



ON A SEQUENCE OF RATIONAL NUMBERS WITH UNUSUAL DIVISIBILITY BY A POWER OF 2

ARTŪRAS DUBICKAS

Received 06 June, 2022

Abstract. In this note we consider the sequence of rational numbers $b_n = \sum_{k=1}^n 2^k/k$. We show that the power of 2 in the expansion of b_n is unusually large, at least $n + 1 - \log_2(n + 1)$, and that this bound is best possible. The sequence $b_n, n = 1, 2, 3, \dots$, is related to the sequence A0031449 in the On-Line Encyclopedia of Integer Sequences.

2010 *Mathematics Subject Classification:* 11B37; 11B83

Keywords: special sequence, power of 2, combinatorial identities

1. INTRODUCTION

In [6] Farhi considered the following sequence $a_1 = 1$,

$$a_n = \frac{na_{n-1}}{2} + (n-1)! \quad (1.1)$$

for $n = 2, 3, \dots$. Set also

$$b_n = \sum_{k=1}^n \frac{2^k}{k} \quad (1.2)$$

for $n \in \mathbb{N}$. Then, by $b_n - b_{n-1} = 2^n/n$ and (1.1), we obtain

$$a_n = \frac{n!}{2^n} b_n \quad (1.3)$$

for each $n \in \mathbb{N}$.

In [6], expressing a_n in terms of Genocchi numbers and Stirling numbers of the first kind, Farhi showed that

$$a_n \in \mathbb{N} \quad (1.4)$$

for each $n \in \mathbb{N}$. This, according to the definition of a_n in (1.1), is nontrivial and in some sense reminds the surprising integrality conditions of so-called Somos sequences (see [14] and also some subsequent work in [8, 11, 17, 18]). The fractional parts of the sequence $\frac{2^n}{n}, n = 1, 2, 3, \dots$, were considered in [4, 5].

Of course, there are several alternative ways to prove (1.4) which are simpler than that in [6]. This was observed by Farhi in a subsequent paper [7]. For example, by the identity

$$\sum_{k=0}^n \frac{k!(n-k)!}{n!} = \sum_{k=0}^n \frac{1}{\binom{n}{k}} = \frac{n+1}{2^n} \sum_{k=0}^n \frac{2^k}{k+1} \quad (1.5)$$

(see [12, 15]), using (1.2) and (1.3) we find that

$$\sum_{k=0}^n k!(n-k)! = \frac{(n+1)!}{2^n} \sum_{k=0}^n \frac{2^k}{k+1} = \frac{(n+1)!}{2^{n+1}} b_{n+1} = a_{n+1},$$

which implies (1.4). In fact, $\sum_{k=0}^n k!(n-k)!$, $n = 1, 2, 3, \dots$, is the sequence A0031449 in [13].

For a prime number p and a positive integer u by $v_p(u)$ we denote the largest nonnegative integer k for which p^k divides u . Likewise, for a rational $r = u/v$, where $u, v \in \mathbb{N}$ are relatively prime integers, we set $v_p(r) = v_p(u) - v_p(v)$.

With this notation in [6, Corollary 2.5] it was also shown that

$$v_2\left(\sum_{k=1}^n \frac{2^k}{k}\right) \geq s_2(n), \quad (1.6)$$

where $s_2(n)$ is the sum of digits of n in base 2. This was improved in [7, Theorem 2.5], where it was shown that

$$v_2\left(\sum_{k=1}^n \frac{2^k}{k}\right) \geq n - \lfloor \log_2 n \rfloor \quad (1.7)$$

for each $n \in \mathbb{N}$.

Now, we will refine the estimates (1.6), (1.7) and obtain a sharp bound.

Theorem 1. *For each $n \in \mathbb{N}$ we have*

$$v_2\left(\sum_{k=1}^n \frac{2^k}{k}\right) \geq n + 1 - \log_2(n+1), \quad (1.8)$$

with equality if and only if $n = 2^k - 1$ for $k \in \mathbb{N}$.

Note that

$$n + 1 - \log_2(n+1) \geq n - \lfloor \log_2 n \rfloor. \quad (1.9)$$

Indeed, choose a unique integer $k \geq 0$ satisfying $2^k \leq n < 2^{k+1}$. Then, $1 + \lfloor \log_2 n \rfloor = k + 1$ and $\log_2(n+1) \leq \log_2(2^{k+1}) = k + 1$, which proves (1.9).

From the above proof of (1.9) we see that the right hand sides of (1.7) and (1.8) are equal only if n has the form $n = 2^k - 1$. We will derive (1.8) from the inequality (2.3) below, which is stronger than (1.8) for many $n \in \mathbb{N}$ that are not of the form $2^k - 1$. The nontrivial part is to show that for $n = 2^k - 1$ one has equality in (1.8). (The proof of (1.7) in [7] is entirely different: it uses (1.5) and some other identity.) The proof

of Theorem 1 is self-contained except that we need any version of the fact that the sequence $v_2(b_n)$, $n = 1, 2, 3, \dots$, is unbounded as $n \rightarrow \infty$.

Let D_n be the least common multiple of $1, 2, 3, \dots, n$. By (1.2), it is clear that $D_n b_n \in \mathbb{N}$ for each $n \in \mathbb{N}$. From Theorem 1 we will derive the following corollary which strengthens (1.4) (since (1.4) holds even if the factor $n!$ in (1.3) is replaced by D_n).

Corollary 1. *For each $n \in \mathbb{N}$ we have*

$$\frac{D_n b_n}{2^n} = D_n \sum_{k=1}^n \frac{2^{k-n}}{k} \in \mathbb{N}. \tag{1.10}$$

Finally, we remark that in [15] Sury proved not only (1.5) but also the identity

$$\sum_{k=1}^n \frac{2^k}{k} = 2 \left(\binom{n}{1} + \frac{1}{3} \binom{n}{3} + \frac{1}{5} \binom{n}{5} + \frac{1}{7} \binom{n}{7} + \dots \right). \tag{1.11}$$

See also [1–3, 9, 10, 16] for some related identities. For example, in [10] Meštrović showed that

$$\sum_{k=1}^n \frac{2^k - 1}{k} = \sum_{k=1}^n \frac{1}{k} \binom{n}{k}. \tag{1.12}$$

We conclude with a simple generalization of (1.11) and (1.12).

Theorem 2. *For each $n \in \mathbb{N}$ and each $z \in \mathbb{C}$ we have*

$$\sum_{k=1}^n \frac{(1+z)^k - (1-z)^k}{k} = 2 \left(z \binom{n}{1} + \frac{z^3}{3} \binom{n}{3} + \frac{z^5}{5} \binom{n}{5} + \frac{z^7}{7} \binom{n}{7} + \dots \right)$$

and

$$\sum_{k=1}^n \frac{(1+z)^k - 1}{k} = \sum_{k=1}^n \frac{z^k}{k} \binom{n}{k}. \tag{1.13}$$

In particular, we give a very simple proof of (1.13), which shows that the identities (1.11), (1.12) and [10, Corollary 1.2] are not so mysterious as it may appear from their proofs in [10]. They come by inserting $z = 1$ into the natural identities of Theorem 2.

2. PROOFS

Proof of Theorem 1. For $n \in \mathbb{N}$ put

$$d_n = v_2(b_n),$$

where b_n is defined in (1.2). Then, $b_n = 2^{d_n} u_n / v_n$, where u_n and v_n are odd coprime positive integers. Note that $d_1 = 1, d_2 = 2, \dots$, etc. We have

$$b_{n+1} - b_n = \frac{2^{n+1}}{n+1} = \frac{2^{n+1-v_2(n+1)}}{(n+1)2^{-v_2(n+1)}},$$

where $n + 1 - v_2(n + 1) > 0$. From

$$b_{n+1} = \frac{2^{d_n} u_n}{v_n} + \frac{2^{n+1-v_2(n+1)}}{(n+1)2^{-v_2(n+1)}}$$

and the fact that the integer $(n + 1)2^{-v_2(n+1)}$ is odd, we find that

$$d_{n+1} = \min(d_n, n + 1 - v_2(n + 1)) \quad (2.1)$$

if

$$d_n \neq n + 1 - v_2(n + 1), \quad (2.2)$$

and

$$d_{n+1} > d_n$$

if

$$d_n = n + 1 - v_2(n + 1).$$

We claim that for each $n \in \mathbb{N}$

$$d_n \geq \min_{k \geq n} (k + 1 - v_2(k + 1)). \quad (2.3)$$

Indeed, if for some $n \in \mathbb{N}$ the inequality opposite to (2.3) holds then d_n is less than $k + 1 - v_2(k + 1)$ for $k = n, n + 1, n + 2, \dots$. Then, by (2.1) and (2.2), we should have $d_n = d_{n+1} = d_{n+2} = \dots$. Thus, the sequence $d_k, k = 1, 2, 3, \dots$, is bounded, which is impossible by (1.6) or (1.7). This proves (2.3).

Next, since $u \geq 2^{v_2(u)}$, for any $u \in \mathbb{N}$ we have $v_2(u) \leq \log_2(u)$. Hence, from (2.3) we get $d_n \geq \min_{k \geq n} (k + 1 - \log_2(k + 1))$. The function $f(x) = x + 1 - \log_2(x + 1)$ is increasing for $x \geq 1$. So, for any $n \in \mathbb{N}$, the smallest value of the function $f(x)$ in the set $x \in \{n, n + 1, n + 2, \dots\}$ is attained at $x = n$. Thus, $d_n \geq n + 1 - \log_2(n + 1)$, which is (1.8).

Further, equality in (1.8) can only hold if $\log_2(n + 1)$ is an integer. For $n \in \mathbb{N}$ this happens for $n = 2^k - 1$, where $k \in \mathbb{N}$, only. So the values $n = 2^k - 1, k = 1, 2, 3, \dots$, are the only values for which equality in (1.8) can possibly be attained. We will show that it is always attained, namely,

$$d_{2^k-1} = v_2(b_{2^k-1}) = 2^k - k \quad (2.4)$$

for every $k \in \mathbb{N}$.

Fix any $k \in \mathbb{N}$. For a contradiction assume that $d_{2^k-1} \neq 2^k - k$, so that (2.2) is true for $n = 2^k - 1$. Then, by (2.1), we must have

$$d_{2^k} = \min(d_{2^k-1}, 2^k - k) \leq 2^k - k.$$

However, by (1.8) and $2^k + 1 < 2^{k+1}$, it follows that

$$d_{2^k} \geq 2^k + 1 - \log_2(2^k + 1) > 2^k + 1 - (k + 1) = 2^k - k,$$

which contradicts to the previous inequality. This rules out the possibility $d_{2^k-1} \neq 2^k - k$ and so proves (2.4). \square

Proof of Corollary 1. Let L_n be the least common multiple of all odd integers between 1 and n . By (1.2), it is clear that $L_n b_n \in \mathbb{N}$ for every $n \in \mathbb{N}$. Furthermore, since L_n is an odd integer, this implies $L_n b_n 2^{-v_2(b_n)} \in \mathbb{N}$. Next, in view of $D_n = L_n 2^{\lfloor \log_2 n \rfloor}$ we obtain

$$D_n b_n 2^{-v_2(b_n) - \lfloor \log_2 n \rfloor} \in \mathbb{N},$$

which implies (1.10) provided that $v_2(b_n) + \lfloor \log_2 n \rfloor \geq n$. However, by Theorem 1, we already know that $v_2(b_n) \geq n + 1 - \log_2(n + 1)$. This completes the proof of the corollary by (1.9). (Of course, as observed by the referee, $v_2(b_n) + \lfloor \log_2 n \rfloor \geq n$ already holds by Farhi's inequality (1.7).) \square

Proof of Theorem 2. In order to prove (1.13) we fix $n \in \mathbb{N}$ and set

$$f(z) = \sum_{k=1}^n \frac{(1+z)^k - 1}{k} - \sum_{k=1}^n \frac{z^k}{k} \binom{n}{k}.$$

Then,

$$f'(z) = \sum_{k=1}^n (1+z)^{k-1} - \sum_{k=1}^n z^{k-1} \binom{n}{k} = \frac{(1+z)^n - 1}{(1+z) - 1} - \frac{(1+z)^n - 1}{z} = 0$$

for $z \neq 0$. Inserting $z = 0$ we obtain $f'(0) = n - \binom{n}{1} = 0$ as well. Hence, $f'(z) = 0$ for each $z \in \mathbb{C}$, which implies that $f(z)$ is a constant. From $f(0) = 0$ we conclude that $f(z) = 0$ for each $z \in \mathbb{C}$. This proves (1.13).

Clearly, (1.13) also implies the first identity of this theorem by subtracting (1.13) with z replaced by $-z$ from (1.13) with z itself. \square

Acknowledgements. I thank the referee for some useful observations.

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Author’s address

Artūras Dubickas

Institute of Mathematics, Faculty of Mathematics and Informatics, Vilnius University, Naugarduko 24, LT-03225 Vilnius, Lithuania

E-mail address: arturas.dubickas@mif.vu.lt