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A Joint Limit Theorem for Epstein and Hurwitz Zeta-Functions

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Abstract: In the paper, we prove a joint limit theorem in terms of the weak convergence of probability measures on $\mathbb{C}^2$ defined by means of the Epstein $\zeta(s;Q)$ and Hurwitz $\zeta(s,\alpha)$ zeta-functions. The limit measure in the theorem is explicitly given. For this, some restrictions on the matrix Q and the parameter α are required. The theorem obtained extends and generalizes the Bohr-Jessen results characterising the asymptotic behaviour of the Riemann zeta-function.

Keywords: Dirichlet L-function; Epstein zeta-function; Hurwitz zeta-function; limit theorem; Haar probability measure; weak convergence

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1. Introduction

Let $\mathbb{P}$, $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, as usual, denote the sets of primes, positive integers, non-negative integers, integers, real, and complex numbers, respectively, $s=\sigma+it$ a complex variable, $n\in\mathbb{N}$, Q a positive-defined $n\times n$ matrix, and $Q[\underline{x}]=\underline{x}^{\mathrm{T}}Q\underline{x}$ for $\underline{x}\in\mathbb{Z}^n$. In [1], Epstein considered a problem to find a zeta-function as general as possible and having a functional equation of the Riemann type. For $\sigma>\frac{n}{2}$, he defined the function

$$\zeta(s;Q)=\sum_{\underline{x}\in\mathbb{Z}^n\setminus\{\underline{0}\}}(Q[\underline{x}])^{-s}.$$

Now, this function is called the Epstein zeta-function. It is analytically continuable to the whole complex plane, except for a simple pole at the point $s=\frac{n}{2}$ with residue

$$\frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})\sqrt{\det Q}},$$

where $\Gamma(s)$ is the Euler gamma-function. Epstein also proved that the function $\zeta(s;Q)$ satisfies the functional equation

$$\pi^{-s}\Gamma(s)\zeta(s;Q)=\sqrt{\det Q}\,\pi^{s-\frac{n}{2}}\Gamma\Big(\frac{n}{2}-s\Big)\zeta\Big(\frac{n}{2}-s;Q\Big)$$

for all $s\in\mathbb{C}$.

It turned out that the Epstein zeta-function is an important object in number theory, with a series of practical applications, for example, in crystallography [2] and mathematical physics, more precisely, in quantum field theory and the Wheeler–DeWitt equation [3,4].

The value distribution of $\zeta(s;Q)$, like that of other zeta-functions, is complicated, and has been studied by many authors including Hecke [5], Selberg [6], Iwaniec [7], Bateman [8], Fomenko [9], and Pańkowski and Nakamura [10]. In Refs. [11,12], the characterisation of the asymptotic behaviour of $\zeta(s;Q)$ was given in terms of probabilistic limit theorems. The latter approach for the Riemann zeta-function

$$\zeta(s)=\sum_{m=1}^{\infty}\frac{1}{m^s},\quad \sigma>1,$$

was proposed by Bohr in [13], and realised in [14,15]. Denote by $\mathcal{B}(\mathbb{X})$ the Borel σ-field of the space $\mathbb{X}$, and by meas$A$ the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. For $A \in \mathcal{B}(\mathbb{C})$, define

$$P^Q_{T,\sigma}(A) = \frac{1}{T}\text{meas}\{t \in [0,T] : \zeta(\sigma+it;Q) \in A\}.$$

Under the restrictions that $Q[\underline{x}] \in \mathbb{Z}$ for all $\underline{x} \in \mathbb{Z}^n \setminus \{0\}$, and $n \geq 4$ is even, it was shown [11] that $P^Q_{T,\sigma}$, for $\sigma > \frac{n-1}{2}$, converges weakly to an explicitly given probability measure P^Q_σ as $T \to \infty$. The discrete version of the latter theorem was given in [12].

The above restrictions on the matrix Q and [9] imply the decomposition

$$\zeta(s;Q) = \zeta(s;E_Q) + \zeta(s;F_Q) \tag{1}$$

with the zeta-function $\zeta(s;E_Q)$ of a certain Eisenstein series, and the zeta-function $\zeta(s;F_Q)$ of a certain cusp form.

Let χ be a Dirichlet character modulo q, and

$$L(s,\chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}, \quad \sigma > 1,$$

the corresponding Dirichlet L-function having analytic continuation to the whole complex plane if χ is a non-principal character, and except for a simple pole at the point $s=1$ if χ is the principal character. Then, (1) and [5,7] lead to the representation

$$\zeta(s;Q) = \sum_{k=1}^{K}\sum_{l=1}^{L} \frac{a_{kl}}{k^s l^s} L(s,\chi_k) L\left(s - \frac{n}{2} + 1, \hat{\chi}_l\right) + \sum_{m=1}^{\infty} \frac{b_Q(m)}{m^s}, \tag{2}$$

where χ_k and $\hat{\chi}_l$ are Dirichlet characters, $a_{kl} \in \mathbb{C}$, $k,l \in \mathbb{N}$, and the series with coefficients $b_Q(m)$ converges absolutely in the half-plane $\sigma > \frac{n-1}{2}$. Thus, the investigation of the function $\zeta(s;Q)$ reduces to that of Dirichlet L-functions which, for $\sigma > 1$, have the Euler product

$$L(s,\chi) = \prod_{p \in \mathbb{P}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

Our aim is to describe in probabilistic terms the joint asymptotic behaviour of the function $\zeta(s;Q)$ and a zeta-function having no Euler product over primes. For this, the most suitable function is the classical Hurwitz zeta-function. Let $0 < \alpha \leq 1$ be a fixed parameter. The Hurwitz zeta-function $\zeta(s,\alpha)$ was introduced in [16], and is defined, for $\sigma > 1$, by

$$\zeta(s,\alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s}.$$

Moreover, $\zeta(s,\alpha)$ has analytic continuation to the whole complex plane, except for a simple pole at the point $s=1$ with residue 1, $\zeta(s,1) = \zeta(s)$, and

$$\zeta\left(s, \frac{1}{2}\right) = \zeta(s)(2^s - 1).$$

The analytic properties of the function $\zeta(s,\alpha)$ depend on the arithmetic nature of the parameter α. Some probabilistic limit theorems for the function $\zeta(s,\alpha)$ can be found, for example, in [17].

The statement of a joint limit theorem for the functions $\zeta(s;Q)$ and $\zeta(s,\alpha)$ requires some notation. Denote two tori

$$\Omega_1 = \prod_{p \in \mathbb{P}} \{s \in \mathbb{C} : |s| = 1\} \quad \text{and} \quad \Omega_2 = \prod_{m \in \mathbb{N}_0} \{s \in \mathbb{C} : |s| = 1\}.$$

With the product topology and pointwise multiplication, Ω_1 and Ω_2 are compact topological Abelian groups. Therefore,

$$\Omega = \Omega_1 \times \Omega_2$$

again is a compact topological group. Hence, on $(\Omega, \mathcal{B}(\Omega))$, the Haar probability measure m_H exists, and we have the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote the elements of Ω by $\omega = (\omega_1, \omega_2)$, where $\omega_1 = (\omega_1(p) : p \in \mathbb{P}) \in \Omega_1$ and $\omega_2 = (\omega_2(m) : m \in \mathbb{N}_0) \in \Omega_2$, and, on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ define, for $\sigma_1 > \frac{n-1}{2}$ and $\sigma_2 > \frac{1}{2}$, the $\mathbb{C}^2$-valued random element

$$\underline{\zeta}(\underline{\sigma}, \omega, \alpha; Q) = (\zeta(\sigma_1, \omega_1; Q), \zeta(\sigma_2, \omega_2, \alpha)),$$

where $\underline{\sigma} = (\sigma_1, \sigma_2)$,

$$\begin{aligned} \zeta(\sigma_1, \omega_1; Q) \quad &= \quad \sum_{k=1}^{K} \sum_{l=1}^{L} \frac{a_{kl}\omega_1(k)\omega_1(l)}{k^{\sigma_1} l^{\sigma_1}} L(\sigma_1, \omega_1, \chi_k) L\Big(\sigma_1 - \frac{n}{2} + 1, \omega_1, \hat{\chi}_l\Big) \\ &+ \quad \sum_{m=1}^{\infty} \frac{b_Q(m)\omega_1(m)}{m^{\sigma_1}}, \end{aligned}$$

with

$$L(\sigma_1, \omega_1, \chi_k) = \prod_{p \in \mathbb{P}} \left(1 - \frac{\chi_k(p)\omega_1(p)}{p^{\sigma_1}}\right)^{-1},$$

$$L\Big(\sigma_1 - \frac{n}{2} + 1, \omega_1, \hat{\chi}_l\Big) = \prod_{p \in \mathbb{P}} \left(1 - \frac{\hat{\chi}_l(p)\omega_1(p)}{p^{\sigma_1 - \frac{n}{2} + 1}}\right)^{-1},$$

$$\omega_1(m) = \prod_{\substack{p^r | m \\ p^{r+1} \nmid m}} \omega_1^r(p), \quad m \in \mathbb{N},$$

and

$$\zeta(\sigma_2, \alpha, \omega_2) = \sum_{m=0}^{\infty} \frac{\omega_2(m)}{(m+\alpha)^{\sigma_2}}, \quad m \in \mathbb{N}.$$

Let

$$L(\mathbb{P}, \alpha) = \{(\log p : p \in \mathbb{P}), (\log(m+\alpha) : m \in \mathbb{N}_0)\}.$$

Moreover, denote by $P^{Q,\alpha}_{\underline{\zeta},\underline{\sigma}}$ the distribution of the random element $\underline{\zeta}(\underline{\sigma}, \omega, \alpha; Q)$, i.e.,

$$P^{Q,\alpha}_{\underline{\zeta},\underline{\sigma}}(A) = m_H\Big\{\omega \in \Omega : \underline{\zeta}(\underline{\sigma}, \omega, \alpha; Q) \in A\Big\}, \quad A \in \mathcal{B}(\mathbb{C}^2).$$

The main result of the paper is the following joint limit theorem of Bohr–Jessen type for the functions $\zeta(s; Q)$ and $\zeta(s, \alpha)$.

For brevity, we set

$$\underline{\zeta}(\underline{\sigma} + it, \alpha; Q) = (\zeta(\sigma_1 + it; Q), \zeta(\sigma_2 + it, \alpha)).$$

Theorem 1. *Suppose that the set $L(\mathbb{P}, \alpha)$ is linearly independent over the field of rational numbers $\mathbb{Q}$, and $\sigma_1 > \frac{n-1}{2}$, $\sigma_2 > \frac{1}{2}$. Then,*

$$P^{Q,\alpha}_{T,\underline{\zeta},\underline{\sigma}}(A) = \frac{1}{T}\mathrm{meas}\Big\{t \in [0,T] : \underline{\zeta}(\underline{\sigma} + it, \alpha; Q) \in A\Big\}, \quad A \in \mathcal{B}(\mathbb{C}^2),$$

converges weakly to the measure $P^{Q,\alpha}_{\underline{\zeta},\underline{\sigma}}$ as $T \to \infty$.

For example, if the parameter α is transcendental, then the set $L(\mathbb{P}, \alpha)$ is linearly independent over $\mathbb{Q}$.

It should be emphasised that the requirements on the matrix Q are related to a possibility of representation of non-negative integers by the quadratic form $\underline{x}^T Q \underline{x}$, $\underline{x} \in \mathbb{Z}^n$.

Let $r(m)$, $m \in \mathbb{N}_0$ denotes the number of $\underline{x} \in \mathbb{Z}^n$ that $\underline{x}^T Q \underline{x} = m$. Then, for even $n \geq 4$, the theta-series

$$\sum_{m=0}^{\infty} r(m) \mathrm{e}^{2\pi i m s}$$

can be expressed as a sum of an Eisenstein series and a cusp form [9], and this leads to the representation (1). Moreover, the requirement on the linear independence over $\mathbb{Q}$ of the set $L(\mathbb{P}, \alpha)$ is necessary for the identification of the limit measure in Theorem 1. This restriction for α is used essentially in the proofs of Lemmas 1 and 5, and thus, in the proof of Theorem 1.

We divide the proof of Theorem 1 into several lemmas, which are limit theorems in some spaces for certain auxiliary objects. The crucial aspect of the proof lies in the identification of the limit measure.

2. Limit Lemma on Ω

For $A \in \mathcal{B}(\Omega)$, set

$$P_{T,\Omega}(A) = \frac{1}{T} \mathrm{meas}\Big\{ t \in [0, T] : \Big(\big(p^{-it}, p \in \mathbb{P}\big), \big((m+\alpha)^{-it}, m \in \mathbb{N}_0\big) \Big) \in A \Big\}.$$

Lemma 1. *Suppose that the set $L(\mathbb{P}, \alpha)$ is linearly independent over the field of rational numbers $\mathbb{Q}$. Then, $P_{T,\Omega}$ converges weakly to the Haar measure m_H as $T \to \infty$.*

Proof. The characters of the torus Ω are of the form

$$\prod_{p \in \mathbb{P}}^{*} \omega_1^{k_p}(p) \prod_{m \in \mathbb{N}_0}^{*} \omega_2^{l_m}(m),$$

where the star "$*$" shows that only a finite number of integers k_p and l_m are non-zero. Therefore, the Fourier transform $F_{T,\Omega}(\underline{k}, \underline{l})$, $\underline{k} = (k_p : k_p \in \mathbb{Z}, p \in \mathbb{P})$, $\underline{l} = (l_m : l_m \in \mathbb{Z}, m \in \mathbb{N}_0)$, is given by

$$F_{T,\Omega}(\underline{k}, \underline{l}) = \int_{\Omega} \left(\prod_{p \in \mathbb{P}}^{*} \omega_1^{k_p}(p) \prod_{m \in \mathbb{N}_0}^{*} \omega_2^{l_m}(m) \right) \mathrm{d}P_{T,\Omega}.$$

Thus, in view of the definition of $P_{T,\Omega}$,

$$\begin{aligned} F_{T,\Omega}(\underline{k}, \underline{l}) &= \frac{1}{T} \int_0^T \left(\prod_{p \in \mathbb{P}}^{*} p^{-itk_p} \prod_{m \in \mathbb{N}_0}^{*} (m+\alpha)^{-itl_m} \right) \mathrm{d}t \\ &= \frac{1}{T} \int_0^T \exp\left\{ -it \left(\sum_{p \in \mathbb{P}}^{*} k_p \log(p) + \sum_{m \in \mathbb{N}_0}^{*} l_m \log(m+\alpha) \right) \right\} \mathrm{d}t. \end{aligned} \tag{3}$$

We have to show that $F_{T,\Omega}(\underline{k}, \underline{l})$ converges to the Fourier transform of the measure m_H as $T \to \infty$ [18], i.e., to

$$F_{\Omega}(\underline{k}, \underline{l}) = \begin{cases} 1 & \text{if } (\underline{k}, \underline{l}) = (\underline{0}, \underline{0}), \\ 0 & \text{otherwise}, \end{cases} \tag{4}$$

where $\underline{0} = (0, \dots, 0, \dots)$. Since the set $L(\mathbb{P}, \alpha)$ is linearly independent over $\mathbb{Q}$,

$$\mathcal{L}(\underline{k}, \underline{l}) \stackrel{def}{=} \sum_{p \in \mathbb{P}}^{*} k_p \log(p) + \sum_{m \in \mathbb{N}_0}^{*} l_m \log(m+\alpha) \neq 0$$

for $(\underline{k}, \underline{l}) \neq (\underline{0}, \underline{0})$. Therefore, in this case, the equality in (3) gives

$$F_{T,\Omega}(\underline{k}, \underline{l}) = \frac{1 - \exp\{-iT\mathcal{L}(\underline{k}, \underline{l})\}}{iT\mathcal{L}(\underline{k}, \underline{l})}.$$

Thus, for $(\underline{k}, \underline{l}) \neq (\underline{0}, \underline{0})$,

$$\lim_{T\to\infty} F_{T,\Omega}(\underline{k}, \underline{l}) = 0.$$

Since, obviously, $F_{T,\Omega}(\underline{0}, \underline{0}) = 1$, this shows that $F_{T,\Omega}(\underline{k}, \underline{l})$ converges to (4) as $T \to \infty$. The lemma is proved. □

Lemma 1 is a starting point for the proof of limit lemmas in $\mathbb{C}^2$ for certain objects given by absolutely convergent Dirichlet series.

3. Absolutely Convergent Series

Let $\beta > \frac{1}{2}$ be a fixed number and, for $N \in \mathbb{N}$, let

$$u_N(m) = \exp\left\{-\left(\frac{m}{N}\right)^{\beta}\right\}, \quad m \in \mathbb{N},$$

and

$$u_N(m, \alpha) = \exp\left\{-\left(\frac{m+\alpha}{N}\right)^{\beta}\right\}, \quad m \in \mathbb{N}_0.$$

Define

$$L_N\Big(s - \frac{n}{2} + 1, \hat{\chi}_l\Big) = \sum_{m=1}^{\infty} \frac{\hat{\chi}_l(m) u_N(m)}{m^{s-\frac{n}{2}+1}},$$

$$L_N\Big(s - \frac{n}{2} + 1, \omega_1, \hat{\chi}_l\Big) = \sum_{m=1}^{\infty} \frac{\hat{\chi}_l(m)\omega_1(m) u_N(m)}{m^{s-\frac{n}{2}+1}},$$

and

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{u_N(m, \alpha)}{(m+\alpha)^s},$$

$$\zeta(s, \omega_2, \alpha) = \sum_{m=0}^{\infty} \frac{\omega_2(m) u_N(m, \alpha)}{(m+\alpha)^s}.$$

Since $u_N(m)$ and $u_N(m, \alpha)$ decrease exponentially with respect to m, the above series are absolutely convergent for $\sigma > \sigma_0$ with arbitrary fixed finite σ_0. For $\sigma_1 > \frac{n-1}{2}$ and $\sigma_2 > \frac{1}{2}$, let

$$\underline{\zeta}_N(\underline{\sigma}, \alpha; Q) = (\zeta_N(\sigma_1; Q), \zeta_N(\sigma_2, \alpha))$$

with

$$\zeta_N(\sigma_1; Q) = \sum_{k=1}^{K} \sum_{l=1}^{L} \frac{a_{kl}}{k^{\sigma_1} l^{\sigma_1}} L(\sigma_1, \chi_k) L_N\Big(\sigma_1 - \frac{n}{2} + 1, \hat{\chi}_l\Big) + \sum_{m=1}^{\infty} \frac{b_Q(m)}{m^{\sigma_1}},$$

and

$$\underline{\zeta}_N(\underline{\sigma}, \omega, \alpha; Q) = (\zeta_N(\sigma_1, \omega_1; Q), \zeta_N(\sigma_2, \omega_2, \alpha))$$

with

$$\begin{aligned} \zeta_N(\sigma_1, \omega_1; Q) \quad &= \quad \sum_{k=1}^{K} \sum_{l=1}^{L} \frac{a_{kl}\omega_1(k)\omega_1(l)}{k^{\sigma_1} l^{\sigma_1}} L(\sigma_1, \omega_1, \chi_k) L_N\Big(\sigma_1 - \frac{n}{2} + 1, \omega_1, \chi_k\Big) \\ &+ \quad \sum_{m=1}^{\infty} \frac{b_Q(m)\omega_1(m)}{m^{\sigma_1}}. \end{aligned}$$

For $A \in \mathcal{B}(\mathbb{C}^2)$, define

$$P_{T,N,\underline{\sigma}}^{Q,\alpha}(A) = \frac{1}{T} \text{meas}\Big\{t \in [0, T] : \underline{\zeta}_N(\underline{\sigma} + it, \alpha; Q) \in A\Big\}$$

and

$$P_{T,N,\underline{\sigma}}^{Q,\alpha,\Omega}(A) = \frac{1}{T} \text{meas}\Big\{t \in [0, T] : \underline{\zeta}_N(\underline{\sigma} + it, \omega, \alpha; Q) \in A\Big\}.$$

This section is devoted to the weak convergence of $P^{Q,\alpha}_{T,N,\underline{\sigma}}$ and $P^{Q,\alpha,\Omega}_{T,N,\underline{\sigma}}$ as $T \to \infty$. Let the mapping $v^{Q,\alpha}_{N,\underline{\sigma}} : \Omega \to \mathbb{C}^2$ be given by

$$v^{Q,\alpha}_{N,\underline{\sigma}}(\omega) = \underline{\zeta}_N(\underline{\sigma}, \omega, \alpha; Q), \quad \sigma_1 > \frac{n-1}{2}, \quad \sigma_2 > \frac{1}{2},$$

and $V^{Q,\alpha}_{N,\underline{\sigma}} = m_H\left(v^{Q,\alpha}_{N,\underline{\sigma}}\right)^{-1}$, where, for $A \in \mathcal{B}(\mathbb{C}^2)$,

$$V^{Q,\alpha}_{N,\underline{\sigma}}(A) = m_H\left(\left(v^{Q,\alpha}_{N,\underline{\sigma}}\right)^{-1} A\right).$$

Since all Dirichlet series in the definition of $\underline{\zeta}_N(\underline{\sigma}, \omega, \alpha; Q)$ are absolutely convergent in the considered region, the mapping $v^{Q,\alpha}_{N,\underline{\sigma}}$ is continuous, hence $(\mathcal{B}(\Omega), \mathcal{B}(\mathbb{C}^2))$-measurable. Therefore, the probability measure $V^{Q,\alpha}_{N,\underline{\sigma}}$ is defined correctly; see, for example, [19], section 5.

Lemma 2. *Under the hypotheses of Theorem 1, $P^{Q,\alpha}_{T,N,\underline{\sigma}}$ and $P^{Q,\alpha,\Omega}_{T,N,\underline{\sigma}}$ both converge weakly to the same probability measure $V^{Q,\alpha}_{N,\underline{\sigma}}$ as $T \to \infty$.*

Proof. We apply the principle of preservation of the weak convergence under continuous mappings (see section 5 of [19]). By the definitions of $P^{Q,\alpha}_{T,N,\underline{\sigma}}$, $P_{T,\Omega}$, and $v^{Q,\alpha}_{N,\underline{\sigma}}$, we have

$$\begin{aligned} P^{Q,\alpha}_{T,N,\underline{\sigma}}(A) \quad = \quad & \frac{1}{T}\text{meas}\left\{t \in [0,T] : \left(\left(p^{-it}, p \in \mathbb{P}\right), \left((m+\alpha)^{-it}, m \in \mathbb{N}_0\right)\right) \in (v^{Q,\alpha}_{N,\underline{\sigma}})^{-1} A\right\} \\ & P_{T,\Omega}\left((v^{Q,\alpha}_{N,\underline{\sigma}})^{-1} A\right) \end{aligned}$$

for every $A \in \mathcal{B}(\mathbb{C}^2)$. Thus, $P^{Q,\alpha}_{T,N,\underline{\sigma}} = P^{Q,\alpha}_{T,\Omega}\left(v^{Q,\alpha}_{N,\underline{\sigma}}\right)^{-1}$. This continuity of $v^{Q,\alpha}_{N,\underline{\sigma}}$, Lemma 1, and Theorem 5.1 of [19] imply that $P^{Q,\alpha}_{T,N,\underline{\sigma}}$ converges to $V^{Q,\alpha}_{N,\underline{\sigma}}$ as $T \to \infty$.

It remains to show that $P^{Q,\alpha,\Omega}_{T,N,\underline{\sigma}}$ also converges to $V^{Q,\alpha}_{N,\underline{\sigma}}$ as $T \to \infty$. Let $\hat{\omega} \in \Omega$, and the mapping $w^{Q,\alpha}_{N,\underline{\sigma}} : \Omega \to \mathbb{C}^2$ be given by

$$w^{Q,\alpha}_{N,\underline{\sigma}}(\omega) = \zeta_N(\underline{\sigma}, \omega\hat{\omega}, \alpha; Q).$$

Thus, we have that

$$w^{Q,\alpha}_{N,\underline{\sigma}}(\omega) = v^{Q,\alpha}_{N,\underline{\sigma}}(\omega)(a(\omega)), \tag{5}$$

where $a : \Omega \to \Omega$ is given by $a(\omega) = \omega\hat{\omega}$. Along the same lines as in the case of $P^{Q,\alpha}_{T,N,\underline{\sigma}}$, we find that $P^{Q,\alpha,\Omega}_{T,N,\underline{\sigma}}$ converges weakly to the measure $W^{Q,\alpha}_{N,\underline{\sigma}} = m_H\left(w^{Q,\alpha}_{N,\underline{\sigma}}\right)^{-1}$. However, by (5) and the invariance of the Haar measure, we obtain

$$W^{Q,\alpha}_{N,\underline{\sigma}} = m_H\left(v^{Q,\alpha}_{N,\underline{\sigma}}(a)\right)^{-1} = \left(m_H a^{-1}\right)\left(v^{Q,\alpha}_{N,\underline{\sigma}}\right)^{-1} = m_H\left(v^{Q,\alpha}_{N,\underline{\sigma}}\right)^{-1} = V^{Q,\alpha}_{N,\underline{\sigma}}.$$

This completes the proof of the lemma. □

4. Approximation Lemmas

In this section, we approximate $\underline{\zeta}(\underline{\sigma} + it, \alpha; Q)$ by $\underline{\zeta}_N(\underline{\sigma} + it, \alpha; Q)$ and $\underline{\zeta}(\underline{\sigma} + it, \omega, \alpha; Q)$ by $\underline{\zeta}_N(\underline{\sigma} + it, \omega, \alpha; Q)$.

Let, for $\underline{z}_1 = (z_{11}, z_{12}), \underline{z}_2 = (z_{21}, z_{22}) \in \mathbb{C}^2$,

$$\rho(\underline{z}_1, \underline{z}_2) = \left(|z_{11} - z_{21}|^2 + |z_{12} - z_{22}|^2\right)^{1/2}.$$

Lemma 3. *For $\sigma_1 > \frac{n-1}{2}$ and $\sigma_2 > \frac{1}{2}$,*

$$\lim_{N\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \rho\Big(\underline{\zeta}(\underline{\sigma}+it,\alpha;Q),\underline{\zeta}_N(\underline{\sigma}+it,\alpha;Q)\Big)\mathrm{d}t=0,$$

and, for almost all $\omega\in\Omega$,

$$\lim_{N\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \rho\Big(\underline{\zeta}(\underline{\sigma}+it,\omega,\alpha;Q),\underline{\zeta}_N(\underline{\sigma}+it,\omega,\alpha;Q)\Big)\mathrm{d}t=0.$$

Proof. The first equality of the lemma is a corollary of the equalities

$$\lim_{N\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T |\zeta(\sigma_1+it;Q)-\zeta_N(\sigma_1+it;Q)|\mathrm{d}t=0$$

and

$$\lim_{N\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T |\zeta(\sigma_2+it,\alpha)-\zeta_N(\sigma_2+it,\alpha)|\mathrm{d}t=0. \tag{6}$$

The first of them was obtained in [11], Lemma 4. Its proof is based on the integral representation

$$L_N\Big(\sigma_1-\frac{n}{2}+1,\hat{\chi}_l\Big)=\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} L\Big(\sigma_1-\frac{n}{2}+1+z,\hat{\chi}_l\Big)l_N(z)\mathrm{d}z$$

with

$$l_N(z)=\frac{1}{\beta}\Gamma\Big(\frac{z}{\beta}\Big)N^z,$$

where $\beta>\frac{1}{2}$ is the same as in the definition of $u_N(m)$, and on the mean square estimate for Dirichlet L-functions in the half-plane $\sigma>\frac{1}{2}$.

For the proof of (6), we use, for $\sigma_2>\frac{1}{2}$, the representation

$$\zeta_N(s,\alpha)=\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\zeta(s+z,\alpha)l_N(z)\mathrm{d}z. \tag{7}$$

Since $\sigma_2>\frac{1}{2}$, there exists $\epsilon>0$ such that $\frac{1}{2}+\epsilon<\sigma_2$. Let $\beta=\sigma_2$ and $\beta_1=\frac{1}{2}+\epsilon-\sigma_2$. The integrand in (7) has simple poles $z=0$ and $z=1-s$ in the strip $\beta_1<\mathrm{Re}z<\beta$. Therefore, by the residue theorem and (7),

$$\zeta_N(\sigma_2+it,\alpha)-\zeta(\sigma_2+it,\alpha)=\frac{1}{2\pi i}\int_{\beta_1-i\infty}^{\beta_1+i\infty}\zeta(\sigma_2+it+z,\alpha)l_N(z)\mathrm{d}z+l_N(1-\sigma_2-it).$$

Hence,

$$\begin{aligned}\zeta_N(\sigma_2+it,\alpha)-\zeta(\sigma_2+it,\alpha)\ &\ll\ \int_{-\infty}^{\infty}\Big|\zeta\Big(\frac{1}{2}+\epsilon+it+i\tau,\alpha\Big)\Big|\Big|l_N\Big(\frac{1}{2}+\epsilon-\sigma_2+i\tau\Big)\Big|\mathrm{d}\tau\\ &\quad+|l_N(1-\sigma_2-it)|\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{T}\int_0^T|\zeta(\sigma_2+it,\alpha)-\zeta_N(\sigma_2+it,\alpha)|\mathrm{d}t\\ &\ll\int_{-\infty}^{\infty}\Bigg(\frac{1}{T}\int_0^T\Big|\zeta\Big(\frac{1}{2}+\epsilon+it+i\tau,\alpha\Big)\Big|\mathrm{d}t\Bigg)\Big|l_N\Big(\frac{1}{2}+\epsilon-\sigma_2+i\tau,\Big)\Big|\mathrm{d}\tau\end{aligned}$$

$$+\frac{1}{T}\int_0^T |l_N(1-\sigma_2-it)|\mathrm{d}t \overset{def}{=} I_1(T,N)+I_2(T,N), \tag{8}$$

where the classical notation $a \ll_\eta b$, $a \in \mathbb{C}$, $b > 0$ means that there exists a constant $c = c(\eta) > 0$ such that $|a| \le cb$. It is well known (see, for example, [17]) that, for $\frac{1}{2} < \sigma < 1$,

$$\int_{-T}^T |\zeta(\sigma+it,\alpha)|^2\mathrm{d}t \ll_{\sigma,\alpha} T.$$

Therefore, for large T,

$$\begin{aligned}
&\frac{1}{T}\int_0^T \left|\zeta\left(\frac{1}{2}+\epsilon+it+i\tau,\alpha\right)\right|\mathrm{d}\tau \ll \left(\frac{1}{T}\int_0^T \left|\zeta\left(\frac{1}{2}+\epsilon+it+i\tau,\alpha\right)\right|^2\mathrm{d}t\right)^{1/2} \\
&\le \left(\frac{1}{T}\int_{-|\tau|}^{T+|\tau|} \left|\zeta\left(\frac{1}{2}+\epsilon+it,\alpha\right)\right|^2\mathrm{d}t\right)^{1/2} \ll_{\epsilon,\alpha} \left(\frac{T+|\tau|}{T}\right)^{1/2} \\
&\ll_{\epsilon,\alpha} (1+|\tau|)^{1/2}.
\end{aligned} \tag{9}$$

For the gamma-function, the estimate

$$\Gamma(\sigma+it) \ll \exp\{-c|t|\}, \quad c > 0, \tag{10}$$

uniformly for σ in every finite interval is valid. Therefore,

$$l_N\left(\frac{1}{2}+\epsilon-\sigma_2+i\tau\right) \ll_{\sigma_2} N^{\frac{1}{2}+\epsilon-\sigma_2}\exp\left\{-\frac{c}{\sigma_2}|\tau|\right\}.$$

This, together with (9), shows that

$$I_1(T,N) \ll_{\epsilon,\sigma_2,\alpha} N^{\frac{1}{2}+\epsilon-\sigma_2}\int_{-\infty}^{\infty}(1+|\tau|)^{1/2}\exp\left\{-\frac{c}{\sigma_2}|\tau|\right\}\mathrm{d}\tau \ll_{\epsilon,\sigma_2,\alpha} N^{\frac{1}{2}+\epsilon-\sigma_2}. \tag{11}$$

By (10) again,

$$l_N(1-\sigma_2-it) \ll_{\sigma_2} N^{1-\sigma_2}\exp\left\{-\frac{c}{\sigma_2}|t|\right\},$$

and thus,

$$I_2(T,N) \ll_{\sigma_2} N^{1-\sigma_2}\int_0^{\infty}\exp\left\{-\frac{c}{\sigma_2}|t|\right\}\mathrm{d}t \ll_{\sigma_2} N^{1-\sigma_2}\frac{\log T}{T}.$$

Since $\frac{1}{2}+\epsilon-\sigma_2 < 0$, this, with (11) and (8), proves (6).

The second equality of the lemma follows from the following two equalities:

$$\lim_{N\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T |\zeta(\sigma_1+it,\omega_1;Q)-\zeta_N(\sigma_1+it,\omega_1;Q)|\mathrm{d}t = 0$$

and

$$\lim_{N\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T |\zeta(\sigma_2+it,\alpha,\omega_2)-\zeta_N(\sigma_2+it,\alpha,\omega_2)|\mathrm{d}t = 0$$

for almost all $\omega_1 \in \Omega_1$ and almost all $\omega_2 \in \Omega_2$, respectively.
The first of these was obtained in [11], Lemma 7, while the second is proved similarly to Equality (6) by using the representation, for $\sigma > \frac{1}{2}$,

$$\zeta_N(s,\alpha,\omega) = \frac{1}{2\pi i}\int\limits_{\beta-i\infty}^{\beta+i\infty} \zeta(s+z,\alpha,\omega) l_N(z)\mathrm{d}z,$$

as well as the bound, for $\frac{1}{2} < \sigma < 1$ and almost all $\omega_2 \in \Omega_2$,

$$\int\limits_{-T}^{T} |\zeta(\sigma+it,\alpha,\omega_2)|^2 \mathrm{d}t \ll_{\sigma,\alpha} T,$$

see, for example, [17]. □

5. Tightness

Let $\{P\}$ be a family of probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. We recall that the family $\{P\}$ is called tight if, for every $\epsilon > 0$, there exists a compact set $K \subset \mathbb{X}$ such that

$$P(K) > 1-\epsilon$$

for all $P \in \{P\}$. The family $\{P\}$ is relatively compact if every sequence $\{P_n\} \subset \{P\}$ contains a subsequence $\{P_n\}$ weakly convergent to a certain probability measure on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ as $n \to \infty$.

A property of relative compactness is useful for the investigation of weak convergence of probability measures. By the classical Prokhorov theorem, see, for example, [19], every tight family $\{P\}$ is relatively compact as well. Therefore, often it is convenient to know the tightness of the considered family. In our case, this concerns the measure $V_N^{Q,\alpha}$, $N \in \mathbb{N}$.

Lemma 4. *The family $\{V_N^{Q,\alpha} : N \in \mathbb{N}\}$ is tight.*

Proof. Consider the marginal measures of the measure $V_N^{Q,\alpha}$, i.e., for $A \in \mathcal{B}(\mathbb{C})$,

$$V_{N,\sigma_1}^{Q}(A) = V_{N,\underline{\sigma}}^{Q,\alpha}(A \times \mathbb{C})$$

and

$$V_{N,\sigma_2}^{\alpha}(A) = V_{N,\underline{\sigma}}^{Q,\alpha}(\mathbb{C} \times A).$$

It is easily seen that the measure V_{N,σ_1}^{Q} appears in the process related to weak convergence of the measure $P_{T,\underline{\sigma}}^{Q}$ and the measure V_{N,σ_2}^{α} is used for study of

$$P_{T,\sigma_2}^{\alpha}(A) = \frac{1}{T}\mathrm{meas}\{t \in [0,T] : \zeta(\sigma_2+it,\alpha) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}).$$

Thus, in [17], the tightness of the family $\{V_{N,\sigma_1}^{Q} : n \in \mathbb{N}\}$ was obtained, i.e., for every $\epsilon > 0$, there exists a compact set $K_1 \subset \mathbb{C}$ such that

$$V_{N,\sigma_1}^{Q}(K_1) > 1 - \frac{\epsilon}{2} \tag{12}$$

for all $N \in \mathbb{N}$. We will prove a similar inequality for V_{N,σ_2}^{α}.

Repeating the proofs of Lemmas 1 and 2 leads to weak convergence of

$$P_{T,N,\sigma_2}^{\alpha}(A) = \frac{1}{T}\mathrm{meas}\{t \in [0,T] : \zeta_N(\sigma_2+it,\alpha) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}),$$

to V_{N,σ_2}^{α} as $T \to \infty$. Let θ_T be a random variable defined on a certain probability space $(\Xi, \mathcal{A}, \mu)$ and uniformly distributed in $[0,T]$, i.e., its density function $p(x)$ is of the form

$$p(x) = \begin{cases} 0, & x \le 0, \\ \frac{1}{T}, & 0 < x \le T, \\ 0, & x > T. \end{cases}$$

Define

$$\xi^{\alpha}_{T,N,\sigma_2} = \xi^{\alpha}_{T,N,\sigma_2}(\sigma) = \zeta_N(\sigma_2 + i\theta_T, \alpha),$$

and denote by $\xrightarrow{D}$ the convergence in distribution. Then, the above remark can be written as

$$\xi^{\alpha}_{T,N,\sigma_2} \xrightarrow[T\to\infty]{D} \xi^{\alpha}_{N,\sigma_2}, \tag{13}$$

where $\xi^{\alpha}_{N,\sigma_2}$ is a random variable with distribution V^{α}_{N,σ_2}. Since the series for $\zeta_N(s,\alpha)$ is absolutely convergent, we have

$$\begin{aligned}\sup_{N\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{T}\int_0^T |\zeta_N(\sigma_2 + it, \alpha)|^2 \mathrm{d}t &= \sup_{N\in\mathbb{N}} \sum_{m=1}^{\infty} \frac{v_N^2(m,\alpha)}{(m+\alpha)^{2\sigma_2}} \leq \sum_{m=1}^{\infty} \frac{1}{(m+\alpha)^{2\sigma_2}} \\ &\leq C_{\alpha,\sigma_2} < \infty.\end{aligned}$$

Then, in view of (13),

$$\begin{aligned}\sup_{N\in\mathbb{N}} \mu\left\{|\xi^{\alpha}_{N,\sigma_2}| \geq \sqrt{C_{\alpha,\sigma_2}\left(\frac{\epsilon}{2}\right)^{-1}}\right\} &= \sup_{N\in\mathbb{N}} \limsup_{T\to\infty} \mu\left\{|\xi^{\alpha}_{T,N,\sigma_2}| \geq \sqrt{C_{\alpha,\sigma_2}\left(\frac{\epsilon}{2}\right)^{-1}}\right\} \\ &\leq \sup_{N\in\mathbb{N}} \frac{1}{C_{\alpha,\sigma_2}} \frac{\epsilon}{2} \limsup_{T\to\infty} \frac{1}{T}\int_0^T |\zeta_N(\sigma_2 + it, \alpha)|^2 \mathrm{d}t \\ &\leq \frac{\epsilon}{2}.\end{aligned} \tag{14}$$

Let $K_2 = \left\{z \in \mathbb{C} : |z| \leq \sqrt{C_{\alpha,\sigma_2}\left(\frac{\epsilon}{2}\right)^{-1}}\right\}$. Then, K_2 is a compact set in $\mathbb{C}$ and, by (14),

$$V^{\alpha}_{N,\sigma_2}(K_1) > 1 - \frac{\epsilon}{2} \tag{15}$$

for all $N \in \mathbb{N}$.

Now, define $K = K_1 \times K_2$. Then, K is a compact set in $\mathbb{C}^2$. Moreover, taking into account (12) and (15) gives

$$V^{Q,\alpha}_{N,\underline{\sigma}}(\mathbb{C}^2 \setminus K)) \leq V^{Q}_{N,\sigma_1}(\mathbb{C} \setminus K_1) + V^{\alpha}_{N,\sigma_2}(\mathbb{C} \setminus K_2) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $N \in \mathbb{N}$. Thus, $V^{Q,\alpha}_{N,\underline{\sigma}}(K) \geq 1 - \epsilon$ for all $N \in \mathbb{N}$, and the proof is complete. □

6. Limit Theorems

Now, we are ready to prove weak convergence for $P_{T,\underline{\zeta},\underline{\sigma}}$ and

$$P^{\Omega}_{T,\underline{\zeta},\underline{\sigma}}(A) = \frac{1}{T}\mathrm{meas}\left\{t \in [0,T] : \underline{\zeta}(\underline{\sigma} + it, \omega, \alpha; Q) \in A\right\}, \quad A \in \mathcal{B}(\mathbb{C}^2).$$

Proposition 1. *Suppose that the set $L(\mathbb{P};\alpha)$ is linearly independent over $\mathbb{Q}$, and $\sigma_1 > \frac{n-1}{2}$, $\sigma_2 > \frac{1}{2}$. Then, $P_{T,\underline{\zeta},\underline{\sigma}}$ and $P^{\Omega}_{T,\underline{\zeta},\underline{\sigma}}$, for almost all $\omega \in \Omega$; both converge to the same probability measure $P_{\underline{\sigma}}$ as $T \to \infty$.*

Proof. Let θ_T be the same random variable as in Section 5. Introduce the $\mathbb{C}^2$-valued random elements

$$\underline{\xi}^{Q,\alpha}_{T,N,\underline{\sigma}} = \underline{\zeta}_N(\underline{\sigma} + i\theta_T, \alpha; Q)$$

and

$$\underline{\xi}^{Q,\alpha}_{T,\underline{\sigma}} = \underline{\zeta}(\underline{\sigma} + i\theta_T, \alpha; Q).$$

Moreover, let $\underline{\xi}^{Q,\alpha}_{N,\underline{\sigma}}$ be a $\mathbb{C}^2$-valued random element having the distribution $V^{Q,\alpha}_{N,\underline{\sigma}}$. Then, the assertion of Lemma 2 for $P^{Q,\alpha}_{T,N,\underline{\sigma}}$ can be written as

$$\underline{\xi}^{Q,\alpha}_{T,N,\underline{\sigma}} \xrightarrow[T\to\infty]{D} \underline{\xi}^{Q,\alpha}_{N,\underline{\sigma}}. \tag{16}$$

By the Prokhorov theorem (see, for example, [19]), every tight family of probability measures is relatively compact. Thus, in view of Lemma 4, the family $\{V^{Q,\alpha}_{N,\underline{\sigma}} : N \in \mathbb{N}\}$ is relatively compact. Hence, we have a sequence $\{V^{Q,\alpha}_{N_r,\underline{\sigma}}\} \subset \{V^{Q,\alpha}_{N,\underline{\sigma}}\}$ and a probability measure $V^{Q,\alpha}_{\underline{\sigma}}$ on $(\mathbb{C}^2, \mathcal{B}(\mathbb{C}^2))$ such that

$$\underline{\xi}^{Q,\alpha}_{N_r,\underline{\sigma}} \xrightarrow[r\to\infty]{D} V^{Q,\alpha}_{\underline{\sigma}}. \tag{17}$$

Now, it is time for the application of Lemma 3. Thus, using Lemma 3, we obtain that, for every $\epsilon > 0$,

$$\begin{aligned}
&\lim_{r\to\infty} \limsup_{T\to\infty} \mu\left\{\rho\left(\underline{\xi}^{Q,\alpha}_{T,\underline{\sigma}}, \underline{\xi}^{Q,\alpha}_{T,N_r,\underline{\sigma}}\right) \geq \epsilon\right\} \\
= &\lim_{r\to\infty} \sup_{T\to\infty} \frac{1}{T}\text{meas}\{t \in [0,T] : \rho(\underline{\zeta}(\underline{\sigma}+it,\alpha;Q), \underline{\zeta}_{N_r}(\underline{\sigma}+it,\alpha;Q)) \geq \epsilon\} \\
\leq &\lim_{r\to\infty} \sup_{T\to\infty} \frac{1}{\epsilon T} \int_0^T \rho(\underline{\zeta}(\underline{\sigma}+it,\alpha;Q), \underline{\zeta}_{N_r}(\underline{\sigma}+it,\alpha;Q))\mathrm{d}t = 0.
\end{aligned}$$

This equality, and relations (16) and (17), show that theorem 4 from [19] can be applied for the random elements $\underline{\xi}^{Q,\alpha}_{T,N_r,\underline{\sigma}}$, $\underline{\xi}^{Q,\alpha}_{N_r,\underline{\sigma}}$, and $\underline{\xi}^{Q,\alpha}_{T,\underline{\sigma}}$. Thus, we have

$$\underline{\xi}^{Q,\alpha}_{T,\underline{\sigma}} \xrightarrow[T\to\infty]{D} V^{Q,\alpha}_{\underline{\sigma}}, \tag{18}$$

in other words, $P_{T,\underline{\zeta},\underline{\sigma}}$ converges weakly to the measure $V^{Q,\alpha}_{\underline{\sigma}}$ as $T \to \infty$.

It remains to prove that $P^{\Omega}_{T,\underline{\zeta},\underline{\sigma}}$, as $T \to \infty$, converges weakly to the measure $V^{Q,\alpha}_{\underline{\sigma}}$ as well. Relation (18) shows that the limit measure $V^{Q,\alpha}_{\underline{\sigma}}$ does not depend on the sequence $\{V^{Q,\alpha}_{N_r,\underline{\sigma}}\}$. Since the family $\{V^{Q,\alpha}_{N,\underline{\sigma}}\}$ is relatively compact, the latter remark implies the relation

$$\underline{\xi}^{Q,\alpha}_{N,\underline{\sigma}} \xrightarrow[N\to\infty]{D} V^{Q,\alpha}_{\underline{\sigma}}. \tag{19}$$

Define the random elements

$$\underline{\xi}^{Q,\alpha}_{T,N,\underline{\sigma}}(\omega) = \underline{\zeta}_N(\underline{\sigma}+i\theta_T,\omega,\alpha;Q)$$

and

$$\underline{\xi}^{Q,\alpha}_{T,\underline{\sigma}}(\omega) = \underline{\zeta}(\underline{\sigma}+i\theta_T,\omega,\alpha;Q).$$

By Lemma 2, for $P^{Q,\alpha,\Omega}_{T,N,\underline{\sigma}}$, the relation

$$\underline{\xi}^{Q,\alpha}_{T,N,\underline{\sigma}}(\omega) \xrightarrow[T\to\infty]{D} \underline{\xi}^{Q,\alpha}_{N,\underline{\sigma}} \tag{20}$$

holds. Moreover, Lemma 3, for every $\epsilon > 0$ and almost all $\omega \in \Omega$, implies

$$\begin{aligned}
&\lim_{N\to\infty} \limsup_{T\to\infty} \mu\left\{\rho\left(\underline{\xi}^{Q,\alpha}_{T,\underline{\sigma}}(\omega), \underline{\xi}^{Q,\alpha}_{T,N,\underline{\sigma}}(\omega)\right) \geq \epsilon\right\} \\
\leq &\lim_{N\to\infty} \limsup_{T\to\infty} \frac{1}{\epsilon T} \int_0^T \rho\left(\underline{\zeta}(\underline{\sigma}+it,\omega,\alpha;Q), \underline{\zeta}_N(\underline{\sigma}+it,\omega,\alpha;Q)\right)\mathrm{d}t = 0.
\end{aligned}$$

This, (19) and (20), and theorem 4.2 of [19] yield, for almost all $\omega \in \Omega$, the relation

$$\underline{\xi}^{Q,\alpha}_{T,\underline{\sigma}}(\omega) \xrightarrow[T\to\infty]{D} V^{Q,\alpha}_{\underline{\sigma}},$$

i.e., that $P^{\Omega}_{T,\underline{\zeta},\underline{\sigma}}$, as $T \to \infty$, converges weakly to $V^{Q,\alpha}_{\underline{\sigma}}$. The proposition is proved. □

7. Proof of Theorem 1

Let $t \in \mathbb{R}$ and $e_t = \big((p^{-it} : p \in \mathbb{P}), ((m+\alpha)^{-it}, m \in \mathbb{N}_0)\big)$. Obviously, e_t is an element of Ω. Using e_t, define a transformation $g_t : \Omega \to \Omega$ by

$$g_t(\omega) = e_t\omega, \quad \omega \in \Omega.$$

In virtue of the invariance of the Haar measure m_H, g_t is a measurable measure preserving transformation on Ω. Then, $\mathcal{G}_t = \{g_t : t \in \mathbb{R}\}$ is the one-parameter group of transformations on Ω. A set $A \in \mathcal{B}(\Omega)$ is invariant with respect to $\mathcal{G}_t$ if for every $t \in \mathbb{R}$ the sets $A_t = g_t(A)$ and A can differ one from another at most by a set of m_H-measure zero. All invariant sets form a σ-subfield of $\mathcal{B}(\Omega)$. We say that the group $\mathcal{G}_t$ is ergodic if its σ-field of invariant sets consists only of sets having m_H-measure 1 or 0.

Lemma 5. *Suppose that the set $L(\mathbb{P},\alpha)$ is linearly independent over $\mathbb{Q}$. Then, the group $\mathcal{G}_t$ is ergodic.*

Proof. We fix an invariant set A of the group $\mathcal{G}_t$, and consider its indicator function I_A. We will prove that, for almost all $\omega \in \Omega$, $I_A(\omega) = 1$ or $I_A(\omega) = 0$. For this, we will use the Fourier transform method.

By the proof of Lemma 1, we know that characters χ of Ω are of the form

$$\chi(\omega) = \prod_{p\in\mathbb{P}}^{*} \omega_1^{k_p}(p) \prod_{m\in\mathbb{N}_0}^{*} \omega_2^{l_m}(m),$$

where the star "$*$" indicates that only a finite number of integers k_p and l_m are non-zero. Hence, if χ is a non-trivial character,

$$\begin{aligned}\chi(g_t) &= \prod_{p\in\mathbb{P}}^{*} p^{-itk_p} \prod_{m\in\mathbb{N}_0}^{*} (m+\alpha)^{-itl_m} \\ &= \exp\left\{-it\left(\sum_{p\in\mathbb{P}}^{*} k_p \log(p) + \sum_{m\in\mathbb{N}_0}^{*} l_m \log(m+\alpha)\right)\right\}.\end{aligned}$$

Since χ is a non-principal character, i.e., $\chi(\omega) \not\equiv 1$. The linear independence of the set $L(\mathbb{P},\alpha)$ shows that

$$\sum_{p\in\mathbb{P}}^{*} k_p \log(p) + \sum_{m\in\mathbb{N}_0}^{*} l_m \log(m+\alpha) \neq 0$$

for $k_p \not\equiv 0$ and $l_m \not\equiv 0$. These remarks imply the existence of $t_0 \neq 0$ such that

$$\chi(g_{t_0}) \neq 1. \tag{21}$$

Moreover, by the invariance of A, for almost all $\omega \in \Omega$,

$$I_A(g_{t_0}) = I_A(\omega). \tag{22}$$

Let $\hat{h}$ denote the Fourier transform of h. Then, by (22), the invariance of m_H, and the multiplicativity of characters

$$\hat{I}_A(\chi) = \int_{\Omega} I_A(\omega)\chi(\omega)\mathrm{d}m_H = \chi(g_{t_0}) \int_{\Omega} I_A(\omega)\chi(\omega)\mathrm{d}m_H = \chi(g_{t_0})\hat{I}_A(\chi).$$

Thus, (21) gives

$$\hat{I}_A(\chi) = 0. \tag{23}$$

Now, suppose that $\chi(\omega) \equiv 1$ and $\hat{I}_A(\chi) = a$. Then,

$$\hat{a}(\chi) = \int_\Omega a(\chi)\chi(\omega)\mathrm{d}m_H = a\int_\Omega \chi(\omega)\mathrm{d}m_H = \begin{cases} a & \text{if } \chi(\omega) \equiv 1, \\ 0 & \text{otherwise,} \end{cases}$$

by orthogonality of characters. This, and (23), gives

$$\hat{I}_A(\chi) = \hat{a}(\chi).$$

The latter equality shows that $I_A(\omega) = a$ for almost all $\omega \in \Omega$. In other words, $a = 1$ or $a = 0$ for almost all $\omega \in \Omega$. Thus, $I_A(\omega) = 1$ or $I_A(\omega) = 0$ for almost all $\omega \in \Omega$. Therefore, $m_H(A) = 1$ or $m_H(A) = 0$, and the proof is complete. □

For convenience, we recall the classical Birkhoff–Khintchine ergodic theorem; see, for example, [20].

Lemma 6. *Suppose that a random process $\xi(t, \hat{\omega})$ is ergodic with finite expectation $\mathbb{E}|\xi(t, \hat{\omega})|$, and we sample paths integrable almost surely in the Riemann sense over every finite interval. Then, for almost all ω,*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \xi(t, \hat{\omega})\mathrm{d}t = \mathbb{E}\xi(0, \hat{\omega}).$$

Proof of Theorem 1. In virtue of Proposition 1, it suffices to identify the limit measure $P_{\underline{\sigma}}$ in it, i.e., to show that $P_{\underline{\sigma}} = P^{Q,\alpha}_{\underline{\zeta},\underline{\sigma}}$.

Let $A \in \mathcal{B}(\mathbb{C}^2)$ be a continuity set of the measure $P_{\underline{\sigma}}$ (A is a continuity set of the measure P if $P(\partial A) = 0$, where ∂A is the boundary of A). Then, by Proposition 1, for almost all $\omega \in \Omega$,

$$\lim_{T\to\infty} \frac{1}{T}\mathrm{meas}\Big\{t \in [0,T] : \underline{\zeta}(\underline{\sigma} + it, \omega, \alpha; Q) \in A\Big\} = P_{\underline{\sigma}}(A). \tag{24}$$

On the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the random variable

$$\xi = \xi(\omega) = \begin{cases} 1 & \text{if } \ \underline{\zeta}(\underline{\sigma}, \omega, \alpha; Q) \in A, \\ 0 & \text{otherwise,} \end{cases}$$

Obviously,

$$\mathbb{E}\xi = \int_\Omega \xi \mathrm{d}m_H = m_H\Big\{\omega \in \Omega : \underline{\zeta}(\underline{\sigma}, \omega, \alpha; Q) \in A\Big\}. \tag{25}$$

By Lemma 5, the random process $\xi(g_t(\omega))$ is ergodic. Therefore, an application of Lemma 6 yields

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \xi(g_t(\omega))\mathrm{d}t = \mathbb{E}\xi \tag{26}$$

for almost all $\omega \in \Omega$. On the other hand, from the definitions of ξ and $\mathcal{G}_t$, we have

$$\frac{1}{T}\int_0^T \xi(g_t(\omega))\mathrm{d}t = \frac{1}{T}\mathrm{meas}\Big\{t \in [0,T] : \underline{\zeta}(\underline{\sigma} + it, \omega, \alpha; Q) \in A\Big\}.$$

Therefore, equalities (25) and (26), for almost all $\omega \in \Omega$, lead to

$$\lim_{T\to\infty} \frac{1}{T}\mathrm{meas}\Big\{t \in [0,T] : \underline{\zeta}(\underline{\sigma} + it, \omega, \alpha; Q) \in A\Big\} = P^{Q,\alpha}_{\underline{\zeta},\underline{\sigma}}(A).$$

This, together with (24), shows that

$$P_{\underline{\sigma}}(A) = P^{Q,\alpha}_{\underline{\zeta},\underline{\sigma}}(A). \tag{27}$$

Since A is an arbitrary continuity set of $P_{\underline{\sigma}}$, equality (27) is valid for all $A \in \mathcal{B}(\mathbb{C}^2)$. This proves the theorem. □

8. Concluding Remarks

Theorem 1 shows that, for sufficiently large T, the value density of the pair $(\zeta(\sigma_1 + it; Q), \zeta(\sigma_2 + it, \alpha))$ is close to a certain probabilistic distribution. Unfortunately, the distribution of $P^{Q,\alpha}_{\underline{\zeta},\underline{\sigma}}$ is too complicated; it is defined only for almost all $\omega \in \Omega$. Hence, it is impossible to give a visualisation of $P^{Q,\alpha}_{\underline{\zeta},\underline{\sigma}}$.

We plan to further investigate the joint value distribution of the Epstein and Hurwitz zeta-functions using probabilistic methods. First, we will prove the discrete version of Theorem 1, i.e., the weak convergence for

$$\frac{1}{N+1}\#\{0 \le k \le N : (\zeta(\sigma_1 + ikh_1; Q), \zeta(\sigma_2 + ikh_2, \alpha)) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}^2),$$

as $N \to \infty$. Here, $\#B$ denotes the cardinality of the set $B \in \mathbb{N}_0$, and h_1, h_2 are fixed positive numbers. Further, we will obtain extensions of limit theorems in the space $\mathbb{C}^2$ for the pair $(\zeta(s; Q), \zeta(s, \alpha))$ to the space $\mathbb{H}^2(D)$, where $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$, and $\mathbb{H}(D)$ is the space of analytic in D functions endowed with the topology of uniform convergence on compacta. Using the limit theorems in $\mathbb{H}^2(D)$, we expect to obtain some results on approximation of a pair of analytic functions by shifts $(\zeta(\sigma_1 + i\tau; Q), \zeta(\sigma_2 + i\tau, \alpha))$. This would be the most important application of probabilistic limit theorems in function theory and practice.

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