

A Discrete Version of the Mishou Theorem Related to Periodic Zeta-Functions

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Abstract. In the paper, we consider simultaneous approximation of a pair of analytic functions by discrete shifts $\zeta_{u_N}(s + ikh_1; \mathbf{a})$ and $\zeta_{u_N}(s + ikh_2, \alpha; \mathbf{b})$ of the absolutely convergent Dirichlet series connected to the periodic zeta-function with multiplicative sequence \mathbf{a} , and the periodic Hurwitz zeta-function, respectively. We suppose that $u_N \rightarrow \infty$ and $u_N \ll N^2$ as $N \rightarrow \infty$, and the set $\{(h_1 \log p : p \in \mathbb{P}), (h_2 \log(m + \alpha) : m \in \mathbb{N}_0), 2\pi\}$ is linearly independent over \mathbb{Q} .

Keywords: Mishou theorem, periodic zeta-function, periodic Hurwitz zeta-function, universality.

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1 Introduction

Let $s = \sigma + it$ be a complex variable, and $0 < \alpha \leq 1$ a fixed parameter. The Riemann zeta-function $\zeta(s)$ and Hurwitz zeta-function $\zeta(s, \alpha)$ are defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} \quad \text{and} \quad \zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m + \alpha)^s},$$

and have analytic continuations to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1. These functions play an important role in pure mathematics, and have various applications in other natural sciences. One of common feature of the functions $\zeta(s)$ and $\zeta(s, \alpha)$ (for some classes of parameter α) is their universality. Let $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$, \mathcal{K} be the class of compact subsets of the strip D with connected complements, $H(K)$, $K \in \mathcal{K}$, class of continuous functions on K and analytic in the interior of K , and $H_0(K)$ the subclass of $H(K)$ of non-vanishing on K functions. Then, it is known [1, 18, 20, 29, 39] that there are infinitely many shifts $\zeta(s + i\tau)$, $\tau \in \mathbb{R}$, approximating every function $f(s) \in H_0(K)$. Similarly, the set of shifts $\zeta(s + i\tau, \alpha)$ with rational or transcendental α approximating a given function $f(s) \in H(K)$ also is infinite [1, 27]. Discrete shifts $\zeta(s + ikh)$ and $\zeta(s + ikh, \alpha)$ with fixed $h > 0$ and $k \in \mathbb{N}$ have an analogical approximation property [1, 15, 16, 19, 33, 37]. The case of algebraic irrational α is more complicated, was discussed in [2], and the best results were obtained in [38].

H. Mishou in [35] obtained a joint universality theorem for $\zeta(s)$ and $\zeta(s, \alpha)$ with transcendental α . Denote by $\text{meas}A$ the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then, the Mishou theorem is the following statement.

Theorem 1. *Suppose that the parameter α is transcendental, $K_1, K_2 \in \mathcal{K}$ and $f_1(s) \in H_0(K_1)$, $f_2(s) \in H(K_2)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K_1} |\zeta(s + i\tau) - f_1(s)| < \varepsilon, \right. \\ \left. \sup_{s \in K_2} |\zeta(s + i\tau, \alpha) - f_2(s)| < \varepsilon \right\} > 0.$$

The problem of algebraic parameter α was discussed in [17].

A discrete analogue of Theorem 1 was proved in [6]. Denote by $\#A$ the cardinality of a set $A \subset \mathbb{R}$, and define the set

$$L(\mathbb{P}, \alpha, h, \pi) = \{(\log p : p \in \mathbb{P}), (\log(m + \alpha) : m \in \mathbb{N}_0), 2\pi/h\},$$

where \mathbb{P} and \mathbb{N}_0 are the sets of all prime and non-negative integers, respectively. Then the main result of [6] is

Theorem 2. *Suppose that the set $L(\mathbb{P}, \alpha, h, \pi)$ is linearly independent over the field of rational numbers \mathbb{Q} . Let $K_1, K_2 \in \mathcal{K}$ and $f_1(s) \in H_0(K)$, $f_2(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N \rightarrow \infty} \frac{1}{N + 1} \# \left\{ 0 \leq k \leq N : \sup_{s \in K_1} |\zeta(s + ikh) - f_1(s)| < \varepsilon, \right. \\ \left. \sup_{s \in K_2} |\zeta(s + ikh, \alpha) - f_2(s)| < \varepsilon \right\} > 0.$$

Generalizations of Theorem 2, including a weighted version, were given in [7, 14] and [34].

The periodic and periodic Hurwitz zeta-functions are generalizations of the Riemann and Hurwitz zeta-functions, respectively. Let $\mathbf{a} = \{a_m : m \in \mathbb{N}\}$

and $\mathbf{b} = \{b_m : m \in \mathbb{N}_0\}$ be two periodic sequences of complex numbers with minimal periods $q_1 \in \mathbb{N}$ and $q_2 \in \mathbb{N}$, respectively. The periodic zeta-function $\zeta(s; \mathbf{a})$ and periodic Hurwitz zeta-function $\zeta(s, \alpha; \mathbf{b})$, $0 < \alpha \leq 1$, are defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s; \mathbf{a}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s} \quad \text{and} \quad \zeta(s, \alpha; \mathbf{b}) = \sum_{m=0}^{\infty} \frac{b_m}{(m + \alpha)^s}.$$

The periodicity of the sequences \mathbf{a} and \mathbf{b} implies, for $\sigma > 1$, the equalities

$$\zeta(s; \mathbf{a}) = \frac{1}{q_1^s} \sum_{l=1}^{q_1} a_l \zeta\left(s, \frac{l}{q_1}\right) \quad \text{and} \quad \zeta(s, \alpha; \mathbf{b}) = \frac{1}{q_2^s} \sum_{l=0}^{q_2-1} b_l \zeta\left(s, \frac{l + \alpha}{q_2}\right),$$

which give the meromorphic continuations for the functions $\zeta(s; \mathbf{a})$ and $\zeta(s, \alpha; \mathbf{b})$ to the whole complex plane, and

$$\operatorname{Res}_{s=1} \zeta(s; \mathbf{a}) = \frac{1}{q_1} \sum_{l=1}^{q_1} a_l \quad \text{and} \quad \operatorname{Res}_{s=1} \zeta(s, \alpha; \mathbf{b}) = \frac{1}{q_2} \sum_{l=0}^{q_2-1} b_l.$$

We recall that the sequence \mathbf{a} is multiplicative if $a_1 = 1$ and $a_{mn} = a_m a_n$ for all $(m, n) = 1$. The case of a multiplicative sequence was treated in [31]. Discrete universality for $\zeta(s; \mathbf{a})$ can be found in [3, 13]. Universality of $\zeta(s, \alpha; \mathbf{b})$ with various types of the parameter α was considered in [8, 11, 28]. A version of the Mishou theorem for periodic zeta-functions $\zeta(s; \mathbf{a})$ and $\zeta(s, \alpha; \mathbf{b})$ was obtained in [12].

Theorem 3. [12]. *Suppose that α is transcendental number, and the sequence \mathbf{a} is multiplicative. Let K_1, K_2 and $f_1(s), f_2(s)$ be the same as in Theorem 1. Then, for every $\varepsilon > 0$,*

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0, T] : \sup_{s \in K_1} |\zeta(s + i\tau; \mathbf{a}) - f_1(s)| < \varepsilon, \right. \\ \left. \sup_{s \in K_2} |\zeta(s + i\tau, \alpha; \mathbf{b}) - f_2(s)| < \varepsilon \right\} > 0.$$

The discrete version of Theorem 3 was presented in [13]. Define the set

$$L(\mathbb{P}; \alpha, h_1, h_2, \pi) = \{(h_1 \log p : p \in \mathbb{P}), (h_2 \log(m + \alpha) : m \in \mathbb{N}_0), 2\pi\},$$

where h_1 and h_2 are positive numbers.

Theorem 4. [13]. *Suppose that the sequence \mathbf{a} is multiplicative, and the set $L(\mathbb{P}; \alpha, h_1, h_2, \pi)$ is linearly independent over \mathbb{Q} . Let K_1, K_2 and $f_1(s), f_2(s)$ be the same as in Theorem 1. Then, for every $\varepsilon > 0$,*

$$\liminf_{N \rightarrow \infty} \frac{1}{N + 1} \# \left\{ 0 \leq k \leq N : \sup_{s \in K_1} |\zeta(s + ikh; \mathbf{a}) - f_1(s)| < \varepsilon, \right. \\ \left. \sup_{s \in K_2} |\zeta(s + ikh, \alpha; \mathbf{b}) - f_2(s)| < \varepsilon \right\} > 0.$$

The aim of this paper, is an extension of Theorem 4 for certain absolutely convergent Dirichlet series related to the functions $\zeta(s; \mathbf{a})$ and $\zeta(s, \alpha; \mathbf{b})$.

Let $\theta > 1/2$ be a fixed number. For $u > 0$, set

$$v_u(m) = \exp\left\{-\left(m/u\right)^\theta\right\}, \quad m \in \mathbb{N},$$

$$v_u(m, \alpha) = \exp\left\{-\left((m + \alpha)/u\right)^\theta\right\}, \quad m \in \mathbb{N}_0,$$

where $\exp\{a\} = e^a$. Define the series

$$\zeta_u(s; \mathbf{a}) = \sum_{m=1}^{\infty} \frac{a_m v_u(m)}{m^s} \quad \text{and} \quad \zeta_u(s, \alpha; \mathbf{b}) = \sum_{m=0}^{\infty} \frac{b_m v_u(m, \alpha)}{(m + \alpha)^s}.$$

Since $v_u(m)$ and $v_u(m, \alpha)$ are exponentially decreasing with respect to m , and a_m and b_m are bounded, the latter series are absolutely convergent for $\sigma > \sigma_0$ with arbitrary finite σ_0 .

The first universality results with certain $u_T \rightarrow \infty$ for $\zeta_{u_T}(s; \{1\})$ were obtained in [21], and discrete version in [32]. The case in short intervals was treated in [23]. A generalization of the above results for $\zeta_{u_T}(s; \mathbf{a})$ with multiplicative sequence \mathbf{a} was presented in [9] and [10]. Similar problems for $\zeta_{u_T}(s, \alpha; \{1\})$ and $\zeta_{u_T}(s, \alpha; \mathbf{b})$ were discussed in [26] and [5]. The papers [22] and [24] are devoted to extension of Mishou’s theorem for absolutely convergent Dirichlet series. In [25], the case of Dirichlet series connected to zeta-functions of certain cusp forms was considered. We also mention the work [30] devoted to the universality of absolutely convergent Dirichlet series with generalized shifts.

We recall a theorem from [4] which extends the Mishou theorem for $\zeta_{u_T}(s; \mathbf{a})$ and $\zeta_{u_T}(s, \alpha; \mathbf{b})$ with $u_T \rightarrow \infty$. For its statement, we need some notation and definitions. Denote $\gamma = \{s \in \mathbb{C} : |s| = 1\}$, and define the sets

$$\Omega_1 = \prod_{p \in \mathbb{P}} \gamma_p \quad \text{and} \quad \Omega_2 = \prod_{m \in \mathbb{N}_0} \gamma_m,$$

where $\gamma_p = \gamma$ for all $p \in \mathbb{P}$ and $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0$. The tori Ω_1 and Ω_2 with the product topology and operation of pointwise multiplication are compact topological Abelian groups. Hence, $\Omega = \Omega_1 \times \Omega_2$ also is a compact topological group, therefore, on $(\Omega, \mathcal{B}(\Omega))$ ($\mathcal{B}(\mathbb{X})$ is the Borel σ -field of the space \mathbb{X}), the probability Haar measure m_H exists, and we have the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega = (\omega_1, \omega_2)$, $\omega_1 = (\omega_1(p) : p \in \mathbb{P})$, $\omega_2 = (\omega_2(m) : m \in \mathbb{N}_0)$, the elements of Ω , and extend the elements $\omega_1(p)$ to the set \mathbb{N} by the formula

$$\omega_1(m) = \prod_{\substack{p^l | m \\ p^{l+1} \nmid m}} \omega_1^l(p), \quad m \in \mathbb{N}.$$

Let $H(D)$ stand for the space of analytic on D functions endowed with topology of uniform convergence on compacta. On the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H^2(D)$ -valued random element

$$\zeta(s, \alpha, \omega_1, \omega_2; \mathbf{a}, \mathbf{b}) = (\zeta(s, \omega_1; \mathbf{a}), \zeta(s, \alpha, \omega_2; \mathbf{b})),$$

where

$$\zeta(s, \omega_1; \mathbf{a}) = \sum_{m=1}^{\infty} \frac{a_m \omega_1(m)}{m^s} \quad \text{and} \quad \zeta(s, \alpha, \omega_2; \mathbf{b}) = \sum_{m=0}^{\infty} \frac{b_m \omega_2(m)}{(m + \alpha)^s}.$$

The main result of [4] is the following theorem.

Theorem 5. [4]. *Suppose that the sequence \mathbf{a} is multiplicative, α is transcendental, and $u_T \rightarrow \infty$ and $u_T \ll T^2$ as $T \rightarrow \infty$. Let $K_1, K_2 \in \mathcal{K}$ and $f_1(s) \in H_0(K_1)$, $f_2(s) \in H(K_2)$. Then the limit*

$$\begin{aligned} & \lim_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K_1} |\zeta_{u_T}(s + i\tau; \mathbf{a}) - f_1(s)| < \varepsilon_1, \right. \\ & \qquad \qquad \qquad \left. \sup_{s \in K_2} |\zeta_{u_T}(s + i\tau, \alpha; \mathbf{b}) - f_2(s)| < \varepsilon_2 \right\} \\ & = m_H \left\{ (\omega_1, \omega_2) \in \Omega : \sup_{s \in K_1} |\zeta(s, \omega_1; \mathbf{a}) - f_1(s)| < \varepsilon_1, \right. \\ & \qquad \qquad \qquad \left. \sup_{s \in K_2} |\zeta(s, \alpha, \omega_2; \mathbf{b}) - f_2(s)| < \varepsilon_2 \right\} \end{aligned}$$

exists and is positive for all but at most countably many $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$.

Here, and in what follows, the notation $x \ll_{\theta} y$, $y > 0$, means that there exists a constant $c = c(\theta) > 0$ such that $|x| \leq cy$.

We extend Theorem 5 to the discrete case by using the set $L(\mathbb{P}; \alpha, h_1, h_2, \pi)$.

Theorem 6. *Suppose that the sequence \mathbf{a} is multiplicative, the set $L(\mathbb{P}; \alpha, h_1, h_2, \pi)$ is linearly independent over \mathbb{Q} , and $u_N \rightarrow \infty$ and $u_N \ll N^2$ as $N \rightarrow \infty$. Let $K_1, K_2 \in \mathcal{K}$ and $f_1(s) \in H_0(K_1)$, $f_2(s) \in H(K_2)$. Then, the limit*

$$\begin{aligned} & \lim_{N \rightarrow \infty} \frac{1}{N + 1} \# \left\{ 0 \leq k \leq N : \sup_{s \in K_1} |\zeta_{u_N}(s + ikh_1; \mathbf{a}) - f_1(s)| < \varepsilon_1, \right. \\ & \qquad \qquad \qquad \left. \sup_{s \in K_2} |\zeta_{u_N}(s + ikh_2, \alpha; \mathbf{b}) - f_2(s)| < \varepsilon_2 \right\} \\ & = m_H \left\{ (\omega_1, \omega_2) \in \Omega : \sup_{s \in K_1} |\zeta(s, \omega_1; \mathbf{a}) - f_1(s)| < \varepsilon_1, \right. \\ & \qquad \qquad \qquad \left. \sup_{s \in K_2} |\zeta(s, \alpha, \omega_2; \mathbf{b}) - f_2(s)| < \varepsilon_2 \right\} \end{aligned}$$

exists and is positive for all but at most countably many $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$.

We observe that the set $L(\mathbb{P}; \alpha, h_1, h_2, \pi)$ is non-empty. We recall that the numbers η_1, \dots, η_r are algebraically independent over \mathbb{Q} if it does not exist any polynomial $p(s, \dots, s_r) \neq 0$ with rational coefficients such that $p(\eta_1, \dots, \eta_r) = 0$. The Nesterenko theorem asserts [36] that the numbers π and e^{π} are algebraically independent over \mathbb{Q} . From the latter theorem, it follows that the set

$L(\mathbb{P}; 1/\pi, h_1, h_2, \pi)$ with rational positive h_1 and h_2 is linearly independent over \mathbb{Q} . The Nesterenko theorem implies the transcendence of the numbers π and e^π . Suppose, on the contrary, that the set $L(\mathbb{P}; 1/\pi, h_1, h_2, \pi)$ is not linearly independent over \mathbb{Q} . Then there exist integers $k_1, \dots, k_{r_1}, \widehat{k}_1, \dots, \widehat{k}_{r_2}$ and \widetilde{k} , not all zeros, such that

$$k_1 h_1 \log p_1 + \dots + k_{r_1} h_1 \log p_{r_1} + \widehat{k}_1 h_2 \log (m_1 + 1/\pi) + \dots + \widehat{k}_{r_2} h_2 \log (m_{r_2} + 1/\pi) + \widetilde{k} \pi = 0.$$

Hence,

$$p_1^{l_1} \dots p_{r_1}^{l_{r_1}} (m_1 + 1/\pi)^{\widehat{l}_1} \dots (m_{r_2} + 1/\pi)^{\widehat{l}_{r_2}} e^{\widetilde{l} \pi} = 1$$

with some integers $l_1, \dots, l_{r_1}, \widehat{l}_1, \dots, \widehat{l}_{r_2}$ and \widetilde{l} , and this contradicts the algebraic independence of the numbers π and e^π . Similarly, the equalities

$$k_1 h_1 \log p_1 + \dots + k_{r_1} h_1 \log p_{r_1} + \widehat{k}_1 h_2 \log (m_1 + 1/\pi) + \dots + \widehat{k}_{r_2} h_2 \log (m_{r_2} + 1/\pi) = 0,$$

$$k_1 h_1 \log p_1 + \dots + k_{r_1} h_1 \log p_{r_1} + \widetilde{k} \pi = 0$$

contradict the transcendence of the numbers π and e^π , respectively. Moreover, it is well known that the set $\{\log p : p \in \mathbb{P}\}$ is linearly independent over \mathbb{Q} , therefore, the equality

$$k_1 h_1 \log p_1 + \dots + k_{r_1} h_1 \log p_{r_1} = 0$$

gives again a contradiction.

A proof of Theorem 6 is probabilistic, it is based on a limit theorem in the space $H^2(D)$ for periodic zeta-functions obtained in [13]. Moreover, the application of a limit theorem requires a certain estimate in the mean for the metric in $H^2(D)$.

2 The main equality

We start with recalling the metric in $H^2(D)$. For $g_1, g_2 \in H(D)$, let

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|},$$

where $\{K_l : l \in \mathbb{N}\} \subset D$ is a sequence of compact embedded set such that D is the union of the sets K_l , and each compact set of D lies in some K_l . Then, ρ is a metric which induces the topology of uniform convergence on compacta in the space $H(D)$. For $\underline{g}_l = (g_{l1}, g_{l2})$, $l = 1, 2$, let

$$\rho_2(\underline{g}_1, \underline{g}_2) = \max(\rho(g_{11}, g_{12}), \rho(g_{21}, g_{22})).$$

Then, ρ_2 is a metric in $H^2(D)$ inducing the product topology.

In this section, we consider the mean value of the distance between $\zeta(s + ik\underline{h}, \alpha; \mathbf{a}, \mathbf{b})$ and $\zeta_{u_N}(s + ik\underline{h}, \alpha; \mathbf{a}, \mathbf{b})$, where

$$\begin{aligned} \zeta(s + ik\underline{h}, \alpha; \mathbf{a}, \mathbf{b}) &= (\zeta(s + ikh_1; \mathbf{a}), \zeta(s + ikh, \alpha; \mathbf{b})), \\ \zeta_{u_N}(s + ik\underline{h}, \alpha; \mathbf{a}, \mathbf{b}) &= (\zeta_{u_N}(s + ikh_1; \mathbf{a}), \zeta_{u_N}(s + ikh, \alpha; \mathbf{b})) \end{aligned}$$

and $\underline{h} = (h_1, h_2)$. For this, we apply the following lemmas.

Lemma 1. *Suppose that $u_N \rightarrow \infty$ and $u_N \ll N^2$ as $N \rightarrow \infty$. Then, for every $h_1 > 0$,*

$$\lim_{N \rightarrow \infty} \frac{1}{N + 1} \sum_{k=0}^N \rho(\zeta(s + ikh_1; \mathbf{a}), \zeta_{u_N}(s + ikh; \mathbf{a})) = 0.$$

The lemma is Lemma 1 from [10].

Lemma 2. *For every fixed $\sigma > 1/2$, $h_2 > 0$ and $t \in \mathbb{R}$, the estimate*

$$\sum_{k=0}^N |\zeta(\sigma + ikh_2 + it, \alpha; \mathbf{b})|^2 \ll_{\sigma, \alpha, \mathbf{b}} N(1 + |t|)$$

is valid.

A proof of lemma is given in [13].

Lemma 3. *Under hypotheses of Lemma 1,*

$$\lim_{N \rightarrow \infty} \frac{1}{N + 1} \sum_{k=0}^N \rho(\zeta(s + ikh_2, \alpha; \mathbf{b}), \zeta_{u_N}(s + ikh_2, \alpha; \mathbf{b})) = 0.$$

Proof. In virtue of the definition of the metric ρ , it is sufficient to show that the equality

$$\lim_{N \rightarrow \infty} \frac{1}{N + 1} \sum_{k=0}^N \sup_{s \in K} |\zeta(s + ikh_2, \alpha; \mathbf{a}) - \zeta_{u_N}(s + ikh_2, \alpha; \mathbf{b})| = 0$$

is true for every compact set $K \subset D$. Using the Mellin formula

$$\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \Gamma(z)b^{-z} dz = e^{-b}, \tag{2.1}$$

where $\Gamma(z)$ denotes the Euler gamma-function, and $a, b > 0$, leads, for $\sigma > 1/2$, to the integral representation

$$\zeta_{u_N}(s, \alpha; \mathbf{b}) = \frac{1}{2\pi i} \int_{\theta-i\infty}^{\theta+i\infty} \zeta(s + z, \alpha; \mathbf{b})l_{u_N}(z) dz, \tag{2.2}$$

where θ comes from the definition of $v_{u_N}(m, \alpha)$, and

$$l_{u_N}(z) = \frac{1}{\theta} \Gamma\left(\frac{z}{\theta}\right) u_N^z.$$

Actually, in view of (2.1),

$$\begin{aligned} \frac{1}{2\pi i} \int_{\theta-i\infty}^{\theta+i\infty} \frac{1}{(m+\alpha)^z} \frac{1}{\theta} \Gamma\left(\frac{z}{\theta}\right) u_N^z dz &= \frac{1}{2\pi i} \int_{\theta-i\infty}^{\theta+i\infty} \Gamma\left(\frac{z}{\theta}\right) \left(\frac{m+\alpha}{u_N}\right)^{(-z/\theta)\theta} dz \\ &= \exp\left\{-((m+\alpha)/u_N)^\theta\right\}. \end{aligned}$$

Therefore, since $\theta + \sigma > 1$ for $\sigma > 1/2$, we have

$$\begin{aligned} \zeta_{u_N}(s, \alpha; \mathbf{b}) &= \frac{1}{2\pi i} \int_{\theta-i\infty}^{\theta+i\infty} \sum_{m=0}^{\infty} \frac{b_m v_{u_N}(m, \alpha)}{(m+\alpha)^{s+z}} l_{u_N}(z) dz \\ &= \frac{1}{2\pi i} \int_{\theta-i\infty}^{\theta+i\infty} \zeta(s+z, \alpha; \mathbf{b}) l_{u_N}(z) dz. \end{aligned}$$

Fix a compact set $K \subset D$. Then, there exists a number $0 < \delta < \frac{1}{6}$ such that $1/2 + 2\delta \leq \sigma \leq 1 - \delta$ for $s = \sigma + it \in K$. Let $\theta = 1/2 + \delta$ and $\theta_1 = 1/2 + \delta - \sigma$. Then, $-1/2 + 2\delta \leq \theta_1 \leq -\delta$. Therefore, the integrand of (2.2), in the strip $\theta_1 \leq \sigma \leq \theta$, has a simple pole at $z = 0$ and a possible simple pole at $z = 1 - s$. Hence, by the residue theorem, we find, for $s \in K$,

$$\zeta_{u_N}(s, \alpha; \mathbf{b}) - \zeta(s, \alpha; \mathbf{b}) = \frac{1}{2\pi i} \int_{\theta_1-i\infty}^{\theta_1+i\infty} \zeta(s+z, \alpha; \mathbf{b}) l_{u_N}(z) dz + R_N(s, \alpha; \mathbf{b}),$$

where

$$R_N(s, \alpha; \mathbf{b}) = \begin{cases} 0 & \text{if } r \stackrel{\text{def}}{=} \operatorname{Res}_{s=1} \zeta(s, \alpha; \mathbf{b}) = 0, \\ r l_{u_N}(1-s) & \text{otherwise.} \end{cases}$$

The latter equality, for $s = \sigma + it \in K$, gives

$$\begin{aligned} &\zeta_{u_N}(s + ikh_2, \alpha; \mathbf{b}) - \zeta(s + ikh_2, \alpha; \mathbf{b}) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \zeta\left(\frac{1}{2} + \delta + it + ikh_2 + i\tau, \alpha; \mathbf{b}\right) l_{u_N}\left(\frac{1}{2} + \delta - \sigma + i\tau\right) d\tau \\ &\quad + R_N(s + ikh_2, \alpha; \mathbf{b}) \\ &\ll \int_{-\infty}^{\infty} \left| \zeta\left(\frac{1}{2} + \delta + ikh_2 + i\tau, \alpha; \mathbf{b}\right) \right| \sup_{s \in K} |l_{u_N}\left(\frac{1}{2} + \delta - s + i\tau\right)| d\tau \\ &\quad + \sup_{s \in K} |R_N(s + ikh_2, \alpha; \mathbf{b})|. \end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{1}{N+1} \sum_{k=0}^N \sup_{s \in K} |\zeta(s + ikh_2, \alpha; \mathbf{b}) - \zeta_{u_N}(s + ikh_2, \alpha; \mathbf{b})| \\ &\ll \int_{-\infty}^{\infty} \left(\frac{1}{N+1} \sum_{k=0}^N |\zeta(1/2 + \delta + ikh_2 + i\tau, \alpha; \mathbf{b})| \right) \\ &\quad \times \sup_{s \in K} |l_{u_N}(1/2 + \delta - s + i\tau)| d\tau \\ &\quad + \frac{1}{N+1} \sum_{k=0}^N \sup_{s \in K} |R_N(s + ikh_2, \alpha; \mathbf{b})| \stackrel{\text{def}}{=} I_N + S_N. \end{aligned} \tag{2.3}$$

For estimating of the integral I_N , we apply Lemma 2. The Cauchy-Schwarz inequality and Lemma 2 yield

$$\begin{aligned} & \frac{1}{N+1} \sum_{k=0}^N |\zeta(1/2 + \delta + ikh_2 + i\tau, \alpha; \mathbf{b})| \\ & \ll \left(\frac{1}{N+1} \sum_{k=0}^N |\zeta(1/2 + \delta + ikh_2 + i\tau, \alpha; \mathbf{b})|^2 \right)^{1/2} \ll (1 + |\tau|)^{1/2}. \end{aligned} \tag{2.4}$$

By the definition of $l_{u_N}(s)$, using the classical bound for the gamma-function

$$\Gamma(\sigma + it) \ll \exp\{-c(1 + |t|)\}, \quad c > 0, \tag{2.5}$$

which is uniform in σ lying in every fixed interval $[\sigma_1, \sigma_2]$, we find that, for $s \in K$,

$$\begin{aligned} l_{u_N}(1/2 + \delta - s + i\tau) & \ll_{\delta} u_N^{1/2+\delta-\sigma} \exp\left\{-\frac{c}{\theta}(1 + |\tau - t|)\right\} \\ & \ll_{\delta, K} u_N^{-\delta} \exp\{-c_1|\tau|\}, \quad c_1 > 0, \end{aligned}$$

because of boundedness of t . This and (2.4) show that

$$I_N \ll_{\delta, h_2, \alpha, \mathbf{b}, K} u_N^{-\delta} \int_{-\infty}^{\infty} (1 + |\tau|)^{1/2} \exp\{-c_1|\tau|\} d\tau \ll_{\delta, h_2, \alpha, \mathbf{b}, K} u_N^{-\delta}. \tag{2.6}$$

By the definitions of $R_N(s, \alpha; \mathbf{b})$ and $l_{u_N}(s)$, and (2.5), for $s \in K$, we have

$$\begin{aligned} R_N(s + ikh_2, \alpha; \mathbf{b}) & \ll_{\delta, \alpha, \mathbf{b}} u_N^{1-\sigma} \exp\{-c_2(1 + kh_2|t|)\} \\ & \ll_{\delta, \alpha, \mathbf{b}, K} u_N^{1/2-2\delta} \exp\{-c_3(1 + kh_2)\}, \quad c_2, c_3 > 0. \end{aligned}$$

Therefore,

$$\begin{aligned} S_N & \ll_{\delta, K} u_N^{1/2-2\delta} \frac{1}{N} \sum_{k=0}^N \exp\{-c_3(1 + kh_2)\} \ll_{\delta, \alpha, \mathbf{b}, K} u_N^{1/2-2\delta} \\ & \times \left(\frac{\log N}{N} + \frac{1}{N} \sum_{k \geq \log N} \exp\{-c_3kh_2\} \right) \ll_{\delta, \alpha, \mathbf{b}, K, h_2} u_N^{1/2-2\delta} \frac{\log N}{N}. \end{aligned}$$

Thus, in view of (2.6),

$$I_N + S_N \ll_{\delta, h_2, \alpha, \mathbf{b}, K} u_N^{-\delta} + u_N^{1/2-2\delta} \frac{\log N}{N}.$$

Since $u_N \rightarrow \infty$ and $u_N \ll N^2$, this shows that

$$\lim_{N \rightarrow \infty} (I_N + S_N) = 0,$$

and, by (2.3), the lemma is proved. \square

Now, we state the main result on the closeness of $\zeta(s, \alpha; \mathbf{a}, \mathbf{b})$, $\zeta_{u_N}(s, \alpha; \mathbf{a}, \mathbf{b})$.

Lemma 4. *Suppose that $u_N \rightarrow \infty$ and $u_N \ll N^2$ as $N \rightarrow \infty$. Then, for every positive h_1 and h_2 ,*

$$\lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{k=0}^N \rho_2 \left(\underline{\zeta}(s + ikh, \alpha; \mathbf{a}, \mathbf{b}), \underline{\zeta}_{u_N}(s + ikh, \alpha; \mathbf{a}, \mathbf{b}) \right) = 0.$$

Proof. By the definition of the metric ρ_2 , it suffices to prove that

$$\lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{k=0}^N \rho \left(\zeta(s + ikh_1; \mathbf{a}), \zeta_{u_N}(s + ikh_1; \mathbf{a}) \right) = 0$$

and

$$\lim_{N \rightarrow \infty} \frac{1}{N+1} \sum_{k=0}^N \rho \left(\zeta(s + ikh_2, \alpha; \mathbf{b}), \zeta_{u_N}(s + ikh_2, \alpha; \mathbf{b}) \right) = 0.$$

Therefore, the lemma is consequence of Lemmas 1 and 3. \square

3 Limit theorems

Recall that $H(D)$ is the space of analytic on D functions. The proof of Theorem 6 relies on a discrete limit theorem for $\underline{\zeta}_{u_N}(s, \alpha; \mathbf{a}, \mathbf{b})$ in the space $H^2(D)$ on weakly convergent probability measures. For brevity, let $P_{\underline{\zeta}}$ be the distribution of the random element $\underline{\zeta}(s, \alpha, \omega_1, \omega_2; \mathbf{a}, \mathbf{b})$, i.e.,

$$P_{\underline{\zeta}}(A) = m_H \{ (\omega_1, \omega_2) \in \Omega : \underline{\zeta}(s, \alpha, \omega_1, \omega_2; \mathbf{a}, \mathbf{b}) \in A \}, \quad A \in \mathcal{B}(H^2(D)).$$

For $A \in \mathcal{B}(H^2(D))$, define

$$P_N(A) = \frac{1}{N+1} \# \{ 0 \leq k \leq N : \underline{\zeta}(s + ikh, \alpha; \mathbf{a}, \mathbf{b}) \in A \}.$$

Lemma 5. [13]. *Suppose that the set $L(\mathbb{P}; \alpha, h_1, h_2, \pi)$ is linearly independent over \mathbb{Q} . Then, P_N converges weakly to $P_{\underline{\zeta}}$ as $N \rightarrow \infty$.*

Lemmas 4 and 5 lead to a limit theorem for $\zeta_{u_N}(s, \alpha; \mathbf{a}, \mathbf{b})$. Let, for $A \in \mathcal{B}(H^2(D))$,

$$P_{N,u_N}(A) = \frac{1}{N+1} \# \left\{ 0 \leq k \leq N : \underline{\zeta}_{u_N}(s + ikh, \alpha; \mathbf{a}, \mathbf{b}) \in A \right\}.$$

Theorem 7. *Suppose that the set $L(\mathbb{P}; \alpha, h_1, h_2, \pi)$ is linearly independent over \mathbb{Q} , and $u_N \rightarrow \infty$ and $u_N \ll N^2$ as $N \rightarrow \infty$. Then, P_{N,u_N} converges weakly to $P_{\underline{\zeta}}$ as $N \rightarrow \infty$.*

Proof. Let ξ_N be a random variable defined on a certain probability space $(\widehat{\Omega}, \mathcal{B}(\widehat{\Omega}), \mu)$ and having the distribution $\mu\{\xi_N=k\}=1/(N+1)$, $k=0, 1, \dots, N$. We will use the equivalent of weak convergence of probability measures in

terms of closed sets, namely, if P and P_n , $n \in \mathbb{N}$, are probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$, then P_n , as $n \rightarrow \infty$, converges weakly to P if and only if

$$\limsup_{n \rightarrow \infty} P_n(F) \leq P(F)$$

for every closed set $F \subset \mathbb{X}$. Fix a closed set $F \subset H^2(D)$, $\varepsilon > 0$, and define the set

$$F_\varepsilon = \left\{ \underline{g} \in H^2(D) : \inf_{\widehat{g} \in F} \{ \rho_2(\underline{g}, \widehat{g}) \leq \varepsilon \} \right\}.$$

Then, the set F_ε is closed as well. Define two $H^2(D)$ -valued random elements

$$\underline{X}_N = \underline{X}_N(s) = \zeta(s + i\xi_N \underline{h}, \alpha; \mathbf{a}, \mathbf{b}), \quad \underline{Y}_N = \underline{Y}_N(s) = \zeta_{u_N}(s + i\xi_N \underline{h}, \alpha; \mathbf{a}, \mathbf{b}).$$

By the definition of the random variable ξ_N , the random elements \underline{X}_N and \underline{Y}_N have the distributions P_N and P_{N,u_N} , respectively. Moreover,

$$\{ \underline{Y}_N \in F_\varepsilon \} \subset \{ \underline{X}_N \in F \} \cup \{ \rho_2(\underline{X}_N, \underline{Y}_N) \geq \varepsilon \}.$$

Hence,

$$\begin{aligned} \mu(F_\varepsilon) &\leq \mu(F) + \mu\{ \rho_2(\underline{X}_N, \underline{Y}_N) \geq \varepsilon \}, \\ P_{N,u_N}(F_\varepsilon) &\leq P_N(F) + \mu\{ \rho_2(\underline{X}_N, \underline{Y}_N) \geq \varepsilon \}. \end{aligned} \tag{3.1}$$

By Lemma 5 and equivalent of weak convergence in terms of closed sets,

$$\limsup_{N \rightarrow \infty} P_N(F) \leq P_\zeta(F). \tag{3.2}$$

Moreover, Lemma 4 implies that

$$\begin{aligned} \limsup_{N \rightarrow \infty} \mu\{ \rho_2(\underline{X}_N, \underline{Y}_N) \geq \varepsilon \} &= \limsup_{N \rightarrow \infty} \frac{1}{N+1} \# \left\{ 0 \leq k \leq N : \right. \\ &\quad \left. \rho_2 \left(\zeta(s + ik\underline{h}, \alpha; \mathbf{a}, \mathbf{b}), \zeta_{u_N}(s + ik\underline{h}, \alpha; \mathbf{a}, \mathbf{b}) \right) \geq \varepsilon \right\} \\ &\leq \limsup_{N \rightarrow \infty} \frac{1}{\varepsilon(N+1)} \sum_{k=0}^N \rho_2 \left(\zeta(s + ik\underline{h}, \alpha; \mathbf{a}, \mathbf{b}), \zeta_{u_N}(s + ik\underline{h}, \alpha; \mathbf{a}, \mathbf{b}) \right) = 0. \end{aligned}$$

Thus, in view of (3.1) and (3.2),

$$\limsup_{N \rightarrow \infty} P_{N,u_N}(F_\varepsilon) \leq P_\zeta(F).$$

Letting $\varepsilon \rightarrow +0$, we obtain $\limsup_{N \rightarrow \infty} P_{N,u_N}(F) \leq P_\zeta(F)$, and this together with equivalent of weak convergence in terms of closed sets proves the theorem. \square

Theorem 7 implies the weak convergence for the corresponding probability measures in the space \mathbb{R}^2 . For $A \in \mathcal{B}(\mathbb{R}^2)$, define

$$Q_{N,u_N}(A) = \frac{1}{N+1} \# \left\{ 0 \leq k \leq N : \left(\sup_{s \in K_1} |\zeta_{u_N}(s + ikh_1; \mathbf{a}) - f_1(s)|, \sup_{s \in K_2} |\zeta_{u_N}(s + ikh_2; \alpha; \mathbf{b}) - f_2(s)| \right) \in A \right\}.$$

Corollary 1. Suppose that the set $L(\mathbb{P}; \alpha, h_1, h_2, \pi)$ is linearly independent over \mathbb{Q} , and $u_N \rightarrow \infty$ and $u_N \ll N^2$ as $N \rightarrow \infty$. Let K_1, K_2 and $f_1(s), f_2(s)$ be as in Theorem 6. Then Q_{N,u_N} converges weakly to the measure

$$m_H \left\{ (\omega_1, \omega_2) \in \Omega : \left(\sup_{s \in K_1} |\zeta(s, \omega_1; \mathbf{a}) - f_1(s)|, \sup_{s \in K_2} |\zeta(s, \alpha, \omega_2; \mathbf{b}) - f_2(s)| \right) \in A \right\}, \quad A \in \mathcal{B}(\mathbb{R}^2), \quad (3.3)$$

as $N \rightarrow \infty$.

Proof. Consider the mapping $u : H^2(D) \rightarrow \mathbb{R}^2$ defined by

$$u(g_1, g_2) = \left(\sup_{s \in K_1} |g_1(s) - f_1(s)|, \sup_{s \in K_2} |g_2(s) - f_2(s)| \right), \quad g_1, g_2 \in H(D).$$

Then, the mapping u is continuous. Actually, suppose that $(g_{N1}, g_{N2}) \rightarrow (g_1, g_2)$ as $N \rightarrow \infty$ in the space $H^2(D)$. Since the convergence in $H(D)$ is uniform on compact sets, we have

$$\lim_{N \rightarrow \infty} \sup_{s \in K_j} |g_{Nj}(s) - g_j(s)| = 0, \quad j = 1, 2.$$

Therefore, using the triangle inequality, we obtain that

$$\left(\sup_{s \in K_j} |g_{Nj}(s) - f_j(s)| - \sup_{s \in K_j} |g_j(s) - f_j(s)| \right) \leq \sup_{s \in K_j} |g_{Nj} - g_j(s)| \xrightarrow{N \rightarrow \infty} 0,$$

for $j = 1, 2$. This proves that

$$\lim_{N \rightarrow \infty} u(g_{N1}, g_{N2}) = u(g_1, g_2),$$

i.e., u is continuous.

By the definitions of u, P_{N,u_N} and Q_{N,u_N} , for $A \in \mathcal{B}(\mathbb{R}^2)$, we have

$$\begin{aligned} Q_{N,u_N}(A) &= \frac{1}{N+1} \# \left\{ 0 \leq k \leq N : \zeta_{u_N}(s + ikh, \alpha; \mathbf{a}, \mathbf{b}) \in u^{-1}A \right\} \\ &= P_{N,u_N}(u^{-1}A) = P_{N,u_N} u^{-1}(A), \end{aligned}$$

i.e., $Q_{N,u_N} = P_{N,u_N} u^{-1}$. Therefore, the continuity of u , Theorem 7 and the preservation of weak convergence under continuous mappings, show that Q_{N,u_N} converges weakly to $P_{\zeta} u^{-1}$, i.e., to the measure (3.3) as $N \rightarrow \infty$. \square

4 Proof of Theorem 6

Theorem 6 follows from Corollary 1, weak convergence in \mathbb{R}^2 , support of the measure P_{ζ} , and the Mergelyan theorem on approximation of analytic functions by polynomials. We recall that the support of the measure P_{ζ} is a minimal closed set S_{ζ} such that $P_{\zeta}(S_{\zeta}) = 1$. The set S_{ζ} consists of all $\underline{g} \in H^2(D)$, for every open neighbourhood G of which the inequality $P_{\zeta}(G) > 0$ is true.

Define $S(\mathbf{a}) = \{g \in H(D) : \text{either } g(s) \neq 0, \text{ or } g(s) \equiv 0\}$ and $S(\mathbf{b}) = H(D)$.

Lemma 6. [13]. *Suppose that the sequence \mathbf{a} is multiplicative, and the set $L(\mathbb{P}; \alpha, h_1, h_2, \pi)$ is linearly independent over \mathbb{Q} . Then the support of the measure P_ζ is the set $S(\mathbf{a}) \times S(\mathbf{b})$.*

The next lemma is a version of the Mergelyan theorem on approximation of analytic functions by polynomials.

Lemma 7. *Suppose that $K \subset \mathbb{C}$ is a compact set with connected complement, and $g(s)$ is a continuous function on K and analytic in the interior of K . Then, for every $\varepsilon > 0$, there exists a polynomial $p_\varepsilon(s)$ such that*

$$\sup_{s \in K} |g(s) - p_\varepsilon(s)| < \varepsilon.$$

Proof. (Proof of Theorem 6). It is well known that the weak convergence of probability measures is equivalent to that of the corresponding distribution functions. Recall that the distribution function $D_n(x_1, x_2)$, as $n \rightarrow \infty$, converges weakly to the distribution function $D(x_1, x_2)$ if

$$\lim_{n \rightarrow \infty} D_n(x_1, x_2) = D(x_1, x_2)$$

for $(x_1, x_2) \in \mathbb{R}^2$ such that x_1 and x_2 are continuity points of the functions $D(x_1, +\infty)$ and $D(+\infty, x_2)$, respectively.

Define the distribution functions

$$F_N(\varepsilon_1, \varepsilon_2) = \frac{1}{N+1} \# \left\{ 0 \leq k \leq N : \sup_{s \in K_1} |\zeta_{u_N}(s + ikh_1; \mathbf{a}) - f_1(s)| < \varepsilon_1, \right. \\ \left. \sup_{s \in K_2} |\zeta_{u_N}(s + ikh_2, \alpha; \mathbf{b}) - f_2(s)| < \varepsilon_2 \right\}$$

$$F(\varepsilon_1, \varepsilon_2) = m_H \left\{ (\omega_1, \omega_2) \in \Omega : \sup_{s \in K_1} |\zeta(s, \omega_1; \mathbf{a}) - f_1(s)| < \varepsilon_1, \right. \\ \left. \sup_{s \in K_2} |\zeta(s, \alpha, \omega_2; \mathbf{b}) - f_2(s)| < \varepsilon_2 \right\}.$$

Then, by Corollary 1, we have that $F_N(\varepsilon_1, \varepsilon_2)$ converges weakly to $F(\varepsilon_1, \varepsilon_2)$ as $N \rightarrow \infty$. Thus,

$$\lim_{N \rightarrow \infty} F_N(\varepsilon_1, \varepsilon_2) = F(\varepsilon_1, \varepsilon_2), \tag{4.1}$$

where ε_1 and ε_2 are continuity points of the distribution functions $F(\varepsilon_1, +\infty)$ and $F(+\infty, \varepsilon_2)$, respectively. Since the distribution functions $F(\varepsilon_1, +\infty)$ and $F(+\infty, \varepsilon_2)$ have at most countable sets of discontinuity points, the equality (4.1) is true for all but at most countably many $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$.

It remains to show that $F(\varepsilon_1, \varepsilon_2) > 0$. For this, we will apply Lemma 7.

By Lemma 7, there exist polynomials $p_1(s)$ and $p_2(s)$ such that

$$\sup_{s \in K_1} |f_1(s) - e^{p_1(s)}| < \frac{\varepsilon_1}{2}, \quad \sup_{s \in K_2} |f_2(s) - p_2(s)| < \frac{\varepsilon_2}{2}. \tag{4.2}$$

Define the set

$$G_{\varepsilon_1, \varepsilon_2} = \left\{ (g_1, g_2) \in H^2(D) : \sup_{s \in K_1} |g_1(s) - e^{p_1(s)}| < \frac{\varepsilon_1}{2}, \right. \\ \left. \sup_{s \in K_2} |f_2(s) - p_2(s)| < \frac{\varepsilon_2}{2} \right\}.$$

The point $(e^{p_1(s)}, p_2(s))$, in view of Lemma 6, is an element of the support of the measure P_{ζ} . Therefore,

$$P_{\zeta}(G_{\varepsilon_1, \varepsilon_2}) > 0. \tag{4.3}$$

Define one more set

$$\mathcal{G}_{\varepsilon_1, \varepsilon_2} = \left\{ (g_1, g_2) \in H^2(D) : \sup_{s \in K_1} |g_1(s) - f_1(s)| < \varepsilon_1, \right. \\ \left. \sup_{s \in K_2} |g_2(s) - f_2(s)| < \varepsilon_2 \right\}.$$

In view of equalities (4.2), we have the inclusion $G_{\varepsilon_1, \varepsilon_2} \subset \mathcal{G}_{\varepsilon_1, \varepsilon_2}$. Therefore, by (4.3),

$$P_{\zeta}(\mathcal{G}_{\varepsilon_1, \varepsilon_2}) \geq P_{\zeta}(G_{\varepsilon_1, \varepsilon_2}) > 0.$$

This and the definitions of P_{ζ} and $\mathcal{G}_{\varepsilon_1, \varepsilon_2}$ gives the positivity of $F(\varepsilon_1, \varepsilon_2)$. The theorem is proved. \square

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