\dagger\delta\boldsymbol{\mathcal{U}}_L+\delta\boldsymbol{Z}_{\chi L}^A\right)_{ji}=&m_j a_{ji}+m_i b_{ji}-m_j\left(\boldsymbol{\mathcal{U}}_R^\dagger\delta\boldsymbol{M}_L\boldsymbol{\mathcal{U}}_L\right)_{ji}\\&-m_i\left(\boldsymbol{\mathcal{U}}_L^\dagger\delta\boldsymbol{M}_R\boldsymbol{\mathcal{U}}_R\right)_{ji},\\-\left(m_i^2-m_j^2\right)\left(\boldsymbol{\mathcal{U}}_R^\dagger\delta\boldsymbol{\mathcal{U}}_R+\delta\boldsymbol{Z}_{\chi R}^A\right)_{ji}=&m_i a_{ji}+m_j b_{ji}-m_i\left(\boldsymbol{\mathcal{U}}_R^\dagger\delta\boldsymbol{M}_L\boldsymbol{\mathcal{U}}_L\right)_{ji}\\&-m_j\left(\boldsymbol{\mathcal{U}}_L^\dagger\delta\boldsymbol{M}_R\boldsymbol{\mathcal{U}}_R\right)_{ji}.\end{aligned}\tag{3.31}$$

Note that here the l.h.s. of both solutions for the field renormalization and the mixing matrix counterterms are proportional to $m_i^2-m_j^2$, while the r.h.s. is not. In turn, in the degenerate mass limit, $m_j\to m_i$, the l.h.s. vanishes, while the r.h.s. in general does not vanish. Even more so, many schemes set the rotated mass counterterms, $\boldsymbol{\mathcal{U}}_R^\dagger\delta\boldsymbol{M}_L\boldsymbol{\mathcal{U}}_L$ and $\boldsymbol{\mathcal{U}}_L^\dagger\delta\boldsymbol{M}_R\boldsymbol{\mathcal{U}}_R$, to be diagonal

(*e.g.* [23, 41, 78, 102, 103]), such that only the field renormalization and the mixing counterterms must be used to account for the UV divergences in a_{ji} and b_{ji}. This causes a problem since to cancel these divergences, the counterterms must be proportional to $1/\left(m_i^2 - m_j^2\right)$ and thus be singular in the degenerate mass limit! While this does not necessarily bring these singularities into the observables, the intermediate steps are guaranteed to have these singularities and may cause problems. The only terms that could be included in $\delta \boldsymbol{Z}^{\boldsymbol{A}}$ and/or $\boldsymbol{\mathcal{U}}^\dagger \delta \boldsymbol{\mathcal{U}}$ without causing practical problems are the gauge-independent terms that are also free of the $m_i^2 - m_j^2$ mass structure.

However, we believe that it is much simpler to set $\delta \boldsymbol{\mathcal{U}} = 0$ and keep the off-diagonal mass counterterms since these can cancel the UV divergences without being singular in the degenerate mass limit. We also think that this solves the problem described in [36], where schemes renormalizing $\tan\beta$ in a gauge-independent way in the Higgs sector of the Minimal Supersymmetric Standard Model (MSSM) were seen to cause numerical instabilities — this is avoided by not renormalizing mixing angles and keeping off-diagonal mass counterterms. In short, the triviality of the mixing matrix counterterm also allows one to achieve the 5th requirement from Section 2.2.

While this concludes the renormalization scheme independent discussion on the triviality of the mixing matrix counterterms, two notes are in order. First, for the final section, we only considered the UV-divergent parts, but a similar discussion can be carried out for the finite parts too. Second, all three sections (basis dependence, gauge dependence, and degenerate mass limits) were done at 1-loop, however, the structures we have seen repeat at every higher order in perturbation theory, for example, we will see that is the case for the gauge-dependent parts once we discuss the all-order extension of our scheme in Section 7. In turn, these arguments hold at all orders, even if presented just at 1-loop.

Having discussed the scheme-independent features of mixing renormalization we may proceed towards showing that points 1 and 4 from Section 2.2 are achievable while having trivial mixing matrix counterterms. In particular, we present the On-Shell renormalization conditions with a specific implementation of the OS conditions leading to a new renormalization scheme. Discussion of the new scheme, both at 1-loop and to all orders, is based on our publication [A3].

4. Setup of the notation

Let us begin by introducing the needed notation, which will be especially useful in discussing the arbitrary orders of our scheme and will make a better connection with the one-loop approach. Of most importance are the three types of self-energies: the bare self-energy computed with bare parameters $\widetilde{\Sigma}(\not{p})$, the bare self-energy computed with renormalized parameters $\Sigma(\not{p})$, and the renormalized self-energy $\Sigma^R(\not{p})$, all of which are understood to be matrices in the flavor space as we drop the bold notation. In terms of Feynman diagrams, the two bare self-energies are the same and only their parameters differ. Neither of the two bare self-energies contains counterterms explicitly. Counterterms explicitly appear only in the renormalized self-energy. All of these self-energies can be related.

The first relation comes from considering the 1-particle irreducible (1PI) generating functional. There the bare term containing the two-point function can be renormalized in analogy with Eq. (2.12)

$$\overline{\psi}_0\widetilde{\Sigma}(\not{p})\psi_0 = \overline{\psi}\gamma^0 Z^\dagger\gamma^0\widetilde{\Sigma}(\not{p})Z\psi = \overline{\psi}\Sigma^R(\not{p})\psi, \tag{4.1}$$

such that the following relation becomes evident

$$\Sigma^R(\not{p}) = \gamma^0 Z^\dagger\gamma^0\widetilde{\Sigma}(\not{p})Z\,. \tag{4.2}$$

The two bare self-energies, Σ and $\widetilde{\Sigma}$, may be related by expanding the parameters λ_k^0 around their renormalized values λ_k in $\widetilde{\Sigma}(\not{p})$. A convenient way to do so is by using the Taylor series operator

$$\mathbf{T}_{\lambda^0} = \sum_{n=0}^{\infty}\frac{(\delta\lambda)^n}{n!}\frac{\partial^n}{\partial(\lambda^0)^n} \tag{4.3}$$

with $(\delta\lambda)^n$ being the nth power of the counterterm. In addition, using that

$$\left.\frac{\partial^n\widetilde{\Sigma}(\not{p})}{\partial(\lambda_k^0)^n}\right|_{\lambda_k^0=\lambda_k} = \frac{\partial^n\Sigma(\not{p})}{\partial\lambda_k^n} \tag{4.4}$$

the relation is as follows

$$\widetilde{\Sigma}(\not{p}) = \left(\prod_k \mathbf{T}_{\lambda_k^0}\right) \widetilde{\Sigma}(\not{p})\Big|_{\lambda_k^0=\lambda_k} = \left(\prod_k \mathbf{T}_{\lambda_k}\right) \Sigma(\not{p}). \tag{4.5}$$

Here the restriction is understood to act on the derivatives as in Eq. (4.4), and the operators $\mathbf{T}_{\lambda_k^0}$ and $\mathbf{T}_{\lambda_k}$ differ only in their derivatives, but the counterterms are unchanged. Eventually, all three self-energies are related as follows

$$\Sigma^R(\not{p}) = \gamma^0 Z^\dagger \gamma^0 \widetilde{\Sigma}(\not{p}) Z = \gamma^0 Z^\dagger \gamma^0 \left(\prod_k \mathbf{T}_{\lambda_k}\right) \Sigma(\not{p}) Z\,. \tag{4.6}$$

It is worth noting that counterterm diagrams in the renormalized self-energy come from the parameter derivative terms due to the Taylor series operator.

Further, nearly everything can be expanded to some loop order in perturbation theory. We count loops with the $\Sigma(\not{p})$ self-energy, which is in terms of the renormalized parameters. Although, to be strict, for our purposes the renormalized parameters should be equal to the physical ones, since otherwise the renormalized parameters could be further expanded as in Eq. (1.20). Then n-loop diagrams in $\Sigma(\not{p})$ give some products of the theory parameters that become associated with the nth order. We indicate the order with a superscript, for example,

$$\Sigma(\not{p}) = \Sigma^{(0)}(\not{p}) + \Sigma^{(1)}(\not{p}) + \Sigma^{(2)}(\not{p}) + \dots\,. \tag{4.7}$$

The two other self-energies can be expanded in these orders as well, but the order no longer matches the number of loops in diagrams, for example, the renormalized self-energy at 1-loop has loop diagrams as well as tree-level diagrams with counterterm insertions. We use the same notation for counterterms, *e.g.*

$$Z = \mathbb{1} + \delta Z = \mathbb{1} + \delta Z^{(1)} + \delta Z^{(2)} + \dots\,. \tag{4.8}$$

The same logic also follows for the series operator. For example, the second order operator is

$$\mathbf{T}^{(2)}_{\lambda^0} = \delta\lambda^{(2)} \frac{\partial}{\partial \lambda^0} + \frac{1}{2}(2\delta\lambda^{(1)}\delta\lambda^{(1)}) \frac{\partial^2}{\partial (\lambda^0)^2}. \tag{4.9}$$

Here the second term comes from squaring the counterterm and then expanding to second order. It is all right to have products like $\delta\lambda^{(1)}\delta\lambda^{(1)}$ at the second order, however, for more complicated expressions we will use a shorthand

notation. For example, explicitly taking the nth order of some product of A and B means summing over all the possible contributions

$$A^{(0)}B^{(n)} + A^{(1)}B^{(n-1)} + \cdots + A^{(n)}B^{(0)} = \sum_{l=0}^{n} A^{(l)}B^{(n-l)}\,, \tag{4.10}$$

which we simply denote as

$$(AB)^{(n)} \equiv \sum_{l=0}^{n} A^{(l)}B^{(n-l)}\,. \tag{4.11}$$

The lower and upper bounds may vary depending on whether the 0th order is included and similar things. For example,

$$(Z\delta Z)^{(n)} = \sum_{l=1}^{n} Z^{(n-l)}\delta Z^{(l)}\,, \tag{4.12}$$

since the renormalization constant includes all orders, while the counterterm starts at the first order. In other words, one has to be conscious of the objects for which this simple notation is used. We have employed the explicit notation in our publication [A3] and there all the derivations are written out in full, while in the thesis we choose to go with the simpler notation.

Finally, we should discuss the decomposition and diagrams included in the self-energies. The decomposition of all the self-energies (Σ^R, $\widetilde{\Sigma}$, Σ) is the same and by-now the standard one

$$\Sigma(\not{p}) = \Sigma^{\gamma L}_{ji}(p^2)\not{p}P_L + \Sigma^{\gamma R}_{ji}(p^2)\not{p}P_R + \Sigma^{sL}_{ji}(p^2)P_L + \Sigma^{sR}_{ji}(p^2)P_R\,, \tag{4.13}$$

with the coefficients $\Sigma^{\gamma L,\gamma R,sL,sR}$ being scalar functions of the squared momentum. As for the diagrams, at 0th order (in the on-shell scheme) all the self-energies are the same

$$\widetilde{\Sigma}^{(0)}(\not{p}) = \Sigma^{(0)}(\not{p}) = \Sigma^{R\,(0)}(\not{p}) = \longrightarrow\!\!\!\!\!\!\longrightarrow = \not{p} - m\,. \tag{4.14}$$

Here the mass matrix m is *not* necessarily diagonal. At next order, the self-energy $\Sigma(\not{p})$ consists of all the 1-loop 1PI diagrams

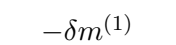
$$\Sigma^{(1)}(\not{p}) = \longrightarrow\!\!\!\!\!\!\bigcirc\!\!\!\!\!\!\longrightarrow + \quad . \tag{4.15}$$

Importantly, here we also include tadpole diagrams, the appearance of which can be justified in two ways. A simple justification comes from simply considering

the structure of the tadpole diagrams. By definition, the reducible diagrams are those, which can be divided into two smaller diagrams by cutting a single propagator $\frac{1}{p^2-m^2}$, which is just a line visually. The tadpole diagrams visually do match the look of 1PR diagrams, but there is no momentum flow through the propagator. In turn, the propagator acts like a simple coupling of $\frac{1}{-m^2}$ so that the actual structure of the tadpole diagrams is that of an irreducible diagram, which justifies the inclusion. Another explanation is in terms of the Fleischer-Jegerlehner (FJ) scheme [46] and can be found in [47].

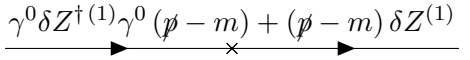
Further 1-loop diagrammatic relations may be written as follows

$$\widetilde{\Sigma}^{(1)}(\not{p}) = \Sigma^{(1)}(\not{p}) + \quad , \tag{4.16}$$



where $\delta m^{(1)}$ is the first order mass counterterm. Analogously, the renormalized self-energy is then

$$\Sigma^{R\,(1)}(\not{p}) = \widetilde{\Sigma}^{(1)}(\not{p}) + \quad , \tag{4.17}$$



where the counterterm diagram now includes the field renormalization and any higher orders follow the same logic.

5. Renormalization conditions on the mass shell

In the following sections we will be occupied with the renormalization of one of the first appearing functions — the 2-point function (*i.e.* the self-energy or the inverse propagator). Nonetheless, correct renormalization of these functions is extremely important, since the field renormalization (or LSZ factors), which appears in all other operators in the Lagrangian, is defined there. As the renormalization procedure implements the physical properties into the theory, first one needs to briefly discuss what are the physical properties of the 2-point functions or, rather, the propagators.

For simplicity, let us assume that we are working in the mass eigenstate basis with the renormalized masses m_i^R, where i is a flavor index, that in general have non-diagonal mass counterterms δm_{ji}, *i.e.* the bare mass is renormalized as

$$m^0_{ji} = m^R_i\delta_{ji} + \delta m_{ji}\,. \tag{5.1}$$

Here δm_{ji} has a chiral structure and can be decomposed into $\delta m^{L,R}$. Then the diagonal component of the renormalized propagator, $i\left(\Sigma^R_{ii}(\not{p})\right)^{-1}$, must have a pole as the momentum approaches the physical mass m_i^P. In other words,

$$\sum_{k,l}\bar{\mathcal{Z}}_{ik}\Sigma^R_{kl}(\not{p})\mathcal{Z}_{li}u_i = 0\,. \tag{5.2}$$

Here k, l are flavor indices and are summed over, $\mathcal{Z}$ and $\bar{\mathcal{Z}}$ are the LSZ factors for the incoming and outgoing particles, respectively, and u_i is the Dirac spinor for which the Dirac equation gives $\not{p} u_i = m_i^P u_i$ (and $p^2 = (m_i^P)^2$). Separating the tree-level contribution relates the renormalized and pole masses via the self-energy

$$\sum_{k,l} \left(\bar{\mathcal{Z}}_{ik} \Sigma^R_{kl}(\not{p}) \mathcal{Z}_{li} \right)^{(>0)} u_i = \left(m_i^R - m_i^P \right) u_i \,, \tag{5.3}$$

where the (> 0) superscript means all orders except tree-level[4]. Of course, this equation is reminiscent of Eqs. (1.17) and (1.19).

In addition, the residue of the propagator at the pole should be unity to resemble the propagator of a free particle

$$\lim_{\not{p} \to m_i^P} \left(\not{p} - m_i^P \right) \sum_{k,l} \frac{i}{\bar{\mathcal{Z}}_{ik} \Sigma^R_{kl}(\not{p}) \mathcal{Z}_{li}} u_i = i u_i \,,$$

although a more convenient form is the inverted one with the imaginary unit removed

$$\lim_{\not{p} \to m_i^P} \frac{1}{\not{p} - m_i^P} \sum_{k,l} \bar{\mathcal{Z}}_{ik} \Sigma^R_{kl}(\not{p}) \mathcal{Z}_{li} u_i = u_i \,. \tag{5.4}$$

Here the LSZ factors are needed to ensure the unit residue condition.

Finally, it is required that the on-shell propagator is diagonal and there is no mixing

$$\sum_{k,l} \bar{\mathcal{Z}}_{jk} \Sigma^R_{kl}(\not{p}) \mathcal{Z}_{li} u_i = 0 \,, \tag{5.5a}$$

$$\sum_{k,l} \bar{u}_j \bar{\mathcal{Z}}_{jk} \Sigma^R_{kl}(\not{p}) \mathcal{Z}_{li} = 0 \,, \tag{5.5b}$$

where $\bar{u}_j$ is the spinor for the outgoing particle and $i \neq j$.

Having these physical conditions we may state the usual on-shell renormalization conditions [45]. These conditions require that the renormalized mass is equal to the pole mass

$$m_i^R = m_i^P \equiv m_i \tag{5.6}$$

and that the LSZ factors are unity

$$\mathcal{Z} = \overline{\mathcal{Z}} = \mathbb{1} \,. \tag{5.7}$$

[4]Depending on the scheme, the renormalized mass could be further expanded, but the r.h.s. is a tree-level contribution.

The former condition simply turns Eq. (5.3) into

$$\sum_{k,l} \left(\bar{\mathcal{Z}}_{ik}\Sigma^R_{kl}(\not{p})\mathcal{Z}_{li}\right)^{(>0)} u_i = 0\,, \tag{5.8}$$

while the latter condition gives the role of LSZ factors to the field renormalization. That this can be done can be seen by, for example, taking Eq. (5.5a) and inserting the expression for the renormalized self-energy from Eq. (4.6)

$$\begin{aligned}\sum_k \Sigma^R_{jk}(\not{p})\mathcal{Z}_{ki}u_i &= \sum_k \left(\gamma^0 Z^\dagger \gamma^0 \widetilde{\Sigma}(\not{p}) Z\right)_{jk} \mathcal{Z}_{ki}u_i \\ &= \sum_{s,r,k} \gamma^0 Z^\dagger_{js}\gamma^0\widetilde{\Sigma}_{sr}(\not{p})Z_{rk}\mathcal{Z}_{ki}u_i\,.\end{aligned} \tag{5.9}$$

By noting the $Z_{rk}\mathcal{Z}_{ki}$ product it is evident that the field renormalization and the LSZ factors have the same role and are degenerate, hence, any one of these can be set to unity. However, this is not possible for unstable particles due to the presence of absorptive parts or, at the very least, not possible without some modifications.

In more detail, hermiticity of the Lagrangian or the generating functional gives requirements on the self-energies and the renormalization constants. Equating Eq. (4.1) with its hermitian conjugate we have

$$\begin{aligned}\overline{\psi}_0\widetilde{\Sigma}(\not{p})\psi_0 = \left(\overline{\psi}_0\widetilde{\Sigma}(\not{p})\psi_0\right)^\dagger &= \overline{\psi}_0\gamma^0\left(\widetilde{\Sigma}(\not{p})\right)^\dagger\gamma^0\psi_0 \\ \Longrightarrow \widetilde{\Sigma}(\not{p}) &= \gamma^0\left(\widetilde{\Sigma}(\not{p})\right)^\dagger\gamma^0\,,\end{aligned}$$

where the hermitian conjugation acts both in spinor and flavor spaces and the resulting relation is called pseudo-hermiticity. Unfortunately, it has been noted that this relation does not hold above particle production thresholds since then the loop functions develop imaginary parts [80, 104, 105]. Depending on the order at which one is working, the available approaches are a little different.

5.1. Renormalization conditions at 1-loop

At 1-loop and in terms of the renormalized Lagrangian, these absorptive parts are not a problem if one sets the field renormalization to unity, $Z = \mathbb{1}$, since then the absorptive parts can be accounted for by the LSZ factors for which there is no pseudo-hermiticity requirement. However, the converse with $\mathcal{Z} = \overline{\mathcal{Z}} = \mathbb{1}$ is not possible since pseudo-hermiticity is required for the field renormalization and the absorptive parts cannot be included without breaking the hermiticity of the Lagrangian [80, 82]. In practice, if one sets $\mathcal{Z} = \overline{\mathcal{Z}} = \mathbb{1}$, and computes

the field renormalization constants from the conditions in Eqs. (5.4), (5.5a), and (5.5b) at 1-loop, one gets expressions for Z and $Z^\dagger$, which are not related by naive hermitian conjugation[5] — this is the overspecification problem.

One proposal to solve this at 1-loop is by breaking the hermiticity of the Lagrangian and having two sets of independent field renormalization constants, namely Z (associated with ψ) and $\bar{Z}$ (associated with $\bar{\psi}$) [80]. While this relaxes the renormalization conditions and allows for enough freedom to fix the field renormalization, this is unsatisfactory in terms of the renormalized Lagrangian. Another proposal can be found in [104], where one sets $\mathcal{Z} = \overline{\mathcal{Z}} = \mathbb{1}$ and the absorptive parts are simply dropped from the loop functions via the $\widetilde{Re}$ operator, such that the field renormalization can be defined without encountering overspecification.

At least at 1-loop, *we take the in-between approach* by keeping the hermiticity of the Lagrangian and the absorptive parts as much as possible. This is done by setting *only* one of the LSZ factors to unity

$$\mathcal{Z} = \mathbb{1} \tag{5.10}$$

such that Eq. (5.5a) becomes

$$\Sigma^{R\,(1)}_{ji}(\not{p})u_i = 0, \tag{5.11}$$

where the $\overline{\mathcal{Z}}$ drops out and accounts for the absorptive parts in Eq. (5.5b). This may be called the "incoming" renormalization scheme as dubbed in [105], with the "outgoing" scheme also being a possible choice.

The absorptive parts also play a role in Eq. (5.8), which defines the diagonal components of the mass counterterm. However, the effects are not as strong and are at most conceptual. The hermiticity of the bare Lagrangian requires that the mass counterterm obeys pseudo-hermiticity

$$\delta m = \gamma^0 \delta m^\dagger \gamma^0\,, \tag{5.12}$$

but this in principle allows for the inclusion of absorptive parts in all components. However, it is more common to keep at least the diagonal components real so that the absorptive parts are dropped from Eq. (5.8)

$$\widetilde{Re}\Sigma^{R\,(1)}_{ii}(\not{p})u_i = 0 \tag{5.13}$$

and imaginary part of $\Sigma^{R\,(1)}_{ii}(\not{p})u_i$ is then identified with the decay width Γ (*e.g.* [80, 106–109]). In contrast, the condition in Eq. (5.4) with $\mathcal{Z} = \overline{\mathcal{Z}} = \mathbb{1}$

[5] In renormalizing the theory we already implemented pseudo-hermiticity by writing $\gamma^0 Z^\dagger \gamma^0$ instead of having an independent renormalization constant $\overline{Z}$ so that here only the hermitian conjugation appears.

and keeping the hermiticity of the Lagrangian defines only the hermitian part of the field renormalization, which must be real. In turn, at least one of the LSZ factors is needed to account for the absorptive parts.

In any case, at 1-loop order our scheme is mostly occupied with the off-diagonal terms and the reader may assume the usual diagonal components. In addition, we do include the absorptive parts in the field renormalization as much as possible, but our scheme does not really rely on the absorptive parts, and they may be moved entirely into the LSZ factors if seen fit by the reader.

The final condition we impose is that the mixing matrix counterterm must be trivial as per Section 3

$$\delta\mathcal{U} = 0\,, \tag{5.14}$$

or, rather, we expect this to be the natural outcome of the scheme without much additional effort.

5.1.1. Renormalization conditions for Majorana fermions

The situation with absorptive parts is even worse for Majorana fermions and requires a separate discussion. The Majorana condition on bare fields, *e.g.* [110],

$$\nu_0 = \nu_0^c = \gamma^0\mathcal{C}\nu_0^\star, \tag{5.15}$$

where $\mathcal{C}$ is the charge conjugation matrix, limits the available degrees of freedom as compared to the Dirac case. If the renormalized fields are also Majorana, then the bare Majorana condition requires an additional relation for the field renormalization constants, namely,

$$Z = \gamma^0\mathcal{C}Z^\star\mathcal{C}^{-1}\gamma^0\,, \tag{5.16}$$

where * marks complex conjugation. More simply put, this relates the left- and right-handed parts of the field renormalization

$$Z_L = Z_R^\star\,. \tag{5.17}$$

Since this relation involves complex conjugation, the absorptive parts already obstruct it if at least one of the LSZ factors is set to unity. In contrast, the symmetry (transposition) properties of Majorana fermions are not affected by the absorptive parts.

There are a few ways of defining the field renormalization constants to keep the Majorana nature of the theory. One way is to use only a selected projection (left or right) of Eq. (5.11), however, as seen in Eq. (3.29) the projections involve both $Z_{L,R}$ and $Z_{R,L} = Z_{L,R}^\star$, which does not lead to anything convenient.

Another possibility is to fully use Eq. (5.11), but then use only one of the solutions, either Z_L or Z_R, and define the remaining one by complex conjugation of the solved one. With this option, the resulting expressions are much more cumbersome than in the Dirac case. We do not choose any of these two options.

Perhaps the easiest way is to simply drop the absorptive parts, such that the no mixing condition in Eq. (5.11) becomes

$$\widetilde{Re}\left[\Sigma^R_{ji}(\not{p})\right] u_i = 0 \tag{5.18}$$

to all orders. This relaxes the no mixing condition, but also preserves the Majorana condition for bare and renormalized fields and also keeps the hermiticity. Of course, this option is the same as having both non-trivial LSZ factors. This seems to be the most convenient option since the expressions we find for Dirac fermions can be easily adapted by simply dropping the absorptive parts — this is the option we choose for Majorana fermions.

5.2. Renormalization conditions at all orders

Beyond 1-loop the situation is more complicated because of two things. First, one often encounters products such as Eq. (4.11) so that at order n the definitions of lower order counterterms become important. For example, if the absorptive parts are not treated well enough at 1-loop, they will obstruct any meaningful discussion at higher orders. Another thing is that beyond 1-loop it makes more sense to discuss the hermiticity of the 1PI generating functional instead of the Lagrangian. Since pseudo-hermiticity may be violated at any order, the 1PI generating functional will not be hermitian. In this regard, a more comprehensive discussion is achieved by *dropping all the absorptive parts from the self-energies* when discussing renormalization at arbitrary orders. Having this, we freely set

$$\mathcal{Z} = \overline{\mathcal{Z}} = \mathbb{1} \tag{5.19}$$

and the field renormalization is sufficient to fully impose the OS renormalization conditions.

Just like at 1-loop, we require that the mixing matrix counterterm is trivial. However, we claim that this result comes out naturally since all the relevant structures repeat at every order.

6. On-Shell fermion renormalization at 1-loop

Next, we consider the implementation of the renormalization conditions in Section 5.1. While we have discussed the problem of over-specification in

terms of the absorptive parts and the field renormalization, there is another problem of degeneracy between the off-diagonal field and mass counterterms. Taking Eq. (5.11) with the mass renormalization as in Eq. (5.1) at 1-loop one arrives at a relation between the anti-hermitian part of the field renormalization and the mass counterterms

$$(m_i^2 - m_j^2)\delta Z_{Lji}^{A\,(1)} - m_j \delta m_{ji}^{L\,(1)} - m_i \delta m_{ji}^{R\,(1)} = f\,(\Sigma) \tag{6.1}$$

with

$$\begin{aligned} f\,(\Sigma) = -\frac{1}{2}\Big(& m_i^2 \Sigma_{ji}^{\gamma L\,(1)}(m_i^2) + m_i m_j \Sigma_{ji}^{\gamma R\,(1)}(m_i^2) \\ & + m_j \Sigma_{ji}^{sL\,(1)}(m_i^2) + m_i \Sigma_{ji}^{sR\,(1)}(m_i^2)\Big) + H.C. \end{aligned} \tag{6.2}$$

There is an analogous relation for the right-handed part of the field renormalization. Importantly, here the off-diagonal field and mass counterterms are both unknowns and a single equation cannot fix them both simultaneously. We solve this problem by *defining* the off-diagonal anti-hermitian part of the field renormalization as the coefficient of the $m_i^2 - m_j^2$ mass structure[6]

$$\begin{aligned} \delta Z_{Lji}^{A\,(1)} \equiv \frac{1}{2}\Big[& -\Big(m_i^2 \Sigma_{ji}^{\gamma L\,(1)}(m_i^2) + m_i m_j \Sigma_{ji}^{\gamma R\,(1)}(m_i^2) \\ & + m_j \Sigma_{ji}^{sL\,(1)}(m_i^2) + m_i \Sigma_{ji}^{sR\,(1)}(m_i^2)\Big) + H.C.\Big]\Big|_{(m_i^2 - m_j^2)}, \\ \delta Z_{Rji}^{A\,(1)} \equiv \frac{1}{2}\Big[& -\Big(m_i^2 \Sigma_{ji}^{\gamma R\,(1)}(m_i^2) + m_i m_j \Sigma_{ji}^{\gamma L\,(1)}(m_i^2) \\ & + m_j \Sigma_{ji}^{sR\,(1)}(m_i^2) + m_i \Sigma_{ji}^{sL\,(1)}(m_i^2)\Big) + H.C.\Big]\Big|_{(m_i^2 - m_j^2)}, \end{aligned} \tag{6.3}$$

where $i \neq j$. The hermitian part is not affected by the degeneracy with the mass counterterms so the usual definition (solution) is sufficient and we do not change it. The definition of the anti-hermitian part of the field renormalization breaks the degeneracy in Eq. (6.1) and one can simply solve for the off-diagonal

[6] Here $[(m_i^2 - m_j^2)^n A + B]\Big|_{(m_i^2 - m_j^2)} = (m_i^2 - m_j^2)^{n-1} A$, for some positive power n and functions A and B.

mass counterterms

$$\begin{aligned}
\delta m_{ji}^{L\,(1)} =& \frac{1}{2}\Big(m_i \Sigma_{ji}^{\gamma R\,(1)}(m_i^2) + \Sigma_{ji}^{sL\,(1)}(m_i^2) \\
&+ m_j \Sigma_{ji}^{\gamma L\,(1)\,\dagger}(m_j^2) + \Sigma_{ji}^{sR\,(1)\,\dagger}(m_j^2)\Big) \\
&+ m_i \delta Z_{Rji}^{A\,(1)} - m_j \delta Z_{Lji}^{A\,(1)}, \\
\delta m_{ji}^{R\,(1)} =& \frac{1}{2}\Big(m_i \Sigma_{ji}^{\gamma L\,(1)}(m_i^2) + \Sigma_{ji}^{sR\,(1)}(m_i^2) \\
&+ m_j \Sigma_{ji}^{\gamma R\,(1)\,\dagger}(m_j^2) + \Sigma_{ji}^{sL\,(1)\,\dagger}(m_j^2)\Big) \\
&+ m_i \delta Z_{Lji}^{A\,(1)} - m_j \delta Z_{Rji}^{A\,(1)}\,.
\end{aligned} \tag{6.4}$$

Here the hermiticity constraint in Eq. (5.12) holds by construction regardless of whether the absorptive parts are included or not.

These definitions of the field and mass counterterms stem from a simple observation of the distinct factor of $m_i^2 - m_j^2$ in Eq. (6.1). Interestingly, these definitions provide a set of useful features

- The counterterms are given in terms of self-energies and restrictions to mass structures and thus are universal — model- and process-independent (as required in Section 2.2, point 4)
- The mass counterterm is gauge-independent by definition (see Section (6.1))

$$\partial_\xi \delta m_{ji} = 0 \tag{6.5}$$

- At 1-loop, the anti-hermitian part of the field renormalization is UV-finite (see Section (6.2))
- There are no left-over UV divergences so the mixing matrix counterterm is naturally trivial (see Section 6.3 and point 1 in Section 2.2)
- The scheme relies on the $m_i^2 - m_j^2$ factor and not on the absorptive parts, which the reader may treat as seen fit
- The scheme also ensures that no $(m_i^2 - m_j^2)^{-1}$ factors appear and the degenerate mass limit is non-singular.

Note that the definitions of the mass and field counterterms are for the off-diagonal components, however, the mass counterterms are almost suitable for the diagonal case, where $i = j$. The diagonal components of the mass counterterms, $\delta m_{ii}^{L,R}$, in our scheme, have the same real part as in [80], but the imaginary parts differ. In turn, the real part of the mass counterterms in Eq. (6.4) may be used for the diagonal case as well.

We dedicate this subsection to the consideration of gauge dependence in our scheme. For this discussion we employ the Nielsen identities [99] just like in Section 3.3, however, here we give more detail. First off, let us write down the most general form of the Nielsen identity [79]

$$\begin{aligned}\partial_\xi \Gamma^R = \left(1+\rho^\xi\right)\mathcal{S}_{\Gamma^R}\left(\frac{\partial\Gamma^R}{\partial\chi}\right) + \sum_i \beta_i^\xi \frac{\partial\Gamma^R}{\partial\lambda_i} \\ + \sum_\phi \gamma_\phi^\xi \mathcal{N}_\phi \Gamma^R + \sum_S \delta_S \int d^4x \frac{\delta\Gamma^R}{\delta S\left(x\right)}\,.\end{aligned} \tag{6.6}$$

Here Γ^R is the renormalized (reduced) 1PI generating functional, $\mathcal{S}_{\Gamma^R}$ is the Slavnov-Taylor operator, χ is the source coupled to the BRST variation of the gauge parameter ξ, λ_i are parameters of the theory, ϕ runs over all the fields in the theory and $\mathcal{N}_\phi$ counts the external ones, while S stands for any scalar that may acquire a vacuum expectation value, finally, ρ^ξ, β_i^ξ, γ_ϕ^ξ, and δ_S parameterize various modifications of the Nielsen Identity due to non-physical renormalization. In particular, the ρ^ξ comes from renormalization of the gauge parameter and only starting at 2-loop level, the β^ξ term comes if the physical parameters are renormalized in a gauge-dependent way, the γ_ϕ term arises if the chosen renormalization scheme is not an on-shell one, δ_S is present if the theory is not at the *true* minimum.

Fortunately, all these modifications are practically irrelevant in our scheme and the situation is much simpler. The ρ^ξ modification is irrelevant at 1-loop and we can assume that the gauge parameters have no counterterms unless stated otherwise. The β^ξ and γ_ϕ modifications are also irrelevant since our scheme is on-shell. Although, strictly speaking, only our two-point functions are explicitly renormalized on-shell, while the requirement that the renormalized parameters are the physical parameters, $\lambda^R = \lambda^P$, is an assumption for higher point functions. The $\delta_S = 0$ modification vanishes since the inclusion of tadpole diagrams ensures that the theory is at the true minimum according to the FJ scheme [46, 47].

Even more so, we are interested in the gauge dependence of the bare self-energies, since the gauge dependence of the renormalized self-energies can be tweaked by choosing the counterterms. Eventually, the truly relevant form of the Nielsen identity is that for the bare self-energy and is much simpler

$$\partial_\xi \Gamma = \mathcal{S}_\Gamma\left(\frac{\partial\Gamma}{\partial\chi}\right) \tag{6.7}$$

and gives rise to the Nielsen identity for two-point functions [79]

$$\partial_\xi \Sigma_{ji}(\not{p}) = \sum_{j'} \Lambda_{jj'} \Sigma_{j'i}(\not{p}) + \sum_{i'} \Sigma_{ji'}(\not{p}) \overline{\Lambda}_{i'i} \,. \tag{6.8}$$

Here $\Lambda_{ji} = -\Gamma_{\chi \bar{\psi}_j \eta_{\psi_i}}$ and $\bar{\Lambda}_{ji} = -\Gamma_{\chi \bar{\eta}_{\psi_j} \psi_i}$, Λ's are functions of $\not{p}$ and have Dirac structure, η_ψ is a source coupled to the BRST variation of the fermion field ψ. This is the all-order version of Eq. (3.20) where we have not explicitly implemented pseudo-hermiticity, hence, the Λ and $\overline{\Lambda}$.

At 1-loop the situation further simplifies, since Λ's vanish at tree-level

$$\begin{aligned} \partial_\xi \Sigma^{(1)}_{ji}(\not{p}) =& \sum_{j'} \Lambda^{(1)}_{jj'} \Sigma^{(0)}_{j'i}(\not{p}) + \sum_{i'} \Sigma^{(0)}_{ji'}(\not{p}) \overline{\Lambda}^{(1)}_{i'i} \\ =& \Lambda^{(1)}_{ji} \left(\not{p} - m_i\right) + \left(\not{p} - m_j\right) \overline{\Lambda}^{(1)}_{ji} \,. \end{aligned} \tag{6.9}$$

By using (4.13) both for Σ and Λ's one easily arrives at

$$\begin{aligned} \partial_\xi \Sigma^{\gamma L\,(1)}_{ji}(p^2) &= -m_i \Lambda^{\gamma L\,(1)}_{ji} - m_j \bar{\Lambda}^{\gamma L\,(1)}_{ji} + \Lambda^{sR\,(1)}_{ji} + \bar{\Lambda}^{sL\,(1)}_{ji} \,, \\ \partial_\xi \Sigma^{\gamma R\,(1)}_{ji}(p^2) &= -m_i \Lambda^{\gamma R\,(1)}_{ji} - m_j \bar{\Lambda}^{\gamma R\,(1)}_{ji} + \Lambda^{sL\,(1)}_{ji} + \bar{\Lambda}^{sR\,(1)}_{ji} \,, \\ \partial_\xi \Sigma^{sL\,(1)}_{ji}(p^2) &= p^2 \Lambda^{\gamma R\,(1)}_{ji} + p^2 \bar{\Lambda}^{\gamma L\,(1)}_{ji} - m_i \Lambda^{sL\,(1)}_{ji} - m_j \bar{\Lambda}^{sL\,(1)}_{ji} \,, \\ \partial_\xi \Sigma^{sR\,(1)}_{ji}(p^2) &= p^2 \Lambda^{L\,(1)}_{ji} + p^2 \bar{\Lambda}^{\gamma R\,(1)}_{ji} - m_i \Lambda^{sR\,(1)}_{ji} - m_j \bar{\Lambda}^{sR\,(1)}_{ji} \,. \end{aligned} \tag{6.10}$$

With this decomposition, we may take the gauge derivative of Eq. (6.3) and get

$$\begin{aligned} \partial_\xi \delta Z^{A\,(1)}_{Lji} &= \left[\left(m_i^2 - m_j^2\right) \left(-m_i \bar{\Lambda}^{\gamma R\,(1)}_{ji}(m_i^2) - \bar{\Lambda}^{sL\,(1)}_{ji}(m_i^2) - H.C. \right) \right] \Big|_{m_i^2 - m_j^2} \\ &= -m_i \bar{\Lambda}^{\gamma R\,(1)}_{ji}\left(m_i^2\right) - \bar{\Lambda}^{sL\,(1)}_{ji}\left(m_i^2\right) - H.C. \,, \\ \partial_\xi \delta Z^{A\,(1)}_{Rji} &= -m_i \bar{\Lambda}^{\gamma L\,(1)}_{ji}\left(m_i^2\right) - \bar{\Lambda}^{sR\,(1)}_{ji}\left(m_i^2\right) - H.C. \,. \end{aligned} \tag{6.11}$$

Gauge dependence of the hermitian part is analogous except for the different sign in front of $H.C.$ Importantly, here we see that the gauge-dependent part necessarily carries a factor of $m_i^2 - m_j^2$ — the same factor we noticed in Eq. (6.1) and Eq. (3.31). This factor cancels even in the standard approach as noticed in [25], while here it helps us define the anti-hermitian part of the field renormalization. This also shows that only the gauge-independent parts can be singular in the degenerate mass limit as the $m_i^2 - m_j^2$ mass structure is not guaranteed for gauge-independent parts. Importantly, it is neither guaranteed nor forbidden, so care should be taken in selecting the gauge-independent parts.

An analogous analysis can be carried out for the mass counterterms, it simply gives

$$\partial_\xi \delta m^{L\,(1)}_{ji} = 0 \qquad \text{and} \qquad \partial_\xi \delta m^{R\,(1)}_{ji} = 0 \tag{6.12}$$

as expected of counterterms associated with physical quantities.

6.2. UV divergences

Further, we showcase the UV properties of the off-diagonal field and mass counterterms in our scheme at 1-loop. In our publication [A3] we gave the discussion by considering how various diagrams contribute to the fermion self-energy and then noticing the various mass structures. However, here we take a slightly different approach and do not involve Passarino-Veltman[111] functions in the discussion.

Let us begin by stating that, since only the UV parts are important in this section, we use the pseudo-hermiticity property, which gives

$$\left(\Sigma^{\gamma L,\gamma R}(p^2)\right)^\dagger = \Sigma^{\gamma L,\gamma R}(p^2), \tag{6.13}$$

$$\left(\Sigma^{sL}(p^2)\right)^\dagger = \Sigma^{sR}(p^2)\,. \tag{6.14}$$

Another important property, once we regard just the UV parts, follows from Eqs. (3.27) and (4.13), where one can notice that that the UV parts of $\Sigma^{\gamma L,\gamma R,sL,sR}$ must not depend on the momentum squared, hence, we are free to drop the p^2 arguments in this discussion. With this the UV-divergent part of Eq. (6.3) gives

$$\begin{aligned}\left[\delta Z^{A\,(1)}_{Lji}\right]_{\text{div.}} = -\frac{1}{2}\Big[&\left(m_i^2+m_j^2\right)\Sigma^{\gamma L\,(1)}_{ji} + 2m_i m_j \Sigma^{\gamma R\,(1)}_{ji} \\ &+2m_j\Sigma^{sL\,(1)}_{ji} + 2m_i\Sigma^{sR\,(1)}_{ji}\Big]_{\text{div.}}\Big|_{m_i^2-m_j^2}\,.\end{aligned} \tag{6.15}$$

Considering the second line of Eq. (6.15), we may split $\Sigma^{sL,sR}$ into hermitian and anti-hermitian parts

$$\begin{aligned}\Sigma^{sL}(p^2) =&\Sigma^{H}(p^2) + \Sigma^{A}(p^2)\,,\\ \Sigma^{sR}(p^2) =&\Sigma^{H}(p^2) - \Sigma^{A}(p^2)\,.\end{aligned} \tag{6.16}$$

This allows to rewrite Eq. (6.15) as

$$\begin{aligned}\left[\delta Z^{A\,(1)}_{Lji}\right]_{\text{div.}} = -\frac{1}{2}\Big[&\left(m_i^2+m_j^2\right)\Sigma^{\gamma L\,(1)}_{ji} + 2m_i m_j \Sigma^{\gamma R\,(1)}_{ji} \\ &+2\left(m_i+m_j\right)\Sigma^{H\,(1)}_{ji} - 2\left(m_i-m_j\right)\Sigma^{A\,(1)}_{ji}\Big]_{\text{div.}}\Big|_{m_i^2-m_j^2}\end{aligned} \tag{6.17}$$

Here, the terms on the first line contain both the hermitian mass structures ($m_i^2+m_j^2$ and $2\,m_i m_j$) and the hermitian self-energy scalar functions ($\Sigma^{\gamma L,\gamma R}_{ji}$). Note that, at least at 1-loop, the couplings enter the self-energies at most to the

second power, hence, it is impossible to form $(m_i^2 - m_j^2)^2$ in $\Sigma_{ji}^{\gamma L,\gamma R}$. In turn, the first line cannot contribute to the UV divergences of the anti-hermitian part of the field renormalization.

The second line of Eq. (6.17) also does not produce the $m_i^2 - m_j^2$ mass structure since the UV parts of $\Sigma^{sL,sR}$ are at most linear in the masses m_i and m_j. This can be noted from Eq. (6.10), where the Λs are multiplied by $m_{i,j}$ or by considering that these functions are of mass dimension 1 and also contribute to the mass renormalization as seen from Eqs. (3.27) and (4.13). This implies that the only available mass structures are $m_i \pm m_j$

$$\Sigma_{ji}^{H\,(1)} \to \Sigma_{ji}^{H\prime} + (m_i + m_j)\,\Sigma_{ji}^{H\prime\prime}\,, \tag{6.18}$$

$$\Sigma_{ji}^{A\,(1)} \to \Sigma_{ji}^{H\prime} + (m_i - m_j)\,\Sigma_{ji}^{A\prime\prime}\,. \tag{6.19}$$

Dropping $\Sigma^{\gamma L,\gamma R}$ terms in Eq.(6.17) and plugging in the decomposition of $\Sigma^{H,A\,(1)}$ we get

$$\begin{aligned}\left[\delta Z_{Lji}^{A\,(1)}\right]_{\text{div.}} =& -\frac{1}{2}\Big[2\left(m_i + m_j\right)\Sigma_{ji}^{H\prime} - 2\left(m_i - m_j\right)\Sigma_{ji}^{A\prime} \\ &+2\left(m_i + m_j\right)^2\Sigma_{ji}^{H\prime\prime} - 2\left(m_i - m_j\right)^2\Sigma_{ji}^{A\prime\prime}\Big]_{\text{div.}}\Big|_{m_i^2 - m_j^2} \\ =&0\,.\end{aligned} \tag{6.20}$$

Here the final equality follows since neither of the terms could provide the $m_i^2 - m_j^2$ structure (in $m_i - m_j$ the masses are not squared). This generally confirms that our definition of the anti-hermitian part of the field renormalization gives a UV finite result. Before, it was noticed in [25] that only the gauge-dependent part is UV-finite.

In contrast, by the same analysis, the $m_i^2 - m_j^2$ appears immediately for the hermitian part of the field renormalization

$$(m_i^2 - m_j^2)\left[\delta Z_{Lji}^{H\,(1)}\right]_{\text{div.}} = -\frac{1}{2}\left(m_i^2 - m_j^2\right)\left[\Sigma_{ji}^{\gamma L\,(1)}\right]_{\text{div.}}\,. \tag{6.21}$$

In addition, since none of the UV divergent terms carry the $m_i^2 - m_j^2$ mass structure, all the UV divergences in Eq. (6.1) are gauge-independent. Naturally, these must be associated with the mass counterterms and we may check whether the same mass structure analysis holds. To do so, we implement the following mass renormalization

$$\begin{aligned}m_0 =&\delta\hat{m}^L P_L + \delta\hat{m}^R P_R \\ &+\left(1 + \delta\hat{Z}_m^L P_L + \delta\hat{Z}_m^R P_R\right) m \left(1 + \delta\hat{Z}_m^L P_R + \delta\hat{Z}_m^R P_L\right)\end{aligned} \tag{6.22}$$

with the hermiticity requirement

$$\left(\delta\hat{Z}_m^L\right)^\dagger = \delta\hat{Z}_m^R\,. \tag{6.23}$$

Then one can relate the counterterm in the l.h.s. of Eq. (5.1) with the hatted counterterms at 1-loop

$$\delta m^{L\,(1)} = \delta\hat{m}^{L\,(1)} + \delta\hat{Z}_m^{L\,(1)} m + m\delta\hat{Z}_m^{R\,(1)}\,, \tag{6.24}$$

$$\delta m^{R\,(1)} = \delta\hat{m}^{R\,(1)} + \delta\hat{Z}_m^{R\,(1)} m + m\delta\hat{m}_m^{L\,(1)}\,. \tag{6.25}$$

Additionally, we split the $\delta\hat{m}^{L,R}$ into hermitian and anti-hermitian parts

$$\delta\hat{m}^L = \delta\hat{m}^H + \delta\hat{m}^A\,, \tag{6.26}$$

$$\delta\hat{m}^R = \delta\hat{m}^H - \delta\hat{m}^A\,, \tag{6.27}$$

which gives

$$\begin{aligned} m_j\delta m_{ji}^{L\,(1)} + m_i\delta m_{ji}^{R\,(1)} = {} & \left(m_i + m_j\right)\delta\hat{m}_{ji}^{H\,(1)} - \left(m_i - m_j\right)\delta\hat{m}_{ji}^{A\,(1)} \\ & - 2m_j m_i\delta\hat{Z}_{mji}^{L\,(1)} - \left(m_i^2 + m_j^2\right)\delta\hat{Z}_{mji}^{R\,(1)}\,. \end{aligned} \tag{6.28}$$

Evidently the mass structures $m_i^2 + m_j^2$, $2m_i m_j$, and $m_i \pm m_j$ from Eq. (6.17) are repeated. In turn, the mass structure analysis holds and the UV divergences of Eq. (6.1) are naturally associated with the mass counterterms as should be.

The important outcome of this section is that all the UV divergences on the r.h.s. of Eq. (6.1) are gauge-independent and can be accounted for by the mass counterterms and leaving the anti-hermitian part of the field renormalization UV-finite.

6.3. Triviality of the mixing matrix counterterm and comparison with other schemes

In the three following subsections, we compare our scheme with three other selected schemes in order to prove that in our scheme the mixing matrix counterterms are trivial. Specifically, we state that there are no UV divergences left to cancel and the mixing matrix counterterm is not necessary, this regards the 1st point from Section 2.2.

6.3.1. The scheme of Denner and Sack

The first scheme we consider is that of Denner and Sack [23], which we already briefly mentioned in Section 2.1. There the authors have a diagonal mass

counterterm and use the OS conditions with $\mathcal{Z} = \bar{\mathcal{Z}} = \mathbb{1}$, such that the field renormalization constants are over specified, but at the time this was not known. Without off-diagonal mass counterterms, the UV divergences in Eqs. (5.11) or (6.1) must be accounted for by the anti-hermitian part of the field renormalization. The field renormalization is defined by two-point functions, but enter all the other terms in the Lagrangian or higher point functions. This, combined with the divergences in the anti-hermitian part of the field renormalization, is what made the W vertex UV divergent and is why the CKM counterterm, Eq. (2.17), which we repeat for convenience

$$\delta V^{\text{CKM}\,(1)} = \delta Z_u^{A\,(1)} V^{\text{CKM}} - V^{\text{CKM}} \delta Z_d^{A\,(1)} \tag{6.29}$$

was *needed*. In addition, since there are no off-diagonal mass counterterms in the scheme of [23], the anti-hermitian part of the field renormalization is singular in the degenerate mass limit as we discussed in Section 3.4. In our scheme, the situation is quite contrary, since the inclusion of off-diagonal mass counterterms allows to have the anti-hermitian part of the field renormalization which is both UV finite and non-singular in the degenerate mass limit. In turn, Eq. (6.29) gives a UV-finite counterterm and the W vertex is automatically UV finite in our scheme, hence, there is no need for a CKM counterterm.

Another problem with the counterterm in Eq. (6.29) is that it is gauge-dependent due to the gauge dependence of the field renormalization. This gauge-dependence breaks the Ward-Takahashi identities as noted in [24], is against the 2nd requirement in Section 2.2, and breaks the gauge-independence of the Wu_jd_i amplitude [80]. In our scheme, the mixing matrix counterterm is trivial so that any gauge-dependence-related problems are simply circumvented. Although, we must note that to have a gauge-independent Wu_jd_i amplitude it is needed to have at least 1 non-trivial LSZ factor [80], as we have in our scheme.

6.3.2. The scheme of Kniehl and Sirlin

Another scheme we choose for comparison is that of Kniehl and Sirlin [29], where the authors deviate from the usual approach (see Section 2.1) by introducing off-diagonal mass counterterms, but also steer back to it by defining a mixing matrix counterterm. The mixing matrix counterterm is defined by diagonalizing the counterterms. For clarity, we briefly give their arguments.

The authors renormalize the masses just like in Eq. (5.1), but also propose an additional rotation by

$$\mathcal{U} = \mathbb{1} + ih^{(1)}\,, \tag{6.30}$$

where $h^{(1)}$ is of 1-loop order and can be decomposed into left- and right-handed parts. Then one requires that the whole mass term is diagonal

$$\gamma^0 U^\dagger \gamma^0 \left(m + \delta m\right) U = \text{diag.}\,, \tag{6.31}$$

which gives

$$ih^{(1)}_{Lji} = \frac{m_j \delta m^{L\,(1)}_{ji} + \delta m^{R\,(1)}_{ji} m_i}{m_i^2 - m_j^2} \qquad \text{for} \qquad i \neq j\,, \tag{6.32a}$$

$$ih^{(1)}_{Rji} = \frac{m_j \delta m^{R\,(1)}_{ji} + \delta m^{L\,(1)}_{ji} m_i}{m_i^2 - m_j^2} \qquad \text{for} \qquad i \neq j\,, \tag{6.32b}$$

$$ih^{(1)}_{Lii} = ih^{(1)}_{Rii} = 0\,. \tag{6.32c}$$

The validity of these equations can be checked by insertion and for more detail we refer to [29, 31].

This additional rotation by $\mathcal{U}$ not only diagonalizes the mass counterterm but also appears in other terms in the Lagrangian. For example, let us consider a chiral interaction with some mixing matrix V, vector A_μ, and fermions f and l

$$\mathcal{L}_{fl} = \bar{f}_L \not{A} V_{fl} l_L. \tag{6.33}$$

Here all the quantities are renormalized, but we omit the counterterms as well as the coupling as they are irrelevant to the discussion. However, note that at this point there is no mixing matrix counterterm at all. Next we rotate the fermions by $\mathcal{U}^{f,l}$ and get

$$\begin{aligned} \mathcal{L}_{fl} &= \left(V_{fl} + \sum_{f'} (-ih^{f\,(1)}_{Lff'}) V_{f'l} + \sum_{l'} V_{fl'} (ih^{l\,(1)}_{Ll'l}) \right) \bar{\hat{f}}_L \not{A} \hat{l}_L \\ &= \left(V_{fl} + \delta V^{h\,(1)}_{fl} \right) \bar{\hat{f}}_L \not{A} \hat{l}_L. \end{aligned} \tag{6.34}$$

Here the fields after the rotation are denoted by hats and we also introduced

$$\delta V^{h\,(1)}_{fl} = \sum_{f'} (-ih^{f\,(1)}_{Lff'}) V_{f'l} + \sum_{l'} V_{fl'} (ih^{l\,(1)}_{Ll'l})\,, \tag{6.35}$$

which the authors in [29] associate with the mixing matrix counterterm. As long as the masses and, in turn, the rotation $\mathcal{U}$ are defined in a gauge-independent way, the resulting counterterm δV^h is also gauge-independent. In particular, Kniehl and Sirlin define the mass counterterms as terms that are free of $\not{p} - m_i$ or $\not{p} - m_j$ in Eq. (5.11). These factors give rise to $m_i^2 - m_j^2$, which we have

employed, and thus lead to a gauge-independent definition of mass counterterms. Even more so, as we discuss in Section 8.3.2, the divergent parts in our scheme and that of [29] are the same up to tadpole contributions. It is not hard to see that this is the case, by rewriting the r.h.s. of Eq. (6.1) in terms of ihs

$$\begin{aligned} f(\Sigma) =&(m_i^2 - m_j^2)\delta Z^{A\,(1)}_{Lji} - m_j \delta m^{L\,(1)}_{ji} - m_i \delta m^{R\,(1)}_{ji} \\ =&(m_i^2 - m_j^2)\left(\delta Z^{A\,(1)}_{Lji} - ih^{(1)}_{Lji}\right) . \end{aligned} \tag{6.36}$$

Since the UV divergences match in our scheme and in [29], the anti-hermitian part of the field renormalization is UV finite in both cases. Despite the slightly different definitions of mass counterterms, the only significant difference between the two schemes is the rotation by $\mathcal{U}$. However, even though $\mathcal{U}$ can lead to a gauge-independent mixing matrix counterterm, we see two other issues with this rotation.

First, the counterterm in Eq. (6.35) is singular in the degenerate mass limit because of the factors of $(m_i^2 - m_j^2)^{-1}$ in Eq. (6.32a). This does not necessarily mean that the physical observables also become singular, as long as these singularities cancel out, but the intermediate steps are bound to be problematic. For example, evaluating the finite parts of δV^H in the degenerate mass limit is cumbersome. In contrast, by not performing the rotation by $\mathcal{U}$ such problems are avoided altogether.

The second issue is that before the rotation by $\mathcal{U}$ the theory is already perfectly well renormalized and there is no need to do anything else. The effect of the rotation is that of relabeling and introducing singular counterterms. In more detail, take note that the $ih^{(1)}$ is of 1-loop order so that the rotation does not change the counterterm-free part of the Lagrangian, which also leaves the self-energies, *i.e.* $f(\Sigma)$ in Eq. (6.36), unchanged. Naively, it would seem that the rotation by $\mathcal{U}$ also removes the mass counterterms from the very same Eq. (6.36), but that is not the case and one must first look at the renormalization condition in Eq. (5.11). There it is important that the rotation first changes the renormalized self-energy by rotating the fields and afterward by rotating the spinors, hence,[7]

$$\left[\Sigma^R_{ji} u_i\right]^{(1)} = \left(\hat{\Sigma}^{R\,(1)}_{ji} + (\not{p} - m_j) \underbrace{\left(ih^{(1)}_{Lji} P_L + ih^{(1)}_{Rji} P_R \right)}_{\delta m \to \text{LSZ}} \right) \hat{u}_i . \tag{6.37}$$

Here the hats denote quantities after the rotation by $\mathcal{U}$, in particular, in $\hat{\Sigma}^{R\,(1)}$ the mass counterterm is purely diagonal, but the self-energy is otherwise unchanged.

[7] We put the square brackets around the whole l.h.s. since the rotation interacts with the tree-level part too, this gives the factor $\not{p} - m_j$.

Importantly, both sides of the above equation result in the same Eq. (6.1) (or Eq. (6.36)), the only difference is that now the mass counterterms become associated with the LSZ factors via the singular quantities $ih_{L,R}$. Further, if one considers an amplitude like Wu_jd_i, one can see that the counterterm δV^h is immediately canceled by the rotation by $\mathcal{U}$ on the external spinors.

Once again, in our scheme, we do not propose to perform such a rotation since it has no effect and is also in disagreement with the conceptual arguments we laid out in Section 3. Nonetheless, we will use the quantities $ih_{L,R}$ for the sake of comparing the UV parts with the ones of [29] in Section 8.2.

6.3.3. The scheme of Baro and Boudjema

Next, we bring our attention to a scheme by Baro and Boudjema [39], which also has similarities with our scheme. The authors considered sfermions (scalars) in the MSSM with OS renormalization conditions, $\mathcal{Z} = \bar{\mathcal{Z}} = \mathbb{1}$, and explicitly use the $\widetilde{Re}$ operator. Importantly, they also employ the $m_i^2 - m_j^2$ mass structure by defining the mass counterterm as the pole of $(m_i^2 - m_j^2)^{-1}$ in the degenerate mass limit and then solving for the field renormalization. In a way, this is opposite to our scheme, where we define the field renormalization by using the $m_i^2 - m_j^2$ factor and solve for the mass counterterms. In both schemes, there are no mixing matrix counterterms, but there are also a few differences and the scheme of [39] cannot be applied to fermions or, at the very least, the needed properties are not reproduced after such application.

In more detail, the authors consider a system of two (scalar) particles $\tilde{f}_{1,2}$ with masses $m^2_{\tilde{f}_{1,2}}$ and form the variables

$$m_{\pm}^2 = \frac{m^2_{\tilde{f}_1} \pm m^2_{\tilde{f}_2}}{2} \,. \tag{6.38}$$

Here m_-^2 is the difference of squared masses, which we also employ. All the scalar self-energies $\Pi\left(m^2_{\tilde{f}_1}, m^2_{\tilde{f}_2}\right)$, which on shell are considered to be functions of the two masses, are then rewritten in terms of the new variables $m_\pm^2$ as $\Pi\left(m_+^2, m_-^2\right)$. Further, one should expand the self-energies in terms of the m_-^2 variable

$$\Pi\left(m_+^2, m_-^2\right) = \Pi\left(m_+^2, 0\right) + m_-^2 \partial_{m_-^2} \Pi\left(m_+^2, 0\right) + \dots \tag{6.39}$$

The first term in this expansion becomes the pole in the degenerate mass limit and is associated with the mass counterterm, while all the other terms proportional to m_-^2 vanish in this limit and are associated with the field renormalization. This procedure seems to work well for scalars, but it cannot be applied to fermions

since the two variables $m_\pm^2$ are not enough. As we have seen, fermions have more mass structures, which are associated with mass counterterms and UV divergences in Eqs. (6.28) and (6.15). For example, if one has a UV divergence proportional to $2m_im_j$ and expresses it in terms of $m_\pm^2$, then one spuriously generates a UV divergence proportional to m_-^2, which is then associated with the field renormalization. As we have seen in Section 6.3.1, UV-finiteness of the anti-hermitian part of the field renormalization is essential in order to have trivial mixing matrix counterterms — application of the methods in [39] to fermions obstructs this. In contrast, the anti-hermitian part of the field renormalization is kept UV-finite in our scheme.

7. On-Shell fermion renormalization to all-orders

In what follows we use the renormalization conditions from Sections 5 and 5.2 to extend our scheme to all orders in perturbation theory. In particular, we show that at every order the mass structure $m_i^2 - m_j^2$ appears and is associated with gauge-dependence such that the anti-hermitian part of the field renormalization can be defined analogously to Eq. (6.3). We remind that for simpler discussion, the absorptive parts are dropped, hence, the LSZ factors are trivial $\mathcal{Z} = \overline{\mathcal{Z}} = \mathbb{1}$ in this section. This also allows us to include the diagonal parts in the discussion, which is a necessity.

Our main driver for the discussion is the use of Nielsen identities we have seen in Eq. (3.20) at 1-loop and in Eq. (6.8) already at all orders. We take that the form of the Nielsen identity in Eq. (6.8) holds for Σ as well as for $\widetilde{\Sigma}$ since all the modifications can only come from the renormalization and theory not being at the minimum. We will perform the discussion in two steps: with the bare gauge parameter ξ_0 and then in terms of the renormalized one ξ. The R_ξ gauge-fixing term should not be renormalized (*e.g.* [112]), but it will help us retrieve all the renormalization-related modifications in Eq. (6.6).

7.1. Bare gauge parameter

We have to investigate both the diagonal and off-diagonal renormalization conditions, but they influence one another. We will start the discussion with the off-diagonal component, which is much simpler, and will assume a certain form of the gauge derivatives of the diagonal components of the field renormalization. Once we deal with the diagonal components, we will see that the assumption is self-fulfilling. One could also reverse the order: begin with the diagonal components and assume a form for the off-diagonal ones without changing the

results.

We take the no mixing condition in Eq. (5.5a) and use Eq. (4.2)

$$\left[\gamma^0 Z^\dagger \gamma^0 \widetilde{\Sigma}(\not{p}) Z\right]^{(n)}_{ji} u_i = 0, \tag{7.1}$$

Here the LSZ factors are trivial and we have switched to the condensed notation from Section 4. Note that since we have dropped the absorptive parts, this also holds when $i = j$. The diagonal case only defines the mass counterterm and we will deal with the unit residue condition separately.

Next, we assume that up to order n all the counterterms are defined and only the nth order counterterms are yet to be fixed. This, along with Eqs. (4.7) and (4.8), allows us to rewrite Eq. (7.1) as follows

$$\begin{aligned}(\not{p} - m_j)\,\delta Z^{(n)}_{ji} u_i &= -\left[\widetilde{\Sigma} + \widetilde{\Sigma}^{(>0)}\delta Z + \gamma^0 \delta Z^\dagger \gamma^0 \left(\widetilde{\Sigma} + \widetilde{\Sigma}\delta Z\right)\right]^{(n)}_{ji} u_i \\ &= -\left[\widetilde{\Sigma} + \widetilde{\Sigma}^{(>0)}\delta Z\right]^{(n)}_{ji} u_i\,.\end{aligned} \tag{7.2}$$

Here we have omitted the $\not{p}$ arguments for brevity and the second equality follows since the $\delta Z^\dagger$ term is always multiplied by the lower order result of Eq. (7.2). For example, at 1st order, the term is not present, at 2nd order the parentheses contain the 1st order renormalization condition, *etc.* An analogous way to rewrite this is

$$\left[\widetilde{\Sigma} Z\right]^{(n)}_{ji} u_i = 0 \tag{7.3}$$

and will be useful to make some terms vanish in further calculations.

Now we may very explicitly see the $m_i^2 - m_j^2$ mass structure by multiplying everything in Eq. (7.2) by $\not{p} + m_j$ on the left

$$\left(m_i^2 - m_j^2\right)\delta Z^{(n)}_{ji} u_i = -\left(\not{p} + m_j\right)\left[\widetilde{\Sigma} + \widetilde{\Sigma}^{(>0)}\delta Z\right]^{(n)}_{ji} u_i\,. \tag{7.4}$$

Here we use the fact that $\not{p}^2 = p^2 \mathbb{1}$, which can be freely commuted through to the spinor where we could use the Dirac equation. What remains to be shown is that on the r.h.s. the same $m_i^2 - m_j^2$ mass structure is present. To do so, let us finally turn to the Nielsen identity, which we rewrite for convenience

$$\partial_{\xi_0}\widetilde{\Sigma} = \widetilde{\Lambda}\widetilde{\Sigma} + \widetilde{\Sigma}\widetilde{\Lambda}\,. \tag{7.5}$$

Then the derivative of Eq. (7.4) is

$$\begin{aligned}\left(m_i^2 - m_j^2\right)\partial_{\xi_0}\delta Z^{(n)}_{ji} u_i = &-\left(\not{p} + m_j\right)\Big[\widetilde{\Lambda}\widetilde{\Sigma} + \widetilde{\Sigma}\widetilde{\Lambda} \\ &+ \left(\widetilde{\Lambda}\widetilde{\Sigma} + \widetilde{\Sigma}\widetilde{\Lambda}\right)\delta Z + \widetilde{\Sigma}^{(>0)}\partial_{\xi_0}\delta Z\Big]^{(n)}_{ji} u_i\,.\end{aligned} \tag{7.6}$$

Here the superscript (>0) of Eq. (7.4) becomes irrelevant for $\widetilde{\Lambda}$s, since the first non-trivial gauge derivative is at 1st order. The $\widetilde{\Lambda}\widetilde{\Sigma}$ terms can be added together to give $\widetilde{\Lambda}\widetilde{\Sigma}Z$, but this vanishes because of Eq. (7.3). We group the remaining terms by splitting the self-energy into the 0th order and higher than 0th order contributions

$$\begin{aligned}\left(m_i^2 - m_j^2\right)\partial_{\xi_0}\delta Z_{ji}^{(n)} u_i =& -\left(\not{p} + m_j\right)\Big[\left(\not{p} - m_j\right)\widetilde{\overline{\Lambda}}Z \\ &+\widetilde{\Sigma}^{(>0)}\left(\widetilde{\overline{\Lambda}}Z + \partial_{\xi_0}\delta Z\right)\Big]_{ji}^{(n)} u_i \\ =& -\left(m_i^2 - m_j^2\right)\left[\widetilde{\overline{\Lambda}}Z\right]_{ji}^{(n)} u_i \\ &-\left(\not{p} + m_j\right)\left[\widetilde{\Sigma}^{(>0)}\left(\widetilde{\overline{\Lambda}}Z + \partial_{\xi_0}\delta Z\right)\right]_{ji}^{(n)} u_i\end{aligned} \tag{7.7}$$

Here the $m_i^2 - m_j^2$ mass structure is evident on both sides, but there is an additional term without it. One can notice that the additional contributions vanish if the terms in the parentheses cancel, *i.e.* if $\partial_{\xi_0}\delta Z = -\widetilde{\overline{\Lambda}}Z$. One can see that this is actually the case by simply inserting $n = 1$, $n = 2$ and so on since $\widetilde{\Sigma}^{(>0)}$ only starts at 1st order. However, most importantly here we have to assume that the gauge-dependent parts of the diagonal components of the field renormalization follow the same form, even though it is fixed by a different renormalization condition. If the assumption holds, then we get

$$\partial_{\xi_0}\delta Z_{ji}^{(n)} u_i = -\left[\widetilde{\overline{\Lambda}}Z\right]_{ji}^{(n)} u_i\,.$$

To justify our assumption let us turn to the unit residue condition in Eq. (5.4), which trivially holds at tree-level, while beyond it we have

$$\lim_{\not{p}\to m_i} \frac{1}{\not{p} - m_i}\left[\gamma^0 Z^\dagger \gamma^0 \widetilde{\Sigma}(\not{p}) Z\right]_{ii}^{(n>0)} u_i = 0\,. \tag{7.8}$$

Here again, the LSZ factors are trivial and we have used Eq. (4.2). Just as for the diagonal case we want to separate the nth order counterterms, but the procedure is not simple due to the presence of the limit and $(\not{p} - m_i)^{-1}$. Let us consider only the nth order field renormalization

$$\lim_{\not{p}\to m_i} \frac{1}{\not{p} - m_i}\left[\gamma^0 \delta Z^\dagger \gamma^0 \left(\not{p} - m_i\right) + \left(\not{p} - m_i\right)\delta Z\right]_{ii}^{(n)} u_i\,. \tag{7.9}$$

Here in the second term the $(\not{p} - m_i)$ simply cancels with the one from the

limit, while in the first term we have

$$\begin{aligned}\lim_{\not{p}\to m_i}\frac{1}{\not{p}-m_i}\left[\gamma^0\delta Z^\dagger\gamma^0\left(\not{p}-m_i\right)\right]_{ii}^{(n)}u_i=&\frac{d}{d\not{p}}\left[\gamma^0\delta Z^\dagger\gamma^0\left(\not{p}-m_i\right)\right]_{ii}^{(n)}u_i\\=&\delta Z_{ii}^{\dagger\,(n)}u_i\,,\end{aligned}\tag{7.10}$$

where the final equality comes from commuting the derivative through $\gamma^0\delta Z^\dagger\gamma^0$. Taking this into account, we see that at nth order one gets twice the hermitian part of the field renormalization. Returning to Eq. (7.8) we may write

$$\begin{aligned}\delta Z_{ii}^{H\,(n)}u_i=-\frac{1}{2}\lim_{\not{p}\to m_i}\frac{1}{\not{p}-m_i}\Big[&\widetilde{\Sigma}+\widetilde{\Sigma}^{(>0)}\delta Z\\&+\gamma^0\delta Z^\dagger\gamma^0\left(\widetilde{\Sigma}^{(>0)}+\widetilde{\Sigma}\delta Z\right)\Big]_{ii}^{(n)}u_i\,.\end{aligned}\tag{7.11}$$

Here we cannot use the other renormalization conditions to make simplifications due to the presence of the limit. Let us then directly take the gauge derivative of the above equation

$$\begin{aligned}\partial_{\xi_0}\delta Z_{ii}^{H\,(n)}u_i=&-\frac{1}{2}\lim_{\not{p}\to m_i}\frac{1}{\not{p}-m_i}\Big[\left(\widetilde{\Lambda}\widetilde{\Sigma}+\widetilde{\Sigma}\widetilde{\Lambda}\right)Z+\widetilde{\Sigma}^{(>0)}\partial_{\xi_0}\delta Z\\&+\gamma^0\partial_{\xi_0}\delta Z^\dagger\gamma^0\left(\widetilde{\Sigma}^{(>0)}+\widetilde{\Sigma}\delta Z\right)+\gamma^0\delta Z^\dagger\gamma^0\left(\widetilde{\Lambda}\widetilde{\Sigma}+\widetilde{\Sigma}\widetilde{\Lambda}\right)Z\\&+\gamma^0\delta Z^\dagger\gamma^0\widetilde{\Sigma}\partial_{\xi_0}\delta Z\Big]_{ii}^{(n)}u_i\,.\\=&-\frac{1}{2}\lim_{\not{p}\to m_i}\frac{1}{\not{p}-m_i}\Big[\gamma^0Z^\dagger\gamma^0\left(\widetilde{\Lambda}\widetilde{\Sigma}+\widetilde{\Sigma}\widetilde{\Lambda}\right)Z\\&+\gamma^0\partial_{\xi_0}\delta Z^\dagger\gamma^0\left(\widetilde{\Sigma}^{(>0)}+\widetilde{\Sigma}\delta Z\right)\\&+\left(\gamma^0\delta Z^\dagger\gamma^0\widetilde{\Sigma}+\widetilde{\Sigma}^{(>0)}\right)\partial_{\xi_0}\delta Z\Big]_{ii}^{(n)}u_i\,.\end{aligned}\tag{7.12}$$

Here we combined some terms to have field renormalization constants instead of the counterterm according to Eq. (4.8) and grouped the gauge derivatives of the counterterms to get the second equality. Next, if a self-energy is either fully on the right or on the left, we separate the 0th order contribution

$$\begin{aligned}\partial_{\xi_0}\delta Z_{ii}^{H\,(n)}u_i=&-\frac{1}{2}\lim_{\not{p}\to m_i}\frac{1}{\not{p}-m_i}\Big[\gamma^0Z^\dagger\gamma^0\widetilde{\Lambda}\left(\not{p}-m_i\right)+\left(\not{p}-m_i\right)\widetilde{\widetilde{\Lambda}}Z\\&+\gamma^0Z^\dagger\gamma^0\widetilde{\Lambda}\left(\widetilde{\Sigma}^{(>0)}+\widetilde{\Sigma}\delta Z\right)+\left(\gamma^0\delta Z^\dagger\gamma^0\widetilde{\Sigma}+\widetilde{\Sigma}^{(>0)}\right)\widetilde{\widetilde{\Lambda}}Z\\&+\gamma^0\partial_{\xi_0}\delta Z^\dagger\gamma^0\left(\widetilde{\Sigma}^{(>0)}+\widetilde{\Sigma}\delta Z\right)\end{aligned}$$

$$
\begin{aligned}
&+\left(\gamma^0\delta Z^\dagger\gamma^0\widetilde{\Sigma}+\widetilde{\Sigma}^{(>0)}\right)\partial_{\xi_0}\delta Z\Big]_{ii}^{(n)}u_i \qquad (7.13)\\
=&-\frac{1}{2}\lim_{\not{p}\to m_i}\frac{1}{\not{p}-m_i}\left[\gamma^0 Z^\dagger\gamma^0\widetilde{\Lambda}\left(\not{p}-m_i\right)+\left(\not{p}-m_i\right)\widetilde{\overline{\Lambda}}Z\right.\\
&+\left(\gamma^0 Z^\dagger\gamma^0\widetilde{\Lambda}+\gamma^0\partial_{\xi_0}\delta Z^\dagger\gamma^0\right)\times\left(\widetilde{\Sigma}^{(>0)}+\widetilde{\Sigma}\delta Z\right)\\
&\left.+\left(\gamma^0\delta Z^\dagger\gamma^0\widetilde{\Sigma}+\widetilde{\Sigma}^{(>0)}\right)\times\left(\widetilde{\overline{\Lambda}}Z+\partial_{\xi_0}\delta Z\right)\right]_{ii}^{(n)}u_i\,.
\end{aligned}
$$

For the second equality, we combined similar terms such that the factors emphasized by the $\times$ symbol become clear.

Let us consider the very first line of the above equation, where we may try to cancel out the factors of $\not{p}-m_i$ just like we did for the field renormalization. In particular, let us take the first term

$$
\lim_{\not{p}\to m_i}\frac{1}{\not{p}-m_i}\left[\gamma^0 Z^\dagger\gamma^0\widetilde{\Lambda}\left(\not{p}-m_i\right)\right]=\lim_{\not{p}\to m_i}\frac{d}{d\not{p}}\left[\gamma^0 Z^\dagger\gamma^0\widetilde{\Lambda}d\not{p}\right]\,. \qquad (7.14)
$$

Here we must commute the momentum derivative to act on the differential $d\not{p}$ as we did in Eq. (7.10), but take care in dealing with $\widetilde{\Lambda}$. Decomposing $\widetilde{\Lambda}$ like the self-energy in Eq. (4.13) and taking just the $\widetilde{\Lambda}^{\gamma L}$ term we have

$$
\begin{aligned}
\lim_{\not{p}\to m_i}\frac{d}{d\not{p}}\left[Z_L^\dagger P_R\widetilde{\Lambda}^{\gamma L}(p^2)\not{p}P_L d\not{p}\right]u_i&=\lim_{\not{p}\to m_i}\left[Z_L^\dagger\widetilde{\Lambda}^{\gamma L}(p^2)\not{p}P_R\right]u_i\\
&=\left[Z_L^\dagger\widetilde{\Lambda}^{\gamma L}\left(m_i^2\right)m_i P_R\right]u_i \qquad (7.15)\\
&=\left[Z_L^\dagger\widetilde{\Lambda}^{\gamma L}\left(p^2\right)\not{p}P_L\right]u_i\,.
\end{aligned}
$$

The effect in $\widetilde{\Lambda}^{\gamma R}$ term is the same, while the $\widetilde{\Lambda}^{sL,sR}$ terms have their projectors exchanged, *i.e.* $P_L\longleftrightarrow P_R$. Next, we note that the absorptive parts are dropped, hence, the pseudo-hermiticity property, $\gamma^0\widetilde{\Lambda}(\not{p})\gamma^0=\left(\widetilde{\overline{\Lambda}}(\not{p})\right)^\dagger$, holds. This property relates the scalar functions in the decomposition so that

$$
\widetilde{\Lambda}^{\gamma L,\gamma R}=\left(\widetilde{\overline{\Lambda}}^{\gamma L,\gamma R}\right)^\dagger\quad\text{and}\quad\widetilde{\Lambda}^{sL,sR}=\left(\widetilde{\overline{\Lambda}}^{sR,sL}\right)^\dagger\,. \qquad (7.16)
$$

These relations allow writing

$$
\begin{aligned}
\lim_{\not{p}\to m_i}\frac{d}{d\not{p}}\left[\gamma^0 Z^\dagger\gamma^0\widetilde{\Lambda}d\not{p}\right]u_i&=\left(Z_L^\dagger\widetilde{\overline{\Lambda}}^{\gamma L\dagger}\not{p}P_L+Z_R^\dagger\widetilde{\overline{\Lambda}}^{\gamma R\dagger}\not{p}P_R\right.\\
&\quad\left.+Z_R^\dagger\widetilde{\overline{\Lambda}}^{sR\dagger}P_R+Z_L^\dagger\widetilde{\overline{\Lambda}}^{sL\dagger}P_L\right)u_i \qquad (7.17)\\
&\equiv\left(\widetilde{\overline{\Lambda}}Z\right)^\dagger u_i\,,
\end{aligned}
$$

such that Eq. (7.13) becomes

$$\begin{aligned}\partial_{\xi_0}\delta Z_{ii}^{H\,(n)}u_i = &-\frac{1}{2}\left[\left(\widetilde{\overline{\Lambda}}Z\right)^\dagger + \widetilde{\overline{\Lambda}}Z\right]_{ii}^{(n)} u_i - \frac{1}{2}\lim_{\not{p}\to m_i}\frac{1}{\not{p}-m_i}\Big[\\ &\left(\gamma^0 Z^\dagger\gamma^0\widetilde{\Lambda} + \gamma^0\partial_{\xi_0}\delta Z^\dagger\gamma^0\right)\times\left(\widetilde{\Sigma}^{(>0)} + \widetilde{\Sigma}\delta Z\right)\\ &+\left(\gamma^0\delta Z^\dagger\gamma^0\widetilde{\Sigma} + \widetilde{\Sigma}^{(>0)}\right)\times\left(\widetilde{\overline{\Lambda}}Z + \partial_{\xi_0}\delta Z\right)\Big]_{ii}^{(n)} u_i\,.\end{aligned} \tag{7.18}$$

Here the bracket without the limit is the Hermitian part of $\widetilde{\overline{\Lambda}}Z$ and we only have to deal with the two other terms, where the limit is still present.

Fortunately, it can be noticed that each factor in the two terms must be a series in powers of $\not{p} - m_i$. First, the terms without gauge derivatives are just the renormalization conditions in Eq. (7.3), but with one of them being (pseudo-)hermitian conjugated and both without the tree-level contribution of $\not{p} - m_i$. Nonetheless, both conditions vanish on-shell at every order when acted on by the spinor u_i on the right or, for the conjugated case, by $\overline{u}_i$ on the left. In turn, these factors must be proportional to powers of $\not{p} - m_i$

$$\left[\widetilde{\Sigma}^{(>0)} + \widetilde{\Sigma}\delta Z\right]_{ki} = \sum_{\beta=1}(A_\beta)_{ki}\,(\not{p} - m_i)^\beta\,. \tag{7.19}$$

Here A_β are some coefficients that can also have a non-trivial Dirac structure, but further determination is not needed. Analogously,

$$\begin{aligned}\left[\gamma^0\delta Z^\dagger\gamma^0\widetilde{\Sigma} + \widetilde{\Sigma}^{(>0)}\right]_{ik} =&\gamma^0\left[\left(\widetilde{\Sigma}Z\right)^\dagger + \widetilde{\Sigma}^{(>0)\dagger}\right]_{ik}\gamma^0\\ =&\sum_{\beta=1}(\not{p} - m_i)^\beta\,\gamma^0(A_\beta)_{ik}^\dagger\gamma^0\,,\end{aligned} \tag{7.20}$$

where for the first equality we have used the pseudo-hermiticity of self-energies.

Similarly, the factors with gauge derivatives in Eq. (7.18) also vanish on-shell as long as $\partial_{\xi_0}\delta Z$ cancels $\widetilde{\overline{\Lambda}}Z$ in the parentheses (and analogously for the conjugated factor). For the time being, we assume that this is the case, since we already started with this assumption and the first line in Eq. (7.18) also implies this form for the diagonal components. In turn, we may write an analogous expansion in powers of $\not{p} - m_i$

$$\left[\widetilde{\overline{\Lambda}}Z + \partial_{\xi_0}\delta Z\right]_{ki} = \sum_{\beta=1}(B_\beta)_{ki}\,(\not{p} - m_i)^\beta \tag{7.21}$$

and

$$\begin{aligned}\left[\gamma^0 Z^\dagger \gamma^0 \widetilde{\Lambda} + \gamma^0 \partial_{\xi_0} \delta Z^\dagger \gamma^0\right]_{ik} =& \gamma^0 \left[\left(\widetilde{\overline{\Lambda}} Z\right)^\dagger + \partial_{\xi_0} \delta Z^\dagger\right]_{ik} \gamma^0 \\ =& \sum_{\beta=1} (\not{p} - m_i)^\beta \gamma^0 (B_\beta)^\dagger_{ki} \gamma^0 \end{aligned} \tag{7.22}$$

with some coefficients B_β.

Plugging this into Eq. (7.18) we have

$$\begin{aligned}\partial_{\xi_0} \delta Z_{ii}^{H\,(n)} u_i =& -\left[\widetilde{\overline{\Lambda}} Z\right]_{ii}^{H(n)} u_i - \frac{1}{2} \lim_{\not{p} \to m_i} \frac{1}{\not{p} - m_i} \Bigg[\\ & \sum_{\beta,\alpha=1} (\not{p} - m_i)^\beta \gamma^0 B_\beta^\dagger \gamma^0 \times A_\alpha (\not{p} - m_i)^\alpha \\ & + \sum_{\beta,\alpha=1} (\not{p} - m_i)^\beta \gamma^0 A_\beta^\dagger \gamma^0 \times B_\alpha (\not{p} - m_i)^\alpha \Bigg]_{ii}^{(n)} u_i \\ =& -\left[\widetilde{\overline{\Lambda}} Z\right]_{ii}^{H(n)} u_i \\ & - \frac{1}{2} \Bigg[\sum_{\beta,\alpha=1} (\not{p} - m_i)^{\beta-1} \gamma^0 B_\beta^\dagger \gamma^0 \times A_\alpha (\not{p} - m_i)^\alpha \\ & + \sum_{\beta,\alpha=1} (\not{p} - m_i)^{\beta-1} \gamma^0 A_\beta^\dagger \gamma^0 \times B_\alpha (\not{p} - m_i)^\alpha \Bigg]_{ii}^{(n)} u_i \\ =& -\left[\widetilde{\overline{\Lambda}} Z\right]_{ii}^{H(n)} u_i \,. \end{aligned} \tag{7.23}$$

In the second equality the limit together with $\frac{1}{\not{p} - m_i}$ simply cancels the $\not{p} - m_i$ factors on the left side, for the third equality the factors of $\not{p} - m_i$ on the right vanish due to the spinor u_i so that the final result is rather simple and of the form we need.

We are almost finished with the bare parameter, but it remains to discuss the diagonal component of the anti-hermitian part of the field renormalization. At 1-loop this component can be chosen arbitrarily and can be simply set to 0, beyond 1-loop this component can enter off-diagonal elements as well so it is best to keep the definitions "uniform" even when the absorptive parts are dropped. A rather simple way to do so is by defining the diagonal component from the off-diagonal one by simply setting $j = i$. If the anti-hermitian part is defined as the coefficient of $m_i^2 - m_j^2$, then taking $j = i$ will not cause any

singular behavior since the problematic mass structure is excluded from the counterterm by definition.

Finally, we conclude that the gauge-dependence of every component of the field renormalization in an on-shell scheme can be described by

$$\partial_{\xi_0}\delta Z_{ji}u_i = -\left[\widetilde{\overline{\Lambda}}Z\right]_{ji}u_i\,. \tag{7.24}$$

This result is analogous to the one found in [79], where the authors considered the LSZ factors with the absorptive parts included. The difference is that our approach is perturbative and explicitly shows the $m_i^2 - m_j^2$ mass structure at every order. In turn, this means that the logic behind our 1-loop definition in Eq. (6.3) can be used at all orders. Then, from Eq. (7.4) we may write down the definition of the anti-hermitian part of the field renormalization

$$\delta Z_{ji}^{A(n)}u_i \equiv -\frac{1}{2}\left((\not{p}+m_j)\left[\widetilde{\Sigma}+\widetilde{\Sigma}^{(>0)}\delta Z\right]_{ji}^{(n)} + H.C.\right)u_i\Big|_{m_i^2-m_j^2}\,, \tag{7.25}$$

where $H.C.$ should be used after acting with $\not{p}$ on the spinor u_i. One can check that for $n=1$ the 1-loop definition in Eq. (6.3) is reproduced.

For completeness, we also provide the definition of the mass counterterm at order n. To do so, we must first separate the nth order mass counterterm from the self-energy

$$\widetilde{\Sigma}^{(n)} = \hat{\widetilde{\Sigma}}^{(n)} - \delta m^{(n)}\,. \tag{7.26}$$

Having this and taking Eq. (7.2) we get

$$\delta m_{ji}^{(n)}u_i = \left[\hat{\widetilde{\Sigma}}+\widetilde{\Sigma}\delta Z\right]_{ji}^{(n)}u_i \tag{7.27}$$

It is rather simple to check that the mass counterterm defined in this way is gauge-independent

$$\partial_{\xi_0}\delta m_{ji}^{(n)}u_i = \left[\widetilde{\Lambda}\widetilde{\Sigma}Z + \widetilde{\Sigma}\widetilde{\overline{\Lambda}}Z - \widetilde{\Sigma}\widetilde{\overline{\Lambda}}Z\right]_{ji}^{(n)}u_i = 0\,. \tag{7.28}$$

Here we have used the Nielsen Identity also for the hatted self-energy and Eq. (7.24) for the field counterterm; the first term vanishes because of Eq. (7.3) and the two other terms simply cancel out. In truth, this required the assumption that the mass counterterm is gauge-independent such that we could use the Nielsen Identity for the hatted self-energy. However, we see that this assumption causes no contradictions and holds. Of course, this is expected since all the available gauge dependence is already accounted for by the field renormalization as should be in an on-shell scheme.

Up to now, we have considered the bare gauge parameter, *i.e.* this matches the case where there are no gauge parameter counterterms and all the counterterms are gauge-independent. In this section, we consider the case where the gauge parameter receives a counterterm. In addition, this also includes the gauge-dependent renormalization of physical quantities. In what follows we re-derive our result that the mass structure $m_i^2 - m_j^2$ appears together with gauge-dependent parts even in the case of a renormalized gauge parameter.

We start the discussion by taking the gauge derivative of Eq. (7.3) w.r.t. the renormalized gauge parameter ξ, where we use Eq. (4.5) to explicitly Taylor expand the bare self-energy

$$\begin{aligned}\partial_\xi \left(\prod_k \mathbf{T}_{\lambda_k} \Sigma Z\right) u_i =& \partial_\xi \left(\prod_k \mathbf{T}_{\lambda_k}\right) \Sigma Z u_i + \left(\prod_k \mathbf{T}_{\lambda_k} \partial_\xi \Sigma\right) Z u_i \\ &+ \widetilde{\Sigma} \partial_\xi Z u_i \\ =& 0\,.\end{aligned} \tag{7.29}$$

In the first term on the r.h.s. the gauge derivative acts on the counterterms present in the series operator and the operator itself acts only on the self-energy, the third term contains the gauge derivative we are after.

Taking the first term of Eq. (7.29) we may use that $Z = \mathbb{1} + \delta Z$. In addition, we restore the bare self-energy $\widetilde{\Sigma}$ by using the second equality of Eq. (4.5) (in reverse) and move the spinor u_i past the restriction to renormalized parameters

$$\begin{aligned}\partial_\xi \left(\prod_k \mathbf{T}_{\lambda_k}\right) \Sigma Z u_i =& \partial_\xi \left(\prod_k \mathbf{T}_{\lambda_k}\right) \Sigma u_i + \partial_\xi \left(\prod_k \mathbf{T}_{\lambda_k}\right) \Sigma \delta Z u_i \\ =& \partial_\xi \left(\prod_k \mathbf{T}_{\lambda_k^0}\right) \widetilde{\Sigma} u_i|_{\lambda_k^0 = \lambda_k} + \partial_\xi \left(\prod_k \mathbf{T}_{\lambda_k}\right) \Sigma \delta Z u_i\,.\end{aligned} \tag{7.30}$$

Here in the first term of the second equality, we may exchange the self-energy for the self-energy and the field renormalization counterterm via a variant of Eq. (7.3), giving

$$\begin{aligned}\partial_\xi \left(\prod_k \mathbf{T}_{\lambda_k}\right) \Sigma Z u_i =& - \partial_\xi \left(\prod_k \mathbf{T}_{\lambda_k^0}\right) \widetilde{\Sigma} \delta Z|_{\lambda_k^0 = \lambda_k} u_i \\ &+ \partial_\xi \left(\prod_k \mathbf{T}_{\lambda_k}\right) \Sigma \delta Z u_i\,.\end{aligned} \tag{7.31}$$

Importantly, the series operator in the first term now acts on the self-energy and the field renormalization counterterm as can be tracked by the restriction to renormalized parameters. Taking this term we can use the fact that the series expansion of the product $\widetilde{\Sigma}\delta Z$ is the same as the product of separate expansions of $\widetilde{\Sigma}$ and δZ. After this separation the gauge derivative acts on both series operators via the Leibniz rule

$$\begin{aligned}\partial_\xi \left(\prod_k \mathbf{T}_{\lambda_k^0}\right) \widetilde{\Sigma}\delta Z|_{\lambda_k^0=\lambda_k} u_i =& \partial_\xi \left(\prod_k \mathbf{T}_{\lambda_k^0}\right) \widetilde{\Sigma}|_{\lambda_k^0=\lambda_k} \prod_l \mathbf{T}_{\lambda_l^0} \delta Z|_{\lambda_l^0=\lambda_l} u_i \\ &+ \prod_k \mathbf{T}_{\lambda_k^0} \widetilde{\Sigma}|_{\lambda_k^0=\lambda_k} \partial_\xi \left(\prod_l \mathbf{T}_{\lambda_l^0}\right) \delta Z|_{\lambda_l^0=\lambda_l} u_i \\ =& \partial_\xi \left(\prod_k \mathbf{T}_{\lambda_k}\right) \Sigma\delta Z u_i \\ &+ \widetilde{\Sigma}\partial_\xi \left(\prod_k \mathbf{T}_{\lambda_k^0}\right) \delta Z|_{\lambda_k^0=\lambda_k} u_i\end{aligned} \tag{7.32}$$

Here for the final equality we have removed the series expansions on which the gauge derivative does not act as well as the restriction in the first term via Eq. (4.5). Finally, we can plug this result back into Eq. (7.31) and get

$$\begin{aligned}\partial_\xi \left(\prod_k \mathbf{T}_{\lambda_k}\right) \Sigma Z u_i =& -\partial_\xi \left(\prod_k \mathbf{T}_{\lambda_k}\right) \Sigma\delta Z u_i - \widetilde{\Sigma}\partial_\xi \left(\prod_k \mathbf{T}_{\lambda_k^0}\right) \delta Z|_{\lambda_k^0=\lambda_k} u_i \\ &+ \partial_\xi \left(\prod_k \mathbf{T}_{\lambda_k}\right) \Sigma\delta Z u_i \\ =& -\widetilde{\Sigma}\partial_\xi \left(\prod_k \mathbf{T}_{\lambda_k^0}\right) \delta Z|_{\lambda_k^0=\lambda_k} u_i \\ =& -\widetilde{\Sigma}\partial_\xi \left(\prod_k \mathbf{T}_{\lambda_k^0}\right) Z|_{\lambda_k^0=\lambda_k} u_i \,.\end{aligned} \tag{7.33}$$

Here the final equality follows since the gauge derivative acting on $\mathbb{1}$ gives 0.

Next, we take the second term in Eq. (7.29) and notice that the gauge derivative acting on the self-energy Σ simply gives the Nielsen identity

$$\begin{aligned}\left(\prod_k \mathbf{T}_{\lambda_k} \partial_\xi \Sigma\right) Z u_i =& \left(\prod_k \mathbf{T}_{\lambda_k} \left(\Lambda\Sigma + \Sigma\overline{\Lambda}\right)\right) Z u_i \\ =& \left(\widetilde{\Lambda}\widetilde{\Sigma} + \widetilde{\Sigma}\widetilde{\overline{\Lambda}}\right) Z u_i = -\widetilde{\Sigma}\partial_{\xi_0} Z u_i \,.\end{aligned} \tag{7.34}$$

Here the second equality follows by simply writing the self-energies and Λ's in terms of bare parameters, the final equality follows by using Eq. (7.3) in the first term and Eq. (7.24) in the second one.

At this point, we have all the ingredients and can return to Eq. (7.29), into which we insert Eqs. (7.33) and (7.34)

$$\partial_\xi \left(\prod_k \mathbf{T}_{\lambda_k} \Sigma Z \right) u_i = \widetilde{\Sigma} \left(-\partial_\xi \left(\prod_k \mathbf{T}_{\lambda_k^0} \right) Z|_{\lambda_k^0=\lambda_k} - \partial_{\xi_0} Z + \partial_\xi Z \right) u_i = 0 \,. \tag{7.35}$$

Here the final equality holds if the parentheses vanish, which gives the desired relation between derivatives w.r.t. the bare and renormalized parameters

$$\partial_\xi Z = \partial_{\xi_0} Z + \partial_\xi \left(\prod_k \mathbf{T}_{\lambda_k^0} \right) Z|_{\lambda_k^0=\lambda_k} \tag{7.36}$$

or, alternatively,

$$\partial_\xi = \partial_{\xi_0} + \partial_\xi \left(\prod_k \mathbf{T}_{\lambda_k^0} \right) \ldots |_{\lambda_k^0=\lambda_k} \,, \tag{7.37}$$

where the $\ldots$ indicates a placeholder. This shows that the bare and renormalized gauge parameter derivatives differ only due to the shift induced by non-physical renormalization of the gauge parameter or other parameters of the theory. In other words, with this one can recover the gauge-dependence arising from the modifications of the Nielsen identity. Interestingly, this shift does not destroy the mass structure $m_i^2 - m_j^2$ such that one can still use the definition of the anti-hermitian part of the field renormalization in Eq. (7.4).

7.3. Practical considerations

As we have presented the definitions and properties of our scheme it is needed to think of how to actually use the scheme in practice. In Section 8, where we apply the scheme to the Grimus–Neufeld model, we will see that at 1-loop one can use a naive approach, which amounts to collecting terms and selecting terms proportional to $m_i^2 - m_j^2$. However, this approach requires explicitly evaluating the self-energies and in some cases even the loop functions. Even more so, this also requires keeping the gauge parameter explicit since otherwise the relevant mass structures may disappear. While doable at 1-loop, it is easy to see that a naive approach becomes cumbersome for higher orders and one needs a better recipe. Such a recipe necessarily requires finding a way of computing

the anti-hermitian part of the field renormalization independently of the mass counterterm, since then one could skip the selection of $m_i^2 - m_j^2$ terms.

Let us consider Eq. (7.24) and treat it as a renormalization condition for $\widetilde{\overline{\Lambda}}(\not{p})$, *i.e.* the gauge derivative of Z renormalizes $\widetilde{\overline{\Lambda}}(\not{p})$. Importantly, Λ's appearing in the Nielsen identity are correlation functions involving BRST sources and can be computed separately, equivalently, this allows us to compute the gauge-dependent part of the field renormalization independently of the mass counterterm. For example, in the SM one could use the Lagrangian from the appendix of [79] to do so. Once the field renormalization is computed, one can plug it into Eq. (7.27) and get the mass counterterm without worrying about $m_i^2 - m_j^2$.

To sketch the procedure in slightly more detail, we may consider the simplest case where all the gauge parameters are equal, $\xi_i = \xi$, and integrate over the anti-hermitian part of Eq. (7.36)

$$\begin{aligned} Z^A_{ji} u_i =& \frac{1}{2}\int d\xi \left[\partial_{\xi_0} Z_{ji} + \partial_\xi \left(\prod_k \mathbf{T}_{\lambda_k^0}\right) Z_{ji}|_{\lambda_k^0=\lambda_k} - H.C.\right] u_i \\ =& \frac{1}{2}\int d\xi \left[-\left(\widetilde{\overline{\Lambda}}(\not{p}) Z\right)_{ji} + \partial_\xi \left(\prod_k \mathbf{T}_{\lambda_k^0}\right) Z_{ji}|_{\lambda_k^0=\lambda_k} - H.C.\right] u_i \,. \end{aligned} \tag{7.38}$$

Here the second equality follows from using Eq. (7.24) in the first term, this first term should eventually be written in terms of renormalized parameters by using the series operator and only then integrated over, and $H.C.$ should be used after using the Dirac equation for terms containing $\not{p}$. Note that the integral is indefinite, but we do not write the gauge-independent integration constant since it is identified with the mass counterterm.

At 1-loop the situation is simpler since the term with the series operator contributes only beyond 1-loop, we then have

$$\begin{aligned} Z^{A\,(1)}_{ji} u_i =& -\frac{1}{2}\int d\xi \left[\widetilde{\overline{\Lambda}}^{(1)}(\not{p}) - H.C.\right] u_i \\ =& -\frac{1}{2}\int d\xi \left[\overline{\Lambda}^{(1)}(\not{p}) - H.C.\right] u_i \,. \end{aligned} \tag{7.39}$$

Here the final equality follows since Λ's vanish at tree level so that expanding $\widetilde{\overline{\Lambda}}^{(1)}$ in terms of renormalized parameters gives no additional contributions at 1-loop level.

A more complicated situation is when there is more than one distinct gauge parameter. In that case, it is still possible to find the gauge-dependent part of the

field renormalization, but the integral has to be modified to account for terms where more than one gauge parameter appears together. Considering two gauge parameters ξ_1 and ξ_2 we have

$$Z^A_{ji}u_i = \left[\int d\xi_1 \partial_{\xi_1} Z^A_{ji} + \int d\xi_2 \partial_{\xi_2} Z^A_{ji} - \int d\xi_2 \int d\xi_1 \partial_{\xi_1}\partial_{\xi_2} Z^A_{ji}\right] u_i \,. \tag{7.40}$$

One can take a simple test function $Z^A_{ji} = \xi_1 + \xi_2 + \xi_1\xi_2$ to see that the integrals recover it from the derivatives and that the equality holds. Importantly, the double integral is essential in accounting for the double counting of terms such as $\xi_1\xi_2$. If there are more gauge parameters the logic can be generalized to account for triple counting, *etc*.

The advantage of this recipe is, as already mentioned, the fact that one can avoid looking for the $m_i^2 - m_j^2$ mass structure. Another convenience is that one can first construct the momentum integrals of $\Lambda(\not{p})$ and integrate over the gauge parameters without integrating over momentum. Once the integration over gauge parameters is completed, the integration over momentum can be performed analytically or numerically, depending on which is more convenient. In addition, after integrating over gauge parameters one may choose a convenient gauge, *e.g.* the Feynman gauge.

A slight disadvantage of this approach is that the anti-hermitian part of the field renormalization retrieved from integrating over the Λ's is purely gauge-dependent. Any gauge-independent term multiplied by the $m_i^2 - m_j^2$ mass structure and that should be included in the anti-hermitian part of the field renormalization according to Eq. (7.4) will be missed. In other words, the practical recipe for computing the anti-hermitian part of the field renormalization does not fully match the definitions of the scheme and differs by possible gauge-independent terms. However, this is not a big issue since including these terms in the mass counterterm does not destroy the on-shell renormalization conditions and does not induce singular degenerate mass limits. Despite the difference between the two approaches, the important part is that in both cases the field renormalization accounts for all the possible gauge dependence (up to absorptive parts) such that the Λ's are renormalized.

8. Applying the proposed renormalization scheme to the Grimus–Neufeld model at 1-loop

In what follows we introduce the so-called Grimus–Neufeld model and provide examples of the proposed scheme within the confines of this model albeit the scheme is not limited to this particular model. All the computations for this model were made by using the Two Higgs Doublet Model (THDM) [113] files [114], which we adapted for the Grimus–Neufeld model, as well as the following software packages: `FeynRules` [115], `FeynArts` [116], `FeynCalc` [117–119], and `PackageX` [120]. The conventions of Passarino-Veltman (PV) functions [111] follow the ones used by the packages, although we also include the factor of $\frac{1}{2^D \pi^{D-2}}$ in the definition of these functions, where the spacetime dimension is $D = 4 - \epsilon$.

Results of this section along with the model files are available in the Lithuanian Electronic Academic Library `elaba.lt` along with the dissertation.

8.1. The Grimus–Neufeld model

8.1.1. The model at tree-level

Already over 30 years ago W. Grimus and H. Neufeld considered extending the Standard Model by additional Higgs doublets as well as left-handed lepton doublets and right-handed singlets [48]. The authors have noticed that depending on the amounts of lepton doublets and singlets there will be massless Majorana neutrinos at tree level, some of which can acquire radiative masses at 1-loop and beyond. The minimal case is an extension by one Higgs doublet and one Majorana neutrino — exactly this we call the Grimus–Neufeld Model (GNM). Parts of the following introduction can be found in [50, 114, 121, A4].

For the scalar potential of the GNM, we take the general THDM potential, but with CP-conservation, such that the coupling constants $\mu_{12},\ \lambda_5,\ \lambda_6,\ \lambda_7$ are real [122, 123]

$$\begin{aligned} -V_{\text{THDM}} = {} & \mu_1 \left(H_1^\dagger H_1\right) + \mu_2 \left(H_2^\dagger H_2\right) + \mu_{12} \left[\left(H_1^\dagger H_2\right) + H.C.\right] \\ & + \lambda_1 \left(H_1^\dagger H_1\right)\left(H_1^\dagger H_1\right) + \lambda_2 \left(H_2^\dagger H_2\right)\left(H_2^\dagger H_2\right) \\ & + \lambda_3 \left(H_1^\dagger H_1\right)\left(H_2^\dagger H_2\right) + \lambda_4 \left(H_1^\dagger H_2\right)\left(H_2^\dagger H_1\right) \\ & + \Big[\lambda_5 \left(H_1^\dagger H_2\right)\left(H_1^\dagger H_2\right) + \lambda_6 \left(H_1^\dagger H_1\right)\left(H_1^\dagger H_2\right) \end{aligned}$$

$$+\lambda_7 \left(H_2^\dagger H_2\right)\left(H_1^\dagger H_2\right) + H.C.\Big] \,. \tag{8.1}$$

Here $H_{1,2}$ are the Higgs doublets, $H.C.$ stands for hermitian conjugation, and parentheses emphasize $SU(2)$ index contractions.

Both the doublets are equivalent and the potential could be rewritten in any other basis of the doublets and this would necessarily introduce a mixing angle. As per our discussion in Section 3 mixing angles are not physical and it is best to choose a convenient, but general basis where the mixing angle between Higgs doublets does not appear. Such a convenient choice is known as the Higgs basis and it is defined by only one of the doublets acquiring a vacuum expectation value (VEV), hence,

$$H_1 = \begin{pmatrix} G^+ \\ \frac{1}{\sqrt{2}}\left(v + h_1 + iG^0\right) \end{pmatrix} \qquad \text{and} \qquad H_2 = \begin{pmatrix} H^+ \\ \frac{1}{\sqrt{2}}\left(h_2 + i\sigma\right) \end{pmatrix} . \tag{8.2}$$

Here v is the VEV, $G^{+,0}$ are the charged and neutral Goldstone bosons, H^+ is the physical charged Higgs with a mass m_+, $h_{1,2}$ are the neutral scalar degrees of freedom and σ is the pseudo-scalar one. All the Goldstone bosons are in the first Higgs doublet due to it containing the VEV and making it identical to the one in the SM.

The neutral degrees of freedom can be rotated to the mass eigenstate basis

$$\begin{pmatrix} h_1 \\ h_2 \\ \sigma \end{pmatrix} = R \begin{pmatrix} h \\ H \\ A \end{pmatrix} \,, \tag{8.3}$$

where h and H are scalars and h is associated with the measured Higgs, A is the pseudo-scalar, and the neutral scalar masses are $m_{h,H,A}$, respectively. As we are dealing with the CP-conserving case there is no mixing between scalars and pseudo-scalars, *i.e.* $\sigma = A$, and the rotation matrix R is simply given by

$$R = \begin{pmatrix} c_\alpha & -s_\alpha & 0 \\ s_\alpha & c_\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \,, \tag{8.4}$$

where α is the mixing angle and

$$s_x \equiv \sin x \qquad \text{and} \qquad c_x \equiv \cos x \,. \tag{8.5}$$

By definition, the matrix R diagonalizes the neutral scalar mass matrix

$$v^2 \begin{pmatrix} 2\lambda_1 & \lambda_6 & 0 \\ \lambda_6 & \frac{\mu_2}{v^2} + \frac{1}{2}\left(\lambda_3 + 2\lambda_5 + \lambda_4\right) & 0 \\ 0 & 0 & \frac{\mu_2}{v^2} + \frac{1}{2}\left(\lambda_3 - 2\lambda_5 + \lambda_4\right) \end{pmatrix} = RM^2R^T \,, \tag{8.6}$$

with $M^2 = \text{diag.}\left(m_h^2,\, m_H^2,\, m_A^2\right)$. The charged Higgs mass is

$$m_+^2 = \mu_2 + \frac{1}{2}v^2\lambda_3 \tag{8.7}$$

and the minimum conditions imply two other relations

$$\mu_1 + v^2\lambda_1 = 0 \tag{8.8}$$
$$2\mu_{12} + v^2\lambda_6 = 0\,. \tag{8.9}$$

With all these relations the couplings λ_i and $\mu_{i(j)}$ can be expressed in terms of the scalar masses and the matrix R

$$\lambda_1 = \frac{1}{2v^2}\left(RM^2R^T\right)_{11}\,, \tag{8.10}$$
$$\lambda_4 = \frac{1}{v^2}\left(\left(RM^2R^T\right)_{22} + \left(RM^2R^T\right)_{33} - 2m_+^2\right)\,, \tag{8.11}$$
$$\lambda_5 = \frac{1}{2v^2}\left(\left(RM^2R^T\right)_{22} - \left(RM^2R^T\right)_{33}\right)\,, \tag{8.12}$$
$$\lambda_6 = \frac{1}{v^2}\left(RM^2R^T\right)_{12}\,, \tag{8.13}$$
$$\mu_1 = -\left(RM^2R^T\right)_{11}\,, \tag{8.14}$$
$$\mu_2 = m_+^2 - \frac{v^2\lambda_3}{2}\,, \tag{8.15}$$
$$\mu_{12} = -\frac{1}{2}\left(RM^2R^T\right)_{12}\,. \tag{8.16}$$

The couplings $\lambda_{2,3,7}$ remain unfixed in terms of the physical masses and will appear in various expressions.

Further, we have the left-handed fermion doublets

$$L_i = \begin{pmatrix} \nu'_{Li} \\ e_{Li} \end{pmatrix}\,, \qquad Q_i = \begin{pmatrix} u_{Li} \\ d_{Li} \end{pmatrix} \tag{8.17}$$

containing the neutrinos ν'_i, charged leptons e_i, and up- and down-type quarks u_i, d_i with the index $i = 1, 2, 3$ numbering the generations. The fermions, except for the neutrinos, are taken to already be in the mass eigenstate basis with $m_i^{l,u,d}$ being the respective masses. These doublets along with the Majorana

singlet N couple to the Higgs doublets through the Yukawa interaction

$$\begin{aligned}
-\mathcal{L}_{\mathrm{H-F}} = & \sum_{j,i=1}^{3} (Y_d)_{ji} \overline{d}_{Rj}.(H_1^\dagger Q_i) + \sum_{j,i=1}^{3} (Y_u)_{ji} \overline{u}_{Rj}.(\tilde{H}_1^T Q_i) \\
& + \sum_{j,i=1}^{3} (Y_l)_{ji} \overline{e}_{Rj}.(H_1^\dagger L_i) + \sum_{i=1}^{3} (Y_\nu)_i \overline{N}.(\tilde{H}_1^T L_i) \\
& + \sum_{j,i=1}^{3} (G_d)_{ji} \overline{d}_{Rj}.(H_2^\dagger Q_i) + \sum_{j,i=1}^{3} (G_u)_{ji} \overline{u}_{Rj}.(\tilde{H}_2^T Q_i) \\
& + \sum_{j,i=1}^{3} (G_l)_{ji} \overline{e}_{Rj}.(H_2^\dagger L_i) + \sum_{i=1}^{3} (G_\nu)_i \overline{N}.(\tilde{H}_2^T L_i) + H.C.
\end{aligned} \tag{8.18}$$

Here Y's and G's are the Yukawa couplings to the first and second Higgs doublet respectively, e_R, u_R, and d_R are the right-handed charged lepton, up- and down-type quark singlets, the dot emphasizes the contraction of fermion indices, i, j are family indices, and $(\tilde{H}_{1,2})_\alpha = \sum_{\beta=1}^{2} \epsilon_{\alpha\beta} (H_{1,2})_\beta$ with $\epsilon_{\alpha\beta}$ being the antisymmetric tensor. Since the charged fermions are already in their mass eigenstate basis, the Yukawa couplings to the first Higgs doublet are diagonal, while the couplings to the second Higgs doublet are not diagonal. Diagonalization of the quark Yukawa couplings or, equivalently, the masses of the quarks introduces the quark mixing matrix in the W vertex. The same goes for the diagonalization of charged and neutral leptons, however, for the time being, the charged leptons are in their mass eigenstate, while the neutral ones are not. In addition, we take the neutral leptons to be rotated by the same transformation that diagonalized the charged leptons such that there is no mixing matrix in the $We\nu$ vertex. A mixing matrix will appear once the neutral leptons are rotated to their mass eigenstates.

Before rotating to the mass eigenstate basis of the neutral leptons, we first introduce the Majorana mass term for the singlet N

$$\mathcal{L}_{\mathrm{M}} = -\frac{1}{2} M_R \overline{N}.N = -\frac{1}{2} M_R N^T.\mathcal{C}^{-1}.N\,, \tag{8.19}$$

where M_R is a real Majorana mass parameter and the second equality follows from the Majorana condition in Eq. (5.15). The mass terms for the neutrinos ν'_L stemming from the Dirac mass term can be rewritten to mimic the form of a

Majorana term due to the Majorana nature of the singlet N

$$\begin{aligned}\sum_{i=1}^{3}(Y_\nu)_i\overline{N}.(\langle\tilde{H}_1^T\rangle L_i) =&\frac{v}{\sqrt{2}}\sum_{i=1}^{3}(Y_\nu)_i\overline{N}.\nu'_{Li}\\ =&\frac{v}{\sqrt{2}}\sum_{i=1}^{3}(Y_\nu)_i N^T.\mathcal{C}^{-1}.\nu'_{Li}\\ =&\frac{v}{\sqrt{2}}\sum_{i=1}^{3}(Y_\nu)_i N_\alpha\mathcal{C}^{-1}_{\alpha\beta}\nu'_{Li\beta}\\ =&-\frac{v}{\sqrt{2}}\sum_{i=1}^{3}(Y_\nu)_i\nu'_{Li\beta}\mathcal{C}^{-1}_{\alpha\beta}N_\alpha\\ =&\frac{v}{\sqrt{2}}\sum_{i=1}^{3}(Y_\nu)_i\nu'^T_{Li}.\mathcal{C}^{-1}.N\,.\end{aligned} \tag{8.20}$$

Here we temporarily used spinor indices α and β, and to get the final equality we have used the fact that the charge conjugation matrix is antisymmetric. By considering the second and final lines as well as Eq. (8.19) we can write

$$\begin{aligned}-\mathcal{L}_{\text{mass}} =&\frac{v}{2\sqrt{2}}\sum_{i=1}^{3}\left((Y_\nu)_i N^T.\mathcal{C}^{-1}.\nu'_{Li}+(Y_\nu)_i\nu'^T_{Li}.\mathcal{C}^{-1}.N\right)\\ &+\frac{1}{2}M_R N^T.\mathcal{C}^{-1}.P_L.N+H.C.\end{aligned} \tag{8.21}$$

From these mass terms, one can read off the full neutrino mass matrix

$$M^\nu=\begin{pmatrix}\mathbb{0}_{3\times3} & \frac{v}{\sqrt{2}}Y_\nu\\ \frac{v}{\sqrt{2}}Y_\nu^T & M_R\end{pmatrix}, \tag{8.22}$$

which is non-diagonal and symmetric.

We may rotate the fields into their mass eigenstates by

$$\begin{pmatrix}\nu'_i\\ N\end{pmatrix}=\begin{pmatrix}(U_L\nu)_i\\ U_R^\star\nu\end{pmatrix}, \tag{8.23}$$

where the matrices U_L and U_R are 3×4 and 1×4, respectively. These matrices can be collected into a single unitary 4×4 matrix

$$U=\begin{pmatrix}U_L\\ U_R^\star\end{pmatrix}. \tag{8.24}$$

The unitarity of U imposes restrictions on $U_{L,R}$

$$\begin{aligned}&U_LU_L^\dagger=\mathbb{1}_{3\times3}, \qquad U_R^\star U_R^T=1, \qquad U_L^\dagger U_L+U_R^T U_R^\star=\mathbb{1}_{4\times4},\\ &U_R^\star U_L^\dagger=\mathbb{0}_{1\times3}\,.\end{aligned} \tag{8.25}$$

The matrix U diagonalizes the full neutrino mass matrix

$$U^T M^\nu U = \tilde{m}^\nu \,. \tag{8.26}$$

Here $\tilde{m}^\nu = \text{diag}(0, 0, m_3, m_4)$, since M^ν is of rank 2 and there are only 2 non-zero eigenvalues. Further, we can write

$$U^T M^\nu = \tilde{m}^\nu U^\dagger \,, \tag{8.27}$$

from which the form of U_R follows

$$U_R = \begin{pmatrix} 0 & 0 & u_{R3} & u_{R4} \end{pmatrix} . \tag{8.28}$$

The 0's of U_R will come in handy, for example,

$$(U_L^\dagger U_L)_{ji} + \cancelto{0}{(U_R^T U_R^*)_{ji}} = \delta_{ji} \tag{8.29}$$

if i and/or j are equal to 1 or 2 and correspond to a massless neutrino.

The masslessness of the two neutrinos has an implication for the matrix $U_L^\dagger$ as well. Namely, the first two rows of $U_L^\dagger$ are not fully determined, since the two massless neutrinos are not distinguishable at tree-level. However, at the loop level, the neutrinos become distinguishable and this can be used to fix the remaining components of $U_L^\dagger$, as we will do explicitly in Section 8.3.

8.1.2. The model at 1-loop level

Let us further consider the neutrinos at 1-loop level as well as their renormalization. The massless neutrinos somewhat obstruct the straightforward application of our renormalization scheme, since the mass structure in Eq. (6.3) no longer appears, and it is not trivial to pick out the terms associated with the field renormalization. Considering the renormalized neutrino self-energy $\Sigma_{ji}^{\nu,R}$, there are three distinct cases:

1. $i, j \geq 3$, both indices correspond to massive neutrinos — this is the fully massive case,

2. $i \leq 2, j \geq 3$ or vice versa, this is the partially massless case since only one of the indices corresponds to the massive neutrino — this is the partially massless case,

3. $i, j \leq 2$, both indices correspond to massless neutrinos — the massless case.

All these cases may also be schematically represented by 2×2 blocks of $\Sigma^{\nu,R}_{ji}$

$$\Sigma^{\nu,R}_{ji} \sim \left(\begin{array}{c|c} m_{j,i} = 0,\, \delta m & m_i \neq 0,\, \delta Z^A,\, \delta m \\ \hline m_j \neq 0,\, \delta Z^A,\, \delta m & m_{j,i} \neq 0,\, \delta Z^A,\, \delta m \end{array} \right) . \tag{8.30}$$

Here δm and δZ^A are shorthand for the presence of the mass and the anti-hermitian part of the field counterterms in each 2×2 block, the equalities with $m_{i,j}$ indicate whether the corresponding masses are equal to 0 or not.

The lower right corner with $i, j \geq 3$ corresponds to the massive case, which is the most straightforward. In this case, the definitions of our scheme in Eqs. (6.3) and (6.4) hold without the need for modifications.

Next, we have the off-diagonal blocks corresponding to the partially massless case with $i \leq 2, j \geq 3$ or vice versa. Here the situation is not as trivial since one of the masses m_i or m_j is 0 and this disrupts the appearance of the $m_i^2 - m_j^2$ mass structure needed for the field renormalization in Eq. (6.3). Fortunately, one can simply arrive at this case by taking one of the masses to 0 once the field renormalization is computed in the massive case. This avoids searching for the non-existent $m_i^2 - m_j^2$ mass structure and also preserves all the UV and gauge properties of the mass and field counterterms.

Finally, in the upper left corner with $i, j \leq 2$, we have the massless case. Here the situation is again simple since the anti-hermitian part of the field renormalization must vanish since both masses are identical and there are no absorptive parts. In addition, there is no need to first compute the massive case and to successively take the masses to 0, instead, this limit can be taken already in Eq. (6.4)

$$\delta m^L_{ji} = \frac{1}{2} \left(\Sigma^{sL}_{ji}(0) + \Sigma^{sR\dagger}_{ji}(0) \right) = \Sigma^{sL}_{ji}(0) \tag{8.31}$$

and

$$\delta m^R_{ji} = \frac{1}{2} \left(\Sigma^{sR}_{ji}(0) + \Sigma^{sL\dagger}_{ji}(0) \right) = \Sigma^{sR}_{ji}(0) \,. \tag{8.32}$$

Both the $\delta m^{L,\,R}$ are UV finite since all the UV divergences must be proportional to the mass structures built out of $m_{i,j}$ or the momentum, none of which are present in $\Sigma^{sL,\,sR}_{ji}(0)$. Even more so, these contributions are also gauge-independent since the $m_i^2 - m_j^2$ mass structure, carried by the gauge-dependent part, is also 0 in the massless limit. One more special property of this 2×2 block is that it can be diagonalized by using the remaining freedom of the matrix U_L. Considering all these properties together we may treat the $\delta m^{L,\,R}$ as radiative masses rather than counterterms since there is nothing to "counter" in this block, this has been noted in [48, 124].

In this section, we begin showcasing the application of our renormalization scheme. Before tackling the full Grimus–Neufeld model, we make things simpler by applying the scheme to the quark sector and including only the Standard Model interactions.

Perhaps of most importance is the ability to pick out the terms corresponding to the anti-hermitian part of the field renormalization as instructed by Eq. (6.3). To highlight this, we simply insert the quark self-energies in the definition to get the anti-hermitian part of the field renormalization for up-type quarks

$$\begin{aligned}
\delta Z^{A,\,u}_{Lji} = &-\sum_{k=1}^{3}\frac{V_{jk}V^{\star}_{ik}}{2v^2}\Big\{-A_0((m^d_k)^2)\left((m^d_k)^2+(D-2)m_W^2+(m^u_j)^2\right)\\
&+\Big((D-3)m_W^2\left((m^d_k)^2+(m^u_i)^2\right)+\left((m^d_k)^2-(m^u_i)^2\right)^2\\
&-(D-2)m_W^4\Big)B_0((m^u_i)^2,(m^d_k)^2,m_W^2)\\
&+\left((m^u_i)^2-(m^u_j)^2\right)\left((m^d_k)^2-(m^u_i)^2+m_W^2\xi_W\right)\\
&\times B_0((m^u_i)^2,(m^d_k)^2,\xi_W m_W^2)\\
&+(m^d_k)^2A_0\left(m_W^2\right)+H.C.\Big\}\Big|_{m_i^2-m_j^2} \qquad (8.33)\\
=&-\sum_{k=1}^{3}\frac{V_{jk}V^{\star}_{ik}}{2v^2}\left[(m^d_k)^2-(m^u_i)^2+\xi_W m_W^2\right]\\
&\times B_0((m^u_i)^2,(m^d_k)^2,\xi_W m_W^2)\\
&+\sum_{k=1}^{3}\frac{V_{jk}V^{\star}_{ik}}{2v^2}\left[(m^d_k)^2-(m^u_j)^2+\xi_W m_W^2\right]\\
&\times B_0^{\star}((m^u_j)^2,(m^d_k)^2,\xi_W m_W^2) \qquad (8.34)
\end{aligned}$$

Here we have included only off-diagonal terms ($i \neq j$) and V is the CKM matrix, to get the second equality we picked the terms containing the $m_i^2 - m_j^2$ mass structure and included the Hermitian-conjugated term. Note that one of the B_0 functions is complex conjugated, which signals the presence of absorptive parts. In addition, the UV parts of B_0 do not depend on the arguments, so it is not hard to check that $\delta Z^{A,\,u}_{Lji}$ is indeed UV-finite as required by our scheme.

Interestingly, applying the definition in Eq. (6.3) for the right-handed part of the field renormalization we get a trivial result

$$\delta Z^{A,\,u}_{Rji} = 0\,. \qquad (8.35)$$

The vanishing of $\delta Z^{A,\,u}_{Rji}$ is a rather welcome outcome, which reflects the chiral nature of the weak interaction. Importantly, such a reflection does not seem to be present in other renormalization schemes, since they include terms that also do not carry the $m_i^2 - m_j^2$ mass structure. Finally, the additional matter content in the full GNM does not change the quark field renormalization w.r.t. the SM, so that $\delta Z^{A,\,u}_{R,\,L}$ are the same in both models. Analogous results also hold for down-type quarks.

Our scheme is not complete without also computing the mass counterterms as defined in Eq. (6.4). The quark mass counterterms in the full GNM can be expressed in terms of PV functions, but the expressions are rather huge and we refer to Appendix B of [A3]. Instead, here we again take just the SM contributions and deal just with the UV parts, which allows for the comparison of UV divergences between our scheme and the schemes discussed in Sections 6.3.1 and 6.3.2.

To explicitly compare the UV parts we form the CKM counterterm via the 1-loop rotation of Section 6.3.2, which relies on the UV divergences of mass counterterms via Eq. (6.32). One may find the divergent parts of $ih_{L,R}$ by using Eq. (6.36) and evaluating the PV functions in Eq.(8.33)

$$\begin{aligned}
[ih^u_{Lji}]_{\text{div.,SM}} &= -\frac{3(V(m^d)^2V^\dagger)_{ji}}{32\pi^2v^2\epsilon_{UV}}\cdot\frac{(m^u_i)^2+(m^u_j)^2}{(m^u_i)^2-(m^u_j)^2},\\
[ih^d_{Lji}]_{\text{div.,SM}} &= -\frac{3(V^\dagger(m^u)^2V)_{ji}}{32\pi^2v^2\epsilon_{UV}}\cdot\frac{(m^d_i)^2+(m^d_j)^2}{(m^d_i)^2-(m^d_j)^2}\,.
\end{aligned} \tag{8.36}$$

Here we have the divergent parts of ih^u_L for up-type quarks and ih^d_L for down-type quarks. From these one may solve for the mass counterterms

$$\begin{aligned}
[\delta m^{L,u}_{ji}]_{\text{div.,SM}} &= -\frac{1}{\epsilon_{UV}}\frac{3m^u_j(V(m^d)^2V^\dagger)_{ji}}{32\pi^2v^2},\\
[\delta m^{L,d}_{ji}]_{\text{div.,SM}} &= -\frac{1}{\epsilon_{UV}}\frac{3m^d_i(V^\dagger(m^u)^2V^{)}_{ji}}{32\pi^2v^2},
\end{aligned} \tag{8.37}$$

with $\delta m^{R,\,u,d} = \left(\delta m^{L,\,u,d}\right)^\dagger$, or form the CKM counterterm as in Eq. (6.35)

$$\begin{aligned}
[\delta V^h_{ji}]_{\text{div.,SM}} &= -\sum_{n\neq j} ih^u_{Ljn}V_{ni} + \sum_{k\neq i} V_{jk}ih^d_{Lki}\\
&= \frac{3}{32\pi^2v^2\epsilon_{UV}}\left[\sum_{k\neq j}(V(m^d)^2V^\dagger)_{jk}\cdot\frac{(m^u_k)^2+(m^u_j)^2}{(m^u_k)^2-(m^u_j)^2}V_{ki}\right.
\end{aligned}$$

$$- \sum_{k \neq i} V_{jk} \frac{(m_i^d)^2 + (m_k^d)^2}{(m_i^d)^2 - (m_k^d)^2} \cdot (V^\dagger (m^u)^2 V)_{ki} \Bigg] . \tag{8.38}$$

One may compare Eq. (8.37) with Eqs. (37), (39), and (42) in [29]. A comparison with Eq. (3.13) in [23] can be done via Eq. (8.38). In both cases, taking the off-diagonal terms one may see that the UV divergences match, which shows both that the computations are correct and that the divergences in the mixing matrix counterterm are fundamentally associated with the mass. In addition, there is an extremely simple and implicit way of arriving at the same result. One can simply notice that, since the anti-hermitian part of the field renormalization is UV finite, all the UV divergences must be accounted for by $ih_{L,R}$ via Eq. (6.36). Finally, we again emphasize that Eq. (8.38) is only needed for comparison with other schemes, while in our scheme it is trivial, *i.e.* $\delta V = 0$ by definition.

8.3. The neutrino sector

Having discussed the more simple quark sector in the SM, let us proceed with the application of our scheme to the more complicated neutrino sector in the Grimus–Neufeld model. We begin the discussion with the simplest case, which is the massless one, follow with the massive case, and finish the section with the partially massless case by taking a relevant limit of the massive case.

8.3.1. The massless case ($i, j \leq 2$)

In the fully massless case, we consider the upper left 2×2 block of Eq. (8.30), where the field-renormalization becomes irrelevant due to the masslessness of the neutrinos. Here we only need to evaluate the 1-loop mass contributions as in Eqs. (8.31) and (8.32), which gives

$$\delta m_{ji}^{L,\nu} = \sum_{a=1}^{4} \frac{(U_L^T G_\nu U_R^\star)_{ja} m_a^\nu}{2} \Big[s_\alpha^2 B_0(0, m_h^2, (m_a^\nu)^2) + c_\alpha^2 B_0(0, m_H^2, (m_a^\nu)^2) - B_0(0, m_A^2, (m_a^\nu)^2) \Big] (U_R^\dagger G_\nu^T U_L)_{ai} , \tag{8.39}$$

$$\delta m_{ji}^{R,\nu} = \sum_{a=1}^{4} \frac{(U_L^\dagger G_\nu^\star U_R)_{ja} m_a^\nu}{2} \Big[s_\alpha^2 B_0(0, m_h^2, (m_a^\nu)^2) + c_\alpha^2 B_0(0, m_H^2, (m_a^\nu)^2) - B_0(0, m_A^2, (m_a^\nu)^2) \Big] (U_R^T G_\nu^\dagger U_L^\star)_{ai} . \tag{8.40}$$

Here the sum runs over the 4 neutrino flavors and in computing these contributions we have used that the PV functions $B_0\left(0, m_1^2, m_2^2\right)$ are real. It is not

difficult to check that the above contributions are UV-finite, since again the UV parts of B_0 functions do not depend on their arguments. Evidently, these results are also gauge-independent.

All these features suggest that these contributions are physical and may be used as radiative masses, which is further reinforced by noticing two additional things. First, as already mentioned at the end of Section 6, when considering only the real part we may use the same expressions also for the diagonal case when $i = j$. Secondly, there is still left-over freedom in the matrix U_L, which we may use to diagonalize the 1-loop contributions. One possibility of doing so is by requiring that one of the rows of U_L^T corresponding to a massless neutrino is orthogonal to G_ν [48]. Perhaps a more simple alternative is to define a new Yukawa coupling to the second Higgs doublet (similar to [50])

$$U_L^T G_\nu \equiv G'_\nu = \begin{pmatrix} 0 \\ g'_2 \\ g'_3 \\ g'_4 \end{pmatrix} . \tag{8.41}$$

Having this definition it is not hard to see that the mass counterterms δm^ν_{ji} in Eqs. (8.39) and (8.40) are $\sim g'_j\, g'_i$, which gives only one non-zero entry with $(g'_2)^2$, when $i = j = 2$, *i.e.* the block is diagonal with $i,\, j \leq 2$. In turn, one of the two massless neutrinos remains massless and the other one acquires a radiative mass, thus we may write

$$m^\nu_1 = \text{Re}[\delta m^{L,\nu}_{11}] = \text{Re}[\delta m^{R,\nu}_{11}] = 0\,, \tag{8.42}$$

$$m^\nu_2 = \text{Re}[\delta m^{L,\nu}_{22}] = \text{Re}[\delta m^{R,\nu}_{22}] = \text{Re}\left[(g'_2)^2 \widetilde{C}\right] . \tag{8.43}$$

Here we have defined the constant

$$\widetilde{C} = \sum_{a=1}^{4} \frac{U^\star_{Ra} U^\star_{Ra} m^\nu_a}{2} \Big[s^2_\alpha B_0(0, m^2_h, (m^\nu_a)^2) + c^2_\alpha B_0(0, m^2_H, (m^\nu_a)^2) - B_0(0, m^2_A, (m^\nu_a)^2) \Big] \tag{8.44}$$

and analogous results can be found in [48, 50].

8.3.2. The massive case ($i, j \geq 3$)

Here we consider the bottom right 2×2 block in Eq. (8.30) with $i,\, j \geq 3$, in which our definitions of the field and mass counterterms fully hold. However, the neutrino sector is much more complicated both due to the sheer amount of

terms and the fact that the anti-hermitian part of the field renormalization is *not* expressible in terms of PV functions.

Yet another complication in the neutrino sector is the apparent ambiguity of mass structures. In the quark sector, there were no such ambiguities, but in the neutrino sector one may find the following structure

$$m_j^\nu \left[3(m_i^\nu)^2 + (m_j^\nu)^2\right] . \tag{8.45}$$

By using the structures $m_i,\ m_j,\ 2m_i m_j,\ m_i^2 \pm m_j^2$ seen in Section 6.2, the same may be written in multiple ways, for example, as

$$m_j^\nu \left(2\left[(m_i^\nu)^2 + (m_j^\nu)^2\right] + \left[(m_i^\nu)^2 - (m_j^\nu)^2\right]\right)$$

or as

$$m_i^\nu \left[2m_i^\nu m_j^\nu\right] + m_j^\nu \left[(m_i^\nu)^2 + (m_j^\nu)^2\right] .$$

Importantly, one variant manifests the $(m_i^\nu)^2 - (m_j^\nu)^2$ mass structure needed for the field renormalization, while the other does not. In turn, such terms are ambiguous and obstruct a clear selection of relevant terms.

Fortunately, the Nielsen identities guarantee that no such ambiguities plague the gauge-dependent terms. In addition, we have found that the ambiguous terms cannot be distributed at will, since that makes the massless limit non-existent. In practice, this means that the gauge-independent terms can be included in the field renormalization only if they manifest the $(m_i^\nu)^2 - (m_j^\nu)^2$ mass structure immediately and without any rearrangements. If such terms appear, they will necessarily be UV-finite as promised by our discussion in Section 6.2.

With this in mind and remembering that for neutrinos we must drop the absorptive parts with the $\widetilde{Re}$ operator (see Section 5.1.1) we may write down the off-diagonal components of the left-handed anti-hermitian part of the field renormalization for neutrinos $\delta Z^{A,\nu}_{Lji}$ in the massive case

$$\begin{aligned}
\delta Z^{A,\nu}_{Lji} = &- \widetilde{Re}\Big\{ \sum_{a=1}^{4}\sum_{s=1}^{3} \frac{(m_a^\nu)^3 R_{1s}^2 (U_L^\dagger U_L)_{ai}(U_L^\dagger U_L)_{aj}}{32\pi^2 v^2 m_j^\nu} \log\left(\frac{(m_s^h)^2}{(m_a^\nu)^2}\right) \\
&+ \sum_{a=1}^{4}\Bigg[\frac{m_a^\nu (U_L^\dagger U_L)_{ai}(U_L^\dagger U_L)_{aj}}{128\pi^2 v^2 (m_j^\nu)^3} \\
&\quad \times \left(2(m_a^\nu)^2(m_j^\nu)^2 - \left((m_a^\nu)^2 - m_Z^2\xi_Z\right)^2 + (m_j^\nu)^4\right) \\
&+ \frac{(U_L^\dagger U_L)_{ai}(U_L^T U_L^\star)_{aj}}{128\pi^2 v^2 (m_j^\nu)^2} \\
&\quad \times \left((m_a^\nu)^4 + (m_i^\nu)^2(m_j^\nu)^2 - m_Z^4\xi_Z^2\right)\Bigg] \log\left(\frac{(m_a^\nu)^2}{m_Z^2\xi_Z}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{3}\frac{U_{Lki}U^{\star}_{Lkj}}{32\pi^2v^2(m_j^{\nu})^2}\\
&\times\left((m_k^l)^4+(m_i^{\nu})^2(m_j^{\nu})^2-m_W^4\xi_W^2\right)\log\left(\frac{(m_k^l)^2}{m_W^2\xi_W}\right)\\
&+\frac{(m_i^{\nu})^2(U_L^TU_L^{\star})_{ij}}{64\pi^2v^2}\left(2\log\xi_W+\log\xi_Z\right)\\
&+\sum_{a=1}^{4}\left[\frac{(U_L^{\dagger}U_L)_{ai}(U_L^TU_L^{\star})_{aj}}{64\pi^2v^2}\left((m_a^{\nu})^2-(m_i^{\nu})^2+m_Z^2\xi_Z\right)\right.\\
&\left.+\frac{m_a^{\nu}(U_L^TU_L^{\star})_{ai}(U_L^TU_L^{\star})_{aj}}{64\pi^2v^2m_i^{\nu}}\left(-(m_a^{\nu})^2+(m_i^{\nu})^2+m_Z^2\xi_Z\right)\right]\\
&\qquad\times\Lambda((m_i^{\nu})^2,m_a^{\nu},m_Z\sqrt{\xi_Z})\\
&+\sum_{k=1}^{3}\frac{U_{Lki}U^{\star}_{Lkj}}{32\pi^2v^2}\left((m_k^l)^2-(m_i^{\nu})^2+m_W^2\xi_W\right)\\
&\qquad\times\Lambda((m_i^{\nu})^2,m_k^l,m_W\sqrt{\xi_W})-H.C.\Big\}
\end{aligned}
\tag{8.46}
$$

Here $R_{ss\prime}$ is the mixing matrix of the scalars from Eq. (8.4), which we left for generality, the index s runs over all the physical scalars giving the masses $m_s^h=\{m_h,m_H,m_A\}$, $\Lambda(m^2,b,c)$ is the `Disc`(ontinuity) function as used in `PackageX` and `FeynCalc`. Note that the terms on the very first line are gauge-independent and come from the Higgs sector. It seems that because of such terms we were not able to express the field renormalization in terms of PV functions, but had to explicitly evaluate them. Perhaps this is not entirely surprising since, unlike quark masses, the neutrino masses arise due to the seesaw mechanism and are a mixture of components of the $SU(2)$ lepton doublets and the Majorana singlet coupled through the Higgs doublets as in Eq. (8.18). In any case, this somewhat complicated situation only showcases the universality of our scheme. Finally, one may get the right-handed part by complex conjugating $\delta Z_L^{A,\nu}$ as in Eq. (5.17).

To complete this case we also present the neutrino mass counterterm, however, it is too large for a sensible presentation. As a compromise, we present only the UV parts

$$
\begin{aligned}
[\delta m_{ji}^{L,\nu}]_{\text{div.}}&=\frac{(U_L^{\dagger}U_L)_{ji}}{16\pi^2v^2\epsilon_{\text{UV}}}m_j^{\nu}\left(2m_+^2-\lambda_3v^2+(m_i^{\nu})^2-\frac{3}{2}\sum_{s=1}^{3}R_{1,s}^2(m_s^h)^2\right)\\
&+\frac{(U_L^{\dagger}U_L)_{ji}}{4\pi^2v^2\epsilon_{\text{UV}}}m_j^{\nu}\sum_{s=1}^{3}\frac{R_{1s}^2}{(m_s^h)^2}\text{Tr.}\{(m^{\nu})^4U_L^{\dagger}U_L+3(m^d)^4+3(m^u)^4+(m_k^l)^4\}
\end{aligned}
$$

$$
- \frac{(U_L^\dagger U_L)_{ji}}{16\pi^2 v^2 \epsilon_{\rm UV}} m_j^\nu \sum_{s=1}^{3} \frac{R_{1s}^2}{(m_s^h)^2} (\lambda_3 m_+^2 v^2 + 6 m_W^4 + 3 m_Z^4)
$$

$$
+ \frac{(U_L^\dagger U_L)_{ji}}{16\pi^2 v^2 \epsilon_{\rm UV}} m_j^\nu \sum_{s,s'=1}^{3} \frac{R_{1s}^2}{(m_s^h)^2} \left(R_{1s'}^2 - 1\right) (m_{s'}^h)^4
$$

$$
- \frac{(U_L^\dagger U_L)_{ji}}{32\pi^2 v^2 \epsilon_{\rm UV}} m_j^\nu \sum_{s,s'=1}^{3} \frac{R_{1s}^2}{(m_s^h)^2} \left(2m_+^2 - \lambda_3 v^2\right) \left(3R_{1s'}^2 - 1\right) (m_{s'}^h)^2
$$

$$
- \frac{\lambda_7 (U_L^\dagger U_L)_{ji}}{32\pi^2 \epsilon_{\rm UV}} m_j^\nu \sum_{s=1}^{3} \frac{R_{1s} R_{2s}}{(m_s^h)^2} 2m_+^2
$$

$$
+ \frac{\lambda_7 (U_L^\dagger U_L)_{ji}}{32\pi^2 \epsilon_{\rm UV}} m_j^\nu \sum_{s,s'=1}^{3} \frac{R_{1s}}{(m_s^h)^2} \left(R_{2s}(R_{1s'}^2 - 1) + 2R_{1s}R_{1s'}R_{2s'}\right) (m_{s'}^h)^2
$$

$$
+ \frac{\sqrt{2}(U_L^\dagger U_L)_{ji}}{16\pi^2 v^2 \epsilon_{\rm UV}} m_j^\nu \sum_{s=1}^{3} \Big[\frac{R_{1s}(R_{2s} - iR_{3s})}{(m_s^h)^2}
$$

$$
\times \mathrm{Tr}.\{(m^\nu)^3 (U_R^T G_\nu^\dagger U_L^\star) + 3(m^d)^3 G_d + 3(m^u)^3 G_u + (m^l)^3 G_l\} + C.C.\Big]
$$

$$
- \frac{\lambda_7 (U_L^T G_\nu U_R^\star)_{ji}}{32\sqrt{2}\pi^2 \epsilon_{\rm UV}} \sum_{s=1}^{3} \frac{v R_{2s}^2}{(m_s^h)^2} \left(3 \sum_{s'=1}^{3} R_{2s'}^2 (m_{s'}^h)^2 + 2m_+^2 + m_A^2\right)
$$

$$
+ \frac{\sqrt{2}(U_L^T G_\nu U_R^\star)_{ji}}{8\pi^2 v \epsilon_{\rm UV}} \sum_{s=1}^{3} \frac{R_{1s} R_{2s}}{(m_s^h)^2}
$$

$$
\times \mathrm{Tr}.\{(m^\nu)^4 U_L^\dagger U_L + 3(m^d)^4 + 3(m^u)^4 + (m_k^l)^4\}
$$

$$
- \frac{\sqrt{2}(U_L^T G_\nu U_R^\star)_{ji}}{32\pi^2 v \epsilon_{\rm UV}} \sum_{s=1}^{3} \frac{R_{1s} R_{2s}}{(m_s^h)} \left(\lambda_3 m_+^2 v^2 + 6 m_W^4 + 3 m_Z^4 + 2(m_s^h)^4\right)
$$

$$
+ \frac{(U_L^T G_\nu U_R^\star)_{ji}}{32\sqrt{2}\pi^2 v \epsilon_{\rm UV}} \sum_{s,s'=1}^{3} \frac{R_{1s} R_{2s}}{(m_s^h)}
$$

$$
\times \left(2\left(R_{1s'}^2 - 1\right)(m_{s'}^h)^4 - \left(2m_+^2 - \lambda_3 v^2\right)\left(3R_{1s'}^2 - 1\right)(m_{s'}^h)^2\right)
$$

$$
+ \frac{(U_L^T G_\nu U_R^\star)_{ji}}{16\pi^2 \epsilon_{\rm UV}} \sum_{s=1}^{3} \frac{(R_{2s} + iR_{3s})}{(m_s^h)} \Big[(R_{2s} - iR_{3s})
$$

$$
\times \mathrm{Tr}.\{(m^\nu)^3 (U_R^T G_\nu^\dagger U_L^\star) + 3(m^d)^3 G_d + 3(m^u)^3 G_u + (m^l)^3 G_l\} + C.C.\Big]
$$

$$
+ \frac{m_j^\nu}{32\pi^2 v^2 \epsilon_{\rm UV}} \left(\frac{1}{2} v^2 \left(U_L^\dagger G_\nu^\star G_\nu^T U_L\right)_{ji} + \frac{1}{2} v^2 \left(U_L^\dagger G_l^\dagger G_l U_L\right)_{ji}\right.
$$

$$
\left. + v^2 \left(U_R^T G_\nu^\dagger G_\nu U_R^\star\right)_{ji} - 3\left(U_L^\dagger (m^l)^2 U_L\right)_{ji} + \left(U_L^\dagger U_L (m^\nu)^2 U_L^\dagger U_L\right)_{ji}\right)
$$

$$- \frac{(U_L^T G_l^T m^l G_\nu U_R^\star)_{ji}}{16\pi^2 \epsilon_{\rm UV}} + (i \leftrightarrow j)\,. \tag{8.47}$$

Here $C.C.$ means complex conjugation, Tr.$\{\dots\}$ denotes traces over family indices, and $i \leftrightarrow j$ means transposition in the i and j indices. The UV parts of $\delta m^{L,\nu}$ are symmetric as should be the case for Majorana fermions, but the full counterterm is not symmetric due to absorptive parts.

For the sake of comparison we select just the "SM" parts, *i.e* without the second Higgs doublet, but with the Majorana neutrino

$$\begin{aligned}
[\delta m_{ji}^{L,\nu}]^{\rm SM}_{\rm div.} &= -\frac{3m_j^\nu \left(U_L^\dagger (m^l)^2 U_L\right)_{ji}}{32\pi^2 v^2 \epsilon_{\rm UV}} + \frac{m_j^\nu \left(U_L^\dagger U_L (m^\nu)^2 U_L^\dagger U_L\right)_{ji}}{32\pi^2 v^2 \epsilon_{\rm UV}} \\
&+ \frac{(U_L^\dagger U_L)_{ji}}{16\pi^2 v^2 \epsilon_{\rm UV}} m_j^\nu (m_i^\nu)^2 - \frac{(U_L^\dagger U_L)_{ji} m_j^\nu}{16\pi^2 v^2 (m_h)^2 \epsilon_{\rm UV}} (6m_W^4 + 3m_Z^4) \\
&+ \frac{(U_L^\dagger U_L)_{ji} m_j^\nu}{4\pi^2 v^2 (m_h)^2 \epsilon_{\rm UV}} \text{Tr.}\{(m^\nu)^4 U_L^\dagger U_L + 3(m^d)^4 + 3(m^u)^4 + (m_k^l)^4\} \\
&+ (i \leftrightarrow j)\,.
\end{aligned} \tag{8.48}$$

We have produced this result from Eq. (8.47) by dropping terms containing couplings to the second Higgs doublet G_x or couplings from the THDM potential λ_x, terms with scalar masses other than m_h were dropped as well, and we have also set $R_{1s} = 1$, while all the other elements of R were set to 0. The remaining terms in Eq. (8.48) can be compared to Eq. (31) of [31], from which one finds that the only difference arises due to our inclusion of *all* tadpole diagrams and otherwise the results are identical. On the other hand, the inclusion of tadpole diagrams containing Goldstone bosons is necessary to achieve a gauge-independent result as is done in [29, 31].

8.3.3. The partially massless case ($i \leq 2, j \geq 3$)

In this section, we deal with the most complicated case, where only one of the indices i or j corresponds to a massless neutrino. This masslessness obstructs a straightforward usage of the definitions in Eq. (6.3), where the $m_i^2 - m_j^2$ mass structure is important. As mentioned, we avoid this complication by simply taking one of the masses to 0 in Eqs. (8.46) and (8.47). For concreteness, we select the $i \leq 2, j \geq 3$ case, where the mass m_i^ν will be taken to 0 and m_j^ν will remain non-zero.

At least for the mass counterterm taking m_i^ν to 0 is not too difficult, since one can use the simplification from Eq. (8.29), which immediately eliminates

terms proportional to $(U_L^\dagger U_L)_{ji}$. In addition, because of Eq. (8.29) we have that

$$\left(U_L^\dagger U_L \mathcal{D} U_L^\dagger U_L\right)_{ij} = \mathcal{D}_{ji} = 0\,, \tag{8.49}$$

where $\mathcal{D}$ is any diagonal matrix and $i \neq j$. Analogous results hold for $\left(U_L^T U_L^\star\right)_{ji}$.

A further simplification follows since the first two components of U_R are 0 (see Eq. (8.28)), hence, matrix products where the matrix U_R on the outside corresponds to the index i can be dropped, *e.g.*

$$(ABCU_R)_{ji} = 0\,, \tag{8.50}$$

for some arbitrary matrices A, B, and C.

Having this, we may write the mass counterterm from Eq. (8.47) in the limit, where $m_i^\nu = 0$ (this limit is denoted by the $\not{m}_i^\nu$ superscript)

$$\begin{aligned}
[\delta m_{ji}^{L,\nu}]^{\not{m}_i^\nu}_{\text{div.}} &= \frac{\sqrt{2}(U_L^T G_\nu U_R^\star)_{ij}}{8\pi^2 v \epsilon_{\text{UV}}} \sum_{s=1}^{3} \frac{R_{1s}R_{2s}}{(m_s^h)^2} \\
&\times \text{Tr.}\{(m^\nu)^4 U_L^\dagger U_L + 3(m^d)^4 + 3(m^u)^4 + (m_k^l)^4\} \\
&- \frac{\lambda_7 (U_L^T G_\nu U_R^\star)_{ij}}{32\sqrt{2}\pi^2 \epsilon_{\text{UV}}} \sum_{s=1}^{3} \frac{v R_{2s}^2}{(m_s^h)^2} \left(3 \sum_{s'=1}^{3} R_{2s'}^2 (m_{s'}^h)^2 + 2m_+^2 + m_A^2\right) \\
&- \frac{\sqrt{2}(U_L^T G_\nu U_R^\star)_{ij}}{32\pi^2 v \epsilon_{\text{UV}}} \sum_{s=1}^{3} \frac{R_{1s}R_{2s}}{(m_s^h)} \left(\lambda_3 m_+^2 v^2 + 6m_W^4 + 3m_Z^4 + 2(m_s^h)^4\right) \\
&+ \frac{(U_L^T G_\nu U_R^\star)_{ij}}{32\sqrt{2}\pi^2 v \epsilon_{\text{UV}}} \sum_{s,s'=1}^{3} \frac{R_{1s}R_{2s}}{(m_s^h)} \\
&\times \left(2\left(R_{1s'}^2 - 1\right)(m_{s'}^h)^4 - \left(2m_+^2 - \lambda_3 v^2\right)\left(3R_{1s'}^2 - 1\right)(m_{s'}^h)^2\right) \\
&+ \frac{(U_L^T G_\nu U_R^\star)_{ij}}{16\pi^2 \epsilon_{\text{UV}}} \sum_{s=1}^{3} \frac{(R_{2s} + iR_{3s})}{(m_s^h)} \Big[(R_{2s} - iR_{3s}) \\
&\times \text{Tr.}\{(m^\nu)^3 (U_R^T G_\nu^\dagger U_L^\star) + 3(m^d)^3 G_d + 3(m^u)^3 G_u + (m^l)^3 G_l\} + C.C.\Big] \\
&+ \frac{m_j^\nu}{32\pi^2 v^2 \epsilon_{\text{UV}}} \left(\frac{1}{2} v^2 \left(U_L^\dagger G_\nu^\star G_\nu^T U_L\right)_{ji} + \frac{1}{2} v^2 \left(U_L^\dagger G_l^\dagger G_l U_L\right)_{ji}\right. \\
&\left. -3\left(U_L^\dagger (m^l)^2 U_L\right)_{ji}\right) - \frac{(U_L^T G_l^T m^l G_\nu U_R^\star)_{ij}}{16\pi^2 \epsilon_{\text{UV}}}\,.
\end{aligned} \tag{8.51}$$

Note that here $(i \leftrightarrow j)$ is no longer present and all the parts are displayed, in turn, some matrix products now have ij as subscripts instead of ji. Another simplification is possible if $i = 1$ since then one may use Eq. (8.41) to drop

all the terms where $U_L^T G_\nu$ on the outside of a matrix product gets the index i. Finally, following the same logic one may check that the mass counterterm is UV finite if both the masses m_i^ν and m_j^ν are taken to 0.

Having dealt with the mass counterterm, we must also take $m_i^\nu \to 0$ in the field renormalization. Most of the terms can be dropped once again by using Eq. (8.29) and Eq. (8.49). However, this is not enough and the situation is slightly more complicated since, at least at first glance, the terms containing $\log\left(\frac{(m_k^l)^2}{m_W^2\xi_W}\right)$ in Eq. (8.46) are singular once $m_i^\nu \to 0$. Fortunately, these singularities simplify with the ones coming from disc functions, which for $m_i^\nu \approx 0$ is approximately

$$\Lambda((m_i^\nu)^2, b, c) \xrightarrow{(m_i^\nu)^2\approx 0} \left[\frac{b^2-c^2}{2(m_i^\nu)^2} - \frac{b^2+c^2}{2(b^2-c^2)}\right]\log\left(\frac{b^2}{c^2}\right) - 1\,. \tag{8.52}$$

With these considerations we first take the small m_i^ν limit of Eq. (8.46), which is followed by taking $m_i^\nu \to 0$ once the singular terms are canceled out

$$\begin{aligned}
\delta Z^{A,\nu}_{Lji,m_i^\nu\approx 0} = -\widetilde{Re}\Big\{ &\sum_{k=1}^{3} U_{Lki}U^\star_{Lkj}\left[\frac{(m_k^l)^2-(m_i^\nu)^2+m_W^2\xi_W}{32\pi^2v^2}\right]\\
\times &\left[\left[\frac{(m_k^l)^2-m_W^2\xi_W}{2(m_i^\nu)^2} - \frac{(m_k^l)^2+m_W^2\xi_W}{2((m_k^l)^2-m_W^2\xi_W)}\right]\log\left(\frac{(m_k^l)^2}{m_W^2\xi_W}\right) - 1\right]\\
+ &\sum_{k=1}^{3}\frac{U_{Lki}U^\star_{Lkj}}{32\pi^2v^2}\left[(m_j^\nu)^2-(m_k^l)^2-m_W^2\xi_W\right]\Lambda^\star((m_j^\nu)^2, m_k^l, \sqrt{\xi_W}m_W)\\
+ &\sum_{k=1}^{3}\frac{\left((m_i^\nu)^2-(m_j^\nu)^2\right)U_{Lki}U^\star_{Lkj}}{64\pi^2v^2(m_i^\nu)^2(m_j^\nu)^2}\\
\times &\left((m_k^l)^4+(m_i^\nu)^2(m_j^\nu)^2-m_W^4\xi_W^2\right)\log\left(\frac{(m_k^l)^2}{m_W^2\xi_W}\right)\Big\}\,,
\end{aligned} \tag{8.53}$$

$$\begin{aligned}
\delta Z^{A,\nu}_{Lji,\cancel{m_i^\nu}} = -\widetilde{Re}\Big\{ &-\frac{(U_L^\dagger(m^l)^2U_L)_{ji}}{32\pi^2v^2} + \sum_{k=1}^{3}\frac{U_{Lki}U^\star_{Lkj}}{64\pi^2v^2}\\
\times &\left[\frac{(m_k^l)^4-m_W^4\xi_W^2-(m_j^\nu)^4}{(m_j^\nu)^2} - \frac{2\left((m_k^l)^4+m_W^4\xi_W^2\right)}{(m_k^l)^2-m_W^2\xi_W}\right]\log\left(\frac{(m_k^l)^2}{m_W^2\xi_W}\right)\\
+ &\sum_{k=1}^{3}\frac{U_{Lki}U^\star_{Lkj}}{32\pi^2v^2}\left[(m_j^\nu)^2-(m_k^l)^2-m_W^2\xi_W\right]\Lambda^\star((m_j^\nu)^2, m_k^l, \sqrt{\xi_W}m_W)\Big\}\,.
\end{aligned} \tag{8.54}$$

Here, as expected, we have arrived at the anti-hermitian part of the field renormalization even though the $m_i^2 - m_j^2$ mass structure cannot be found once one

of the masses is 0. Nonetheless, the field renormalization is still UV finite and contains all the relevant gauge-dependent parts as should be the case in our renormalization scheme.

Having taken $m_i^\nu \to 0$ both for the mass counterterm in Eq. (8.51) and the field renormalization in Eq.(8.54), we may conclude that the presented renormalization scheme is valid under a wide range of circumstances. We note that taking these limits can be a lot easier depending on the model at hand. For example, taking $m_i^u \to 0$ in the quark field renormalization in Eq. (8.34) is rather simple and there is no need to deal with singular terms.

We have not presented the hermitian part of the field renormalization, since it does not change w.r.t. the usual approach, or the neutrino mass counterterm except for the UV parts. Nonetheless, these are included in a `Mathematica` notebook, which is available in Lithuanian Electronic Academic Library `elaba.lt` along with the dissertation.

A large part of this thesis is concerned with the renormalization of fermion 2-point functions in the presence of particle mixing, however, in this part, we turn to a slightly different aspect of renormalization. The point of renormalization is to make the theory predictive, but the ability to quickly compare predictions between models is also important. For example, one may want to compare the predictions of the Grimus–Neufeld model to those of the Standard Model or even some other model. One way of comparison is to compute the needed radiative corrections in each model separately and then simply compare the numerical results for a given observable. While this is a perfectly valid way of comparison, it is not necessarily quick. If one is concerned with the electroweak (EW) sector and the models under consideration are "similar enough" one may use the formalism of the so-called oblique parameters to perform a much faster, yet precise enough, comparison. However, problems arise if the models differ too much, in particular, if the custodial symmetry protecting the well-known tree-level relation between the W and Z boson masses through the cosine of the Weinberg angle θ_W, $m_W = \cos\theta_W m_Z$, is violated in one model, but holds in the other. For example, such a mismatch obstructs the comparison of predictions between the GNM or SM and a model with a Higgs triplet, which acquires a vacuum expectation value. Hence, in this part of the thesis, we discuss the oblique parameters in the context of custodial symmetry breaking and base our discussion on [A5].

9. Oblique corrections

In more detail, when talking of radiative corrections one may discern two categories:

1. the direct corrections coming from the box, vertex, and external leg corrections,

2. and non-direct corrections coming from the corrections to gauge-boson propagators.

The latter category is usually referred to as oblique corrections, which is an important concept in EW physics [125]. The reason for having these two categories is the fact that the direct corrections depend on the process, while the oblique (non-direct) ones are process-independent. This means that the oblique corrections enter all the relevant processes rather universally, which allows one to perform the computations only once. Even more so, precision

measurements are sensitive enough to be compared with the computations of oblique corrections in a given model. In this way, one can look for hints of new physics, *i.e.* physics that differ from the Standard Model.

The history of oblique corrections is rather old, already more than 50 years ago it was noticed that the effects of particles heavier than W and Z gauge-bosons may be visible at low energies [126]. For example, the Veltman ρ parameter [127] is sensitive to heavy physics effects. These non-decoupling effects in the oblique corrections were more systematically studied in [53, 57, 128, 129], where more observables other than the ρ parameter were considered. A common way of parameterizing the oblique corrections and capturing the heavy-physics effects is via the so-called oblique parameters $\mathcal{S}$, T, and $\mathcal{U}$ introduced in [53, 54]. While this is the most common parameterization used to express corrections to various EW observables, equivalent formulations are also available [56, 58, 59, 130].

The $\mathcal{STU}$ parameters alone are not sufficient to also cover the effects of light new physics. To do so, one has to update the definitions of the three parameters and add three more, such that a total of six oblique parameters, namely $STUVWX$, are needed [55, 131]. This is the parameterization we have focused on in [A5] and use in the thesis as well. The SU parameters become the initial $\mathcal{SU}$ parameters in the limit of only heavy new physics. We note that in principle there should be a seventh parameter related to the self-energy of the photon evaluated at the Z pole, however, this is neglected in [55]. A seven-parameter formulation in the limit of heavy physics can be found in [132], where the observables are considered at the energies of the Large Electron-Positron (LEP) collider.

Whichever the formulation, the oblique parameters consider the difference between two sufficiently similar models. The usual case is where one of the models is the SM and the other one is an extension of the SM with extra matter (fermions and/or scalars). To be sufficiently similar, both models should have the same $SU(2) \times U(1)$ gauge symmetry (although exceptions are possible [132]) and the custodial $SU(2)$ symmetry, which ensures that $m_W = \cos\theta_W m_Z$ at tree-level. This relation may be expressed via the Veltman $\widehat{\rho}$ parameter [126, 127]

$$\widehat{\rho} \equiv \frac{\widehat{m}_W^2}{\widehat{c}^2 \widehat{m}_Z^2} \begin{cases} = 1 & \text{with custodial symmetry,} \\ \text{is not fixed} & \text{without custodial symmetry,} \end{cases} \tag{9.1}$$

where $\widehat{c} \equiv \cos\widehat{\theta}_W$ is the cosine of the Weinberg angle $\widehat{\theta}_W$ and here we introduced hats over the bare parameters. Indeed, since the SM has a fixed value

of $\widehat{\rho} = 1$ there is a certain number of fixed parameters, while in a model with a free value for $\widehat{\rho}$ there is an additional free parameter, which requires for an additional renormalization condition. The difference in the amounts of free parameters in the EW sector obstructs a consistent comparison of the models.

In more detail, a model with custodial symmetry (such as the SM or GNM) has only three free parameters that require renormalization conditions. As is usual, we take these three parameters to be the fine structure constant α, the Fermi coupling constant $G_{F(\text{charged})}$ that is measured in muon decay, and m_Z. In contrast, if the custodial symmetry is broken and $\widehat{\rho}$ is no longer fixed, then four instead of three parameters are needed to fully renormalize the EW sector. We will take this fourth parameter to be the mass of the W boson. Models with broken custodial symmetry have become more important after the measurement of the m_W by the CDF Collaboration [61], which disagrees with the SM prediction by seven standard deviations and hints that $\widehat{m}_W = \cos\widehat{\theta}_W \widehat{m}_Z$ may no longer hold at tree-level. An additional motivation for such models or, more precisely, for models with scalar triplets is also the possibility to explain the 95 GeV excess [133–137] measured by the CMS Collaboration [138–141]

A particularly interesting consequence of the fact that there is an additional free parameter in models, where $\widehat{m}_W \neq \cos\widehat{\theta}_W \widehat{m}_Z$, is the UV divergence of the T parameter, as has been found by many authors [60, 62–65, 67–70, 142]. Naturally, the divergent parameter cannot be used in parameterization of observables and must be removed such that eventually only five oblique parameters remain. We note that these implications were noticed in some initial papers considering the oblique corrections [54, 60, 143].

Since different models might need different amounts of input parameters, we may single out a few cases relevant to the computation of oblique parameters. However, beforehand we set up the nomenclature:

- The SM and the New Physics Model (NPM) have $\widehat{\rho} = 1$; the NPM is identical to the SM plus some extra matter.
- The Base Model (BM) and the Beyond Base Model (BBM) do not have fixed $\widehat{\rho}$; the BBM is identical to the BM plus some extra matter.

Then the possible combinations for comparison of EW predictions are:

1. SM *vs.* NPM, *i.e.* two models that have $\widehat{\rho} = 1$.
2. BM *vs.* BBM, *i.e.* two models that have free $\widehat{\rho}$.
3. SM *vs.* BBM, *i.e.* one model has $\widehat{\rho} = 1$ and the other one has $\widehat{\rho}$ free.

The first case is the most widely encountered and in terms of the oblique parameters it was studied already in [54] as well as in many other references. As far as we know, there is no other genuine oblique parameter formalism for the second case, except for [A5], and no genuine oblique parameter formalism at all for the third case. Without a doubt, the third case is the most interesting phenomenologically and there have been attempts at comparing the SM with models, where $\widehat{\rho}$ is not fixed. For example, the T parameter is sometimes modified to include the tree-level contributions from the model with a free $\widehat{\rho}$, but the loop corrections are computed in the limit with $\widehat{\rho} = 1$ [64, 67, 144, 145]. However, this only works as an approximation but is not a genuine formalism. For example, if one were to compute the loop corrections without assuming $\widehat{\rho} = 1$, one would quickly find that the T parameter is UV divergent. In turn, this implies that the parameter T depends on the renormalization scheme and should be renormalized, which is in contrast to the usual formalism of oblique parameters. Approaches, where the oblique parameters are renormalized were taken in [146] for the T parameter and in [147] in a different context for the S parameter. In addition, in this third case, other parameters are gauge-dependent if the formalism of the first case is blindly applied, thus they are not physical and cannot be used to parameterize the observables.

In [A5] as well as in this thesis we work out the second case, *i.e.* we develop an oblique parameter formalism for the case, where both models have $\widehat{\rho}$ free. It is true, that this case is not as important phenomenologically, but we hope that it is a significant stepping-stone towards a genuine formalism for the third case.

9.1. Definitions in the $SU(2) \times U(1)$ gauge sector

For the oblique parameters to work, all the models must have the $SU(2) \times U(1)$ gauge symmetry, hence, we briefly review the conventions and definitions.

The covariant derivative is

$$D_\mu = \partial_\mu - i\,\widehat{g} \sum_{a=1}^{3} T^a W_\mu^a - i\,\widehat{g}\,' Y B_\mu \,. \tag{9.2}$$

Here $\widehat{g}$, T^a, and W_μ^a are the coupling constant, generators, and the fields of the $SU(2)$ gauge group, respectively. Analogously, $\widehat{g}\,'$, Y (the hypercharge), and B_μ are the coupling constant, generator, and field corresponding to the $U(1)$ gauge group. For brevity, we do not include hats over the fields and, for example, write B_μ instead of $\widehat{B}_\mu$.

Once the EW symmetry is spontaneously broken by the vacuum expectation value (for example, as in Eq. 8.2) down to $U(1)$, the gauge bosons acquire

masses. Then the mass eigenstates of the W bosons with the mass m_W can be expressed in terms of the gauge eigenstates

$$W_\mu^\pm = \frac{W_\mu^1 \mp i W_\mu^2}{\sqrt{2}} \quad \Leftrightarrow \quad \begin{array}{rcl} W_\mu^1 &=& \frac{W_\mu^- + W_\mu^+}{\sqrt{2}} \\ W_\mu^2 &=& \frac{W_\mu^- - W_\mu^+}{i\sqrt{2}} \end{array} , \tag{9.3}$$

The neutral fields of the massless photon A_μ and the Z-boson Z_μ with the mass m_Z are related to the gauge eigenstates as

$$\begin{array}{rcl} A_\mu &=& \widehat{c}\, B_\mu + \widehat{s}\, W_\mu^3 \\ Z_\mu &=& \widehat{c} W_\mu^3 - \widehat{s}\, B_\mu \end{array} \quad \Leftrightarrow \quad \begin{array}{rcl} B_\mu &=& \widehat{c}\, A_\mu - \widehat{s}\, Z_\mu \\ W_\mu^3 &=& \widehat{c}\, Z_\mu + \widehat{s}\, A_\mu \end{array} , \tag{9.4}$$

where $\widehat{s}$ and $\widehat{c}$ are the sine and cosine of the Weinberg angle $\widehat{\theta}_W$, respectively. The gauge couplings and the Weinberg angle are related to the electric charge $\widehat{e}$ or, equivalently, with the fine structure constant $\widehat{\alpha}$ (not to be confused with the scalar mixing angle in Eq. (8.4))

$$\widehat{e} = \widehat{g}\,\widehat{s} = \widehat{g}'\,\widehat{c} = \sqrt{4\pi\widehat{\alpha}}\,. \tag{9.5}$$

Having this we may also write down the covariant derivative in the mass eigenstate basis

$$D_\mu = \partial_\mu - i\widehat{g}\left(T^- W_\mu^- + T^+ W_\mu^+\right) - i\frac{\widehat{g}}{\widehat{c}}\left(T^3 - s^2 Q\right) Z_\mu - i\widehat{e} Q A_\mu\,. \tag{9.6}$$

Here the corresponding generators are

$$T^\pm = \frac{T^1 \pm i T^2}{\sqrt{2}}\,, \tag{9.7}$$

$$Q = T^3 + Y\,. \tag{9.8}$$

Given a scalar multiplet Φ with isospin J_Φ and hypercharge Y_Φ, its neutral component acquires a vacuum expectation value, which gives the following quadratic terms of the gauge-bosons

$$\begin{aligned} \left(D_\mu\langle\Phi\rangle\right)^\dagger D_\mu\langle\Phi\rangle =& \langle\Phi\rangle^\dagger \Big(\widehat{g}^2 \left(T^+T^- + T^-T^+\right) W_\mu^+ W_\mu^- \\ &+ \frac{\widehat{g}^2}{\widehat{c}^2}\left(T^3 - \widehat{s}^2 Q\right)^2 Z_\mu Z_\mu + \widehat{e}^2 Q^2 A_\mu A_\mu \\ &+ \frac{\widehat{g}\widehat{e}}{\widehat{c}}\left(Q\left(T^3 - \widehat{s}^2 Q\right) + \left(T^3 - \widehat{s}^2 Q\right) Q\right) Z_\mu A_\mu \Big) \langle\Phi\rangle \\ =& \langle\Phi\rangle^\dagger \widehat{g}^2 \left(T^+T^- + T^-T^+\right) W_\mu^+ W_\mu^- \langle\Phi\rangle \\ &+ \langle\Phi\rangle^\dagger \frac{\widehat{g}^2}{\widehat{c}^2} Y_\Phi^2 Z_\mu Z_\mu \langle\Phi\rangle\,. \end{aligned} \tag{9.9}$$

Here we dropped all the terms linear in $T^{\pm}$ since $\langle\Phi\rangle^{\dagger}T^{\pm}\langle\Phi\rangle = 0$, to get the second equality we have used that the vacuum expectation values are electrically neutral, namely, $Q = 0$ and $T^3 = -Y$ via Eq. (9.8). The remaining terms provide the masses for the W and Z bosons, while the photon A remains massless due to the unbroken $U(1)$ gauge symmetry.

The W boson mass may be written in a different form by noticing that

$$T^{\pm}T^{\mp} = \frac{1}{2}\left((T^1)^2 + (T^2)^2 \mp i\left[T^1, T^2\right]\right) \tag{9.10}$$

and

$$\begin{aligned} T^+T^- + T^-T^+ =& (T^1)^2 + (T^2)^2 \\ =& (\vec{T})^2 - (T^3)^2\,. \end{aligned} \tag{9.11}$$

Here $(\vec{T})^2 = (T^1)^2 + (T^2)^2 + (T^3)^2$ is the total isospin squared with the eigenvalue of $J_\Phi(J_\Phi + 1)$ for the components of the multiplet Φ. With this we have

$$\begin{aligned} \left(D_\mu\langle\Phi\rangle\right)^{\dagger} D_\mu\langle\Phi\rangle =& \langle\Phi\rangle^{\dagger}\widehat{g}^2\left(J_\Phi(J_\Phi+1) - (Y_\Phi)^2\right)W_\mu^+W_\mu^-\langle\Phi\rangle \\ &+ \langle\Phi\rangle^{\dagger}\frac{\widehat{g}^2}{\widehat{c}^2}Y_\Phi^2 Z_\mu Z_\mu\langle\Phi\rangle\,. \end{aligned} \tag{9.12}$$

Setting $\langle\Phi\rangle = \widehat{v}_\Phi/\sqrt{2}$ as usual and assuming an arbitrary number of multiplets we read off the following squared masses of the gauge bosons

$$\widehat{m}_Z^2 = \frac{\widehat{g}^2}{\widehat{c}^2}\sum_\Phi Y_\Phi^2\widehat{v}_\Phi^2\,, \tag{9.13}$$

$$\widehat{m}_W^2 = \frac{\widehat{g}^2}{2}\sum_\Phi\left(J_\Phi(J_\Phi+1) - Y_\Phi^2\right)\widehat{v}_\Phi^2\,. \tag{9.14}$$

With this one may express the Veltman ρ parameter of Eq. (9.1) as

$$\widehat{\rho} = \frac{\widehat{m}_W^2}{\widehat{c}^2\widehat{m}_Z^2} = \frac{\sum_\Phi\left(J_\Phi(J_\Phi+1) - Y_\Phi^2\right)\widehat{v}_\Phi^2}{2\sum_\Phi Y_\Phi^2\widehat{v}_\Phi^2}\,, \tag{9.15}$$

which is a well-known result [148, 149].

This expression allows us to more precisely tell which models have $\widehat{\rho} = 1$ or, equivalently, which models have the gauge boson masses related via $\widehat{m}_W = \widehat{c}\,\widehat{m}_Z$. Setting the l.h.s. of Eq. (9.15) to 1 we get that for every multiplet the isospin and the hypercharge must be related via the following relation

$$J_\Phi(J_\Phi+1) - 3Y_\Phi^2 = 0 \tag{9.16}$$

or, equivalently,

$$(2J_\Phi + 1)^2 - 12Y_\Phi^2 = 1\,. \tag{9.17}$$

For example, such a relation is satisfied in the SM by the Higgs doublet with $J_\Phi = Y_\Phi = 1/2$. Other, but less common, options are also available, for example, the septuplet with $J_\Phi = 3$ and $Y_\Phi = 2$ or the 26-plet with $J_\Phi = 25/2$ and $Y_\Phi = 15$ provide $\widehat{\rho} = 1$ [150]. Any additional scalar multiplet preserves this relation if it does not acquire a vacuum expectation value. Importantly, for all such models, one is free to compare the EW predictions with those of the SM by using the usual oblique parameter formalism.

There is one more way of ensuring that $\widehat{\rho} = 1$ without selecting specific quantum numbers by instead tuning the vacuum expectation values. For example, by singling out some multiplet ϕ with a non-trivial $\widehat{v}_\phi$ and by using Eq. (9.15) one may get

$$\widehat{v}_\phi^2 = \frac{\sum_{\Phi\neq\phi}\left(J_\Phi(J_\Phi+1) - 3Y_\Phi^2\right)\widehat{v}_\Phi^2}{J_\phi(J_\phi+1) - 3Y_\phi^2}\,. \tag{9.18}$$

For example, such a relation is employed in the well-known Georgi-Machacek model [146, 151–158]. By tuning the VEVs one may get $\widehat{\rho} = 1$, but this is not a stable relation as there is no mechanism, that would require the VEVs to be related in this particular way, but rather it is an *ad hoc* relation imposed by hand. Models with tuned VEVs trace a path of enhanced symmetry, in particular, the custodial symmetry that provides $\widehat{\rho} = 1$, but the full parameter space does not possess this symmetry. In turn, the naive expectation to use the usual oblique parameter formalism and to compare the EW predictions with the SM fails, since once again the T parameter becomes UV-divergent [146, 158, 159].

Depending on whether $\widehat{\rho}$ is free or fixed to be 1, there are different definitions of the Weinberg angle through the Fermi constant $\widehat{G}_{F(\text{charged})}$. Considering the tree-level muon decay at low momenta and relating it to the effective four-fermion Lagrangian gives

$$\sqrt{2}\,\widehat{G}_{F(\text{charged})} = \frac{\pi\widehat{\alpha}}{\widehat{s}^2\widehat{m}_W^2}\,, \tag{9.19}$$

where $\widehat{m}_W$ comes from the W boson propagator at low momenta. This relation is possible if $\widehat{\rho}$ is not fixed. Otherwise, if $\widehat{\rho} = 1$, we may use that $\widehat{m}_W = \widehat{c}\,\widehat{m}_Z$ and get the more common relation

$$\sqrt{2}\,\widehat{G}_{F(\text{charged})} = \frac{\pi\widehat{\alpha}}{\widehat{s}^2\widehat{c}^2\widehat{m}_Z^2}\,. \tag{9.20}$$

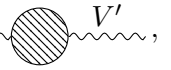
Any of Eqs. (9.19) or (9.20) may be used as a definition of the bare Weinberg angle θ_W in terms of $\widehat{G}_{F(\text{charged})}$, which is very well measured, for models with $\widehat{\rho} = 1$. The choice eventually depends on the input parameters, if one uses $\{\alpha, G_{F(\text{charged})}, m_W\}$ as inputs, then Eq. (9.19) is appropriate. If $\{\alpha, G_{F(\text{charged})}, m_Z\}$ is used, then Eq. (9.20) is the correct choice. On the other hand, for models where $\widehat{\rho}$ is not fixed, only Eq. (9.19) can be used to define the Weinberg angle.

9.2. The oblique parameters

Having introduced the gauge sector and the possible cases regarding the Veltman parameter, we now present the definitions of the oblique parameters.

The oblique corrections originate from the bare self-energies of the gauge bosons V and $V\prime$

$$\Pi^{\mu\nu}_{VV'}\left(p^2\right) = \quad , \tag{9.21}$$



where $V, V' = W, Z, A$ and p is the momentum transfer. The self-energy can be decomposed as

$$\Pi^{\mu\nu}_{VV'}\left(p\right) = g^{\mu\nu}\,\Pi_{VV'}\left(p^2\right) + p^\mu p^\nu \text{ terms}\,. \tag{9.22}$$

The oblique parameters are defined solely in terms of the coefficients of the metric $g^{\mu\nu}$, while the $p^\mu p^\nu$ terms do not enter. In addition, we define the following helpful abbreviations

$$\widetilde{\Pi}_{VV'}\left(p^2\right) \equiv \frac{\Pi_{VV'}\left(p^2\right) - \Pi_{VV'}\left(0\right)}{p^2}\,, \tag{9.23}$$

$$\Pi'_{VV'}\left(p^2\right) \equiv \frac{\mathrm{d}\Pi_{VV'}\left(p^2\right)}{\mathrm{d}p^2}\,. \tag{9.24}$$

The self-energies depend on the intricacies of a particular model, hence, we will add superscripts indicating the model, for example, $\Pi^{\text{SM}}_{VV'}$ is the self-energy computed in the Standard Model.

The oblique parameters [54, 55] are usually defined with self-energies subtracted as $\Pi^{\text{new}}_{VV'} = \Pi^{\text{NPM}}_{VV'} - \Pi^{\text{SM}}_{VV'}$, however, for our purposes it is more enlightening to define the non-subtracted (model) oblique parameters:

$$S^{\text{M}} = \frac{4s^2c^2}{\alpha}\left[\widetilde{\Pi}^{\text{M}}_{ZZ}\left(m_Z^2\right) + \frac{s^2-c^2}{sc}\,\Pi'^{\text{M}}_{ZA}\left(0\right) - \Pi'^{\text{M}}_{AA}\left(0\right)\right]\,, \tag{9.25a}$$

$$T^{\text{M}} = \frac{1}{\alpha}\left[\frac{\Pi^{\text{M}}_{WW}\left(0\right)}{m_W^2} - \frac{\Pi^{\text{M}}_{ZZ}\left(0\right)}{m_Z^2} - \frac{2s}{c}\,\frac{\Pi^{\text{M}}_{ZA}\left(0\right)}{m_Z^2}\right]\,, \tag{9.25b}$$

$$U^{\mathrm{M}} = \frac{4s^2}{\alpha}\left[\widetilde{\Pi}^{\mathrm{M}}_{WW}\left(m_W^2\right) - c^2\,\widetilde{\Pi}^{\mathrm{M}}_{ZZ}\left(m_Z^2\right) - 2sc\,\Pi'^{\mathrm{M}}_{ZA}\left(0\right) - s^2\,\Pi'^{\mathrm{M}}_{AA}\left(0\right)\right], \tag{9.25c}$$

$$V^{\mathrm{M}} = \frac{1}{\alpha}\left[\Pi'^{\mathrm{M}}_{ZZ}\left(m_Z^2\right) - \widetilde{\Pi}^{\mathrm{M}}_{ZZ}\left(m_Z^2\right)\right], \tag{9.25d}$$

$$W^{\mathrm{M}} = \frac{1}{\alpha}\left[\Pi'^{\mathrm{M}}_{WW}\left(m_W^2\right) - \widetilde{\Pi}^{\mathrm{M}}_{WW}\left(m_W^2\right)\right], \tag{9.25e}$$

$$X^{\mathrm{M}} = \frac{sc}{\alpha}\left[\Pi'^{\mathrm{M}}_{ZA}\left(0\right) - \widetilde{\Pi}^{\mathrm{M}}_{ZA}\left(m_Z^2\right)\right]. \tag{9.25f}$$

Here the non-hatted parameters are defined in terms of input parameters and we define them more precisely in the following section. In addition, we introduce a shorthand for the linear combination of oblique parameters that comes in handy

$$K^{\mathrm{M}} \equiv \frac{S^{\mathrm{M}}}{2c^2} + \frac{s^2 - c^2}{4s^2c^2}\,U^{\mathrm{M}} \tag{9.26}$$

$$= \frac{1}{\alpha}\left[\frac{s^2 - c^2}{c^2}\,\widetilde{\Pi}^{\mathrm{M}}_{WW}\left(m_W^2\right) + \widetilde{\Pi}^{\mathrm{M}}_{ZZ}\left(m_Z^2\right) - \frac{s^2}{c^2}\,\Pi'^{\mathrm{M}}_{AA}\left(0\right)\right]. \tag{9.27}$$

It is important to note that the model parameters we have defined here are neither UV finite nor gauge-independent [160] and cannot be used to parameterize the observables. Only subtractions of such parameters between appropriate models give UV finite and gauge-independent results that can be used to parameterize the observables in the EW sectors. For example, referring to the cases we have given above, some oblique parameter O subtracted between models M1 and M2

$$O = O^{\mathrm{M2}} - O^{\mathrm{M1}} \tag{9.28}$$

is UV-finite and gauge-independent only in the first case (M1=SM, M2=NPM) and in the second case (M1=BM, M2=BBM), but not in the third case (M1=SM, M2=BBM).

10. Corrections to selected observables in the three cases

10.1. Input parameters in the electroweak sector

As already outlined in this thesis in Section 1, one needs to renormalize the theory to make it predictive, *i.e.* one has to input information from the experiment. In the previous parts, we have done so by using counterterms, but in this part, we circumvent or, rather, hide this procedure. We do so by computing loop corrections to some bare parameter $\widehat{\lambda}$ and equating it to the value measured in the experiment (observable) λ

$$\lambda = \widehat{\lambda}^{\mathrm{M}}\left(1 + \mathrm{loops}^{\mathrm{M}}\right). \tag{10.1}$$

Here we have added the superscript on M, to indicate that both the bare parameter and the loop corrections depend on relations particular to some model M. By inverting this equation we may express $\widehat{\lambda}^{\mathrm{M}}$ in terms of loop corrections and the measured values (input parameters). Such an expression of $\widehat{\lambda}^{\mathrm{M}}$ may further be used to express other observables in terms of loop corrections and inputs.

In a renormalizable theory, the amount of needed input parameters is finite, hence, it is not uncommon to find that some observable *not* belonging to the input parameter set may be expressed in terms of the inputs. In that case, such a parameter κ becomes a prediction of some model M

$$\kappa^{\mathrm{M}} = f\left(\widehat{\lambda}^{\mathrm{M}}\right)\left(1+\mathrm{loops}'^{\mathrm{M}}\right) = f\left(\lambda\right)\left(1+\mathrm{loops}'^{\mathrm{M}} - \lambda\partial f\left(\lambda\right)\mathrm{loops}^{\mathrm{M}}\right) . \tag{10.2}$$

Here $\widehat{\kappa} \equiv f(\widehat{\lambda}^{\mathrm{M}})$ defines the bare parameter, $\partial f(\lambda) = \partial f/\partial\widehat{\lambda}^{\mathrm{M}}|_{\widehat{\lambda}^{\mathrm{M}}=\lambda}$, the minus sign comes from inverting Eq. (10.1), and we have dropped terms of order loops$\times$loops or higher. In Section 1 we were able to make predictions of the same observable but at a different scale. Here the same is still possible, but we also emphasize that predictions to completely different observables such as κ may be made in terms of the inputs λ. Of course, one is free to choose, which parameters are to be measured and taken as inputs, and which ones are to be predicted and then compared with measurements.

Turning to the electroweak sector, we have already mentioned that for EW models where $\widehat{\rho} = 1$ it is enough to provide three input parameters. These three parameters are chosen to be the ones, which are the most accurately measured and are [161]

- the fine structure constant at 0 momentum $\alpha(0) = 1/137.035999084$,
- the Fermi constant $G_{F(\text{charged})} = 1.1663788 \times 10^{-5}$ GeV^{-2},
- mass of the Z boson $m_Z = 91.1876$ GeV.

If a model does not have a fixed $\widehat{\rho}$, then a fourth parameter is needed, which we take to be

- the mass of the W boson $m_W^{\mathrm{PDG}} = 80.377$ GeV from [161] or $m_W^{\mathrm{CDF}} = 80.4335$ GeV from [61].

While these are the inputs, it is more convenient and common to work with the sine of the Weinberg angle instead of the Fermi constant. This can be done by employing Eqs. (9.19) and (9.20) to define the sines in terms of input parameters. For models where $\widehat{\rho}$ is a free parameter we use Eq. (9.19) and

define

$$s^2 = \frac{\pi\alpha(0)}{\sqrt{2}G_{F(\text{charged})}m_W^2} \qquad \Longrightarrow \qquad s^2 = 0.214826\,, \tag{10.3}$$

where we have taken the value for m_W from [61]. Analogously, for models with $\widehat{\rho} = 1$ we have

$$\bar{s}^2\bar{c}^2 = \frac{\pi\alpha(0)}{\sqrt{2}G_{F(\text{charged})}m_Z^2} \qquad \Longrightarrow \qquad \bar{s}^2 = 0.212152\,, \tag{10.4}$$

where we have added bars over s and c to distinguish between the definitions. It should be noted that appropriate sines and cosines should be used in the definitions of the oblique parameters in Eq. (9.25) depending on whether $\widehat{\rho} = 1$ or is free in a given model. The same note applies to the rest of this section.

Having selected our input parameters we may write them in terms of the bare parameters and loop corrections as in Eq. (10.1). The fine structure constant may then be written as [78, 104, 162, 163]

$$\alpha = \widehat{\alpha}^{\text{M}}\left[1 + \Pi_{AA}^{\text{M}\prime}(0) + \frac{2s}{c}\,\frac{\Pi_{ZA}^{\text{M}}(0)}{m_Z^2}\right]. \tag{10.5}$$

It is important to note that via Ward Identities this includes all the 1-loop corrections including external leg and vertex corrections, but written entirely in terms of self-energies. More importantly, it contains the $Z - A$ mixing term, which vanishes upon subtraction

$$\Pi_{ZA}(0) \equiv \Pi_{ZA}^{\text{M}'}(0) - \Pi_{ZA}^{\text{M}}(0) = 0\,, \tag{10.6}$$

but separately neither $\Pi_{ZA}^{\text{M}'}(0)$ nor $\Pi_{ZA}^{\text{M}}(0)$ are vanishing. This poses a problem, since a non-zero $\Pi_{ZA}^{\text{M}',\text{M}}(0)$ implies mass terms for the photon. The masslessness of the photon may simply be ensured via a counterterm in the on-shell scheme as in [164]. However, when working with oblique parameters, one expects to avoid using the counterterms explicitly, hence we follow [125] and introduce a 1-loop order shift of the bare gauge coupling

$$\widehat{g}^{\text{M}} \to \widehat{g}^{\text{M}}\left(1 - \frac{1}{sc}\,\frac{\Pi_{ZA}^{\text{M}}(0)}{m_Z^2}\right). \tag{10.7}$$

This shift ensures that the photon remains massless at 1-loop without introducing an explicit counterterm. Another effect is the redefinition of the gauge-boson self-energies

$$\Pi_{ZA}^{\text{M}}\left(p^2\right) \to \Pi_{ZA}^{\text{M}}\left(p^2\right) - \Pi_{ZA}^{\text{M}}(0)\,, \tag{10.8a}$$

$$\Pi^{\mathrm{M}}_{ZZ}\left(p^2\right) \to \Pi^{\mathrm{M}}_{ZZ}\left(p^2\right) - \frac{2c}{s}\,\Pi^{\mathrm{M}}_{ZA}\left(0\right), \tag{10.8b}$$

$$\Pi^{\mathrm{M}}_{WW}\left(p^2\right) \to \Pi^{\mathrm{M}}_{WW}\left(p^2\right) - \frac{2c}{s}\left(\frac{m_W^2}{c^2 m_Z^2}\right)\Pi^{\mathrm{M}}_{ZA}\left(0\right). \tag{10.8c}$$

Note that the oblique parameters in Eq. (9.25) already include this shift. However, only the T parameter in Eq. (9.25b) is sensitive to this shift and all the other parameters are as in [55].

Now we are ready to write down all the input parameters in terms of the bare parameters and the loop corrections with the shift in Eq. (10.7) included:

$$\alpha = \widehat{\alpha}^{\mathrm{M}}\left[1 + \Pi^{\mathrm{M}\prime}_{AA}\left(0\right)\right], \tag{10.9a}$$

$$G_{F(\text{charged})} = \widehat{G}^{\mathrm{M}}_{F(\text{charged})}\left[1 - \frac{\Pi^{\mathrm{M}}_{WW}\left(0\right)}{m_W^2} + \frac{2}{sc}\,\frac{\Pi^{\mathrm{M}}_{ZA}\left(0\right)}{m_Z^2} + \delta^{\mathrm{M}}_{Gc}\right], \tag{10.9b}$$

$$m_Z^2 = \left(\widehat{m}^{\mathrm{M}}_Z\right)^2\left[1 + \frac{\Pi^{\mathrm{M}}_{ZZ}\left(m_Z^2\right)}{m_Z^2} - \frac{2c}{s}\,\frac{\Pi^{\mathrm{M}}_{ZA}\left(0\right)}{m_Z^2}\right], \tag{10.9c}$$

$$\left.\begin{array}{ll} \text{free } \widehat{\rho} & , m_W^2 \\ \widehat{\rho} = 1 & , \left(m^{\mathrm{M}}_W\right)^2 \end{array}\right\} = \left(\widehat{m}^{\mathrm{M}}_W\right)^2\left[1 + \frac{\Pi^{\mathrm{M}}_{WW}\left(m_W^2\right)}{m_W^2} - \frac{2}{sc}\,\frac{\Pi^{\mathrm{M}}_{ZA}\left(0\right)}{m_Z^2}\right]. \tag{10.9d}$$

Here δ^{M}_{Gc} stands for the external leg, vertex, and box corrections to muon decay from which the Fermi constant is measured. Eventually, these non-oblique corrections are assumed to be small and will be dropped as usual. In Eq. (10.9d) we see that the mass of the W is an input parameter if $\widehat{\rho}$ is free, otherwise it is a model prediction, hence, the M superscript.

For convenience, we exchange $G_{F(\text{charged})}$ for s or $\bar{s}$ depending on the model, hence, it makes sense to write down these parameters in the form of Eq. (10.1) or Eqs. (10.9). To do so, for $\widehat{\rho}$ being free, we take Eq. (10.3) and insert the input parameters from Eq. (10.9)

$$\begin{aligned} s^2 =& (\widehat{s}^{\mathrm{M}})^2\,\frac{\left[1 + \Pi^{\mathrm{M}\prime}_{AA}\left(0\right)\right]}{\left[1 - \frac{\Pi^{\mathrm{M}}_{WW}(0)}{m_W^2} + \frac{2}{sc}\,\frac{\Pi^{\mathrm{M}}_{ZA}(0)}{m_Z^2} + \delta^{\mathrm{M}}_{Gc}\right]\left[1 + \frac{\Pi^{\mathrm{M}}_{WW}\left(m_W^2\right)}{m_W^2} - \frac{2}{sc}\,\frac{\Pi^{\mathrm{M}}_{ZA}(0)}{m_Z^2}\right]} \\ \approx& (\widehat{s}^{\mathrm{M}})^2\left[1 + \Pi^{\mathrm{M}\prime}_{AA}\left(0\right) + \frac{\Pi^{\mathrm{M}}_{WW}\left(0\right)}{m_W^2} - \frac{2}{sc}\,\frac{\Pi^{\mathrm{M}}_{ZA}\left(0\right)}{m_Z^2} - \delta^{\mathrm{M}}_{Gc}\right. \\ & \left. - \frac{\Pi^{\mathrm{M}}_{WW}\left(m_W^2\right)}{m_W^2} + \frac{2}{sc}\,\frac{\Pi^{\mathrm{M}}_{ZA}\left(0\right)}{m_Z^2}\right] \\ =& (\widehat{s}^{\mathrm{M}})^2\left[1 + \Pi^{\mathrm{M}\prime}_{AA}\left(0\right) - \widetilde{\Pi}_{WW}\left(m_W^2\right) - \delta^{\mathrm{M}}_{Gc}\right]. \end{aligned} \tag{10.10}$$

Here we dropped the terms of order $(\Pi_{VV'})^2$ or higher and defined

$$(\widehat{s}^{\rm M})^2 = \frac{\pi\widehat{\alpha}^{\rm M}}{\sqrt{2}\widehat{G}^{\rm M}_{F({\rm charged})}(\widehat{m}^{\rm M}_W)^2}\,. \tag{10.11}$$

Analogously, when $\widehat{\rho}=1$, we may get the expression of $\bar{s}$ from Eq. (10.4)

$$\begin{aligned}\bar{s}^2 =&(\widehat{s}^{\rm M})^2\left[1+\frac{\bar{c}^2}{\bar{s}^2-\bar{c}^2}\left(\frac{\Pi^{\rm M}_{WW}(0)}{m_W^2}-\frac{\Pi^{\rm M}_{ZZ}(0)}{m_Z^2}-\frac{2\bar{s}}{\bar{c}}\frac{\Pi^{\rm M}_{ZA}(0)}{m_Z^2}\right.\right.\\ &\left.\left.+\Pi^{\rm M\prime}_{AA}(0)-\widetilde{\Pi}^{\rm M}_{ZZ}\left(m_Z^2\right)-\delta^{\rm M}_{Gc}\right)\right]\\ =&(\widehat{s}^{\rm M})^2\left[1+\frac{\bar{c}^2}{\bar{s}^2-\bar{c}^2}\left(\alpha\bar{T}^{\rm M}+\Pi^{\rm M\prime}_{AA}(0)-\widetilde{\Pi}^{\rm M}_{ZZ}\left(m_Z^2\right)-\delta^{\rm M}_{Gc}\right)\right]\,.\end{aligned} \tag{10.12}$$

Here $\alpha\bar{T}^{\rm M}$ is as in Eq. (9.25b), but with $s,\,c\to\bar{s},\,\bar{c}$ and $(\widehat{s}^{\rm M})^2$ is the same as in Eq. (10.11), but one should additionally use $\widehat{m}_W=\widehat{c}\,\widehat{m}_Z$. Of course, in both cases, one may also find analogous expressions for c and $\bar{c}$, but we leave this to the reader.

Also, note that the fine structure constant in Eq. (10.9a) is measured at low momenta, but one can also find definitions, where α is set at the Z pole [64, 72, 165, 166]. In addition, mixed approaches, where the photon and $Z-A$ self-energies are taken at 0 momentum, but the input value for alpha is taken at the Z pole, *e.g.* [131]. However, to the best of our knowledge, there is no direct measurement of α at the Z pole[8] and the value is acquired by running from $\alpha(0)$, which includes only known SM physics. If one assumes that the new physics is heavy, then there is no problem with the running, but in the presence of light new physics, such running is model-dependent. Since we are using the $STUVWX$ parameters, which are applicable for light physics, we choose to keep $\alpha\equiv\alpha(0)$ as our input parameter. On the other hand, the reader is free to input $\alpha(m_Z)$ instead and to follow the more common mixed approach.

10.2. Oblique corrections to the Veltman ρ parameter

Having set up all the definitions and the renormalization scheme, let us compute corrections to the Veltman ρ parameter. The measured ρ parameter may be defined analogously to Eq. (9.1) by removing the hats, but the definition depends on whether $\widehat{\rho}$ is equal to 1 or free. If $\widehat{\rho}=1$, we have

$$\bar{\rho}^{\rm M}\equiv\frac{\left(m^{\rm M}_W\right)^2}{\bar{c}^2m_Z^2}\,, \tag{10.13}$$

[8] There are measurements of the running of $\alpha\left(Q^2\right)$, but not through the Z pole [167, 168]

if $\widehat{\rho}$ is free, then

$$\rho \equiv \frac{m_W^2}{c^2 m_Z^2} \,. \tag{10.14}$$

In the latter equation, the ρ parameter is defined solely in terms of input parameters of Eq. (10.9), while in the former equation, a model dependence develops through Eq. (10.9d) and we have added a bar to emphasize the difference of definitions of the Weinberg angle. In other words, for $\widehat{\rho} = 1$ the Veltman parameter is a prediction, while if $\widehat{\rho}$ is free, the Veltman parameter might as well be used as an input parameter.

Let us take Eq. (10.13) and insert Eqs. (10.9c), (10.9d), and (10.12)

$$\begin{aligned} \bar{\rho}^{\rm M} =& \widehat{\rho}^{\rm M} \left[1 + \frac{\bar{c}^2}{\bar{c}^2 - \bar{s}^2} \Big(\alpha T^{\rm M} - \widetilde{\Pi}^{\rm M}_{ZZ} \left(m_Z^2 \right) + \frac{\bar{s}^2}{\bar{c}^2} \, \Pi^{\rm M\prime}_{AA} \left(0 \right) \right. \\ & \left. - \frac{\bar{s}^2}{\bar{c}^2} \, \delta^{\rm M}_{Gc} + \frac{\bar{c}^2 - \bar{s}^2}{\bar{c}^2} \, \widetilde{\Pi}^{\rm M}_{WW} \left(m_W^2 \right) \Big) \right] \\ =& \widehat{\rho}^{\rm M} \left[1 + \frac{\bar{c}^2}{\bar{c}^2 - \bar{s}^2} \left(\alpha T^{\rm M} - \alpha K^{\rm M} \right) - \frac{\bar{s}^2}{\bar{c}^2 - \bar{s}^2} \, \delta^{\rm M}_{Gc} \right] . \end{aligned} \tag{10.15}$$

Although we have kept $\widehat{\rho}^{\rm M}$ for generality, this is a prediction, which is UV finite and gauge-independent, for the Veltman parameter when $\widehat{\rho}^{\rm M} = 1$. Having this, we may compare the predictions of two models, say, SM and an NPM

$$\frac{\bar{\rho}^{\rm NPM}}{\bar{\rho}^{\rm SM}} = \frac{\widehat{\rho}^{\rm NPM}}{\widehat{\rho}^{\rm SM}} \left[1 + \frac{\bar{c}^2}{\bar{c}^2 - \bar{s}^2} \left(\alpha \bar{T} - \alpha \bar{K} \right) + \frac{\bar{s}^2}{\bar{s}^2 - \bar{c}^2} \left(\delta^{\rm NPM}_{Gc} - \delta^{\rm SM}_{Gc} \right) \right] . \tag{10.16}$$

Here $\bar{T}$ and $\bar{K}$ are the subtracted oblique parameters as in Eqs. (9.25b), (9.26), and (9.28), but with the substitution $s,\ c \to \bar{s},\ \bar{c}$. Next, we assume that

$$\delta^{\rm M\prime}_{Gc} - \delta^{\rm M}_{Gc} \tag{10.17}$$

is negligible [54] and set $\widehat{\rho}^{\rm SM} = \widehat{\rho}^{\rm NPM} = 1$ to get

$$\bar{\rho}^{\rm NPM} = \bar{\rho}^{\rm SM} \left[1 + \frac{\bar{c}^2}{\bar{c}^2 - \bar{s}^2} \left(\alpha \bar{T} - \alpha \bar{K} \right) \right] . \tag{10.18}$$

Here we have a prediction for the Veltman parameter in the NPM in terms of the SM prediction and the subtracted oblique parameters. Importantly, we have dropped all the non-oblique corrections, but the final result is still UV-finite and gauge-independent since the subtracted oblique corrections (parameters) form a UV-finite and gauge-independent set.

Using Eq. (10.13) and the fact that $\bar{c}$ and m_Z are treated as input parameters, it is not hard to see that Eq. (10.18) is equivalent to the prediction of the W

mass in Eq. (10.9d)

$$m_W^{\rm NPM} = m_W^{\rm SM}\left[1+\frac{\bar{c}^2}{2\left(\bar{c}^2-\bar{s}^2\right)}\,\alpha\bar{T}+\frac{\alpha\bar{S}}{4\left(\bar{s}^2-\bar{c}^2\right)}+\frac{\alpha\bar{U}}{8\bar{s}^2}\right]. \tag{10.19}$$

Here we have inserted the definition of K from Eq. (9.26) and arrived at the usual parameterization for the prediction of the W boson mass in NPM models with $\widehat{\rho}=1$.

Turning to models, where $\widehat{\rho}$ is free, and taking Eq. (10.14) we insert all the inputs from Eq. (10.9) and get

$$\rho = \widehat{\rho}^{\rm M}\left(1+\alpha T^{\rm M}-\alpha K^{\rm M}-\frac{s^2}{c^2}\,\delta_{Gc}^{\rm M}\right). \tag{10.20}$$

One may also invert this relation and find

$$\widehat{\rho}^{\rm M} = \rho\left(1-\alpha T^{\rm M}+\alpha K^{\rm M}+\frac{s^2}{c^2}\,\delta_{Gc}^{\rm M}\right). \tag{10.21}$$

Since ρ does not depend on the model, a comparison between a BM and a BBM gives a relation between the bare parameters and the oblique parameters

$$\frac{\widehat{\rho}^{\rm BBM}}{\widehat{\rho}^{\rm BM}} = 1-\alpha T+\alpha K\,. \tag{10.22}$$

Here we have also dropped the non-oblique corrections $\sim\left(\delta_{Gc}^{\rm BBM}-\delta_{Gc}^{\rm BM}\right)$.

These relations only confirm that $\rho=\rho^{\rm BM}=\rho^{\rm BBM}$ is treated as an input parameter since the bare parameters fully cancel all the loop corrections. Nonetheless, Eq. (10.22) will be very useful in deriving a general substitution rule that gets rid of the subtracted T parameter, which is UV divergent in the BM *vs.* BBM case, from the parameterization of predictions.

10.3. Oblique corrections to the $\rho_\star$ parameter

The ratio between the charged and neutral currents is defined as

$$\rho_\star^{\rm M} \equiv \frac{G_{F(\text{neutral})}^{\rm M}}{G_{F(\text{charged})}}\,. \tag{10.23}$$

Here $G_{F(\text{neutral})}^{\rm M}$ is a model prediction for the low-energy neutrino scattering mediated by the Z boson

$$G_{F(\text{neutral})}^{\rm M} = \widehat{G}_{F(\text{neutral})}^{\rm M}\left[1-\frac{\Pi_{ZZ}^{\rm M}(0)}{m_Z^2}+\frac{2}{sc}\,\frac{\Pi_{ZA}^{\rm M}(0)}{m_Z^2}+\delta_{Gn}^{\rm M}\right], \tag{10.24}$$

with $\delta^{\rm M}_{Gn}$ representing the non-oblique and process-dependent corrections.

Plugging the definitions of $G_{F(\text{charged})}$ and $G^{\rm M}_{F(\text{neutral})}$ into Eq. (10.23), we get

$$\rho^{\rm M}_\star = \widehat{\rho}^{\,\rm M}\,\left(1+\alpha T^{\rm M}+\delta^{\rm M}_{Gn}-\delta^{\rm M}_{Gc}\right)\,. \tag{10.25}$$

To get this, we have used the definition of $\widehat{G}_{F(\text{charged})}$ in Eq. (9.19), analogously defined (dropping M for clarity)

$$\widehat{G}_{F(\text{neutral})} = \frac{\pi\,\widehat{\alpha}}{\sqrt{2}\,\widehat{s}^{\,2}\widehat{c}^{\,2}\widehat{m}_Z^2} \tag{10.26}$$

and used

$$\widehat{\rho}^{\,\rm M} = \frac{\widehat{G}^{\rm M}_{F(\text{neutral})}}{\widehat{G}^{\rm M}_{F(\text{charged})}}\,. \tag{10.27}$$

Let us consider the SM *vs.* NPM case and compare the predictions for $\rho_\star$. We get [54]

$$\frac{\rho^{\rm NPM}_\star}{\rho^{\rm SM}_\star} = \frac{\widehat{\rho}^{\,\rm NPM}}{\widehat{\rho}^{\,\rm SM}}\left(1+\alpha\bar{T}\right) = 1+\alpha\bar{T}\,, \tag{10.28}$$

where we again neglected the non-oblique corrections and set $\widehat{\rho}^{\,\rm SM}=1$ and $\widehat{\rho}^{\,\rm NPM}=1$.

On the other hand, considering the case, where $\widehat{\rho}$ is not fixed, we get

$$\frac{\rho^{\rm BBM}_\star}{\rho^{\rm BM}_\star} = \frac{\widehat{\rho}^{\,\rm BBM}}{\widehat{\rho}^{\,\rm BM}}\,(1+\alpha T) = 1+\alpha K\,, \tag{10.29}$$

where we have used Eq. (10.22) to get the final equality. Importantly, the linear combination of the oblique parameters denoted by K is UV-finite and gauge-independent, unlike the parameter T in this case. In other words, by introducing the fourth input parameter, we have managed to get rid of the divergent T parameter and replace it with a suitable combination of the subtracted oblique parameters S and U. Even more so, this is a general replacement rule. Having parameterizations for observables of the $\widehat{\rho}=1$ case, one may easily get the free $\widehat{\rho}$ case by replacing

$$T^{\rm M}\to K^{\rm M} \qquad (\text{or} \quad \bar{T}\to K) \qquad \text{and} \qquad \bar{s},\bar{c}\to s,c\,. \tag{10.30}$$

The rule may be explicitly confirmed by consulting the appendices of [A5], which contain essential derivations of the parameterizations for both $\widehat{\rho}$ cases and summarizing tables for the $\widehat{\rho}=1$ case.

In the previous sections, we have presented the oblique parameters, when $\widehat{\rho} = 1$, and developed a genuine oblique parameter formalism when $\widehat{\rho}$ is free in both models under comparison. Equivalently, referring to Section 9, we have considered the first case of SM *vs.* NPM and the second case of BM *vs.* BBM, while we briefly cover the third case of SM *vs.* BBM here.

To see the problem, let us return to the Veltman ρ parameter and try to compare the predictions of the SM (model with $\widehat{\rho} = 1$) and some BBM (model with a free $\widehat{\rho}$). We can do so by taking the ratio of Eqs. (10.13) and (10.14)

$$\frac{\rho}{\bar{\rho}^{\text{SM}}} = \frac{m_W^2}{c^2 m_Z^2}\left[1 - \frac{\bar{c}^2}{\bar{c}^2 - \bar{s}^2}\left(\alpha T^{\text{SM}} - \alpha K^{\text{SM}}\right) + \frac{\bar{s}^2}{\bar{c}^2 - \bar{s}^2}\delta_{Gc}^{\text{SM}}\right], \quad (10.31)$$

where we have used Eq. (10.15) and set $\widehat{\rho}^{\text{SM}} = 1$. Such a comparison is perfectly valid and is UV-finite and gauge-independent, but at the same time, this is not a genuine oblique parameter formalism, since there is a process-dependent piece δ_{Gc}^{SM}. In addition, this parameterization also contains the non-subtracted model parameters of the Standard Model. If one assumes, that the non-oblique corrections are small and neglects them, then one automatically ruins the gauge independence, since the gauge dependence cancels between the (model) oblique parameters and the universal parts of vertex and box corrections contained in δ_{Gc}^{SM}. Of course, in this comparison, part of the problem is that in the SM the Veltman parameter is a prediction and gets corrections, while in the BBM ρ is equivalent to an input parameter and does not get corrected. In turn, it is impossible to form subtracted oblique parameters and such a comparison (if δ_{Gc}^{SM} is dropped) is bound to be at least gauge-dependent[9].

Let us see what happens if we take an observable, which is a prediction in both the SM and the BBM. Such an observable is $\rho_\star$ and by using Eqs. (10.21) and (10.25) we have

$$\begin{aligned}\frac{\rho_\star^{\text{BBM}}}{\rho_\star^{\text{SM}}} = \rho\Big[1 + \alpha K^{\text{BBM}} - \alpha T^{\text{SM}} \\ + \left(\delta_{Gn}^{\text{BBM}} - \delta_{Gn}^{\text{SM}}\right) - \left(\frac{c^2 - s^2}{c^2}\delta_{Gc}^{\text{BBM}} - \delta_{Gc}^{\text{SM}}\right)\Big].\end{aligned} \quad (10.32)$$

Here the comparison is again perfectly valid but contains non-oblique corrections and the subtractions do not occur due to different parameterizations, except for the $(\delta_{Gn}^{\text{BBM}} - \delta_{Gn}^{\text{SM}})$ term. However, even in this seeming subtraction, no cancelations between the gauge-dependent parts occur, simply because the definitions

[9] Certain gauges, for example, where all the gauge parameters are equal, produce UV finite model parameters [160].

of the Weinberg angle or, equivalently, of the W boson mass are different. For example, in $\delta_{Gn}^{\mathrm{BBM}}$ some gauge-dependent term might be proportional to m_W, while in $\delta_{Gn}^{\mathrm{SM}}$ an analogous term will be proportional to cm_Z, which not only look different but are also numerically different. Even if one neglects the non-oblique terms, the remaining oblique corrections still do not provide the needed cancelations both because of different parameterization and because of different numerical values. In turn, it follows, that one cannot compare predictions of models with different sets of renormalization conditions ($\widehat{\rho} = 1$ *vs.* $\widehat{\rho}$ free) by using just the oblique parameters and at the moment the full calculation is needed.

We are not the first ones to notice this and, as a solution, it has been proposed to include the universal parts of $\delta_{Gn,Gc}^{\mathrm{M}}$ in the self-energies [60, 143, 169]. Such an inclusion would make the oblique model parameters UV finite and gauge-independent. However, this is not a genuine oblique parameter formalism, since it is needed to compute vertex and box diagrams, which is more involving than computing just the self-energies. Whether such an effort is worthwhile remains an open question.

CONCLUSIONS

In this thesis we have discussed various aspects of renormalization, out of which three main topics emerge:

1. renormalization of mixing matrices,

2. development of an on-shell renormalization scheme for fermions and its application to the Grimus-Neufeld model,

3. renormalization in the electroweak sector in terms of the oblique corrections, when $m_W \neq \cos\theta_W m_Z$ at tree-level.

The discussions of our **first topic** are based on [A1], where we have discussed the renormalization of mixing matrices for a general system of scalars. In this thesis, we have opted for the fermionic case to show the universality of our arguments. Interestingly, the generality and universality of the discussion allowed for simplicity since it was sufficient to consider only two-point functions and to only assume that particles mix. In contrast, the renormalization of mixing matrices is usually discussed by considering 3-point (vertex) functions.

The basic question tackled in this part is whether one should renormalize mixing matrices and, if so, how it should be done. We have approached the question by considering two arbitrary bases (one of which was also written in two forms — with and without rotation matrices) related by a rotation, which for a specific selection of the initial and final bases can be identified with the usual mixing matrix. Further, in each basis, we have introduced counterterms for the fields, masses, and the rotation matrix, but no particular scheme was selected and only the *set* of counterterms was important. Since the bases are related by rotations in the bare Lagrangian, we have performed various basis changes in the renormalized Lagrangian. However, that led to various inconsistencies, for example, by rotating back by the renormalized rotation matrix to the initial basis the rotation matrix counterterm was left without the renormalized parameter. In turn, the rotation matrix counterterm became an obstruction to recover the bare Lagrangian from the renormalized one. Another inconsistency is that the presence of the rotation matrix counterterm changed the form of the basis transformations for the anti-hermitian part of the field renormalization.

Apart from these conceptual inconsistencies, we have also found practical problems. For example, there are no natural renormalization conditions for the counterterms of rotation (mixing) matrices and often the *ad-hoc* conditions produce gauge-dependent counterterms, while, quite contrary, gauge-independent counterterms are sought after since particle mixing is a physical process. In

addition, the presence of the rotation matrix counterterm along with the usual diagonal mass matrix counterterm leads to a singular degenerate mass limit, which may cause numerical problems.

Interestingly, all of these problems point to the *presence* of the rotation matrix counterterm, if, on the other hand, the counterterm is trivially 0, none of the problems exist. Even more so, in such a case one trivially satisfies all the requirements of mixing matrix renormalization listed in Section 2.2. The triviality of the counterterm follows from its arbitrariness: the same bare Lagrangian can be defined both with and without a rotation matrix. In other words, a rotation matrix selects a preferred basis completely by hand and, in turn, it is a non-physical and basis-dependent object, which should not receive a counterterm. An alternative explanation may be formulated by noticing that there is nothing special neither about basis rotations nor about renormalization and the two procedures should commute. The only way to preserve this commutation is to have a trivial rotation matrix counterterm, such that no basis is preferred. In conclusion, rotation (mixing) matrices should not have counterterms for consistency and to avoid practical problems. However, note that non-physicality and trivial counterterms do not imply that mixing matrices cannot be measured, quite contrary, mixing matrices are derived from some initial renormalized mass matrix, hence, measurement of the initial mass matrix is equivalent to the measurement of the new mass matrix and the mixing matrix.

Having shown that it is consistent to have trivial mixing matrix counterterms, we could begin the discussion of the **second topic** — the development and application of an on-shell scheme with trivial mixing matrix counterterms [A2–A4]. However, explicitly implementing such a counterterm is, on the contrary, not trivial as one quickly runs into problems with degeneracy — the off-diagonal components of the mass and the anti-hermitian part of the field counterterms cannot be solved for independently. In our scheme, this degeneracy is broken in a fairly distinct way by defining the anti-hermitian part of the field renormalization as the coefficient of the $m_i^2 - m_j^2$ mass structure and afterward solving for the mass counterterms in terms of self-energies and the anti-hermitian part of the field renormalization. As we have checked, such a definition indeed allows for a trivial mixing matrix counterterm, while satisfying all the mixing renormalization requirements and also being model- and process-independent. In addition, at least at 1-loop, we took care of including the absorptive parts to the maximum possible extent for Dirac fermions, while Majorana fermions do not allow for such inclusion at all. Further, by extensive use of Nielsen Identities, we have shown that the scheme can be constructed to all orders in perturbation theory, although, with the absorptive parts dropped.

We have explicitly applied our scheme at 1-loop to the Grimus-Neufeld model, which provided plenty of scenarios to test our scheme against. In particular, the Grimus-Neufeld model contains quarks and charged leptons that, despite coupling to the second Higgs doublet, are like in the Standard Model as well as 4 neutrinos, two of which are massless at tree level. SM-like quarks provided a fairly simple case and we could apply our scheme without difficulty and also compare to selected schemes of Denner and Sack, and Kniehl and Sirlin. On the other hand, the neutrino sector was more complicated as there were three distinct blocks, depending on which neutrinos were used to compute the self-energies. We had the massive case if both neutrinos were massive, the partially massless case if one of the neutrinos was massless, and the massless case if both considered neutrinos were massless. In the massive and massless cases, we could apply our scheme without much difficulty, while in the partially massless case our definition of the anti-hermitian part of the field renormalization was no longer applicable since the $m_i^2 - m_j^2$ mass structure does not exist if m_i or m_j is 0. Fortunately, we were able to overcome this by taking the massless limit ($m_i \to 0$ or $m_j \to 0$) of the massive case, without producing any unwanted behavior and keeping the properties of the scheme. In turn, we could conclude that our scheme indeed holds despite the peculiarities of the Grimus-Neufeld model. In addition, we have achieved satisfactory renormalization of the two-point functions in the fermion sector with the hope that a similar procedure may be carried out in the scalar sector as well.

For our **third topic**, based on [A5], we have investigated the oblique corrections in the electroweak sector in the case when the W and Z boson masses are not related through the Weinberg angle at tree level, namely, $\widehat{m}_W \neq \cos\widehat{\theta}_W \widehat{m}_Z$. Alternatively, this may be phrased in terms of the tree-level Veltman $\widehat{\rho}$ parameter: if $\widehat{m}_W = \cos\widehat{\theta}_W \widehat{m}_Z$ holds, then $\widehat{\rho} = 1$, if not, we say that $\widehat{\rho}$ is a free parameter. This additional freedom in the latter case implies that an input parameter (renormalization condition) in addition to the fine-structure constant, Fermi constant, and mass of the Z boson is needed in the EW sector. We have taken the mass of the W boson to be this fourth input parameter, the numerical value of which may be taken from the newest CDF measurement. By employing four input parameters we have found that instead of the six oblique parameters $STUVWX$ of the $\widehat{\rho} = 1$ case, only five parameters $SUVWX$ remain and the T parameter is removed by the renormalization procedure. This is a welcome achievement since it is well-known in the literature that the T parameter is UV-divergent and cannot be used to parametrize observables when $\widehat{\rho}$ is free. As far as we know, the $SUVWX$ formalism we have developed is the only genuine oblique parameter formalism when $\widehat{\rho}$ is free.

However, rather unfortunately, the $SUVWX$ formalism comes with limitations analogous to the $STUVWX$ formalism, which we were unable to overcome. Namely, in the $STUVWX$ formalism *both* models under consideration must have $\widehat{\rho} = 1$, while in the $SUVWX$ formalism, *both* must have a free $\widehat{\rho}$. It seems that just the oblique corrections (parameters) are insufficient if one wants to compare the predictions of a model with $\widehat{\rho} = 1$ (*e.g.* the SM or GNM) with those of a model with a free $\widehat{\rho}$ (*e.g.* the triplet model, where the triplet acquires a vacuum expectation value). At the moment, it seems that to perform such a comparison one needs to do a full calculation and include the process-dependent non-oblique loop corrections.

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SANTRAUKA LIETUVIŲ KALBA

Įvadas

Pirmosiomis kvantinės lauko teorijos (angl. *quantum field theory*, QFT) dienomis buvo susidurta su ultravioletinės (angl. *ultraviolet*, UV) katastrofos variantu — 1930-aisiais ir 1940-aisiais perturbatyviai suskaičiuoti stebiniai kvantinėje elektrodinamikoje (angl. *quantum electrodynamics*, QED) buvo begaliniai. Tomonaga, Schwinger'is, Feynman'as bei Dyson'as šią kliūtį įveikė apibrėždami renormalizacijos procedūrą [1–10], kuri davė baigtinius rezultatus. Tiesa, net ir su tokiu pasisekimu, fizikų bendruomenė kiek beviltiškai juokavo [12] ir renormalizaciją laikė tam tikru triuku, kuris „sušluoja begalybes po kilimu" [11]. Šiais laikais renormalizacijos procedūra jau daug tvirtesnės reputacijos, ypač 1970-aisiais atsiradusios renormalizacijos grupės (angl. *renormalization group*, RG) idėjų dėka [12–19], bei sutinkama įprastuose kvantinės lauko teorijos vadovėliuose bei kursuose.

UV divergencijų (begalybių) prigimtis grindžiama kvantinio lauko šiurkštumu ties mažais atstumais arba naivaus tikėjimosi, jog mūsų teorijos yra teisingos bet kokiems atstumų (energijų) masteliams, negaliojimu. Kad ir kaip būtų, norint dirbti su QFT, reikia turėti nuoseklų UV divergencijų aprašymą, tad įprastai teorijos *reguliarizuojamos* (angl. *regulate, regularize*), pavyzdžiui, tiesiog nukertant integralus ties didelėmis judesio kiekio vertėmis [20] ar naudojant kitokius metodus, tačiau stebiniai turi nepriklausyti nuo ranka parinkto reguliatoriaus. Iš renormalizacijos grupės žinome, jog skirtingi masteliai atsiskiria — nebūtina žinoti apie kvarkus, kad būtų nusakytas vandenilio spektras, nebūtina žinoti, kaip išsidėsčiusios gatvės Vilniaus senamiestyje, kad apskaičiuotume Jupiterio orbitą ir t.t. Iš to seka, jog reguliatorius, atitinkantis aukštas energijas, paprasčiausiai išsiprastina dviejų (mažos energijos) stebinių palyginime — tai ir yra renormalizacijos procedūra. Kitaip tariant, stebinių prognozės išreikštos per ankstesnių stebinių (matavimų) rinkinį nepriklauso nuo reguliatoriaus ir yra UV-baigtinės.

Dauguma renormalizacijos principų yra puikiai suprasti, tačiau velnias slypi detalėse — yra atvejų, kai nėra visiškai aišku, kaip reikėtų atlikti renormalizacijos procedūrą. Dėl to literatūroje galima sutikti daugybę renormalizacijos schemų turinčių savų pliusų ir minusų, tačiau šiuo metu nėra visų priimto recepto. Ypač dominantis atvejis, apimantis Standartinį Modelį (SM) bei įvairius jo plėtinius, yra teorijos, kuriose dalelės gali tarpusavyje maišytis ir maišymosi matricų renormalizacija nėra akivaizdi. Standartiniame Modelyje dalelių maišymasis pasireiškia kvarkų sektoriuje, o atitinkama maišymosi matrica yra

Cabbibo-Kobayashi-Maskawa (CKM) [21, 22] matrica. Ko gero pats pirmasis bandymas renormalizuoti maišymosi matricą gali būti surastas [23], kur renormalizuojama būtent CKM matrica. Vėliau buvo pastebėta, jog [23] pritaikyta renormalizacijos schema turi trūkumų, pavyzdžiui, CKM matricos atsvaros narys (angl. *counterterm*) priklauso nuo kalibruotės [24, 25] — nenorimas bruožas, kadangi dalelių maišymasis yra fizikinis efektas. Per pastaruosius 30 metų buvo gana nemažai bandymų renormalizuoti besimaišančius fermionus [24, 26–31] bei pašalinti pačio pirmojo būdo trūkumus. Maišymosi renormalizacijos problemos nėra susijusios išskirtinai tik su fermionais, bet taip pat yra svarbios ir išplėstiniuose skaliariniuose sektoriuose, kuriuose beveik neišvengiamai egzistuoja dalelių maišymasis, dėl to literatūroje galima rasti daug renormalizacijos schemų skaliariniams sektoriams [32–44].

Pagrindinė šios disertacijos tema yra renormalizacija tuo atveju, kai dalelės tarpusavyje gali maišytis.

Svarbiausia paminėti, jog sukonstravome renormalizacijos schemą, kuri savo prigimtimi yra išskirtinai konceptuali — reikalaujame trivialų atsvaros narių maišymosi matricoms (kampams), kadangi tai yra nuo bazės pasirinkimo priklausomi dydžiai. Schemai apibrėžti naudojame jau įprastomis tapusias renormalizacijos sąlygas ant masės apvalko (angl. *On-Shell*, OS) [45], tačiau jos yra išpildytos neįprastu būdu ir taip pat naudojant Fleischer-Jegerlehner (FJ) buožgalvių (angl. *tadpole*) schemą. Schemos neįprastumas taip pat pasireiškia ir nediagonalių lauko renormalizacijos konstantų apibrėžimu, kuris šias konstantas parenka kaip $m_i^2 - m_j^2$ masių struktūros koeficientus, kur i ir j yra aromato (angl. *flavor*), taip pat kartos arba šeimos, indeksai. Įdomu, jog toks apibrėžimas leidžia turėti trivialų maišymosi matricos atsvaros narį, t.y. prilyginti jį nuliui, ir taip iškart patenkinti visus maišymosi renormalizacijos reikalavimus suformuluotus [27, 36, 43]. Dar vienas svarbus aspektas tas, jog parodome, kad mūsų schema gali būti apibrėžta kiekvienai perturbacijų teorijos eilei.

Neskaitant bendrų ir nuo modelių nepriklausomų argumentų, pagrindinis įrankis schemos pristatymui yra Grimus-Neufeld modelis (GNM) [48–52] — Standartinis modelis išplėstas papildomu Higgs'o dubletu bei Majorana singletu. Šis modelis įdomus tiek iš techninės, tiek iš fizikinės pusės. Iš techninės pusės modelyje yra du neutrinų mases generuojantys mechanizmai: sverto (angl. *seesaw*) bei radiatyviųjų pataisų. Ties medžio lygmeniu tik 2 iš keturių Majorana neutrinų masės tikrinių būsenų bazėje yra masyvūs ir savo mases įgauna sverto mechanizmu. Ties 1-kilpa vienas iš prieš tai buvusių nemasyvių neutrinų įgauna masę per radiatyvias pataisas, o kitas neutrinas išlieka be masės. Tiesa, šis nemasyvus neutrinas aukštesnėse eilėse taip pat gali įgauti masę, kadangi nėra

jokios nuo to saugančios simetrijos. Šie mechanizmai ir ypač bemasės būsenos yra įdomūs išbandant renormalizacijos schemą, kuri remiasi masių struktūromis. Iš fizikinės pusės nenulinės masės neutrinai yra viena iš nedaugelio užuominų apie fiziką už Standartinio Modelio ribų, tad GNM, pasinaudodamas sverto bei radiatyviųjų masių mechanizmais, suteikia galimą neutrinų masių (ir jų mažumo) paaiškinimą. Papildomai, Majorana singletas, kuris atsiskiria ties mažomis energijomis, taip pat gali būti ir tamsiosios materijos kandidatas. Modelis turi būti renormalizuotas, kad būtų galima daryti prognozes ir taip patikrinti modelio teisingumą.

Svarbu ne tik daryti prognozes, bet ir sugebėti jas lengvai palyginti su kitų modelių prognozėmis bei eksperimentų duomenimis — tai kita šios disertacijos tema. Gan dažnas stebinių prognozių tarp modelių palyginimo būdas yra per taip vadinamus netiesioginius (angl. *oblique*) S, T, U parametrus įvestus [53, 54] ir vėliau papildytus X, V, W [55]. Verta paminėti, jog egzistuoja ir daugiau netiesioginių pataisų parametrizavimo būdų [56–60], kurie tarpusavyje yra susiję. Kai galima atmesti dėžių (angl. *box*) bei viršūninės (angl. *vertex*) pataisas ir užtenka įtraukti kalibruotinių (angl. *gauge*) bozonų propagatorių pataisas, netiesioginiai parametrai tampa naudingu įrankiu parametrizuojant elektrosilpnojo (angl. *electroweak*, EW) sektoriaus stebinių vienos kilpos pataisas. Svarbu pastebėti, jog netiesioginiai parametrai yra UV baigtiniai ir nepriklauso nuo kalibruotės, todėl gali būti naudojami parametrizuojant stebinius. Įprastai netiesioginiams parametrams apibrėžti reikalinga parinkti tris įvesties parametrus iš EW sektoriaus, kitaip tariant — juos renormalizuoti. Tuomet netiesioginiai parametrai naudojami palyginant SM ir kokio nors kito modelio, kuris turi tą pačią kalibruotinę grupę bei apsauginę $SU(2)$ simetriją, kuri išlaiko gerai žinomą medžio lygmens sąryšį $m_W = \cos\theta_W m_Z$ tarp W ir Z bozonų masių per Weinberg'o kampą θ_W. Iš to seka, kad Grimus-Neufeld modelis gali būti palygintas su SM arba net paimtas kaip pamatinis modelis, su kuriuo lyginami kiti modeliai. Kita vertus, atsižvelgiant į naujus CDF kolaboracijos atliktus W bozono masės matavimus [61], kurie numato didesnę masę nei SM prognozės, modeliai, kuriuose medžio lygmens lygybė $m_W = \cos\theta_W m_Z$ nėra išlaikoma, tampa vertais dėmesio. Daugelis autorių pastebėjo, jog tokiu atveju T parametras nebėra UV-baigtinis [60, 62–70], todėl nebegali būti naudojamas parametrizuojant stebinius. Šią kliūtį galima pašalinti renormalizuojant papildomą EW sektoriaus parametrą, kadangi tai eliminuoja T parametrą, tačiau sumažintas netiesioginių parametrų rinkinys gali būti panaudotas tik modeliams, kuriuose $m_W \neq \cos\theta_W m_Z$ medžio lygmenyje.

Pagrindiai tikslai ir uždaviniai

Pagrindinis šios disertacijos tikslas yra priartėti prie pilnos Grimus-Neufeld modelio renormalizacijos. Pagrindinė kliūtis renormalizacijoje yra modelyje esantis dalelių maišymasis tiek fermionų, tiek ir skaliarų sektoriuose, kuris reikalauja apibrėžti nuoseklią renormalizacijos schemą. Kad tai atliktume, įveikėme keletą uždavinių:

1. parodėme, jog atsvaros nariai neturėtų būti priskiriami maišymosi kampams,
2. ties 1-kilpa sukonstravome renormalizacijos schemą, kurioje maišymosi kampų atsvaros nariai yra išreikštai trivialūs; schema paremta įprastomis renormalizacijos sąlygomis ant masės apvalko [45],
3. ties 1-kilpa Grimus-Neufeld modelyje renormalizavome fermionų laukus, mases bei maišymosi matricas,
4. schemą išplėtėme visoms eilėms perturbacijų teorijoje.

Kitas reikšmingas tikslas yra parametrizuoti stebinius, taip kad jie galėtų būti greitai palyginami su eksperimentu. Netiesioginiai parametrai — vienas populiariausių metodų, tačiau jis nėra iškart tinkamas visiems modeliams. Konkrečiau, modeliuose, kuriuose $m_W \neq \cos\theta_W m_Z$, T parametras turi ultravioletines divergencijas. Kad išspręstume šią problemą atlikome šiuos uždavinius:

1. elektrosilpnajame sektoriuje renormalizavome papildomą parametrą — W bozono masę,
2. elektrosilpnojo sektoriaus stebinius parametrizavome sumažintu netiesioginių parametrų rinkiniu, kuriame nebėra T parametro.

Naujumas ir svarba

Tam tikrais atžvilgiais renormalizacijos procedūra yra puikiai suprasta, tačiau yra keletas niuansų, iš kurių pastebimiausias yra renormalizacija, kai dalelės maišosi. Per pastaruosius 30 metų atsirado nemažai pasiūlymų kaip renormalizuoti kvarkų maišymosi matricą Standartiniame Modelyje bei maišymosi matricas išplėstiniuose skaliariniuose sektoriuose. Iki šiol nėra visiems priimtino metodo, tad literatūroje galima sutikti daugybę varijuojančio patogumo ir nuoseklumo schemų. Šioje disertacijoje pateikiame nuoseklų bei konceptualų priėjimą prie maišymosi matricų (kampų) renormalizacijos ir pateikiame naują renormalizacijos schemą, kuri išreikštai įgyvendina konceptualius aspektus.

Dar vienas naujas šios disertacijos rezultatas renormalizacijos prasme yra nuoseklus netiesioginių parametrų pritaikymas modeliuose, kuriuose negalioja gerai žinomas medžio lygmens sąryšis $m_W = \cos\theta_W m_Z$. Pasirodo, kad T parametras, kurio divergencijos tokiu atveju buvo patvirtintos daugelio autorių, gali būti pašalintas iš įprasto netiesioginių parametrų rinkinio įvedant papildomą renormalizacijos sąlygą elektrosilpnajame sektoriuje, konkrečiai kalbant, W bozono masės renormalizaciją. Nors tai ir leidžia parametrizuoti stebinius panaudojat sumažintą netiesioginių parametrų rinkinį, tai tuo pat metu nurodo, jog naudojant netiesioginius parametrus negalima palyginti Standartinio Modelio ir modelio, kuriame išlaikoma medžio lygmens lygybė $m_W \neq \cos\theta_W m_Z$, prognozių.

Ginamieji teiginiai

1. Sukimų (maišymosi) kampų atsvaros nariai kliudo bazės sukimų ir renormalizacijos procedūros komutatyvumui, t.y. atsiranda skirtumas priklausomai nuo to, kurioje bazėje buvo atlikta renormalizacijos procedūra. Šia prasme, sukimų (maišymosi) kampų atsvaros nariai yra nesuderinami ir todėl jų neturėtų būti.

2. Pasiūlyta fermionų renormalizacijos schema yra išreikštas schemos ant masės apvalko pavyzdys, kuriame maišymosi kampai negauna atsvaros narių. Svarbu pabrėžti, jog schema galioja visoms perturbacijų teorijos eilėms.

3. Naudojant netiesioginius parametrus, neįmanoma palyginti stebinių tarp Standartinio Modelio (arba Grimus-Neufeld modelio) ir modelio, kuriame medžio lygmens lygybė $m_W = \cos\theta_W m_Z$ nėra išlaikoma. Palyginimo pasirinkimai yra tokie:

    - jei abu modeliai *išlaiko* $m_W = \cos\theta_W m_Z$ medžio lygmenyje, tuomet įprasti S, T, U (kartais išplėsti su X, V, W) netiesioginiai parametrai gali būti naudojami,
    - jei abu modeliai *pažeidžia* $m_W = \cos\theta_W m_Z$ medžio lygmenyje, tuomet T parametras gali būti pašalintas iš įprasto netiesioginių parametrų rinkinio panaudojant paprastą taisyklę.
    - tarpiniais variantais — vienas modelis išlaiko, o kitas pažeidžia $m_W = \cos\theta_W m_Z$ medžio lygmenyje — negalima naudoti netiesioginių parametrų ir reikalinga atlikti pilnus pataisų skaičiavimus kiekviename modelyje prieš darant palyginimus.

Literatūroje su maišymosi matricų renormalizacija susiduriama jau daugiau nei 30 metų ir egzistuoja atitinkamas skaičius renormalizacijos schemų, tačiau didžioji schemų dauguma seka keliais žingsniais, kurie sutinkami ir, mūsų žiniomis, pačiame pirmajame darbe [23]:

i. nepridengtame (angl. *bare*) Lagranžiane atliekamas bazės keitimas iš pradinės (aromato) bazės į masės tikrinių verčių bazę,

ii. Lagranžianas renormalizuojamas diagonaliais masės atsvaros nariais, o lauko atsvaros nariai nėra diagonalūs,

iii. maišymosi matricos atsvaros nariai parenkami, taip kad būtų išprastintos UV begalybės atsirandančios tokiose amplitudėse kaip $W\overline{u}d$.

Šie žingsniai neišvengiamai veda prie maišymosi matricų sąryšio su lauko renormalizacija, todėl schemose ant masės apvalko maišymosi matricų atsvaros nariai tampa priklausomi nuo kalibruotės [25]. Tokia priklausomybė nėra geras bruožas, kadangi dalelių maišymasis yra fizikinis procesas [24, 79, 80]. Siekiant išspręsti šią ir kitas problemas buvo apibrėžta daug schemų [24, 25, 27, 29, 31, 81–83], tačiau jos visos seka tuos pačius žingsnius, o problemos sprendžiamos pridedant pritemptas ar sunkiai įgyvendinamas sąlygas.

Nepriklausomybė nuo kalibruotės nėra vienintelė bėda ir egzistuoja 5 maišymosi renormalizacijos reikalavimų rinkinys, kuris akumuliavosi per pastaruosius 20 metų [27, 36, 43]:

1. Maišymosi matricų atsvaros nariai turėtų išprastinti visas UV divergencijas likusias po masių, laukų ir sąveikos konstantų renormalizacijos.

2. Maišymosi matricų atsvaros nariai turi nepriklausyti nuo kalibruotės.

3. Renormalizacijos procedūra turėtų išlaikyti pamatinę teorijos struktūrą, ypač, maišymosi matricos neturėtų pažeisti teorijos unitarumo.

4. Maišymosi renormalizacija turėtų būti simetriška maišymosi laisvės laipsnių atžvilgiu ir būti nepriklausoma nuo konkretaus fizikinio proceso.

5. Maišymosi matricų renormalizacija neturėtų sugadinti skaitmeninio perturbatyvaus skleidinio stabilumo:

   (a) Išsigimusių besimaišančių dalelių masių riboje, fizikiniuose stebiniuose neturėtų atsirasti singuliarumų,

(b) Parametrų erdvėje neturėtų būti „mirusių kampų", kai renormalizuotas parametras artėja prie begalybės.

Kiek detaliau aptarkime šį reikalavimų sąrašą. Pirmasis reikalavimas iš ties gali atrodyti net per daug trivialus, kad jį būtų verta užrašyti, kadangi vienas iš esminių renormalizacijos bruožų ar tikslų yra būtent UV-baigtinumo užtikrinimas. Kad ir kaip būtų, pirmajame reikalavime esminis žodis yra „likusių" — maišymosi renormalizacija turi pasirūpinti tik likusiomis UV begalybėmis, tačiau, jei tokių begalybių paprasčiausiai nėra, maišymosi matricų atsvaros nariai gali būti baigtiniai. Mūsų sukonstruotoje renormalizacijos schemoje būtent taip ir yra pagal apibrėžimą.

Antrasis reikalavimas seka iš pastebėjimo, kad schemose ant masės apvalko $W\overline{u}d$ amplitudė UV diverguoja, bet nepriklauso nuo kalibruotės, jei CKM atsvaros narys nėra įtraukiamas [24, 83, 84]. Kitaip tariant, jei maišymosi matricos atsvaros nariai priklauso nuo kalibruotės, tai būtinai sukelia tokią priklausomybę ir fizikinių procesų amplitudėse, kurios nuo kalibruotės neturėtų priklausyti. Analogiškai galima pasakyti, jog maišymosi matricos parametrizuoja fizikinius procesus, todėl dažnai pačios yra laikomos fizikiniais dydžiais ir dėl to neturėtų priklausyti nuo kalibruotės.

Trečiasis reikalavimas kyla iš įvairių pamatinių aspektų [27]: Becchi-Rouet-Stora-Tyutin (BRST) [85, 86] simetrija ir Ward-Takahashi tapatybės [87, 88] būtų pažeistos, taip pat su unitarumo neišsaugančia maišymosi matrica unitarumo trikampis prarastų savo prasmę už medžio lygmens ribų.

Ketvirtasis reikalavimas yra tiek konceptualus, tiek ir praktinis. Savaime maišymosi matricos neteikia pirmenybės nei vienam aromatui, todėl natūralu to paties tikėtis ir iš atitinkamų atsvaros narių. Tačiau taip nėra automatiškai, pavyzdžiui, ši savybė sugadinama, jei renormalizacijos schema yra paremta konkrečiu fizikiniu procesu. Iš praktinės pusės, ties 1-kilpa ar aukštesnėse eilėse gali būti sunku suskaičiuoti neuniversalius atsvaros narius. Taip pat, tokias schemas gali būti sunku pritaikyti skirtinguose modeliuose, kadangi dalelių turinys gali būti skirtingas.

Paskutinysis reikalavimas svarbus tiek norint išsaugoti perturbatyvaus skleidinio prasmę, tiek ir praktiniais sumetimais. Skaitmeninis nestabilumas gali turėti keletą šaltinių. Pavyzdžiui, neuniversalios bei nuo proceso priklausančios schemos gali sukelti nenatūralias skaitmenines pataisas (pvz.: [41]), tačiau nebūtinai (pvz.: [36]). Kitas šaltinio pavyzdys yra gan dažnai maišymosi renormalizacijoje pasirodantis $1/(m_i^2 - m_j^2)$ daugiklis, kuris tampa singuliariu, kai abi masės m_i ir m_j tampa lygiomis, t.y. $m_j \to m_i$ riboje. Žinoma, parametrų skenavimas šio singuliarumo aplinkoje tampa keblia užduotimi.

Antrasis, trečiasis bei penktasis reikalavimai gali būti patenkinti automatiškai, jei maišymosi matricos atsvaros narys yra lygus nuliui. Kita vertus, pirmasis ir ketvirtasis reikalavimai neseka tiesiogiai iš maišymosi matricos atsvaros nario trivialumo — tam reikia aptarti konkrečios renormalizacijos schemos įgyvendinimą.

Kaip parodėme [A1] ir aptarėme 3 skyriuje, maišymosi matricų trivialumas seka iš argumentų nepriklausomų nuo konkretaus modelio ar konkrečios schemos. Bendrai kalbant, literatūroje galima pamatyti dvi kryptis: nuo bazės pasirinkimo priklausančius ir nuo bazės pasirinkimo nepriklausančius metodus. Ties medžio lygmeniu nuo bazės nepriklausanti kryptis atrodo siektina ir literatūroje sutinkami darbai, kuriuose formuojami bazės invariantai, kuriais gali būti išreiškiami stebiniai [89–97]. Svarbu pastebėti, jog ties medžio lygmeniu galima atsikratyti maišymosi kampų, kadangi jie yra nuo bazės pasirinkimo priklausomi dydžiai (nebent konkreti bazė tampa išskirta dėl papildomų reikalavimų) [94, 95, 98]. Iš kitos pusės, dirbant ties 1-kilpa ar aukštesnėse eilėse yra įprasta renormalizuoti (priskirti atsvaros narį) ir maišymosi matricoms, visai kaip ir jau minėtame įprastame maišymosi renormalizacijos scenarijuje. Taigi, galima pastebėti, jog literatūroje egzistuoja tam tikra įtampa tarp dviejų krypčių — vienoje jų maišymosi matricos gali būti pašalintos iš aprašymo, o kitoje jos yra svarbios ir renormalizuojamos. Įtampos tarp šių krypčių pašalinimas veda prie trivialių maišymosi matricos atsvaros narių.

Kalbant kiek detaliau, pasirinkome nagrinėti vieną paprasčiausių atvejų — besimaišančių fermionų sistemos kinetinį narį $\mathcal{K}$, kurį nepridengtame Lagranžiane užrašėme keletu skirtingų būdų

$$\begin{aligned}\mathcal{K} =& \overline{\boldsymbol{\psi}}_0 \left(i\not{\partial} - \boldsymbol{M}_0\right) \boldsymbol{\psi}_0 && \text{(1a)}\\ =& \overline{\boldsymbol{\chi}}_0 \left(i\not{\partial} - \gamma^0 \boldsymbol{\mathcal{U}}_0^\dagger \gamma^0 \boldsymbol{M}_0 \boldsymbol{\mathcal{U}}_0\right) \boldsymbol{\chi}_0 && \text{(1b)}\\ =& \overline{\boldsymbol{\chi}}_0 \left(i\not{\partial} - \widetilde{\boldsymbol{M}}_0\right) \boldsymbol{\chi}_0 \,. && \text{(1c)}\end{aligned}$$

Čia nepridengtų fermionų laukų vektoriai $\boldsymbol{\psi}_0$ ir $\boldsymbol{\chi}_0$ susieti per unitarinę transformaciją

$$\boldsymbol{\psi}_0 = \boldsymbol{\mathcal{U}}_0 \boldsymbol{\chi}_0 \,, \tag{2}$$

o masių matricos susijusios taip

$$\widetilde{\boldsymbol{M}}_0 = \gamma^0 \boldsymbol{\mathcal{U}}_0^\dagger \gamma^0 \boldsymbol{M}_0 \boldsymbol{\mathcal{U}}_0 \,, \tag{3}$$

kur γ^0 yra viena iš Dirako gama matricų. Visus tris kinetinio nario variantus renormalizuojame, kiekvienam parametrui $(\boldsymbol{M}_0, \widetilde{\boldsymbol{M}}_0, \boldsymbol{\mathcal{U}}_0)$ bei laukams priskirdami atsvaros narius, kuriuos išskaidome į ermitines (H) ir antiermitines (A)

dalis

$$\begin{aligned}\mathcal{K} =&\overline{\psi}\left\{i\not{\partial} - \boldsymbol{M} + \gamma^0\delta\boldsymbol{Z}_\psi^H\gamma^0\left(i\not{\partial} - \boldsymbol{M}\right) + \left(i\not{\partial} - \boldsymbol{M}\right)\delta\boldsymbol{Z}_\psi^H\right.\\ &\left.-\gamma^0\left[\gamma^0\boldsymbol{M}, \delta\boldsymbol{Z}_\psi^A\right] - \delta\boldsymbol{M}\right\}\psi\,,\end{aligned} \tag{4a}$$

$$\begin{aligned}\mathcal{K} =&\overline{\boldsymbol{\chi}}\left\{i\not{\partial} - \widetilde{\boldsymbol{M}} + \gamma^0\delta\boldsymbol{Z}_\chi^H\gamma^0\left(i\not{\partial} - \widetilde{\boldsymbol{M}}\right) + \left(i\not{\partial} - \widetilde{\boldsymbol{M}}\right)\delta\boldsymbol{Z}_\chi^H\right.\\ &\left.-\gamma^0\left[\gamma^0\widetilde{\boldsymbol{M}}, \boldsymbol{\mathcal{U}}^\dagger\delta\boldsymbol{\mathcal{U}} + \delta\boldsymbol{Z}_\chi^A\right] - \gamma^0\boldsymbol{\mathcal{U}}^\dagger\gamma^0\delta\boldsymbol{M}\boldsymbol{\mathcal{U}}\right\}\boldsymbol{\chi}\,,\end{aligned} \tag{4b}$$

$$\begin{aligned}\mathcal{K} =&\overline{\boldsymbol{\chi}}\left\{i\not{\partial} - \widetilde{\boldsymbol{M}} + \gamma^0\delta\boldsymbol{Z}_\chi^H\gamma^0\left(i\not{\partial} - \widetilde{\boldsymbol{M}}\right) + \left(i\not{\partial} - \widetilde{\boldsymbol{M}}\right)\delta\boldsymbol{Z}_\chi^H\right.\\ &\left.-\gamma^0\left[\gamma^0\boldsymbol{M}, \delta\boldsymbol{Z}_\chi^A\right] - \delta\widetilde{\boldsymbol{M}}\right\}\boldsymbol{\chi}\,.\end{aligned} \tag{4c}$$

Čia laužtiniais skliausteliais pažymėjome komutatorių.

Svarbu, jog visi trys užrašymo variantai atitinka tą patį kinetinį narį nors ir užrašytą skirtingose bazėse, lygtys (1a) ir (1b) arba (4a) ir (4c), arba toje pačioje bazėje, bet užrašytoje skirtingai, lygtys (1b) ir (1c) arba (4b) ir (4c). Kitas svarbus momentas, jog nei vienoje bazėje nereikalavome diagonalios masės matricos.

Palyginę visus užrašymo būdus galėjome sukonstruoti keletą su bazės nepriklausomumu susijusių argumentų ar pavyzdžių, kurie veda prie išvados, jog maišymosi (transformacijų) matricų atsvaros nariai turi būti trivialūs. Pavyzdžiui, vienas pirmųjų ir gana akivaizdžių pastebėjimų yra tas, kad, priklausomai nuo transformacijos matricos $\boldsymbol{\mathcal{U}}$ atsvaros nario, lygtyse (4) skiriasi kinetinio nario *forma*. Šis formos skirtumas (arba $\delta\boldsymbol{\mathcal{U}}$) taip pat implikuoja išskirtinę bazės transformacijų taisyklę antiermitinei lauko renormalizacijos daliai

$$\delta\boldsymbol{Z}_\psi^A \stackrel{!}{=} \delta\boldsymbol{\mathcal{U}}\boldsymbol{\mathcal{U}}^\dagger + \boldsymbol{\mathcal{U}}\delta\boldsymbol{Z}_\chi^A\boldsymbol{\mathcal{U}}^\dagger\,, \tag{5}$$

nors visi kiti dydžiai transformuojasi pagal taisyklę, kurioje nėra nario su $\delta\boldsymbol{\mathcal{U}}$, pvz., kaip lygtyje (3). Antiermitinė lauko pernormavimo dalis nėra niekuo ypatinga, todėl neturėtų būti ir išskirtinumo bazės transformacijų atžvilgiu. Taip pat, lygtis (5) negalioja automatiškai, kadangi tai priklauso nuo transformacijos matricos renormalizacijos sąlygų parinkimo.

Kitas ir stipresnis argumentas seka iš pastebėjimo, jog nei bazės transformacijos, nei renormalizacija nėra kažkuo ypatingos ar viena kitą apribojančios procedūros, todėl turėtume galėti atlikti bazės transformacijas net ir po renormalizacijos. Lygtyje (4b) atlikus pasukimą su $\boldsymbol{\mathcal{U}}^{-1}$, t.y. atliekant sukimą *atgal* su renormalizuota transformacijos matrica, galime iš Lagranžiano panaikinti matricą $\boldsymbol{\mathcal{U}}$, tačiau bendru atveju, tai neleidžia panaikinti atsvaros nario $\delta\boldsymbol{\mathcal{U}}$. Iš to seka, jog galima prieiti prie situacijos, kuomet atsvaros narys lieka be

atitinkamo renormalizuoto parametro, o tai savo ruoštu trukdo atstatyti pradinį nepridengtą Lagranžianą, t.y. pakeičiamas pradinis teoriją apibrėžiantis (nepridengtas) Lagranžianas. Kaip pastebėjome [A1–A3], taip neturėtų būti ir renormalizacija bei *bazės transformacijos turėtų tarpusavyje komutuoti*, o tai reiškia, jog nepridengta ir renormalizuota transformacijų matricos turėtų būti sutapatintos, t.y. $\boldsymbol{\mathcal{U}}_0 = \boldsymbol{\mathcal{U}}$ ir $\delta\boldsymbol{\mathcal{U}} = 0$.

Ko gero pagrindinis tokio sutapatinimo rezultatas yra galimybė laisvai keisti bazę tiek prieš, tiek ir po renormalizacijos niekaip nepakeičiant teorijos ar transformacijos taisyklių. Schematiškai galime pasakyti, kad sutapatinimo atveju turime nuo bazės nepriklausomų atsvaros narių *rinkinį*

$$\{\delta\boldsymbol{Z}_\psi, \delta\boldsymbol{M}, \delta\lambda\} \xrightarrow{\text{bazės keitimas}} \left\{\delta\boldsymbol{Z}_\chi, \delta\widetilde{\boldsymbol{M}}, \delta\widetilde{\lambda}\right\}, \tag{6}$$

bet ne

$$\{\delta\boldsymbol{Z}_\psi, \delta\boldsymbol{M}, \delta\lambda\} \xrightarrow{\text{bazės keitimas}} \left\{\delta\boldsymbol{Z}_\chi, \delta\boldsymbol{\mathcal{U}}, \delta\widetilde{\boldsymbol{M}}, \delta\widetilde{\lambda}\right\}, \tag{7}$$

kur $\delta\lambda$ ir $\delta\widetilde{\lambda}$ reiškia bet kokius kitus parametrus atitinkamose bazėse.

Iš šio sutapatinimo taip pat seka ir filosofinė ar pamatinė formuluotė. Renormalizacijos procedūros esmė yra atskaitinio matavimo įvedimas į teoriją, kad būtų galima daryti tolesnes prognozes. Tai padeda atlikti jau standartiniu įrankiu tapę atsvaros nariai. Kadangi stebiniai pagal apibrėžimą nepriklauso nuo bazės, būtų patogu, jei ir atsvaros nariai atkartotų šį bruožą. Dėl to nuo bazės nepriklausomų atsvaros narių rinkinys arba $\delta\boldsymbol{\mathcal{U}} = 0$, yra sveikintinas aprašymo bruožas bei žingsnis nepriklausomybės nuo bazės kryptimi. Papildomai, nesunku pastebėti, jog trivialus transformacijos (maišymosi) matricos atsvaros narys išties tenkina 2 ir 3 reikalavimus, kadangi 0 akivaizdžiai nepriklauso nuo kalibruotės ir niekaip nekeičia teorijos unitarumo.

Tiesa, čia reikalingas atsakymas į kontrargumentą, jog tokios maišymosi matricos kaip CKM matrica yra matuojamos eksperimentuose ir todėl fizikinės. Atsakymas seka iš pastebėjimo, jog dalelių maišymasis iš ties yra fizikinis efektas, tačiau yra daug būdų ar bazių pasirinkimų, kurie gali jį parametrizuoti. Taip pat nesunku atsekti maišymosi matricų kilmę — jos atsiranda diagonalizuojant pradinę masės matricą, todėl maišymosi matricos yra išvestiniai dydžiai, priklausantys būtent nuo pradinės masių matricos elementų. Kitaip tariant, matuojant dalelių maišymąsi išties matuojama pradinė masės matrica, nors procesą ir parametrizuojame naudodami maišymosi matricas. Svarbiausia pastebėti, jog tai masės matricų renormalizacija priveda prie nuo bazės nepriklausomų atsvaros narių rinkinio, bet ne maišymosi matricų, kurios priklauso nuo bazes, renormalizacija.

Neskaitant visų konceptualių argumentų, taip pat nagrinėjome ir praktinius maišymosi renormalizacijos aspektus. Jie seka iš (4b) lygties, kur komutatoriuje turime išsigimimą tarp anti-ermitinės lauko renormalizacijos dalies bei transformacijos matricos atsvaros nario. Žinoma, tai reiškia, jog transformacijos matricos atsvaros narys negali būti fizikinis, kadangi lauko renormalizacija nėra fizikinė. Panaudojant Nielsen tapatybes [79, 99] galima nesunkiai rasti, jog transformacijos matricų atsvaros nariai yra natūraliai priklausomi nuo kalibruotės, o tai prieštarauja antrajam maišymosi renormalizacijos reikalavimui. Žinoma, priklausomybę nuo kalibruotės galima pašalinti, tačiau neatrodo, jog egzistuoja pakankamai patogus ir natūralus būdas. Papildomai nagrinėjome ir išsigimusių masių ribą $m_j \to m_j$. Pastebėjome, jog maišymosi matricų atsvaros narių nuo kalibruotės nepriklausomos dalys bendru atveju diverguoja dėl atsirandančio $1/(m_i^2 - m_j^2)$ daugiklio, o tai sukelia potencialius skaitmeninius iššūkius (penktasis reikalavimas). Ši problema nesunkiai pašalinama paprasčiausiai įvedus nediagonalius masės atsvaros narius. Kaip bebūtų, abi praktinės problemos lengviausiai išsprendžiamos tiesiog sutapatinus nepridengtą bei renormalizuotas transformacijų (maišymosi) matricas.

2-taškų funkcijų renormalizacija

Bendro pobūdžio argumentais parodėme, jog maišymosi matricų atsvaros nariai turi būti trivialūs, tačiau svarbus ir šių idėjų įgyvendinimas, todėl kita svarbi disertacijos tema yra dviejų taškų funkcijų renormalizacija. Šių funkcijų renormalizacija apibrėžia masės bei lauko renormalizacijos atsvaros narius, kurie taip pat susiję su maišymosi matricų atsvaros nariais. Kitaip tariant, tinkama masių bei laukų renormalizacija turėtų vesti prie trivialaus maišymosi matricos atsvaros nario.

Visų pirma, bet kokia schema turi išlaikyti šias fizikines savybes:

1. renormalizuota dviejų taškų funkcija išnyksta (renormalizuotas propagatorius turi polių) ties poliaus mase m_i^P

$$\sum_{k,l} \left(\bar{\mathcal{Z}}_{ik} \Sigma^R_{kl}(\not{p}) \mathcal{Z}_{li}\right)^{(>0)} u_i = \left(m_i^R - m_i^P\right) u_i \,, \tag{8}$$

2. ant poliaus propagatoriaus rezidiumas lygus vienetui

$$\lim_{\not{p} \to m_i^P} \frac{1}{\not{p} - m_i^P} \sum_{k,l} \bar{\mathcal{Z}}_{ik} \Sigma^R_{kl}(\not{p}) \mathcal{Z}_{li} u_i = u_i \,. \tag{9}$$

3. ant poliaus nėra dalelių maišymosi

$$\sum_{k,l} \bar{\mathcal{Z}}_{jk} \Sigma^R_{kl}(\not{p}) \mathcal{Z}_{li} u_i = 0 \,, \tag{10a}$$

$$\sum_{k,l} \bar{u}_j \bar{\mathcal{Z}}_{jk} \Sigma^R_{kl}(\not{p}) \mathcal{Z}_{li} = 0\,. \tag{10b}$$

Čia $\Sigma^R(\not{p})$ yra renormalizuota fermionų dviejų taškų funkcija, $\bar{\mathcal{Z}}$ ir $\mathcal{Z}$ yra Lehmann-Symanzik-Zimmermann (LSZ) [75, 76] daugikliai susiję su išeinančiomis ir įeinančiomis dalelėmis, viršutinysis indeksas (> 0) nurodo, jog paimami tik aukštesni nei medžio lygmens nariai, o u_i yra išoriniai spinoriai, kuriems galioja Dirako lygtis $\not{p} u_i = m_i^P$.

Tam tikras šių savybių išpildymas apibrėžia tam tikrą renormalizacijos schemą, pavyzdžiui, viena tokių schemų yra renormalizacijos ant masės apvalko schema [45], kuri reikalauja, kad:

1. renormalizuota ir poliaus masės sutaptų

$$m_i^R = m_i^P \equiv m_i\,,$$

2. LSZ daugikliai būtų trivialūs

$$\mathcal{Z} = \bar{\mathcal{Z}} = \mathbb{1}\,.$$

Antroji sąlyga nurodo, kad LSZ daugiklių funkcija perkeliama į lauko renormalizacijos konstantas. Tačiau toks funkcijos perkėlimas ne visada korektiškas, kadangi LSZ daugikliai iš principo gali būti nepriklausomi, o lauko renormalizacijos konstantos susijusios per pseudoermitiškumą kaip galima pamatyti (4) lygtyje. Tad, jei savosios energijos (dviejų taškų funkcijos) netenkina pseudoermitiškumo sąlygos

$$\widetilde{\Sigma}(\not{p}) = \gamma^0 \left(\widetilde{\Sigma}(\not{p})\right)^\dagger \gamma^0\,, \tag{11}$$

yra būtina turėti bent vieną netrivialų LSZ daugiklį, kad būtų galima užtikrinti fizikines savybes. Pseudoermitiškumas pažeidžiamas taip vadinamų absorptyvių (kilpų funkcijų menamų) dalių, kurios gali atsirasti, jei teorijoje yra nestabilių dalelių.

Renormalizacija ant masės apvalko ties 1-kilpa

Atsižvelgiant į pseudoermitiškumą ir ties 1-kilpa daugiausiai dirbant tik su nediagonaliais nariais, parinkome tokias sąlygas:

1. renormalizuota ir poliaus masės sutampa

$$m_i^R = m_i^P \equiv m_i\,, \tag{12}$$

2. įeinančių dalelių LSZ daugiklis trivialus

$$\mathcal{Z} = \mathbb{1}\,. \tag{13}$$

Šios sąlygos išsaugo tiek Lagranžiano ermitiškumą, tiek ir fizikines savybes. Tiesa, šios sąlygos pilnai taikytinos tik nediagonaliems nariams, tačiau būtent šie nariai ir yra svarbūs apibrėžiant maišymosi matricų atsvaros narius.

Fizikinėms savybėms ties 1-kilpa pritaikę parinktas renormalizacijos sąlygas gauname tokį sąryšį tarp lauko ir masės atsvaros narių

$$(m_i^2 - m_j^2)\delta Z_{Lji}^{A\,(1)} - m_j \delta m_{ji}^{L\,(1)} - m_i \delta m_{ji}^{R\,(1)} = f\left(\Sigma\right)\,, \tag{14}$$

kur

$$\begin{aligned} f\left(\Sigma\right) = -\frac{1}{2}\Big(& m_i^2 \Sigma_{ji}^{\gamma L\,(1)}(m_i^2) + m_i m_j \Sigma_{ji}^{\gamma R\,(1)}(m_i^2) \\ & + m_j \Sigma_{ji}^{sL\,(1)}(m_i^2) + m_i \Sigma_{ji}^{sR\,(1)}(m_i^2)\Big) + H.C. \end{aligned} \tag{15}$$

Čia viršutinysis indeksas (1) reiškia 1-kilpos įnašą, $H.C.$ reiškia ermitinį jungtinumą, pasinaudojome įprasta savųjų energijų bei masių dekompozicija

$$\Sigma(\not{p}) = \Sigma_{ji}^{\gamma L}(p^2)\not{p}P_L + \Sigma_{ji}^{\gamma R}(p^2)\not{p}P_R + \Sigma_{ji}^{sL}(p^2)P_L + \Sigma_{ji}^{sR}(p^2)P_R\,, \tag{16}$$

bei masės atsvaros narį išskaidėme į kairinę (L) ir dešininę (R) dalis (analogiškai pasielgėme ir su lauko renormalizacija)

$$\delta m = \delta m^L P_L + \delta m^R P_R\,. \tag{17}$$

Svarbu, jog turime „du" nežinomuosius — lauko ir masės atsvaros narius — bet tik „vieną" lygtį, todėl iš pirmo žvilgsnio galime rasti atsvaros narius tik vienas kito atžvilgiu. Čia gelbsti pastebėjimas, jog lauko atsvaros narį daugina išskirtinis daugiklis $m_i^2 - m_j^2$, kuris turėtų susiformuoti ir dešinėje (14) lygties pusėje. Išnaudodami šį pastebėjimą, išsigimimą tarp masės ir lauko atsvaros narių panaikiname *apibrėždami* antiermitinius lauko renormalizacijos atsvaros

narius kaip $m_i^2 - m_j^2$ masės struktūros koeficientus[10]

$$\begin{aligned}
\delta Z^{A\,(1)}_{Lji} \equiv \frac{1}{2}\Big[&- \Big(m_i^2 \Sigma^{\gamma L\,(1)}_{ji}(m_i^2) + m_i m_j \Sigma^{\gamma R\,(1)}_{ji}(m_i^2) \\
&+ m_j \Sigma^{sL\,(1)}_{ji}(m_i^2) + m_i \Sigma^{sR\,(1)}_{ji}(m_i^2)\Big) + H.C.\Big]\Big|_{(m_i^2 - m_j^2)}, \\
\delta Z^{A\,(1)}_{Rji} \equiv \frac{1}{2}\Big[&- \Big(m_i^2 \Sigma^{\gamma R\,(1)}_{ji}(m_i^2) + m_i m_j \Sigma^{\gamma L\,(1)}_{ji}(m_i^2) \\
&+ m_j \Sigma^{sR\,(1)}_{ji}(m_i^2) + m_i \Sigma^{sL\,(1)}_{ji}(m_i^2)\Big) + H.C.\Big]\Big|_{(m_i^2 - m_j^2)}.
\end{aligned} \tag{18}$$

Pašalinus išsigimimą, masės atsvaros narius galime gauti paprasčiausiai išsprendę (14) lygtį

$$\begin{aligned}
\delta m^{L\,(1)}_{ji} =& \frac{1}{2}\Big(m_i \Sigma^{\gamma R\,(1)}_{ji}(m_i^2) + \Sigma^{sL\,(1)}_{ji}(m_i^2) \\
&+ m_j \Sigma^{\gamma L\,(1)\,\dagger}_{ji}(m_j^2) + \Sigma^{sR\,(1)\,\dagger}_{ji}(m_j^2)\Big) \\
&+ m_i \delta Z^{A\,(1)}_{Rji} - m_j \delta Z^{A\,(1)}_{Lji}, \\
\delta m^{R\,(1)}_{ji} =& \frac{1}{2}\Big(m_i \Sigma^{\gamma L\,(1)}_{ji}(m_i^2) + \Sigma^{sR\,(1)}_{ji}(m_i^2) \\
&+ m_j \Sigma^{\gamma R\,(1)\,\dagger}_{ji}(m_j^2) + \Sigma^{sL\,(1)\,\dagger}_{ji}(m_j^2)\Big) \\
&+ m_i \delta Z^{A}_{Lji} - m_j \delta Z^{A}_{Rji}\,.
\end{aligned} \tag{19}$$

Šie atsvaros narių apibrėžimai (renormalizacijos schema) yra vienas pagrindinių disertacijos rezultatų, todėl verta išvardinti svarbiausius bruožus:

- atsvaros nariai yra išreikšti per savąsias energijas ir apribojimus masės struktūroms bei nepriklauso nuo modelio ar proceso, o tai tenkina ketvirtąjį maišymosi renormalizacijos reikalavimą.

- pagal apibrėžimą masės atsvaros narys nepriklauso nuo kalibruotės

$$\partial_\xi \delta m_{ji} = 0\,. \tag{20}$$

- nepriklausomybės nuo kalibruotės užtikrinimui yra būtina dirbti FJ schemoje [46, 47], todėl į savasias energijas yra įtraukiamos buožgalvių diagramos

[10] Čia $[(m_i^2 - m_j^2)^n A + B]\Big|_{(m_i^2 - m_j^2)} = (m_i^2 - m_j^2)^{n-1} A$, teigiam laipsniui n ir funkcijoms A ir B.

- pagal apibrėžimą antiermitinė lauko atsvaros nario dalis yra UV baigtinė ir visos (14) lygties UV begalybės yra įtraukiamos į masės atsvaros narį. Iš to seka, jog nebėra jokių UV begalybių, kurios turėtų būti išprastinos maišymosi matricų atsvaros narių, tad išreikštai tenkinamas ir pirmasis maišymosi renormalizacijos reikalavimas.

- schema remiasi $m_i^2 - m_j^2$ masių struktūra, kuri niekaip nepriklauso nuo absorptyvių dalių, tad skaitytojas gali laisvai pasirinkti, ar tokias dalis nori įtraukti nagrinėjant Dirako fermionus. Kita vertus, nagrinėjant Majorana fermionus tokias dalis privaloma atmesti, dėl Majorana sąlygos apribojimų.

- schemoje apibrėžtuose atsvaros nariuose nėra singuliaraus $1/(m_i^2 - m_j^2)$ daugiklio, o tai užtikrina skaitmeninį stabilumą.

Renormalizacija ant masės apvalko visoms perturbacijų teorijos eilėms

Mūsų renormalizacijos schemą taip pat galima išplėsti ir už vienos kilpos ribų, nors tai reikalauja keletos kompromisų. Kad galėtume nuosekliai įtraukti ir diagonalius komponentus atmetame visas absorptyvias dalis — atstatome pseudoermitiškumą, todėl galime pilnai naudoti [45] renormalizacijos sąlygas ant masės apvalko. Neradome būdo, kaip už vienos kilpos ribų parodyti ar antiermitinė lauko renormalizacijos dalis yra UV baigtinė, todėl šio atsakymo neturime.

Iš kitos pusės, tai nėra esminiai klausimai, kadangi naudodami Nielsen tapatybes [79, 99] parodėme $m_i^2 - m_j^2$ masių struktūros svarbą kiekvienai perturbacijų teorijos eilei. Kitaip tariant, ir n-toje eilėje galime apibrėžti antiermitinę lauko renormalizacijos dalį panaudodami $m_i^2 - m_j^2$ daugiklį

$$\delta Z_{ji}^{A(n)} u_i \equiv -\frac{1}{2}\left((\not{p} + m_j)\left[\widetilde{\Sigma} + \widetilde{\Sigma}^{(>0)}\delta Z\right]_{ji}^{(n)} + H.C.\right) u_i \Big|_{m_i^2 - m_j^2} . \quad (21)$$

Čia $\widetilde{\Sigma}$ yra nepridengtos savosios energijos suskaičiuotos naudojant nepridengtus parametrus, t.y. savosios energijos bent jau diagramatiškai neturinčios jokių atsvaros narių. Galima patikrinti, jog gaunamas 1-kilpos apibrėžimas, kai $n = 1$. Taip pat turime ir apibrėžimą ermitinei daliai, kuris nesiskiria nuo apibrėžimo įprastoje schemoje ant masės apvalko

$$\begin{aligned} \delta Z_{ii}^{H\,(n)} u_i = -\frac{1}{2} \lim_{\not{p}\to m_i} \frac{1}{\not{p} - m_i} \Big[& \widetilde{\Sigma} + \widetilde{\Sigma}^{(>0)}\delta Z \\ & + \gamma^0 \delta Z^\dagger \gamma^0 \left(\widetilde{\Sigma}^{(>0)} + \widetilde{\Sigma}\delta Z \right) \Big]_{ii}^{(n)} u_i \,. \end{aligned} \quad (22)$$

Naudojantis Nielsen tapatybe nepridengtoms savosioms energijoms

$$\partial_{\xi_0}\widetilde{\Sigma} = \widetilde{\Lambda}\widetilde{\Sigma} + \widetilde{\Sigma}\widetilde{\overline{\Lambda}}\,, \tag{23}$$

galima gauti itin paprastą priklausomybės nuo nepridengto kalibruotės parametro (ξ_0) išraišką lauko renormalizacijos atsvaros nariams

$$\partial_{\xi_0}\delta Z_{ji}u_i = -\left[\widetilde{\overline{\Lambda}}Z\right]_{ji}u_i\,. \tag{24}$$

Turėdami lauko renormalizacijos konstantas taip pat apibrėžėme masės atsvaros narį

$$\delta m^{(n)}_{ji}u_i = \left[\hat{\widetilde{\Sigma}} + \widetilde{\Sigma}\delta Z\right]^{(n)}_{ji}u_i\,, \tag{25}$$

kur

$$\widetilde{\Sigma}^{(n)} = \hat{\widetilde{\Sigma}}^{(n)} - \delta m^{(n)}\,. \tag{26}$$

Vėl pasinaudojus Nielsen tapatybėmis ir žinoma lauko renormalizacijos priklausomybe nuo kalibruotės, galima įsitikinti, jog masės atsvaros narys nepriklauso nuo kalibruotės, todėl atitinka fizikinio dydžio atsvaros narį.

Turėdami šiuos lauko bei masės atsvaros narių apibrėžimus, teigiame, jog maišymosi matricos atsvaros narys trivialus kiekvienoje perturbacijų teorijos eilėje, kadangi kiekvienoje eilėje galioja ta pati 1-kilpos logika. Ties viena kilpa tiek bendrai, tiek ir išreikštai (mūsų schemoje) gavome, jog maišymosi matricos atsvaros narys yra lygus 0. Papildomai, schemos apibendrinimas išlaiko ir kitas svarbias savybes, kurias sutikome ir vienos kilpos atveju, pavyzdžiui, schema išlieka universali — nepriklausoma nei nuo modelio, nei nuo proceso, singuliarus $1/(m_i^2 - m_j^2)$ daugiklis nepasirodo atsvaros nariuose ir t.t.

Grimus-Neufeld modelio renormalizacija

Turėdami schemos apibrėžimą galime pereiti prie jos taikymo Grimus-Neufeld modeliui ties 1-kilpa. W. Grimus'as ir H. Neufeld'as daugiau nei prieš 30 metų pasiūlė klasę modelių, kurie išplečia Standartinį Modelį papildomais Higgs'o dubletais bei Majorana singletais. Mes Grimus-Neufeld modeliu vadiname minimalų atvejį, kai yra tik vienas papildomas dubletas ir vienas Majorana singletas, kitaip tariant turime dviejų Higgs'o dubletų modelį (angl. *two Higgs doublet model*, THDM) [113] su papildomu Majorana singletu. Įvairios šio modelio dalys aprašytos [50, 114, 121, A4].

Skaliarinis GNM potencialas yra CP simetriją išsaugantis bendras THDM potencialas, todėl konstantos μ_{12}, λ_5, λ_6, λ_7 yra realios [122, 123]

$$-V_{\text{THDM}} = \mu_1\left(H_1^\dagger H_1\right) + \mu_2\left(H_2^\dagger H_2\right) + \mu_{12}\left[\left(H_1^\dagger H_2\right) + H.C.\right]$$

$$
\begin{aligned}
&+\lambda_1\left(H_1^\dagger H_1\right)\left(H_1^\dagger H_1\right)+\lambda_2\left(H_2^\dagger H_2\right)\left(H_2^\dagger H_2\right)\\
&+\lambda_3\left(H_1^\dagger H_1\right)\left(H_2^\dagger H_2\right)+\lambda_4\left(H_1^\dagger H_2\right)\left(H_2^\dagger H_1\right)\\
&+\Big[\lambda_5\left(H_1^\dagger H_2\right)\left(H_1^\dagger H_2\right)+\lambda_6\left(H_1^\dagger H_1\right)\left(H_1^\dagger H_2\right)\\
&\quad+\lambda_7\left(H_2^\dagger H_2\right)\left(H_1^\dagger H_2\right)+H.C.\Big]\,.
\end{aligned}
\tag{27}
$$

Čia $H_{1,2}$ yra Higgs'o dubletai, o skliausteliai žymi $SU(2)$ indeksų sumavimą.

Abu dubletai yra ekvivalentūs, todėl galime laisvai pasirinkti patogią bazę, kurioje nebūtų maišymosi kampų tarp šių dubletų. Tokia bazė vadinama Higgs'o baze, joje tikėtina vakuumo vertė (angl. *vacuum expectation value*, VEV) v yra tik viename iš dubletų

$$
H_1=\begin{pmatrix} G^+ \\ \frac{1}{\sqrt{2}}\left(v+h_1+iG^0\right)\end{pmatrix}\qquad\text{and}\qquad H_2=\begin{pmatrix} H^+ \\ \frac{1}{\sqrt{2}}\left(h_2+i\sigma\right)\end{pmatrix}\,. \tag{28}
$$

Čia pirmasis Higgs'o dubletas yra identiškas SM dubletui, $G^{+,0}$ yra krūvį turintis bei neutralusis Goldstone'o bozonai, H^+ fizikinis krūvį turintis Higgs'o bozonas su mase m_+, $h_{1,2}$ yra skaliariniai laisvės laipsniai, o σ pseudoskaliarinis su mase m_A. $h_{1,2}$ gali būti pasukti į masės tikrinių verčių bazę, taip gautume fizikinius skaliarus h ir H su atitinkamomis masėmis m_h ir m_H. Mūsų nagrinėjamu atveju pseudoskaliariniai ir skaliariniai laisvės laipsniai nesimaišo, tačiau bendru atveju taip gali būti.

Kitas svarbus ingredientas tai Higgs'o dubletų ir fermionų Yukawa sąveikos Lagranžianas

$$
\begin{aligned}
-\mathcal{L}_{\mathrm{H-F}} = &\sum_{j,i=1}^{3}(Y_d)_{ji}\overline{d}_{Rj}.(H_1^\dagger Q_i)+\sum_{j,i=1}^{3}(Y_u)_{ji}\overline{u}_{Rj}.(\tilde{H}_1^T Q_i)\\
&+\sum_{j,i=1}^{3}(Y_l)_{ji}\overline{e}_{Rj}.(H_1^\dagger L_i)+\sum_{i=1}^{3}(Y_\nu)_i\overline{N}.(\tilde{H}_1^T L_i)\\
&+\sum_{j,i=1}^{3}(G_d)_{ji}\overline{d}_{Rj}.(H_2^\dagger Q_i)+\sum_{j,i=1}^{3}(G_u)_{ji}\overline{u}_{Rj}.(\tilde{H}_2^T Q_i)\\
&+\sum_{j,i=1}^{3}(G_l)_{ji}\overline{e}_{Rj}.(H_2^\dagger L_i)+\sum_{i=1}^{3}(G_\nu)_i\overline{N}.(\tilde{H}_2^T L_i)+H.C.
\end{aligned}
\tag{29}
$$

Čia L_i ir Q_i yra kairiniai leptonų ir kvarkų $SU(2)$ dubletai, e_R, u_R ir d_R yra dešininiai krūvį turinčių leptonų, kylančių (angl. *up*) bei krintančių (angl. *down*) tipo kvarkų singletai, N Majorana singletas, Y ir G yra Yukawa sąveikos atitinkamai su pirmuoju ir antruoju Higgs'o dubletu, taškais pažymėtas fermionų

indeksų sumavimas, i ir j yra šeimos indeksai, $(\tilde{H}_{1,2})_\alpha = \sum_{\beta=1}^{2} \epsilon_{\alpha\beta}(H_{1,2})_\beta$, kur $\epsilon_{\alpha\beta}$ yra antisimetrinis tenzorius. Laikome, jog krūvį turinčių leptonų bei kvarkų masių matricos jau diagonalizuotos, todėl ir atitinkamos Yukawa sąveikos konstantos Y su pirmuoju Higgs'o dubletu yra diagonalios.

Neutrinų masės matrica kyla iš Yukawa sąveikos su pirmuoju Higgs'o dubletu bei iš Majorana masės nario, kuris nėra draudžiamas

$$\begin{aligned} -\mathcal{L}_{\text{mass}} =& \frac{v}{2\sqrt{2}} \sum_{i=1}^{3} \left((Y_\nu)_i N^T.\mathcal{C}^{-1}.\nu'_{Li} + (Y_\nu)_i \nu'^T_{Li}.\mathcal{C}^{-1}.N \right) \\ &+ \frac{1}{2} M_R N^T.\mathcal{C}^{-1}.P_L.N + H.C. \end{aligned} \tag{30}$$

Čia ν'_{Li} yra neutrinai iš L_i dubleto, M_R Majorana masės parametras, kurį paprastumo dėlei laikome realiu. Iš $\mathcal{L}_{\text{mass}}$ nuskaitome pilną neutrinų masės matricą

$$M^\nu = \begin{pmatrix} \mathbb{0}_{3\times 3} & \frac{v}{\sqrt{2}} Y_\nu \\ \frac{v}{\sqrt{2}} Y_\nu^T & M_R \end{pmatrix} , \tag{31}$$

kuri yra simetriška, nors ir nediagonali. Šią matricą diagonalizuojame unitarinės transformacijos pagalba

$$U = \begin{pmatrix} U_L \\ U_R^\star \end{pmatrix} , \tag{32}$$

kur U_L ir U_R atitinkamai yra 3×4 ir 1×4 matricos. Pasinaudoję unitarumu galime parašyti

$$\begin{aligned} &U_L U_L^\dagger = \mathbb{1}_{3\times 3}, \qquad U_R^\star U_R^T = 1, \qquad U_L^\dagger U_L + U_R^T U_R^\star = \mathbb{1}_{4\times 4}, \\ &U_R^\star U_L^\dagger = \mathbb{0}_{1\times 3} \, . \end{aligned} \tag{33}$$

Diagonalizavus M^ν turime

$$U^T M^\nu U = \tilde{m}^\nu = \text{diag}(0, 0, m_3, m_4) \, . \tag{34}$$

Svarbiausia, jog tik dvi tikrinės vertės nėra lygios nuliui medžio lygmenyje. Papildomai, kuo didesnis parametras M_R, tuo didesnė masė m_4 ir tuo mažesnė m_3 — turime sverto mechanizmą.

Pereinant prie modelio vienoje kilpoje, verta pastebėti, jog bemasiai neutrinai sukelia šiek tiek keblumų, kadangi dėl jų nebeįmanoma suformuoti $m_i^2 - m_j^2$ masių struktūros. Kalbant apie renormalizuotą neutrinų savąją energiją galime išskirti tris atvejus:

1. $i, j \geq 3$; abu indeksai atitinka masyvius neutrinus,

2. $i \leq 2, j \geq 3$ arba priešingai; vienas iš neutrinų masyvus, o kitas bemasis,

3. $i, j \leq 2$; abu indeksai atitinka bemasius neutrinus.

Visus šiuos atvejus galime schematiškai atvaizduoti 2×2 blokais

$$\Sigma_{ji}^{\nu,R} \sim \left(\begin{array}{c|c} m_{j,i} = 0,\, \delta m & m_i \neq 0,\, \delta Z^A,\, \delta m \\ \hline m_j \neq 0,\, \delta Z^A,\, \delta m & m_{j,i} \neq 0,\, \delta Z^A,\, \delta m \end{array} \right) . \tag{35}$$

Čia δm ir δZ^A trumpai pažymi, ar atitinkamame bloke egzistuoja masės ir lauko atsvaros nariai, taip pat pažymėjome, kurios masės lygios ar nelygios 0.

Apatiniame dešiniajame bloke turime atvejį, kai abu indeksai atitinka masyvius neutrinus — tai paprasčiausias atvejis ir jame iškart galime taikyti mūsų renormalizacijos schemos apibrėžimus.

Blokai virš ir kairiau masyvaus atvejo atitinka tarpinį variantą, kai vienas iš indeksų atitinką masyvų neutriną, o kitas — bemasį. Čia tiesiogiai nebegalime taikyti schemos apibrėžimų, kadangi viena iš masių lygi nuliui, tačiau šią kliūtį galima apeiti apibrėžimus panaudojus masyviu atveju ir po to paėmus vienos iš masių ribą į nulį. Toks metodas išsaugo visas schemos savybes.

Galiausiai, viršutiniame kairiajame bloke turime atvejį, kai abu indeksai atitinka bemasius neutrinus. Iš vienos pusės, čia taip pat neegzistuoja reikalingos masių struktūros, tad turėtume skaičiavimus atlikti masyviam atvejui ir po to paimti ribą. Iš kitos pusės, šias ribas patogiai galime imti iškart apibrėžimuose, pavyzdžiui, masės atsvaros nariai bemasiu atveju yra

$$\delta m_{ji}^L = \frac{1}{2} \left(\Sigma_{ji}^{sL}(0) + \Sigma_{ji}^{sR\dagger}(0) \right) = \Sigma_{ji}^{sL}(0) \tag{36}$$

ir

$$\delta m_{ji}^R = \frac{1}{2} \left(\Sigma_{ji}^{sR}(0) + \Sigma_{ji}^{sL\dagger}(0) \right) = \Sigma_{ji}^{sR}(0) \,. \tag{37}$$

Svarbiausia, jog šie atsvaros nariai UV baigtiniai, kadangi visos divergencijos proporcingos masėms, ir nepriklauso nuo kalibruotės. Papildomai, šiame bloke galima diagonalizuoti δm pasinaudojus matrica U_L, kuri medžio lygmenyje nėra pilnai fiksuota. Apjungiant visas šias savybes, $\delta m^{L,R}$ galima laikyti radiacinėmis masėmis kaip buvo pastebėta [48, 124].

Paminėkime keletą išreikštų renormalizacijos schemos taikymo rezultatų. Schemoje išskirtiniausias yra antiermitinės lauko renormalizacijos apibrėžimas, tad jo taikymas įdomiausias. Paprasčiausias atvejis — kvarkų savosios energijos,

tad (18) lygtyje paėmę kylančiuosius kvarkus ir atmetę sąveikas su antruoju Higgs'o dubletu gauname

$$\begin{aligned}\delta Z^{A,\,u}_{Lji} =& -\sum_{k=1}^{3}\frac{V_{jk}V^{\star}_{ik}}{2v^2}\Big\{ - A_0((m^d_k)^2)\left((m^d_k)^2 + (D-2)m^2_W + (m^u_j)^2\right)\\ &+\left((D-3)m^2_W\left((m^d_k)^2+(m^u_i)^2\right)+\left((m^d_k)^2-(m^u_i)^2\right)^2\right.\\ &\left.-(D-2)m^4_W\right)B_0((m^u_i)^2,(m^d_k)^2,m^2_W)\\ &+\left((m^u_i)^2-(m^u_j)^2\right)\left((m^d_k)^2-(m^u_i)^2+m^2_W\xi_W\right)\\ &\times B_0((m^u_i)^2,(m^d_k)^2,\xi_W m^2_W)\\ &+(m^d_k)^2A_0\left(m^2_W\right)+H.C.\Big\}\Big|_{m^2_i-m^2_j} \qquad (38)\\ =& -\sum_{k=1}^{3}\frac{V_{jk}V^{\star}_{ik}}{2v^2}\left[(m^d_k)^2-(m^u_i)^2+\xi_W m^2_W\right]\\ &\times B_0((m^u_i)^2,(m^d_k)^2,\xi_W m^2_W)\\ &+\sum_{k=1}^{3}\frac{V_{jk}V^{\star}_{ik}}{2v^2}\left[(m^d_k)^2-(m^u_j)^2+\xi_W m^2_W\right]\\ &\times B^{\star}_0((m^u_j)^2,(m^d_k)^2,\xi_W m^2_W) \qquad (39)\end{aligned}$$

Čia V yra CKM matrica, A_0 ir B_0 yra Passarino-Veltman kilpų funkcijos [111] apibrėžtos kaip ir mūsų naudotuose programiniuose paketuose (`FeynRules` [115], `FeynArts` [116], `FeynCalc` [117–119] ir `PackageX` [120]), tačiau papildomai į funkcijas įtraukėme $\frac{1}{2^D\pi^{D-2}}$ daugiklį, kur $D = 4-\epsilon$ yra erdvėlaikio dimensija. Paskutinę lygybę gavome palikę narius, kurie proporcingi $m^2_i - m^2_j$ — iš tikrųjų pavyko nesunkiai pritaikyti apibrėžimą. Taip pat galima nesunkiai įsitikinti, jog antiermitinė lauko renormalizacijos dalis tikrai yra UV baigtinė kaip ir buvo galima tikėtis iš bendrų argumentų. Papildomai, čia neįtraukėme sąveikų su antruoju Higgs'o dubletu, tačiau paskutinės lygybės jie nekeičia ir rezultatas galioja pilname GNM. Dar vienas įdomus schemos aspektas, jog skaičiuojant dešininę antiermitinę lauko renormalizacijos dalį kvarkams trivialiai

$$\delta Z^{A,\,u}_{Rji} = 0\,, \qquad (40)$$

o tai atspindi chiralinę $SU(2)$ kalibruotinės simetrijos prigimtį.

Kitas nesudėtingas rezultatas, tai neutrinų masės atsvaros nariai, kai abu indeksai atitinka bemasius neutrinus, tad gauname

$$\delta m^{L,\nu}_{ji} = \sum_{a=1}^{4}\frac{(U^T_L G_\nu U^{\star}_R)_{ja}m^{\nu}_a}{2}\Big[s^2_\alpha B_0(0,m^2_h,(m^{\nu}_a)^2)+c^2_\alpha B_0(0,m^2_H,(m^{\nu}_a)^2)$$

$$- B_0(0, m_A^2, (m_a^\nu)^2)\Big](U_R^\dagger G_\nu^T U_L)_{ai}\,, \tag{41}$$

$$\delta m_{ji}^{R,\nu} = \sum_{a=1}^{4} \frac{(U_L^\dagger G_\nu^\star U_R)_{ja} m_a^\nu}{2}\Big[s_\alpha^2 B_0(0, m_h^2, (m_a^\nu)^2) + c_\alpha^2 B_0(0, m_H^2, (m_a^\nu)^2)$$
$$- B_0(0, m_A^2, (m_a^\nu)^2)\Big](U_R^T G_\nu^\dagger U_L^\star)_{ai}\,. \tag{42}$$

Kaip minėjome, matrica U_L nėra pilnai apibrėžta ties medžio lygmeniu, todėl galime apibrėžti naują Yukawa sąveikos konstantą

$$U_L^T G_\nu \equiv G'_\nu = \begin{pmatrix} 0 \\ g'_2 \\ g'_3 \\ g'_4 \end{pmatrix}\,, \tag{43}$$

kuri diagonalizuoja δm_{ji}^ν, kai $i, j \leq 2$. Imant tik realią dalį šį diagonalizuotą bloką galime laikyti radiacinėmis masėmis, tad turime

$$m_1^\nu = \mathrm{Re}[\delta m_{11}^{L,\nu}] = \mathrm{Re}[\delta m_{11}^{R,\nu}] = 0\,, \tag{44}$$
$$m_2^\nu = \mathrm{Re}[\delta m_{22}^{L,\nu}] = \mathrm{Re}[\delta m_{22}^{R,\nu}] = \mathrm{Re}\left[(g'_2)^2 \widetilde{C}\right]\,, \tag{45}$$

kur

$$\widetilde{C} = \sum_{a=1}^{4} \frac{U_{Ra}^\star U_{Ra}^\star m_a^\nu}{2}\Big[s_\alpha^2 B_0(0, m_h^2, (m_a^\nu)^2) + c_\alpha^2 B_0(0, m_H^2, (m_a^\nu)^2)$$
$$- B_0(0, m_A^2, (m_a^\nu)^2)\Big]\,. \tag{46}$$

Analogiškus rezultatus galima rasti [48, 50].

Kiek sudėtingesni schemos taikymo pavyzdžiai yra masyvaus atvejo antiermitinė lauko renormalizacijos dalis

$$\begin{aligned}
\delta Z_{Lji}^{A,\nu} = &- \widetilde{Re}\Big\{ \sum_{a=1}^{4}\sum_{s=1}^{3} \frac{(m_a^\nu)^3 R_{1s}^2 (U_L^\dagger U_L)_{ai}(U_L^\dagger U_L)_{aj}}{32\pi^2 v^2 m_j^\nu} \log\left(\frac{(m_s^h)^2}{(m_a^\nu)^2}\right) \\
&+ \sum_{a=1}^{4}\Bigg[\frac{m_a^\nu (U_L^\dagger U_L)_{ai}(U_L^\dagger U_L)_{aj}}{128\pi^2 v^2 (m_j^\nu)^3} \\
&\quad \times \Big(2(m_a^\nu)^2(m_j^\nu)^2 - \big((m_a^\nu)^2 - m_Z^2\xi_Z\big)^2 + (m_j^\nu)^4\Big) \\
&+ \frac{(U_L^\dagger U_L)_{ai}(U_L^T U_L^\star)_{aj}}{128\pi^2 v^2 (m_j^\nu)^2} \\
&\quad \times \big((m_a^\nu)^4 + (m_i^\nu)^2(m_j^\nu)^2 - m_Z^4\xi_Z^2\big)\Bigg] \log\left(\frac{(m_a^\nu)^2}{m_Z^2\xi_Z}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{3}\frac{U_{Lki}U^{\star}_{Lkj}}{32\pi^2v^2(m^{\nu}_j)^2}\\
&\times\left((m^l_k)^4+(m^{\nu}_i)^2(m^{\nu}_j)^2-m^4_W\xi^2_W\right)\log\left(\frac{(m^l_k)^2}{m^2_W\xi_W}\right)\\
&+\frac{(m^{\nu}_i)^2(U^T_LU^{\star}_L)_{ij}}{64\pi^2v^2}\left(2\log\xi_W+\log\xi_Z\right)\\
&+\sum_{a=1}^{4}\left[\frac{(U^{\dagger}_LU_L)_{ai}(U^T_LU^{\star}_L)_{aj}}{64\pi^2v^2}\left((m^{\nu}_a)^2-(m^{\nu}_i)^2+m^2_Z\xi_Z\right)\right.\\
&\left.+\frac{m^{\nu}_a(U^T_LU^{\star}_L)_{ai}(U^T_LU^{\star}_L)_{aj}}{64\pi^2v^2m^{\nu}_i}\left(-(m^{\nu}_a)^2+(m^{\nu}_i)^2+m^2_Z\xi_Z\right)\right]\\
&\qquad\times\Lambda((m^{\nu}_i)^{\,2},m^{\nu}_a,m_Z\sqrt{\xi_Z})\\
&+\sum_{k=1}^{3}\frac{U_{Lki}U^{\star}_{Lkj}}{32\pi^2v^2}\left((m^l_k)^2-(m^{\nu}_i)^2+m^2_W\xi_W\right)\\
&\qquad\times\Lambda((m^{\nu}_i)^2,m^l_k,m_W\sqrt{\xi_W})-H.C.\Big\}
\end{aligned}
\tag{47}
$$

Čia $\Lambda(m^2,b,c)$ yra `Disc` funkcija pagal `PackageX` ir `FeynCalc` apibrėžimus, $\widetilde{Re}$ operatorius atmetantis absorptyvias kilpų funkcijų dalis reikalingas [104] Majorana dalelėms. Vienas įdomesnių bruožų — UV baigtiniai ir nuo kalibruotės nepriklausomi nariai, kurie turėjo $m^2_i-m^2_j$ masių struktūrą ir todėl turėjo būti įtraukti į lauko renormalizaciją. Kitas pastebėjimas — nepavyko išreikšti lauko renormalizacijos tik per Passarino-Veltman funkcijas ir jos turėjo būti analitiškai suskaičiuotos, kad galėtume pritaikyti schemos apibrėžimus, taip galėjo nutikti dėl neįprastos neutrinų masės prigimties. Lyginant su kvarku atveju, lauko renormalizacijos išraiška daug sudėtingesnė, tačiau tai tik parodo, kad schemą galima taikyti ir sudėtingais atvejais.

Iš masyvaus atvejo taip pat pavyko pereiti į atvejį, kuomet vienas iš indeksų atitinka bemasį neutriną, konkretumo dėlei $m_i=0$, todėl turime

$$
\begin{aligned}
\delta Z^{A,\nu}_{Lji,\cancel{m_i}}=-\widetilde{Re}\Big\{&-\frac{(U^{\dagger}_L(m^l)^2U_L)_{ji}}{32\pi^2v^2}+\sum_{k=1}^{3}\frac{U_{Lki}U^{\star}_{Lkj}}{64\pi^2v^2}\\
&\times\left[\frac{(m^l_k)^4-m^4_W\xi^2_W-(m^{\nu}_j)^4}{(m^{\nu}_j)^2}-\frac{2\left((m^l_k)^4+m^4_W\xi^2_W\right)}{(m^l_k)^2-m^2_W\xi_W}\right]\log\left(\frac{(m^l_k)^2}{m^2_W\xi_W}\right)\\
&+\sum_{k=1}^{3}\frac{U_{Lki}U^{\star}_{Lkj}}{32\pi^2v^2}\left[(m^{\nu}_j)^2-(m^l_k)^2-m^2_W\xi_W\right]\Lambda^{\star}((m^{\nu}_j)^2,m^l_k,\sqrt{\xi_W}m_W)\Big\}.
\end{aligned}
\tag{48}
$$

Svarbu, jog tokia riba, kai viena iš dalelių neturi masės, išties egzistuoja.

Taip pat skaičiavome ir masės atsvaros narius, tačiau Grimus-Neufeld modelyje pilnas rezultatas per didelis, kad būtų prasminga jį pristatyti, todėl kaip pavyzdį pateikiame tik UV diverguojančias dalis ir atmetę Yukawa sąveikas su antruoju Higgs'o dubletu

$$
\begin{aligned}
[\delta m_{ji}^{L,\nu}]^{\text{SM}}_{\text{div.}} &= -\frac{3m_j^\nu \left(U_L^\dagger (m^l)^2 U_L\right)_{ji}}{32\pi^2 v^2 \epsilon_{\text{UV}}} + \frac{m_j^\nu \left(U_L^\dagger U_L (m^\nu)^2 U_L^\dagger U_L\right)_{ji}}{32\pi^2 v^2 \epsilon_{\text{UV}}} \\
&+ \frac{(U_L^\dagger U_L)_{ji}}{16\pi^2 v^2 \epsilon_{\text{UV}}} m_j^\nu (m_i^\nu)^2 - \frac{(U_L^\dagger U_L)_{ji} m_j^\nu}{16\pi^2 v^2 (m_h)^2 \epsilon_{\text{UV}}} (6m_W^4 + 3m_Z^4) \\
&+ \frac{(U_L^\dagger U_L)_{ji} m_j^\nu}{4\pi^2 v^2 (m_h)^2 \epsilon_{\text{UV}}} \text{Tr.}\{(m^\nu)^4 U_L^\dagger U_L + 3(m^d)^4 + 3(m^u)^4 + (m_k^l)^4\} \\
&+ (i \leftrightarrow j) \ . \qquad (49)
\end{aligned}
$$

Tiek UV dalys, tiek ir čia nepateiktos baigtinės dalys nepriklauso nuo kalibruotės bei nėra singuliarios išsigimusių masių riboje, kaip ir turėtų būti mūsų schemoje. Papildomai, šį rezultatą galima nesunkiai palyginti ir su (31) lygtimi iš [31], kur vieninteliai skirtumai atsiranda dėl to, jog įtraukėme visas buožgalvių diagramas.

Radiacinės pataisos elektrosilpnajame sektoriuje

Didžioji disertacijos dalis nagrinėja dviejų taškų funkcijų renormalizaciją besimaišančių dalelių kontekste, tačiau taip pat aptarėme ir kiek kitą renormalizacijos aspektą — netiesioginius parametrus elektrosilpnajame sektoriuje. Netiesioginiai parametrai yra dažnai sutinkama aproksimacija, tačiau ją galima taikyti ne visada, kadangi lyginami modeliai turi turėti tą pačią $SU(2) \times U(1)$ kalibruotinę simetriją bei apsauginę $SU(2)$ simetriją, kuri užtikrina medžio lygmens sąryšį $m_W = \cos\theta_W m_Z$. Jei toks sąryšis negalioja, tuomet T parametras tampa UV diverguojančiu [60, 62–65, 67–70, 142] ir nebegali būti panaudotas elektrosilpnojo sektoriaus stebinių parametrizavimui. Kita vertus, modeliai, kuriuose T parametras diverguoja, šiomis dienomis tapo svarbūs tiek dėl CDF kolaboracijos naujų W masės matavimų [61], tiek dėl potencialios galimybės paaiškinti 95 GeV perteklių [133–137] išmatuotą CMS kolaboracijos [138–141].

Kiek patogiau medžio lygmens sąryšį tarp W ir Z bozono masių išreikšti per Veltman'o $\widehat{\rho}$ parametrą [126, 127]

$$
\widehat{\rho} \equiv \frac{\widehat{m}_W^2}{\widehat{c}^2 \widehat{m}_Z^2} \begin{cases} = 1 & \text{su apsaugine simetrija,} \\ \text{nefiksuotas} & \text{be apsauginės simetrijos,} \end{cases} \qquad (50)
$$

kur kepurėlės nurodo nepridengtus parametrus ir $\widehat{c} \equiv \cos\widehat{\theta}_W$. Jei $\widehat{\rho} = 1$ kaip SM arba GNM, tuomet modelyje yra apribota laisvė ir todėl užtenka 3 įvesties parametrų (renormalizacijos sąlygų). Dažniausiai tokiais įvesties parametrais parenkami smulkiosios struktūros konstanta α, Fermi konstanta $G_{F(\text{charged})}$ išmatuota miuono skilime ir Z bozono masė m_Z. Kita vertus, jei $\widehat{\rho}$ nėra fiksuotas, tuomet atsiradusią laisvę reikia kompensuoti įvedant papildomą įvesties parametrą. Būtent tai ir padarėme renormalizuodami W bozono masę.

Įveskime keletą sąvokų. Kadangi netiesioginiai parametrai palygina du modelius, naudinga vadinti

- SM ir naujos fizikos modeliu (angl. *new physics model*, NPM), tokius modelius, kuriuose $\widehat{\rho} = 1$; NPM yra SM su papildomais medžiagos sektoriais.

- pamatiniu modeliu (angl. *base model*, BM) ir viršpamatiniu modeliu (angl. *beyond base model*, BBM), tokius modelius, kuriuose $\widehat{\rho}$ nėra fiksuotas; BBM yra BM su papildomais medžiagos sektoriais.

Tuomet galime suformuoti tokius galimų palyginimų variantus

1. SM prieš NPM, t.y. abu modeliai su $\widehat{\rho} = 1$.

2. BM prieš BBM, t.y. abiejuose modeliuose $\widehat{\rho}$ nėra fiksuotas.

3. SM prieš BBM, t.y. viename modelyje $\widehat{\rho} = 1$, o kitame $\widehat{\rho}$ nėra fiksuotas.

Pirmasis variantas jau plačiai aptartas literatūroje, būtent šiam atvejui ir buvo apibrėžti netiesioginiai parametrai. Mūsų žiniomis, nei antrajam, nei trečiajam atvejui nėra tikro netiesioginių parametrų formalizmo. Trečiasis atvejis yra neabejotinai pats įdomiausias fenomenologiniu atžvilgiu, todėl literatūroje galima sutikti bandymų taikyti įvairiai modifikuotus netiesioginius parametrus, pavyzdžiui, į T parametrą įtraukiant medžio lygmens įnašą, o kilpos skaičiavimus atliekant $\widehat{\rho} = 1$ riboje [64, 67, 144, 145]. Kitas įmanomas variantas yra pačių netiesioginių parametrų renormalizavimas kaip buvo atlikta [146, 147]. Tokios modifikacijos išties apeina T parametro divergencijas, tačiau tai nėra tikri netiesioginių parametrų formalizmai, nes netiesioginiai parametrai neturi priklausyti nuo renormalizacijos schemos. Šioje disertacijoje pristatėme netiesioginių parametrų formalizmą antrajam atvejui, tačiau toks formalizmas trečiuoju atveju lieka atviru klausimu.

Netiesioginiai parametrai apibrėžti [54, 55], tačiau mūsų tikslams patogiau naudoti neatimtus parametrus, todėl kiekvienas parametras priklauso nuo

modelio M

$$S^{\mathrm{M}} = \frac{4s^2c^2}{\alpha}\left[\widetilde{\Pi}^{\mathrm{M}}_{ZZ}\left(m_Z^2\right) + \frac{s^2-c^2}{sc}\,\Pi'^{\mathrm{M}}_{ZA}\left(0\right) - \Pi'^{\mathrm{M}}_{AA}\left(0\right)\right], \tag{51a}$$

$$T^{\mathrm{M}} = \frac{1}{\alpha}\left[\frac{\Pi^{\mathrm{M}}_{WW}\left(0\right)}{m_W^2} - \frac{\Pi^{\mathrm{M}}_{ZZ}\left(0\right)}{m_Z^2} - \frac{2s}{c}\,\frac{\Pi^{\mathrm{M}}_{ZA}\left(0\right)}{m_Z^2}\right], \tag{51b}$$

$$U^{\mathrm{M}} = \frac{4s^2}{\alpha}\left[\widetilde{\Pi}^{\mathrm{M}}_{WW}\left(m_W^2\right) - c^2\,\widetilde{\Pi}^{\mathrm{M}}_{ZZ}\left(m_Z^2\right) - 2sc\,\Pi'^{\mathrm{M}}_{ZA}\left(0\right) - s^2\,\Pi'^{\mathrm{M}}_{AA}\left(0\right)\right], \tag{51c}$$

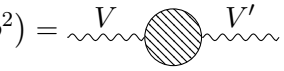
$$V^{\mathrm{M}} = \frac{1}{\alpha}\left[\Pi'^{\mathrm{M}}_{ZZ}\left(m_Z^2\right) - \widetilde{\Pi}^{\mathrm{M}}_{ZZ}\left(m_Z^2\right)\right], \tag{51d}$$

$$W^{\mathrm{M}} = \frac{1}{\alpha}\left[\Pi'^{\mathrm{M}}_{WW}\left(m_W^2\right) - \widetilde{\Pi}^{\mathrm{M}}_{WW}\left(m_W^2\right)\right], \tag{51e}$$

$$X^{\mathrm{M}} = \frac{sc}{\alpha}\left[\Pi'^{\mathrm{M}}_{ZA}\left(0\right) - \widetilde{\Pi}^{\mathrm{M}}_{ZA}\left(m_Z^2\right)\right]. \tag{51f}$$

Čia

$$\Pi^{\mu\nu}_{VV'}\left(p^2\right) = \tag{52}$$



kalibruotinių bozonų ($V, V' = W, Z, A$) savosios energijos, kurioms pritaikėme tokią dekompoziciją

$$\Pi^{\mu\nu}_{VV'}\left(p\right) = g^{\mu\nu}\,\Pi_{VV'}\left(p^2\right) + p^\mu p^\nu \text{ nariai}\,. \tag{53}$$

Taip pat apibrėžėme

$$\widetilde{\Pi}_{VV'}\left(p^2\right) \equiv \frac{\Pi_{VV'}\left(p^2\right) - \Pi_{VV'}\left(0\right)}{p^2}\,, \tag{54}$$

$$\Pi'_{VV'}\left(p^2\right) \equiv \frac{\mathrm{d}\Pi_{VV'}\left(p^2\right)}{\mathrm{d}p^2}\,. \tag{55}$$

Galiausiai apibrėžiame ir naudingą keletos netiesioginių parametrų tiesinę kombinaciją

$$K^{\mathrm{M}} \equiv \frac{S^{\mathrm{M}}}{2c^2} + \frac{s^2-c^2}{4s^2c^2}\,U^{\mathrm{M}} \tag{56}$$

$$= \frac{1}{\alpha}\left[\frac{s^2-c^2}{c^2}\,\widetilde{\Pi}^{\mathrm{M}}_{WW}\left(m_W^2\right) + \widetilde{\Pi}^{\mathrm{M}}_{ZZ}\left(m_Z^2\right) - \frac{s^2}{c^2}\,\Pi'^{\mathrm{M}}_{AA}\left(0\right)\right]. \tag{57}$$

Svarbu pastebėti, jog modelio parametrai nėra savaime tinkami stebinių parametrizavimui, kadangi jie gali būti tiek UV begaliniai, tiek ir priklausomi nuo kalibruotės [160]. Tik atėmus modelio netiesioginius parametrus tarp atitinkamų modelių gauname parametrus tinkamus stebinių parametrizavimui

$$O = O^{\mathrm{M2}} - O^{\mathrm{M1}}\,, \tag{58}$$

kur O yra koks nors netiesioginis parametras ir M1, M2=SM, NPM arba M1, M2=BM, BBM.

Priklausomai nuo nagrinėjamo atvejo turime tokius įvesties parametrus (renormalizacijos sąlygas)

$$\alpha = \widehat{\alpha}^{\mathrm{M}} \left[1 + \Pi_{AA}^{\mathrm{M}\prime}\left(0\right)\right] , \tag{59a}$$

$$G_{F(\text{charged})} = \widehat{G}_{F(\text{charged})}^{\mathrm{M}} \left[1 - \frac{\Pi_{WW}^{\mathrm{M}}\left(0\right)}{m_W^2} + \frac{2}{sc}\,\frac{\Pi_{ZA}^{\mathrm{M}}\left(0\right)}{m_Z^2} + \delta_{Gc}^{\mathrm{M}}\right] , \tag{59b}$$

$$m_Z^2 = \left(\widehat{m}_Z^{\mathrm{M}}\right)^2 \left[1 + \frac{\Pi_{ZZ}^{\mathrm{M}}\left(m_Z^2\right)}{m_Z^2} - \frac{2c}{s}\,\frac{\Pi_{ZA}^{\mathrm{M}}\left(0\right)}{m_Z^2}\right] , \tag{59c}$$

$$\left.\begin{array}{ll} \text{laisvas } \widehat{\rho} & , m_W^2 \\ \widehat{\rho} = 1 & , \left(m_W^{\mathrm{M}}\right)^2 \end{array}\right\} = \left(\widehat{m}_W^{\mathrm{M}}\right)^2 \left[1 + \frac{\Pi_{WW}^{\mathrm{M}}\left(m_W^2\right)}{m_W^2} - \frac{2}{sc}\,\frac{\Pi_{ZA}^{\mathrm{M}}\left(0\right)}{m_Z^2}\right] . \tag{59d}$$

Čia δ_{Gc}^{M} yra išorinių kojų, viršūnių ir dėžių pataisos, viršutiniai indeksai M nurodo ar dydis priklauso nuo modelio, (59d) lygtyje m_W yra įvesties parametras, kai $\widehat{\rho} = 1$, o m_W^{M} yra modelio prognozė, kai $\widehat{\rho}$ nėra fiksuotas. Kitaip tariant, vienu atveju negalime naudoti $\widehat{m}_W = \widehat{c}\,\widehat{m}_Z$, o kitu — galime. Taip pat svarbu pastebėti, jog šios renormalizacijos sąlygos užrašytos jau įvedus silpnosios sąveikos postūmį [125]

$$\widehat{g}^{\mathrm{M}} \to \widehat{g}^{\mathrm{M}} \left(1 - \frac{1}{sc}\,\frac{\Pi_{ZA}^{\mathrm{M}}\left(0\right)}{m_Z^2}\right) , \tag{60}$$

kuris užtikrina, kad ties 1-kilpa fotonas neturėtų masės.

Nors $G_{F(\text{charged})}$ yra vienas iš įvesties parametrų, kiek patogiau naudoti Weinberg'o kampo sinusą (ar kosinusą), kuriam turime

$$(\widehat{s}^{\mathrm{M}})^2 = \frac{\pi \widehat{\alpha}^{\mathrm{M}}}{\sqrt{2}\widehat{G}_{F(\text{charged})}^{\mathrm{M}} (\widehat{m}_W^{\mathrm{M}})^2} , \tag{61}$$

kai $\widehat{\rho}$ nėra fiksuotas, ir

$$(\widehat{s}^{\mathrm{M}})^2 = \frac{\pi \widehat{\alpha}^{\mathrm{M}}}{\sqrt{2}\widehat{G}_{F(\text{charged})}^{\mathrm{M}} (\widehat{c}\widehat{m}_Z^{\mathrm{M}})^2} , \tag{62}$$

kai $\widehat{\rho} = 1$. Atitinkamai, kiekvienam atvejui turime tokias renormalizacijos sąlygas:

$$s^2 = (\widehat{s}^{\mathrm{M}})^2 \left[1 + \Pi_{AA}^{\mathrm{M}\prime}\left(0\right) - \widetilde{\Pi}_{WW}\left(m_W^2\right) - \delta_{Gc}^{\mathrm{M}}\right] \tag{63}$$

ir

$$\bar{s}^2 = (\widehat{s}^{\,\mathrm{M}})^2 \left[1 + \frac{\bar{c}^2}{\bar{s}^2 - \bar{c}^2}\left(\alpha \bar{T}^{\mathrm{M}} + \Pi_{AA}^{\mathrm{M}\prime}(0) - \widetilde{\Pi}_{ZZ}^{\mathrm{M}}\left(m_Z^2\right) - \delta_{Gc}^{\mathrm{M}}\right)\right] . \quad (64)$$

Čia naudojame brūkšnius, kad būtų galima atskirti sinusų apibrėžimus: s atitinka atvejį, kai $\widehat{\rho}$ yra laisvas, o $\bar{s}$, kai $\widehat{\rho} = 1$. Analogiškai brūkšnius naudojame ir kitiems dydžiams.

Trumpai panagrinėkime Veltman'o parametrą visais trimis modelių palyginimo atvejais. Pirmuoju atveju (SM prieš NPM) turime

$$\bar{\rho}^{\mathrm{M}} \equiv \frac{\left(m_W^{\mathrm{M}}\right)^2}{\bar{c}^2 m_Z^2} , \quad (65)$$

todėl Veltman'o parametras tampa modelio prognoze dėl W bozono masės. Palyginus modelius turėsime

$$\bar{\rho}^{\,\mathrm{NPM}} = \bar{\rho}^{\,\mathrm{SM}} \left[1 + \frac{\bar{c}^2}{\bar{c}^2 - \bar{s}^2}\left(\alpha \bar{T} - \alpha \bar{K}\right)\right] . \quad (66)$$

Čia turime atimtus netiesioginius parametrus, kurie yra UV baigtiniai ir nepriklausomi nuo kalibruotės, taip pat atmetėme visas tiesiogines pataisas.

Kita vertus, antruoju atveju turime, jog Veltman'o parametras nebėra modelio prognozė ir yra pilnai apibrėžtas įvesties parametrų

$$\rho^{\mathrm{BM}} = \rho^{\mathrm{BBM}} = \rho \equiv \frac{m_W^2}{c^2 m_Z^2} . \quad (67)$$

Šiuo atveju taip pat nėra jokių UV divergencijų ar priklausomybės nuo kalibruotės. Taip pat čia galima pastebėti, jog norint pereiti iš pirmojo atvejo parametrizavimo į antrojo, užtenka atlikti tokius pakeitimus

$$T^{\mathrm{M}} \to K^{\mathrm{M}} \qquad (\text{arba} \quad \bar{T} \to K) \qquad \text{ir} \qquad \bar{s}, \bar{c} \to s, c . \quad (68)$$

Disertacijos tekste tai parodėme detaliau, tačiau tokia paprasta taisyklė tarp parametrizacijų išties galioja, todėl turint stebinių parametrizavimus pirmu atveju nesudėtinga pereit į antrąjį atvejį.

Paskutiniuoju atveju (SM prieš BBM) gauname

$$\frac{\rho}{\bar{\rho}^{\mathrm{SM}}} = \frac{m_W^2}{c^2 m_Z^2}\left[1 - \frac{\bar{c}^2}{\bar{c}^2 - \bar{s}^2}\left(\alpha T^{\mathrm{SM}} - \alpha K^{\mathrm{SM}}\right) + \frac{\bar{s}^2}{\bar{c}^2 - \bar{s}^2}\delta_{Gc}^{\mathrm{SM}}\right] , \quad (69)$$

tačiau tokiu atveju nesusiformuoja atimti netiesioginiai parametrai, o modelio parametrai yra tiek UV begaliniai, tiek ir priklausomi nuo kalibruotės. Tiesa, šie nefizikiniai įnašai yra išprastinami tiesioginių pataisų, tačiau tokios pataisos nėra universalios, sunkiau suskaičiuojamos bei išeina už netiesioginių parametrų ribų. Trumpai tariant, trečiuoju atveju netiesioginių parametrų formalizmas neegzistuoja ir reikalinga atlikti pilnus skaičiavimus.

Išvados

Šioje disertacijoje aptarėme įvairius renormalizacijos aspektus, kuriuos galime suskirstyti į tris pagrindines temas:

1. maišymosi matricų renormalizacija,
2. fermionams skirtos renormalizacijos schemos sukonstravimas ir pritaikymas Grimus-Neufeld modelyje,
3. elektrosilpnojo sektoriaus renormalizacija per netiesioginius parametrus, kai $m_W \neq \cos\theta_W m_Z$ medžio lygmenyje.

Pirmosios temos aptarimas paremtas [A1], kur nagrinėjome maišymosi matricų renormalizaciją apibendrintoje skaliarų sistemoje. Šioje disertacijoje pasirinkome pristatyti fermioninį atvejį ir taip pabrėžti argumentų universalumą. Būtent bendrumas ir universalumas užtikrino ir paprastumą, kadangi pakako nagrinėti 2 taškų funkcijų renormalizaciją padarius prielaidą, kad dalelės tarpusavyje maišosi. Kita vertus, maišymosi matricų renormalizacija dažniausiai aptarinėjama kalbant apie 3 taškų funkcijas.

Pagrindinis šios dalies klausimas buvo suprasti, ar apskritai reikia renormalizuoti maišymosi matricas ir, jei reikia, kaip tai padaryti. Prie klausimo priėjome nagrinėdami dvi apibendrintas bazes (vieną kurių užrašėme dviem būdais — su ir be sukimų matricos) susietas sukimu, kuris tam tikru pradinės ir galinės bazės pasirinkimu gali būti sutapatintas su įprasta maišymosi matrica. Toliau, kiekvienoje bazėje įvedėme laukų, masių ir sukimų atsvaros narius, tačiau neparinkome konkrečios renormalizacijos schemos, nes svarbus buvo tik pats atsvaros narių *rinkinys*. Kadangi nepridengtame Lagranžiane bazės yra susijusios sukimais, taip pat atlikome įvairius sukimus ir renormalizuotame Lagranžiane. Tačiau tai greitai privedė prie prieštarų, pavyzdžiui, atliekant pasukimą su renormalizuota sukimų matrica atgal į pradinę bazę, sukimų matricos atsvaros narys liko be atitinkamo renormalizuoto parametro. Taip sukimų matricos atsvaros narys tapo kliūtimi atstatant nepridengtą Lagranžianą iš renormalizuotojo. Dar vienas nesuderinamumas, kylantis iš sukimo matricos atsvaros nario buvimo, yra bazės transformacijų taisyklės pakitimas antiermitinei lauko renormalizacijos daliai.

Nepaisant šių konceptualių trūkumų, taip pat pastebėjome ir praktinius. Pavyzdžiui, kadangi nėra jokios natūralios maišymosi matricų renormalizacijos sąlygos, įvairūs *ad-hoc* metodai dažniausiai duoda nuo kalibruotės priklausančius atsvaros narius, nors dalelių maišymasis yra fizikinis procesas ir norima turėti nuo kalibruotės nepriklausančius atsvaros narius. Taip pat, jei naudojami įprasti diagonalūs masės atsvaros nariai, sukimų matricų atsvaros narių buvimas

veda prie singuliarios išsigimusių masių ribos, o tai gali sukelti skaitmeninių problemų.

Įdomu, jog visi šie trūkumai susiję su sukimų matricos atsvaros narių *buvimu*. Kita vertus, jei toks atsvaros narys trivialiai lygus nuliui, tuomet nebelieka nei vieno trūkumo. Tokiu atveju taip pat trivialiai patenkinami visi maišymosi matricų renormalizacijos reikalavimai, kuriuos surašėme 2.2 skyrelyje. Atsvaros nario trivialumas seka iš jo sutartinumo: nepridengtas Lagranžianas gali būti apibrėžtas tiek su sukimų matrica, tiek ir be jos. Kitaip tariant, sukimų matricos dirbtinai išskiria kokią nors bazę, todėl sukimų matricos neturėtų turėti atsvaros narių, nes jos yra nuo bazės priklausomi bei nefizikiniai objektai. Alternatyvų paaiškinimą galime suformuluoti pastebėję, jog nei renormalizacija, nei bazės transformacijos savaime nėra niekuo ypatingos, todėl abi procedūros turėtų komutuoti. Vienintelis būdas išlaikyti šį komutavimą — turėti trivialų sukimų matricos atsvaros narį, kad nebūtų išskirta nei viena bazė. Apibendrinant galime pasakyti, kad sukimų (maišymosi) matricos neturėtų turėti atsvaros narių, kad būtų išvengta tiek konceptualių, tiek ir praktinių bėdų. Kita vertus, nefizikiškumas ir atsvaros narių trivialumas nereiškia, kad maišymosi matricos negali būti išmatuotos. Yra priešingai — maišymosi matricos gaunamos iš kokios nors pradinės masių matricos, todėl pradinės masių matricos išmatavimas yra ekvivalentus naujos masių matricos *ir* maišymosi matricos išmatavimui.

Parodę, jog trivialūs maišymosi matricų atsvaros nariai yra tinkami, galėjome pereiti prie **antrosios temos** — renormalizacijos schemos ant masės apvalko su trivialiais maišymosi matricos atsvaros nariais konstravimo bei taikymo [A2–A4]. Kiek priešingai, išreikštas tokios schemos įgyvendinimas nėra trivialus, kadangi greitai susiduriama su išsigimimu — nediagonalūs masių matricos bei antiermitinės lauko renormalizacijos dalies atsvaros nariai neturi nepriklausomų sprendinių. Mūsų schemoje šis išsigimimas panaikinamas gan neįprastu būdu apibrėžiant antiermitinę lauko renormalizacijos dalį kaip $m_i^2 - m_j^2$ masių struktūros koeficientą ir po to paprasčiausiai gaunant masės atsvaros narius kaip sprendinius užrašytus per savasias energijas bei lauko atsvaros narius. Patikrinome, jog toks apibrėžimas išties veda prie trivialių maišymosi matricos atsvaros narių, tenkina visus maišymosi renormalizacijos reikalavimus ir tuo pat metu yra nepriklausomas nei nuo modelio, nei nuo konkretaus fizikinio proceso. Papildomai, bent jau ties 1-kilpa pasirūpinome kaip tik įmanoma didesniu absorptyvių dalių įtraukimu Dirako fermionams, nors Majorana dalelėms tai nėra įmanoma. Be to, plačiai panaudoję Nielsen'o tapatybes, parodėme, jog mūsų schema gali būti sukonstruota visoms perturbacijų teorijos eilėms, nors ir atmetus absorptyvias dalis.

Ties 1-kilpa savo schemą pritaikėme Grimus-Neufeld modeliui, kuris suteikė

pakankamai scenarijų schemos išbandymui. Kiek konkrečiau Grimus-Neufeld modelis talpina kvarkus ir krūvį turinčius leptonus, kurie, nepaisant sąveikos su antruoju Higgs'o dubletu, panašūs į Standartinio Modelio atitikmenis, bei 4 neutrinus, iš kurių du ties medžio lygmeniu neturi masių. Kvarkai panašūs į SM sutinkamus suteikė pakankamai paprastą atvejį, tad savo schemą galėjome pritaikyti be jokių sunkumų ir taip pat savo rezultatus palyginome su Denner'io ir Sack'o bei Kniehl'o ir Sirlin'o schemomis. Kita vertus, neutrinų sektorius gerokai sudėtingesnis, kadangi jame galima išskirti tris atvejus priklausomai nuo to, kuriems neutrinams buvo suskaičiuotos savosios energijos. Turėjome masyvųjį atvejį, jei abu neutrinai turėjo mases, dalinai masyvų atvejį, jei tik vienas iš dviejų neutrinų turėjo masę, ir bemasį atvejį, jei abu neutrinai buvo bemasiai. Masyviuoju ir bemasiu atvejais galėjome savo schemą taikyti be didesnių sunkumų, kai tuo tarpu dalinai masyviuoju atveju nebegalėjome tiesiogiai taikyti antiermitinės lauko renormalizacijos dalie apibrėžimo, kadangi šiuo atveju neegzistuoja $m_i^2 - m_j^2$ masių struktūra. Laimei, šią kliūtį sugebėjome apeiti paprasčiausiai paimdami bemasę masyviojo atvejo ribą ($m_i \to 0$ arba $m_j \to 0$), o tai nesukėlė jokių papildomų bėdų ir nesugriovė schemos savybių. Taip įsitikinome, jog mūsų schema veikia nepaisant Grimus-Neufeld modelio savitumo. Papildomai, pavyko pasiekti patenkinamą 2 taškų funkcijų renormalizacija fermionų sektoriuje ir viliamės, jog panaši procedūra taip pat gali būti pritaikyta skaliariniuose sektoriuose.

Trečiajai temai nagrinėjome netiesioginius parametrus elektrosilpnajame sektoriuje, kai W ir Z bozonų masės ties medžio lygmeniu nėra susietos per Weinberg'o kampą gerai žinomu sąryšiu $\widehat{m}_W \neq \cos\widehat{\theta}_W \widehat{m}_Z$. Šį sąryšį galime suformuluoti ir per Veltman'o $\widehat{\rho}$ parametrą: jei $\widehat{m}_W = \cos\widehat{\theta}_W \widehat{m}_Z$ išlaikomas, tuomet $\widehat{\rho} = 1$, jei ne — sakome, kad $\widehat{\rho}$ yra laisvas parametras. Antruoju atveju ši papildoma laisvė implikuoja, jog prie smulkiosios struktūros konstantos, Fermi konstantos ir Z bozono masės EW sektoriuje reikia papildomo įvesties parametro (renormalizacijos sąlygos). Ketvirtuoju įvesties parametru pasirinkome W bozono masę, kurio skaitinė vertė gali būti paimta iš naujausio CDF matavimo. Panaudoję 4 įvesties parametrus radome, jog vietoje 6 netiesioginių parametrų $STUVWX$ iš $\widehat{\rho} = 1$ atvejo lieka tik penki $SUVWX$ parametrai ir renormalizacijos procedūra pašalina T parametrą. Tai norimas rezultatas, kadangi būtent T parametras UV diverguoja ir negali būti panaudotas stebinių parametrizavimui, kai $\widehat{\rho}$ yra laisvas parametras. Mūsų žiniomis mūsų išvestas $SUVWX$ formalizmas yra vienintelis tikras netiesioginių parametrų formalizmas tinkantis atvejui, kai $\widehat{\rho}$ nėra fiksuotas.

Deja, bet $SUVWX$ formalizmo apribojimai analogiški $STUVWX$ formalizmo apribojimams ir mums nepavyko jų apeiti. $STUVWX$ formalizme

abu lyginami modeliai privalo turėti $\widehat{\rho} = 1$, o tuo tarpu $SUVWX$ formalizmas reikalauja, kad *abiejuose* modeliuose $\widehat{\rho}$ būtų laisvas parametras. Panašu, jog tik netiesioginių pataisų (parametrų) nepakanka, jei norima palyginti modelius, vienas kurių turi $\widehat{\rho} = 1$ (pvz., SM arba GNM), o kitame $\widehat{\rho}$ laisvas parametras (pvz., tripleto modelis, kuriame tripletas įgauna netrivialią tikėtiną vakuumo vertę). Atrodo, jog šiuo metu tokiam palyginimui reikalinga kiekviename modelyje atlikti pilnus skaičiavimus įtraukiant ir nuo proceso priklausančias tiesiogines kilpų pataisas.

CURRICULUM VITAE

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Education

Ariogala gymnasium	2010 - 2014
Vilnius University, Physics Faculty	2014 – 2018
University of Oxford, Mathematical Institute	2018 – 2019
Vilnius University, Physics Faculty	2019 – 2024

Conferences, Talks, and Posters

CERN/CMS Computing Technologies Workshop	December 2016
Open Readings 2018	March 2018
44th Theoretical Physics Conference Matter to the Deepest	September 2021
Open Readings 2021	March 2021
1st CERN Baltic Conference	June 2021
44th Lithuanian National Physics Conference	October 2021
COST Action: Particleface webinar series	October 2021
Workshop on Multi-Higgs Models	September 2022
2nd CERN Baltic Conference	October 2022
EPS-HEP Conference 2023	August 2023
45th Theoretical Physics Conference Matter to the Deepest	September 2023
3rd CERN Baltic Conference	October 2023
45th Lithuanian National Physics Conference	October 2023

- Part of the organizing committee

Schools

CERN Summer School	June 2017 - August 2017
CERN Spring Campus	April 2018
GGI Lectures on the Theory of Fundamental Interactions	January 2020
PREFIT20: Precision Effective Field Theory School	March 2020

Computer Algebra and Particle Physics April 2021

Baltic School of High-Energy Physics and Accelerator Technologies August 2021

PSI Particle Physics Summer School – Vision and Precision August 2022

Teaching

- Tutorials for the *Classical Mechanics* course for 2nd year bachelor students since 2021
- Tutorials for the *Quantum Mechanics* course for 3rd year bachelor students since 2022
- Tutorials for the *Quantum Field Theory II/II* course for 1st year master students since 2022
- In 2021 supervised two 2nd and 3rd year bachelor students, who made a poster presentation in the student conference Open Readings 2021 (online)
- As of 2022 also supervising students working on their bachelor theses

Outreach

- Contributed to the *CERN* installation at the exhibition *School 2016* in *LitExpo*
- Moderator of practical tasks in the *CERN/CMS masterclasses* at Vilnius university since 2018
- Moderator at the *Researcher's Night* in September 2019
- Helping with Wilson chamber demonstrations to pupils since 2019
- Giving talks at the science festival *Spaceship Earth* since 2020
- Moderator at the *Hadron Therapy masterclass* in March 2023
- Supervised a group of lithuanian teachers for the *CERN Baltic Teacher Programme* in April 2023 and March 2024

Other Activities

- One of the organizers of the 2nd and 3rd *Lithuanian Particle Physics Meetings*
- Clarinet player during Bachelor and PhD studies in the *Vilnius University Wind Orchestra Oktava*

- Clarinet player during Master studies in the *Oxford University Symphonic Band*
- Practicing aikido since 2022

ACKNOWLEDGEMENTS/PADĖKOS

First and foremost I sincerely thank my supervisor asoc. prof. Thomas Gajdosik, who has guided me through doctoral studies. I am mostly grateful to Thomas for being an example of honesty, openness, and kindness.

I am thankful to Valstybinis Studijų Fondas, which provided me with a scholarship for my master's studies at Oxford University — this helped immensely by preparing me well for doctoral studies.

Esu dėkingas ir pirmajam savo vadovui dr. Andriui Juodagalviui, kuris dar bakalauro studijų metais itin aktyviai įtraukė mane į su dalelių fizika susijusias veiklas ir išties daug išmokė.

Už bendrą darbą, diskusijas, patarimus ir draugiškumą labai dėkoju Vytautui Dūdėnui.

Dėkoju visai mūsų savanoriškų seminarų dalyvių grupelei tiek už galimybę išmokti naujų dalykų, tiek ir už uždavinių sprendimų pristatinėjimo pramogą.

Itin dėkoju visai šeimai, kuri visokeriopai palaikė mane tiek iki doktorantūros, tiek ir jos metu bei nekantriai laukia gynimo dienos.

Didelis ačiū taip pat keliauja visiems draugams, kurių dėka pavyko išsaugoti (ne?-)sveiką protą. Branginu visus gimtadienius, stalo žaidimų vakarus, grilius, baidares, koncertus, pasivaikščiojimus ir visą kitą laiką, juoką bei palaikymą. Jūsų išties daug, bet pabandysiu išvardinti: Justinas, Ieva, Ignas ir Greta, Marijus ir Aneta, Domantas ir Gintarė, Martynas ir Akvilė, Agnė ir Aistė, Madita. Taip pat ačiū ir visiems *Mikrobams* — best.day.ever.

Turėjau gerą dešimtmetį džiaugsmo grojant orkestre „Oktava", tad dėkoju vadovui Albinui Kučinskui už progą tobulėti tiek grojimo, tiek ir muzikinio skonio prasme. Žinoma, vertinu ir kiekvieną pažintį orkestre!

Esu itin dėkingas treneriui Vytautui Ribikauskui ir visam „Sūkurio" kolektyvui už gerai praleistą laiką, treniruotes ir svarbiausia už daugybę stovyklose peržengtų ribų.

Ko gero šis sąrašas nėra išbaigtas ir dar galėčiau daug kam padėkoti. Apskritai, esu dėkingas už kiekvieną gerą patirtį, kad ir kokia mažytė ji buvo, bei manau, kad dažnai mane tiesiog lydėjo sėkmė. Kita vertus, A. Copland'o operoje „The Second Hurricane" yra žodžiai „if it happens to you, if you come out alright, it's a good thing that it did", tiek ir lietuvių liaudies išmintis byloja, jog „viskas gerai, kas gerai baigiasi", tad lygiai taip pat vertinu ir sunkumus, kurie mane sustiprino.

LIST OF PUBLICATIONS

$*$ marks publications in WoS published during doctoral studies

[A1] $*$ S. Draukšas, "Relations between basis sets of fields in the renormalization procedure", *The European Physical Journal C*, vol. 83, no. 10, p. 962, Oct. 25, 2023, ISSN: 1434-6052. DOI: 10.1140/epjc/s10052-023-12121-6. arXiv: 2307.01642 [hep-ph].

[A2] $*$ S. Draukšas, "On-shell Renormalization of Fermion Masses, Fields, and Mixing Matrices at 1-loop", *Acta Physica Polonica B Proceedings Supplement*, vol. 15, no. 2, p. 1, Feb. 28, 2022, ISSN: 1899-2358, 2082-7865. DOI: 10.5506/APhysPolBSupp.15.2-A15.

[A3] $*$ S. Draukšas, "On-shell renormalization of fermion masses, fields, and mixing matrices at 1-loop", *The European Physical Journal Plus*, vol. 138, no. 7, p. 671, Jul. 31, 2023, *Reproduced with permission from Springer Nature*, ISSN: 2190-5444. DOI: 10.1140/epjp/s13360-023-04226-4.

[A4] $*$ S. Draukšas, V. Dūdėnas, T. Gajdosik, A. Juodagalvis, P. Juodsnukis, and D. Jurčiukonis, "The Grimus-Neufeld model with FlexibleSUSY at one-loop", *Symmetry*, vol. 11, no. 11, p. 1418, Nov. 16, 2019, ISSN: 20738994. DOI: 10.3390/sym11111418.

[A5] S. Draukšas, V. Dūdėnas, and L. Lavoura. "Oblique corrections when $m_W \neq m_Z \cos\theta_W$ at tree level". arXiv: 2305.14050 [hep-ph]. (May 23, 2023), [Online]. Available: http://arxiv.org/abs/2305.14050, preprint.

LIST OF PRESENTATIONS

[B1] S. Draukšas, "On the On-Shell Renormalization of Fermion Masses, Fields, and Mixing Matrices at 1-loop", Talk, presented at the COST: Particleface Webinar Series (Online), Oct. 12, 2021. [Online]. Available: https://indico.cern.ch/event/1071300/.

[B2] S. Draukšas, "Apie Fermionų Masių, Laukų ir Maišymosi Matricų Pernormavimą ant Masės Apvalko ties 1-kilpa", Talk, presented at the 44 Lietuvos Nacionalinė Fizikos Konferencija (Vilnius), Oct. 7, 2021. [Online]. Available: http://lnfk.ftmc.lt/wp-content/

uploads/2021/10/LNFK44-039-Simonas-Drauksas-VU-TFAI.pdf.

[B3] S. Draukšas, “On the On-Shell Renormalization of Fermion Masses, Fields, and Mixing Matrices at 1-loop”, Talk, presented at the Matter To The Deepest 2021 (Online), Sep. 15, 2021. [Online]. Available: https://indico.if.us.edu.pl/event/9/contributions/274/.

[B4] S. Draukšas, “On the On-Shell Renormalization of Fermion Masses, Fields, and Mixing Matrices at 1-loop”, Talk, presented at the 1st CERN Baltic Conference (Online), Jun. 29, 2021. [Online]. Available: https://indico.cern.ch/event/970609/timetable/?layout=room#53-on-the-on-shell-renormaliza.

[B5] S. Draukšas and T. Gajdosik, “Renormalizing the neutrino mixing matrix in the Grimus-Neufled Model”, Poster, presented at the 64th International Student Conference Open Readings (Vilnius), Mar. 17, 2021. [Online]. Available: https://openreadings.eu/wp-content/abstracts/abstract-2021.pdf.

[B6] S. Draukšas, “On-Shell renormalization of scalar sectors”, Talk, presented at the 2nd CERN Baltic Conference (Vilnius), Oct. 11, 2022. [Online]. Available: https://indico.cern.ch/event/1147717/timetable/#43-on-shell-renormalization-of.

[B7] S. Draukšas, “On-Shell renormalization of scalar sectors”, Talk, presented at the Workshop on Multi-Higgs Models (Lisbon), Sep. 2, 2022. [Online]. Available: http://cftp.tecnico.ulisboa.pt/~2hdmwork/Slides/2022/Day4/Simonas_Drauksas.pdf.

[B8] S. Draukšas, “Sąryšiai tarp laukų bazių pernormavimo procedūroje”, Talk, presented at the 45 Lietuvos Nacionalinė Fizikos Konferencija (Vilnius), Oct. 26, 2023. [Online]. Available: https://ff.vu.lt/external/ff/files/LNFK45/LNFK_abstracts/05_Zodinis_Drauksas_2023-09-06_18-01-04_S__rysiai_tarp_lauku_baziu_pernormavimo_proceduroje_lnfk45-tezes-sdrauksas.pdf.

[B9] S. Draukšas, “Relations between basis sets of fields in the renormalization procedure”, Talk, presented at the Matter to the Deepest 2023 (Ustron), Sep. 20, 2023. [Online]. Available: https://indico.if.us.edu.pl/event/18/contributions/455/.

[B10] S. Draukšas, “Relations between basis sets of fields in the renormalization procedure”, Talk, presented at the 3rd CERN Blatic Conference (Riga), *Reproduced with permission from Springer Nature*, Sep. 10, 2023. [Online]. Available: https://indico.cern.ch/event/1288731/contributions/5625721/.

[B11] S. Draukšas, “Relations between basis sets of fields in the renormalization procedure”, Poster, presented at the EPS-HEP 2023 (Hamburg), Aug. 22, 2023. [Online]. Available: https://indico.desy.de/event/34916/contributions/147837/.

NOTES

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