VILNIAUS UNIVERSITY

Jūratė Petrauskienė

POISSON TYPE APPROXIMATIONS FOR SUMS OF DEPENDENT VARIABLES

Doctoral Dissertation

Physical Sciences, Mathematics (01 P)

Vilnius, 2011

The scientific work was carried out during 2006-2010 at Vilnius Univerity

Scientific supervisor:
Prof. Dr. Habil. Vydas Čekanavičius (Vilnius University, Physical Sciences, Mathematics - 01 P)

VILNIAUS UNIVERSITETAS

Jūratė Petrauskienė

PRIKLAUSOMŲ ATSITIKTINIŲ DYDŽIŲ SUMŲ APROKSIMAVIMAS PUASONO TIPO MATAIS

Daktaro disertacija

Fiziniai mokslai, matematika (01 P)

Vilnius, 2011

Disertacija rengta 2006-2010 metais Vilnius universitete

Mokslinis vadovas:
Prof. habil. dr. Vydas Čekanavičius (Vilnius universitetas, fiziniai mokslai, matematika - 01 P)

Contents

Notation

$\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N} = \{1, 2, \dots\}$.

$\mathbb{Z}$ denotes the set of integers, $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$.

I_a denotes the distribution concentrated at real a, $I = I_0$.

$U = I_1 - I$.

$z = \widehat{U}(t) = \mathrm{e}^{\mathrm{i}t} - 1$.

Let V, M be two finite (signed) measure concentrated on $\mathbb{Z}$. Then

the total variation norm of M is denoted by

$$\|M\| = \sum_{k=-\infty}^{\infty} |M\{k\}|,$$

the local norm of M is denoted by

$$\|M\|_\infty = \sup_{k\in\mathbb{Z}} |M\{k\}|,$$

the uniform Kolmogorov norm is denoted by

$$|M| = \sup_{k\in\mathbb{Z}} |M\{(-\infty, k]\}|,$$

the Wasserstein norm is denoted by

$$\|M\|_{\mathrm{W}} = \sum_{k=-\infty}^{\infty} |M\{(-\infty, k]\}|.$$

$\widehat{M}(t) = \sum_{k\in\mathbb{Z}} M\{k\}\mathrm{e}^{\mathrm{i}t}$ denotes the Fourier transform of M, $(t \in \mathbb{R})$.

$\widehat{\mathrm{E}}Y_1Y2\cdots Y_k$ is defined recursively by

$$\widehat{\mathrm{E}}Y_1Y_2\cdots Y_k = \mathrm{E}Y_1Y_2\cdots Y_k - \sum_{j=1}^{k-1} \widehat{\mathrm{E}}Y_1\cdots Y_j\mathrm{E}Y_{j+1}\cdots Y_k,$$

where $Y_1, Y_2, \dots$ form a sequence of arbitrary complex-valued random variables.

$C, C_1, C_2, \dots$ denote positive absolute constants.

Θ denotes the finite signed measure on $\mathbb{Z}$ satisfying $\|\Theta\| \leqslant 1$.

θ stands for any complex or real number, satisfying $|\theta| \leqslant 1$.

ν_k denotes the kth factorial moment by

$$\nu_k = \mathrm{E}X(X-1)\cdots(X-k+1).$$

$\mathcal{L}(S)$ denotes the distribution of $S = \eta_1 + \eta_2 + \cdots + \eta_n$, where $\eta_j = \xi_j\xi_{j-1}$ and ξ_j, $j = 0, 1, 2, \dots, n$ are independent identically distributed Bernoulli variables.

G_2 and G_3 denote measures used for approximations of $\mathcal{L}(S)$.

F_n denotes the distribution of $X_1 + X_2 + \cdots + X_n$, where $X_1, X_2, \dots, X_n$ are identically distributed 1-dependent random variables concentrated on $\mathbb{Z}$.

D_1, D_2 and D_3 denote measures used for approximations of F_n.

P_n denotes the distribution of $\hat{X}_1 + \hat{X}_2 + \cdots + \hat{X}_n$, where $\hat{X}_1, \hat{X}_2, \dots, \hat{X}_n$ is a triangular array of 1-dependent identically distributed three-point random variables.

B^n denotes measure used for approximations of P_n.

$\tilde{S} = \tilde{X}_1 + \tilde{X}_2 + \cdots + \tilde{X}_n$, where $\tilde{X}_j$, $j = 1, 2, \dots, n$ is a sequence of 1-dependent not identically distributed Bernoulli variables.

M_1, M_2 and M_3 denote compound Poisson measures used for approximations of N_n.

$a(i_1, i_2, \dots, i_m) = \frac{\mathrm{E}X_1(X_1-1)\cdots(X_1-i_1+1)\cdots X_m(X_m-1)\cdots(X_m-i_m+1)}{i_1!i_2!\cdots i_m!}$.

$Z \sim Be(p)$ means that Z is Bernoulli random variable, $\mathrm{P}(Z = 1) = p = 1 - \mathrm{P}(Z = 0)$.

Pois(λ) denotes Poisson distribution with the mean λ.

Chapter 1

Introduction

As it follows from the title, our aim is to investigate Poisson type approximations to the sums of dependent integer-valued random variables. In this thesis, only one type of dependence is considered, namely m-dependent random variables. We recall that the sequence of random variables (X_k), $k = 1, \dots$ is called m-dependent if, for $1 < s < t < \infty$, $t - s > m$, the σ-algebras generated by $X_1, \dots, X_s$ and $X_t, X_{t+1} \dots$ are independent.

1.1 Metrics

In this thesis, the accuracy of approximation is measured in the total variation, local, uniform (Kolmogorov) and Wasserstein metrics. For an integer-valued measure M, the definitions of these metrics are given in the notation section at the beginning of our thesis. In particularly, when we have two integer-valued random variables ξ and ζ, the following relations are true. For the total variation, we have

$$\|\mathcal{L}(\xi) - \mathcal{L}(\zeta)\| = \sum_{k=-\infty}^{\infty} |\mathrm{P}(\xi = k) - \mathrm{P}(\zeta = k)| = 2 \sup_A |\mathrm{P}(\xi \in A) - \mathrm{P}(\zeta \in A)|.$$

Here supremum is taken over all Borel sets. For the uniform (Kolmogorov), local (point) and Wasserstein metrics, we have

$$|\mathcal{L}(\xi) - \mathcal{L}(\zeta)| = \sup_{k \in \mathbb{Z}} |\mathrm{P}(\xi \leqslant k) - \mathrm{P}(\zeta \leqslant k)|, \quad \|\mathcal{L}(\xi) - \mathcal{L}(\zeta)\|_\infty = \sup_{k \in \mathbb{Z}} |\mathrm{P}(\xi = k) - \mathrm{P}(\zeta = k)|,$$

$$\|\mathcal{L}(\xi) - \mathcal{L}(\zeta)\|_{\mathrm{W}} = \sum_{k=-\infty}^{\infty} |\mathrm{P}(\xi \leqslant k) - \mathrm{P}(\zeta \leqslant k)|,$$

respectively. Note that there exist many alternative definitions of the metrics given in above. Let f be a function defined on $\mathbb{Z}$. Then, for example, it is possible to define the total variation, local and Wasserstein metric as $\sup_f |\mathrm{E} f(\xi) - \mathrm{E} f(\zeta)|$, where supremum is taken over all $\sup_k |f(k)| \leqslant 1$ or $\sup_k |f(k+1) - f(k)| \leqslant 1$ or $\sum_{k \in \mathbb{Z}} |f(k)| \leqslant 1$, respectively, see [8]. Note also that, all metrics by no means are restricted to the discrete case, though, of course, the definitions then should be formulated in more general terms. Note also that, in the literature, the Wasserstein metric is also called the Fortet-Mourier metric and is a partial case of the Kantorovich metric, see [35].

Let I_a denote the distribution concentrated at real a and set $I = I_0$. Let V and M be two finite signed measures concentrated on integers $\mathbb{Z}$. Products and powers of V and M are understood in the convolution sense, i.e, $VM\{A\} = \sum_{k=-\infty}^{\infty} V\{A - k\} M\{k\}$ for a set $A \subseteq \mathbb{Z}$; further $M^0 = I$. Using the simple equality

$$\|(I_1 - I)M\|_{\mathrm{W}} = \|M\|, \tag{1.1}$$

it is possible to switch from the Wasserstein norm to the total variation norm. Note also that

$$\|M\|_\infty \leqslant \|M\|, \qquad |M| \leqslant \|M\|, \quad \|VM\|_\infty \leqslant \|V\| \|M\|_\infty, \quad \|VM\| \leqslant \|V\| \|M\|.$$

1.2 Poisson type approximation

Now we discuss what, in this thesis, is called the Poisson type approximation. We recall that if $\zeta \sim P(\lambda)$ then

$$\mathrm{P}(\zeta = k) = \frac{\lambda^k}{k!}\mathrm{e}^{-\lambda}, \quad (k = 0, 1, 2, \dots)$$

and the characteristic function of ζ is equal to $\exp\{\lambda(\mathrm{e}^{\mathrm{i}t} - 1)\}$. The compound Poisson distribution can be defined in many alternative ways. For exmple, we can view the compound Poisson random variable as random sum of independent identically distributed random variables Z_j, when the number of summands $\zeta \sim P(\lambda)$ is independent of Z_j, that is as the distribution of

$$\sum_{j=1}^{\zeta} Z_j, \quad \zeta \sim P(\lambda).$$

It is not difficult to check that the characteristic function of the compound Poisson distribution is equal to $\exp\{\lambda(f(t) - 1)\}$, where $f(t)$ is the characteristic function of Z_1. If Z_1 is concentrated on non-negative integers the characteristic function can be written in the form:

$$\exp\Big\{\sum_{j=1}^{\infty} \lambda_j(\mathrm{e}^{j\mathrm{i}t} - 1)\Big\}. \tag{1.2}$$

Here λ_j are some positive quantities. As it follows from (1.2), if the compound Poisson distribution is concentrated on integers, then it can be viewed as the distribution of a sum of independent Poisson variables ζ_j. Here ζ_j is concentrated on the lattice with the maximum span j. Compound Poisson distributions are used as approximations in this thesis. However, sometimes we apply (1.2) with some negative λ_j, that is we deal with the compound Poisson structured *signed* measures. In this case, it is very inconvenient to write the definition in terms of random variables. Therefore, for our purposes, we use the measure notation. We define the (signed) compound Poisson measure as exponential of some measure M:

$$\mathrm{e}^M = \exp\{M\} = \sum_{k=0}^{\infty} \frac{1}{k!} M^k.$$

Note that the Fourier-Stieltjes transform of exponential measure is equal to

$$\widehat{\exp}\{M\}(t) = \exp\{\widehat{M}(t)\}.$$

By Poisson type approximation we call the exponential measure $\exp\{M\}$, where M is chosen to match some factorial moments of approximated distribution. As a rule, M has a quite simple structure. Let us explain the construction of approximating Poisson type measure in detail. Let $\widehat{F}(t)$ be a characteristic function of random variable taking non-negative integer values, and let κ_j be its jth factorial cumulant. We then can write formally

$$\widehat{F}(t) = \exp\{\ln \widehat{F}(t)\} = \exp\Big\{\kappa_1(\mathrm{e}^{\mathrm{i}t} - 1) + \frac{\kappa_2(\mathrm{e}^{\mathrm{i}t} - 1)^2}{2!} + \frac{\kappa_3(\mathrm{e}^{\mathrm{i}t} - 1)^3}{3!} + \dots\Big\}.$$

If, in the exponent, we drop all summands except the first one, the resulting characteristic function is that of the Poisson law. The idea of (signed) compound Poisson approximation is to leave more than one factorial cumulant in the exponent. In this case, the resulting Fourier-Stieltjes transform corresponds to a compound Poisson-structured signed measure which can have negative Poisson parameters. It matches more than one moment of the initial distribution and can be viewed as a special kind of asymptotic expansion.

The construction of the Poisson type approximation is more complicated when the approximated distribution is concentrated on negative and positive integers. We then can apply the idea of Kruopis

[28] and use the formal expansion

$$
\begin{aligned}
\widehat{F}(t) &= 1+\sum_{j=1}^{\infty}F\{j\}(\mathrm{e}^{\mathrm{i}tj}-1)+\sum_{k=1}^{\infty}F\{-k\}(\mathrm{e}^{-\mathrm{i}tk}-1)\\
&= \exp\Big\{\ln\Big(1+\sum_{j=1}^{\infty}F\{j\}(\mathrm{e}^{\mathrm{i}tj}-1)+\sum_{k=1}^{\infty}F\{-k\}(\mathrm{e}^{-\mathrm{i}tk}-1)\Big)\Big\}\\
&= \exp\Big\{\sum_{j,k=1}^{\infty}\kappa_{jk}(\mathrm{e}^{\mathrm{i}t}-1)^j(\mathrm{e}^{-\mathrm{i}t}-1)^k\Big\}. \qquad (1.3)
\end{aligned}
$$

If we choose M_s in such a way that $\widehat{M}(t)=\sum_{j,k=1}^{s}\kappa_{jk}(\mathrm{e}^{\mathrm{i}t}-1)^j(\mathrm{e}^{-\mathrm{i}t}-1)^k$ then we can expect $\exp\{M_s\}$ to be close to F. Of course, (1.3) gives just an idea of the construction of approximation. Application of this idea in practice depends on the assumed dependence of summands.

If M has complicated structure, then $\exp\{M\}$ is even more complicated. Therefore, we also used *second order* approximations to the Poisson type measures, that is asymptotic expansion of the form

$$\exp\{M\}(I+A(I_1-I)^k).$$

1.3 Known results

Literature on Poisson approximations is enormous, we just mention [8], [11], [23], [29], [34] and the references therein.

Compound and signed compound Poisson approximations are commonly applied in insurance models and in limit theorems; see [9], [10], [3], [20], [24], [27], [32], [38], and the references therein. Note that the discussed approach of constructing approximations does not necessarily result in *signed* measure. Under certain conditions one can get compound Poisson *distribution*. Roughly the main benefits of such approximations are the following:

1) the accuracy of approximation is of the same or better order than can be obtained by the normal approximation,

2) the estimates hold for the total variation, which is impossible for the normal law due to the differences in supports,

3) unlike the Edgeworth expansion no additional smoothing terms are needed for asymptotics.

Thus, when dealing with (signed) compound Poisson measures, we usually investigate some discrete alternatives to the normal law. We recall some of the results relevant to the results of our thesis. We are primarily interested in the order of accuracy. Therefore, we usually assume some additional restriction allowing for simpler form of estimate. To make expressions shorter we also set $U=I_1-I$.

We begin from the classical Poisson approximation to the so-called Poisson binomial distribution. Let $Z_j\sim Be(p_j)$, $(j=1,2,\dots,n)$ be independent Bernoulli random variables, $0\leqslant p_j\leqslant 1$. Let $n\bar{p}:=\sum_{j=1}^{n}p_j$ and let $\mathrm{Pois}(n\bar{p})$ denote Poisson distribution, that is $\mathrm{Pois}(n\bar{p})=\exp\{n\bar{p}U\}$. The following estimates hold

$$\frac{1}{16(1\wedge n\bar{p})}\sum_{j=1}^{n}p_j^2\leqslant\|\prod_{j=1}^{n}(1+p_jU)-\mathrm{Pois}(n\bar{p})\|\leqslant\frac{2(1-\mathrm{e}^{-n\bar{p}})}{n\bar{p}}\sum_{j=1}^{n}p_j^2, \qquad (1.4)$$

see [7]. Approximation of the binomial and Poisson binomial distributions by the Poisson distribution has a long history and many names to it, the first result being that of Prokhorov [34], see introduction of [8]. In the case when $p_j\equiv p<1/2$, we have

$$\frac{1}{16}\min(np^2,p)\leqslant\|(I+pU)^n-\mathrm{Pois}(np)\|\leqslant 2\min(np^2,p). \qquad (1.5)$$

It is obvious, that (1.5) is sharp for small p only and trivial for $p = O(1)$. Now let us write the Berry-Esseen estimate for the binomial distribution. Let $Y \sim N(np, np(1-p))$, $0 < p < 1/2$. Then

$$|(I + pU)^n - \mathcal{L}(Y)| \leqslant \frac{C_1}{\sqrt{np}}. \tag{1.6}$$

Here C_1 is an absolute constant. Therefore, we see that the normal approximation, as expected holds for the weaker uniform norm and is sharper, if $p > n^{-1/3}$. The best order of accuracy $O(n^{-1/2})$ is achieved when $p = O(1)$. If $p < n^{-1/3}$ the better order of accuracy gives (1.5). It is remarkable, that Poisson *limit* occurs, if $p = O(n^{-1})$ only. Thus, we see that some prelimiting distribution can be much sharper then the limiting one.

The accuracy in (1.4) can be improved by the second order asymptotic expansion. However, this improvement is insignificant if $p_i = O(1)$. Indeed, let $p_j \leqslant 1/2$, then

$$\Big\| \prod_{j=1}^{n}(1 + p_j U) - \mathrm{Pois}(n\bar{p})\Big(I - \frac{1}{2}\sum_{j=1}^{n} p_j^2 U^2\Big)\Big\| \leqslant C_2 \sum_{j=1}^{n} p_j^3 \min\Big(1, (n\bar{p})^{-1}\Big). \tag{1.7}$$

The estimate (1.7) follows from the more general asymptotic expansion in [2]. For the binomial distribution the estimate becomes $C\min(np^3, p^2)$. Therefore, some improvements are available, for $p = o(1)$ only.

If we apply signed compound Poisson measure matching two moments of the Poisson binomial distribution (just like in the Normal approximation) then we get much better accuracy. Let again $p_j < 1/2$, then

$$\Big\| \prod_{j=1}^{n}(1 + p_j U) - \exp\Big\{n\bar{p}U - \frac{1}{2}\sum_{j-1}^{n} p_j^2 U^2\Big\}\Big\| \leqslant C_3 \sum_{j=1}^{n} p_j^3 \min\Big(1, (n\bar{p})^{-3/2}\Big). \tag{1.8}$$

The estimate follows from a more general theorem 3 in [27], see also theorem 4.1 in [6]. For the binomial distribution we get the following estimate

$$\Big\|(I + pU)^n - \exp\Big\{npU - \frac{np^2}{2}U^2\Big\}\Big\| \leqslant C_4 \min\Big(np^3, \frac{p\sqrt{p}}{\sqrt{n}}\Big). \tag{1.9}$$

Note that (1.9) was proved earlier than (1.8) in [32].

Comparing (1.9) with (1.5) and (1.6) we see that it is uniformly better than the Poisson or Normal approximations. The accuracy in (1.9) is always at least of the order $O(n^{-1/2})$ and always at least of the order $O(p^2)$.

The total variation norm is the main metric considered in this thesis. Therefore, we note only that, in the local metric, the accuracy of approximation is of better order and, in the Wasserstein metric, is of worse order, than can be obtained for the total variation norm . For example,

$$\begin{aligned} \|(I + pU)^n - \mathrm{Pois}(np)\|_\infty &\leqslant C_5 \min(np^2, p(np)^{-1/2}), \\ \|(I + pU)^n - \mathrm{Pois}(np)\|_{\mathrm{W}} &\leqslant C_5 \min(np^2, p(np)^{1/2}), \end{aligned}$$

see Introduction of [8] and [5]. The accuracy of compound Poisson approximation increases dramatically if the initial distribution is symmetric. We give just one example of this phenomena. Let $p < 1/4$, then

$$\|((1-2p)I + pI_1 + pI_{-1})^n - \exp\{np(I_1 + I_{-1} - 2I)\}\| \leqslant C_6 \min(np^2, n^{-1}). \tag{1.10}$$

The estimate can be obtained, for example, from the general Theorem 1 in [45].

Application of the Poisson type approximation to the sum of random variables, which are more general than Bernoulli ones, usually requires fulfilment of some additional restrictive condition. One of the best known of such condition was introduced by Franken [22]. Let X be random variable concentrated on non-negative integers and having distribution F. We then denote its kth factorial moment by

$$\nu_k = \mathrm{E}X(X-1)\cdots(X-k+1).$$

If

$$\nu_1 - \nu_2 - \nu_1^2 > 0, \tag{1.11}$$

then

$$|F^n - \mathrm{Pois}(n\nu_1)| \leqslant C_7 \frac{n(\nu_2 + \nu_1^2)}{1 \vee n(\nu_1 - \nu_2 - \nu_1^2)}. \tag{1.12}$$

The estimate (1.12) is partial case of more general Franken's result for non-identically distributed summands, see [22]. Assumption (1.11) is called *Franken's condition*. In principle, Franken's condition means that almost all probability mass of F is concentrated at zero and unity. It is easy to check that any Bernoulli variable satisfies (1.11). Therefore, we can view (1.12) as a generalization of the Poisson approximation to the binomial law. Though in (1.12) the weaker uniform metric is used, it is not difficult to prove similar result in total variation, applying, for example Theorem 1 from [45]. By adding second factorial cumulant to the Poisson approximation in the exponent we obtain analogue of (1.9). If $\nu_3 < \infty$ and condition (1.11) is satisfied, then

$$\Big|F^n - \exp\Big\{n\nu_1 U + \frac{1}{2}n(\nu_2 - \nu_1^2)U^2\Big\}\Big| \leqslant C_8 \frac{n(\nu_1^3 + \nu_1\nu_2 + \nu_3)}{(1 \vee n(\nu_1 - \nu_2 - \nu_1^2))^{3/2}}. \tag{1.13}$$

The estimate (1.13) follows from Theorem 3 in [28]. The estimate (1.13) can be easily generalized for the total variation norm by summing non-uniform estimates from [13], Theorem 2 and applying estimate for concentration function from p. 117,

$$\begin{aligned}&\Big\|F^n - \exp\Big\{n\nu_1 U + \frac{1}{2}n(\nu_2 - \nu_1^2)U^2\Big\}\Big\| \\ &\qquad\leqslant\; C_9 \frac{n(\nu_1^3 + \nu_1\nu_2 + \nu_3)}{(1 \vee n(\nu_1 - \nu_2 - \nu_1^2))^{3/2}} \Big(1 + \frac{n\nu_1}{n(\nu_1 - \nu_2 - \nu_1^2)}\Big)^2. \end{aligned} \tag{1.14}$$

Let $\nu_i \asymp C$, then (1.14) is of order $O(n^{-1/2})$. Note that, in this case, Poisson approximation (1.12) is of the order $O(1)$ as $n \to \infty$. Any standard Poisson asymptotic expansion gives the same trivial order with respect to n. Thus, two-parametric compound Poisson approximation has some similarities to the normal approximation. One benefit of (1.14) over the standard normal approximation is that it holds for stronger total variation distance.

Roughly our goal is to obtain some analogues of (1.8)–(1.14) in the case of weakly dependent summands. Therefore, let us discuss first what results are already known for m-dependent random variables. Normal approximation to the sum of m-dependent random variables has been thoroughly investigated; see, for example, [25], [26], [41], [43], [44], [46], and the references therein. For the completeness we formulate one result of Sunklodas.

Let $Y_1, Y_2, \ldots$ be m-dependent random variables and $\mathrm{E}Y_i = 0$, $i = 1, \ldots, n$, $d = \max_{1 \le i \le n} \mathrm{E}|Y_i|^s < \infty$, where $2 < s \le 3$. Let

$$S_n = \sum_{i=1}^n Y_i, \quad B_n^2 = \mathrm{E}S_n^2, \quad Z_n = S_n/B_n, \quad F_n(x) = \mathrm{P}(Z_n < x),$$

$$\Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^x \mathrm{e}^{-u^2/2} \mathrm{d}u.$$

Let $B_n^2 \geqslant c_0 n$, where $0 < c_0 < \infty$ and $m + 1 \leqslant (B_n^2/c_0^{1/2})$. Then

1. if $d \geqslant c_0^{s/2}$, then for all $n \geqslant 1$ and $m \geqslant 0$

$$\sup_x |F_n(x) - \Phi(x)| \leqslant C(m+1)^{s-1} \frac{d}{c_0 B_n^{s-2}}, \tag{1.15}$$

2. if $0 < d < c_0^{s/2}$, then for all $n \geqslant 1$ and $m + 1 \geqslant \ln 4\frac{n}{\ln n} = m_0$ we have the same inequality (1.15),

see Theorem 3 from [43].

We see that, in principle, normal approximation has almost the same properties as in the famous Berry-Esseen theorem, that is, it holds for uniform metric and under mild assumptions when $s = 3$ and we do not consider the triangle array it ensures the accuracy of approximation of order $O(n^{-1/2})$. When approximating lattice variables in the total variation metric one can not expect to get non-trivial result due to the differences of supports.

There are numerous papers dealing with Poisson approximations for dependent summands. After Chen's seminal paper [18] the vast majority of them uses the Stein method. Poisson approximation is one-parametric approximation and usually its parameter is chosen to match the mean of approximated distribution. Therefore, the order of accuracy depends on the assumption that almost all probability mass of approximated distribution is concentrated at zero. As a rule, the summands form triangular array and are Bernoulli variables. For some results and discussion; see [40], [8], [1], [4]. We formulate one general result from [1].

Let $X_i \sim Be(p_i)$, $(i = 1, \dots, n)$ be a sequence of Bernoulli random variables and $S = \sum_{i=1}^n X_i$. Assume that the random variables X_i are dependent only in a certain neighborhood defined by

$$N_i := \{X_j : j \neq i, X_j \text{ and } X_i \text{ are dependent}\} \tag{1.16}$$

so that X_i and X_j are nearly independent for $j \notin N_i$. Let $n\bar{p} = \sum_{i=1}^n$, and let $\mathrm{Pois}(n\bar{p})$ denote Poisson distribution with the mean $n\bar{p}$. Let

$$\begin{aligned} b_1 &= \sum_{i=1}^n \sum_{j \in N_i \cup \{i\}} p_i p_j, \\ b_2 &= \sum_{i=1}^n \sum_{j \in N_i} \mathbb{E}(X_i X_j), \\ b_3 &= \sum_{i=1}^n \mathbb{E}|\mathbb{E}(X_j - p_j | S_i)|, \end{aligned}$$

Here $S_i = S - X_i$. Then

$$\|\mathcal{L}(S) - \mathrm{Pois}(n\bar{p})\| \leqslant 2\left[\frac{1 - \mathrm{e}^{-n\bar{p}}}{n\bar{p}}(b_1 + b_2) + b_3\left(1 \wedge \frac{1.4}{\sqrt{n\bar{p}}}\right)\right]. \tag{1.17}$$

There is no general result for two-parametric (signed) compound Poisson approximations to the sums of dependent indicators. There are many studies of the approximation to the Markov binomial distribution, see [14], [15], [48] and the references therein. Note that Markov binomial distribution is a sum of not m-dependent random variables and, in this thesis, is not considered.

The best investigated case of compound Poisson approximations for m-dependent random variables is k-runs statistic. The run statistic was introduced in [30]. It plays important role in reliability theory. The negative binomial approximation to k-runs distribution is investigated in [47]; see also [12], [21]. Translated Poisson approximation to 2-runs is used in [36]. Compound Poisson approximation to 2-runs is applied in [6]. Two-runs statistic has the arguably the simplest dependence structure and is also thoroughly investigated in this thesis. Therefore, we formulate relevant results.

Let $\xi_j \sim Be(p_i)$, $j = 0, 1, 2, \dots, n$, $\eta_j = \xi_j \xi_{j-1}$, $S = \eta_1 + \eta_2 + \cdots + \eta_n$. It is obvious, that η_j are 1-dependent random variables. To take care of the edge effect it is also assumed that ξ_1 depends on ξ_1 (that is, ξ_n is treated as ξ_0). Let

$$\begin{aligned} \tilde{G}_2 &= \exp\{bU + aU^2/2\}, \quad b = \sum_{i=1}^n p_{i-1} p_i, \qquad (1.18) \\ a &= \sum_{i=1}^n p_{i-1} p_i \big[(1 - p_{i-1}) p_{i-2} - (1 - p_i) p_{i+1} + p_{i-1} p_i\big], \\ \gamma &= \sum_{i=1}^n (1 + p_{i+1})^2 p_i (1 - p_i) p_{i-1} - 6 \max_{1 \le j \le n} (1 - p_{j+1})^2 p_j (1 - p_j) p_{j-1}. \end{aligned}$$

If $|a|/b < \frac{1}{2}$, then

$$\|\mathcal{L}(S) - \tilde{G}_2\| \leqslant \frac{9.2}{(b-2|a|)\sqrt{\gamma}} \times \sum_{i=1}^{n} \left[3p_{i-2}p_{i-1}p_ip_{i+1} + p_{i-1}^3p_i^3 + 4p_{i-1}^2p_i^2p_{i+1} + 4p_{i-2}p_{i-1}^2p_i^2 + 7p_{i-3}p_{i-2}^2p_{i-1}^2p_i\right].$$

In particular, if $p_i = p < 1/4$, $n > 7$, then

$$\|\mathcal{L}(S) - \tilde{G}_2\| \leqslant \frac{27.6p + 73.6(p^2+p^3)}{(1-2p(2-3p))\sqrt{(n-6)(1-p)^3}}, \tag{1.19}$$

see Theorem 5.2 from [6]. For small p the order of accuracy, in (1.19), is $Cpn^{-1/2}$. Thus, the accuracy of approximation is always at least of the same order (and for $p = o(1)$ of a better order) than in the Berry-Esseen theorem and better than in Poisson approximation, see (1.20) below. Consequently, we have analogue of (1.9). Note that, in the case $p_i = p$,

$$\tilde{G}_2 = \exp\Big\{np^2U + \frac{1}{2}np^3(2-3p)U^2\Big\}.$$

Though parameters of $\tilde{G}_2$ are chosen to match two factorial cumulants of S, for $p \leqslant 2/3$, the resulting approximation is compound Poisson distribution not a signed measure. Indeed, one can check that

$$\tilde{G}_2 = \exp\Big\{np^2((1-p)^2 + 2p^2)(I_1 - I) + \frac{1}{2}np^3(2-3p)(I_2 - I)\Big\}.$$

The accuracy of Poisson approximation to S can be obtained as a partial case of (1.17). Indeed, we get

$$\|\mathcal{L}(S) - \mathrm{Pois}(np^2)\| \leqslant 2(2+3p)\min(np^3, p). \tag{1.20}$$

We see that Poisson approximation is applicable for small p only. Moreover, the accuracy does not depend on the number of summands n.

Barbour and Xia result was generalized to k-runs in [48]. However, talking about Poisson type approximation for m-dependent integer-valued random variables, we are unaware about

- any three - or more -parametric approximation,
- any lower bound result,
- any non-uniform estimate for discrete approximation,
- estimates in other than total variation metric,
- application of two-parametric approximations, when the summands are not Bernoulli variables.
- any approximation, when the symmetry of distribution is taken into account.

In this thesis, we partially solve all mentioned problems.

1.4 Actuality

Aims and problems

The main problems considered in this thesis are the following:

1. Construction of two and three-parametric Poisson type approximations to the distribution of two-runs statistic.
2. Establishing of the lower bound estimates for two-runs statistic that are of the same order as the upper bound estimates.

3. Calculation of asymptotically sharp constants.

4. Obtaining of non-uniform estimates.

5. Application of (signed) compound Poisson measures for approximation of sums of 1-dependent integer-valued random variables under analogue of Franken's condition.

6. Application of Poisson-type approximation for sums of dependent random variables, when the symmetry of approximated distribution is taken into account;

7. Investigation of possibility to extend results to the case of non-identically distributed random variables.

Methods

The characteristic function method (Heinrich's method) is used in the proofs.

Novelty

All results of the thesis are new. Considering Poisson and (signed) compound Poisson approximations to the sum of 1-dependent random variables, we obtained first lower bound estimates, first asymptotically sharp constants, first nonuniform results for discrete approximations, and first results when symmetry of distribution is taken into account.

Statements presented for the defence

1. Approximation of two-runs statistic by Poisson and compound Poisson distributions has direct similarities to approximation of the binomial distribution: two-parametric compound Poisson approximation is sharper than second order Poisson approximation and can be improved by asymptotic expansion. The accuracy of approximation is estimated in the total variation and the local metrics. For a special case, asymptotically sharp constants are calculated.

2. Lower bound estimates obtained for Kolmogorov metric demonstrates that a) upper bound estimates are of the right order, b) the same order of accuracy can be achieved for the total variation and Kolmogorov norms.

3. Integer random variables, satisfying analogue of Franken's condition can be used for transition from m-dependent to 1-dependent random variables. Signed compound Poisson approximations are of the same order of accuracy as known results for the sums of similar independent random variables.

4. When random variables are symmetric, the accuracy of compound Poisson approximation is much better than in nonsymmetric case.

5. The sum of 1-dependent non-identically distributed Bernoulli variables ia a direct generalization of the Poisson binomial distribution. The accuracy of its approximation by the two-parametric Poisson-type measure is similar to the one, when all summands are independent. This can be said about the local, total variation and Wasserstein norms.

Approbation

Several presentations at conferences were given on the topic of this thesis:

1. V. Čekanavičius, J. Petrauskienė, *On lower bounds for Poisson apporoximation to Two-runs statistic*. LI Conference of the Lithuanian Mathematical Society held at Institute of Mathematics and Informatic on 17-18 June 2010 in Šiauliai, Lithuania.

2. V. Čekanavičius, J. Petrauskienė, *Poisson-type approximation for sums of 1-dependent indicators.* L Conference of the Lithuanian Mathematical Society held at Institute of Mathematics and Informatic on 18-19 June 2009 in Vilnius, Lithuania.

3. V. Čekanavičius, J. Kelmelytė, *Poisson-type approximation for sums of 1-dependent indicators.* XLIX Conference of the Lithuanian Mathematical Society held at Vytautas Magnus University on 25-26 2008 June in Kaunas, Lithuania.

4. V. Čekanavičius, J. Kelmelytė, *Poisson-type approximation for sums of 1-dependent indicators.* Conference "Number theory and probability theory" held on 2007 September in Druskininkai, Lithuania.

Principal publications

The main results on the thesis are published in the following papers:

1. J. Kelmelytė and V. Čekanavičius, Poisson-type approximation for sums of 1-dependent indicators, *Lith. Math. J. 48/49*, (spec. nr.), 395–400, 2008.
2. J. Petrauskienė and V. Čekanavičius, Poisson-type approximation for sums of 1-dependent indicators, *Lith. Math. J. 50*, (spec. nr.), 431–436, 2009.
3. J. Petrauskienė and V. Čekanavičius, On lower bounds for Poisson approximation to 2-runs statistic. *Lith. Math. J. 51*, (spec. nr.), 470–474, 2010.
4. J. Petrauskienė and V. Čekanavičius, Compound Poisson approximations for sums of 1-dependent random variables I, Lith. Math. J. 50(3), 323–336, 2010.
5. J. Petrauskienė and V. Čekanavičius, Compound Poisson approximations for sums of 1-dependent random variables II. *To appear in Lith. Math. J.*, 2010.

Acknowledgments

My first, and most earnest, acknowledgement must go to my scientific supervisor Professor V. Čekanavičius. I would like to express deep and sincere gratitude to Professor for his constant attention, care and belief.
I owe my loving thanks to my parents and husband. Without their encouragement and understanding it would have been impossible for me to finish this work.

1.5 Thesis structure

All results are divided into four parts. The first (and largest) part is devoted to 2-runs, when $p_i = p$. We generalize (1.19) in two directions: by estimating the second order asymptotic expansion and asymptotic expansion in the exponent. Moreover, lower bound estimates are established, proving the optimality of upper bound estimates. Since, the method of proof does not allow to get small constants, in certain cases, we calculate asymptotically sharp constants.

In the second part, we consider sums of 1-dependent random variables, concentrated on non-negative integers and satisfying analogue of Franken's condition. This case is more general than approximation of 2-runs statistic, since the case of independent random variables is also included. All result of this part are comparable to the known results for independent summands.

In the third part, we consider Poisson type approximations for sums of 1-dependent symmetric three-point distributions. Our goal is to prove analogue of (1.10). As already mentioned in above, we are unaware about any Poisson-type approximation result for dependent random variables, when symmetry of the distribution is taken into account. We know about numerous Poisson-type approximations that are obtained via the Stein method. However, the Stein method is applicable to non-negative random variables only. Thus, it can not be applied in our case.

In the last part, we consider 1-dependent non-identically distributed Bernoulli random variables. It is shown, that even for this simple generalization of the Poisson binomial model, very elaborative calculations are needed. However, we succeed to obtain partial generalization of (1.8).

The rest of thesis is devoted to proofs. For the proofs, we use Heinrich's method, which is a version of the characteristic function method, see [25].

Chapter 2

Results

2.1 Compound Poisson approximations for 2-runs statistic

In this section, we generalize and extend known results for 2-runs. Let ξ_j, $j = 0, 1, 2, \dots, n$ be independent identically distributed Bernoulli variables, $P(\xi_1 = 1) = p$, $P(\xi_1 = 0) = 1 - p$. Let $\eta_j = \xi_j \xi_{j-1}$, $S = \eta_1 + \eta_2 + \cdots + \eta_n$. It is obvious, that η_j are 1-dependent random variables. We recall that $\mathcal{L}(S)$ denotes the distribution of S; $C_1, C_2, \dots$ are positive absolute constants, I_a denotes the distribution concentrated at real a, set $I = I_0$. To make expressions shorter we use notation $U = I_1 - I$. All products of measures are understood in convolution sense.

In this section we use the following approximating distributions and measures:

$$G_1 = \mathrm{Pois}(\gamma_1) = \exp\{\gamma_1 U\}, \quad G_2 := \exp\{\gamma_1 U + \gamma_2 U^2\}, \quad G_3 := \exp\{\gamma_1 U + \gamma_2 U^2 + \gamma_3 U^3\}.$$

Here

$$\gamma_1 = np^2, \ \gamma_2 = \frac{np^3(2-3p) - 2p^3(1-p)}{2}, \ \gamma_3 = \frac{np^4(3 - 12p + 10p^2) - 6p^4(1-p)(1-2p)}{3}.$$

Note that G_2 slightly differs from $\tilde{G}_2$ used in (1.19). We allow for the edge effect. Indeed, all η_j ($j = 2, 3, \dots, n-1$) depend on two random variables (η_{j-1} and η_{j+1}). On the other hand, each of variables η_1 and η_n depends on one neighboring random variable only. In (1.19), ξ_0 is treated as ξ_n. Consequently, η_1 is treated as dependent on η_n. We do not assume this simplification. Note that the dependence of variables changes the main probabilistic characteristics of S. For example, though we investigate sum of Bernoulli variables, we have $\mathrm{E}S < \mathrm{Var}S$.

We begin with a demonstration that one can not expect much of an improvement if Poisson approximation is replaced by a standard second order asymptotic expansion.

Theorem 2.1.1 *Let $p \leqslant 1/5$, $n \geqslant 3$. Then*

$$\|\mathcal{L}(S) - G_1(I + \gamma_2 U^2)\| \leqslant C_1 \min(np^4, p^2), \tag{2.1}$$

$$\|\mathcal{L}(S) - G_1(I + \gamma_2 U^2)\|_\infty \leqslant C_2 \min\left(np^4, \frac{p}{\sqrt{n}}\right). \tag{2.2}$$

It is easy to check, that Theorem 2.1.1 is a direct analogous of (1.7) reformulated for 2-runs. Comparing (2.1) with (1.20) we see that both estimates are trivial, if $p = O(1)$. The situation is different for G_2. For the completeness of results we formulate an analogue of (1.19) and add the local estimate.

Theorem 2.1.2 *Let $p \leqslant 1/5$, $n \geqslant 3$. Then*

$$\|\mathcal{L}(S) - G_2\| \leqslant C_3 \min\left(np^4, \frac{p}{\sqrt{n}}\right), \tag{2.3}$$

$$\|\mathcal{L}(S) - G_2\|_\infty \leqslant C_4 \min\left(np^4, \frac{1}{n}\right).$$

As expected, G_2 approximates $\mathcal{L}(S)$ with the same order of accuracy as $\tilde{G}_2$. Since, for $p = Const$, the accuracy in (2.3) is of order $O(n^{-1/2})$, it can be treated as discrete version of the normal approximation.

Unlike the Stein method, which was used for (1.19), we apply the characteristic function method (Heinrich's method); see [25], [26]. As a consequence, we do not get reasonably small constants. Asymptotically sharp constants give an impression of their magnitude. For the completeness of the results we also give asymptotically sharp constants for Poisson approximation. Let

$$\tilde{C}_1 = \frac{4}{\sqrt{2\pi \mathrm{e}}} = 0,967883, \quad \tilde{C}_2 = \frac{1}{\sqrt{2\pi}} = 0,398942, \quad \tilde{C}_3 = \sqrt{\frac{2}{\pi}}(1 + 4\mathrm{e}^{-3/2}) = 1,51,$$

$$\tilde{C}_4 = \sqrt{\frac{3}{\pi}} \exp\Big\{\sqrt{\frac{3}{2}} - \frac{3}{2}\Big\}\sqrt{3 - \sqrt{6}} = 0,550588.$$

Theorem 2.1.3 *Let $p \leqslant 1/5$, $np^2 \geqslant 1$. Then*

$$\begin{aligned}
\Big|\|\mathcal{L}(S) - G_1\| - \tilde{C}_1 p\Big| &\leqslant C\Big(p^2 + \frac{1}{\sqrt{n}}\Big), \\
\Big|\|\mathcal{L}(S) - G_1\|_\infty - \frac{\tilde{C}_2}{\sqrt{n}}\Big| &\leqslant C\Big(\frac{p}{\sqrt{n}} + \frac{1}{np}\Big), \\
\Big|\|\mathcal{L}(S) - G_2\| - \tilde{C}_3 \frac{p}{\sqrt{n}}\Big| &\leqslant C\Big(\frac{p^2}{\sqrt{n}} + \frac{1}{n}\Big), \\
\Big|\|\mathcal{L}(S) - G_2\|_\infty - \frac{\tilde{C}_4}{n}\Big| &\leqslant C\Big(\frac{p}{n} + \frac{1}{n\sqrt{np^2}}\Big).
\end{aligned} \tag{2.4}$$

Corollary 2.1.1 *Let $p \to 0$, $np^2 \to \infty$. Then*

$$\begin{aligned}
\|\mathcal{L}(S) - G_1\| &\sim \tilde{C}_1 p, & \|\mathcal{L}(S) - G_1\|_\infty &\sim \frac{\tilde{C}_2}{\sqrt{n}}. \\
\|\mathcal{L}(S) - G_2\| &\sim \frac{\tilde{C}_3 p}{\sqrt{n}}, & \|\mathcal{L}(S) - G_2\|_\infty &\sim \frac{\tilde{C}_4}{n}.
\end{aligned}$$

The accuracy of approximation can be improved by asymptotic expansions. We can further develop the idea of exponential expansion.

Theorem 2.1.4 *Let $p \leqslant 1/5$, $n \geqslant 3$. Then*

$$\|\mathcal{L}(S) - G_3\| \leqslant C_5 \min\left(np^5, \frac{p}{n}\right), \tag{2.5}$$

$$\|\mathcal{L}(S) - G_3\|_\infty \leqslant C_6 \min\left(np^5, \frac{1}{n\sqrt{n}}\right). \tag{2.6}$$

It can be checked that G_3 is a compound Poisson distribution. Therefore, it can be viewed not as asymptotic expansion, but rather as more sharp probabilistic approximation. However, it is difficult to calculate probabilities of compound Poisson measures with complicated compounding distributions. Therefore, we formulate a second order asymptotic expansion to G_2, which has a more common form.

Theorem 2.1.5 *Let $p \leqslant 1/5$, $n \geqslant 3$. Then*

$$\|\mathcal{L}(S) - G_2(I + \gamma_3 U^3)\| \leqslant C_7 \min\left(np^5, \frac{p}{n}\right), \tag{2.7}$$

$$\|\mathcal{L}(S) - G_2(I + \gamma_3 U^3)\|_\infty \leqslant C_8 \min\left(np^5, \frac{1}{n\sqrt{n}}\right). \tag{2.8}$$

If $p = Const$, then the accuracy in (2.7) is of order $O(n^{-1})$, the same order as the one expected from the appropriate Edgeworth expansion. However, there is one very important difference. As a rule, Edgeworth expansions for lattice distributions incorporate one additional term, which compensates the differences of supports. No such term is needed in (2.7), since *both* distributions are concentrated on integers.

As far as we know, no lower bound estimates were proved for Poisson-type approximations for m-dependent random variables. We obtain lower bound estimates, for Poisson approximation, a second order asymptotic expansion and two-parametric compound Poisson approximation. The estimates are obtained for the uniform and local metrics.

Theorem 2.1.6 *Let $p \leqslant 1/5$, $n \geqslant 3$. Then*

$$|\mathcal{L}(S) - G_1| \geqslant C_9 \min(np^3, p), \tag{2.9}$$

$$\|\mathcal{L}(S) - G_1\|_\infty \geqslant C_{10} \min\left(np^3, \frac{1}{\sqrt{n}}\right). \tag{2.10}$$

Thus, we see that (1.20) is of the correct order.

Theorem 2.1.7 *Let $p \leqslant 1/5$, $n \geqslant 3$. Then*

$$|\mathcal{L}(S) - G_1(I + \gamma_2 U^2)| \geqslant C_{11} \min(np^4, p^2), \tag{2.11}$$

$$\|\mathcal{L}(S) - G_1(I + \gamma_2 U^2)\|_\infty \geqslant C_{12} \min\left(np^4, \frac{p}{\sqrt{n}}\right). \tag{2.12}$$

Theorem 2.1.8 *Let $p \leqslant 1/5$, $n \geqslant 3$. Then*

$$|\mathcal{L}(S) - G_2| \geqslant C_{13} \min\left(np^4, \frac{p}{\sqrt{n}}\right), \tag{2.13}$$

$$\|\mathcal{L}(S) - G_2\|_\infty \geqslant C_{14} \min\left(np^4, \frac{1}{n}\right). \tag{2.14}$$

Since $|M| \leqslant \|M\|$, we see, that, in general, upper bound estimates are of the right order. Moreover, the order of the accuracy of approximation can not be improved if the weaker uniform Kolmogorov metric is used instead of the total variation norm. We also draw conclusion that the accuracy of approximation essentially depends on the chosen form of expansion. Expansion in the exponent is significantly more accurate.

We end this section by formulating non-uniform estimates.

Theorem 2.1.9 *Let $p \leqslant 1/5$, $np^2 \geqslant 1$. Then, for $m = 0, 1, 2, \ldots$,*

$$\Big|(\mathcal{L}(S) - G_1)\{[0,m)\}\Big|\left(1 + \frac{|m - np^2|}{\sqrt{np}}\right) \leqslant C_9 p, \tag{2.15}$$

$$\Big|(\mathcal{L}(S) - G_2)\{[0,m)\}\Big|\left(1 + \frac{(m - np^2)^2}{np^2}\right) \leqslant C_{10} \frac{p}{\sqrt{n}}, \tag{2.16}$$

$$\Big|(\mathcal{L}(S) - G_3)\{[0,m)\}\Big|\left(1 + \frac{(m - np^2)^2}{np^2}\right) \leqslant C_{11} \frac{p}{n}. \tag{2.17}$$

We see that (2.15)–(2.17) give estimates comparable to the total variation estimates in (1.20)–(2.5). Of course, when m is far from the mean, one can not expect our result to be very accurate. Then some large deviation result is needed.

Summing up nonuniform estimates leads us to the following corollary.

Corollary 2.1.2 *If $p \leqslant 1/5$ and $np^2 \geqslant 1$, then*

$$\|\mathcal{L}(S) - G_2\|_{\mathrm{W}} \leqslant Cp^2,$$

$$\|\mathcal{L}(S) - G_3\|_{\mathrm{W}} \leqslant C\frac{p^2}{\sqrt{n}}.$$

As expected the order of accuracy is worse than in total variation.

2.2 Compound Poisson approximations for sums of 1-dependent random variables under analogue of Franken's condition

When we deal with 2-runs statistic, the corresponding random variables η_j have very explicitly defined 1-dependence.In this section, we consider sums of more general 1-dependent random variables, concentrated on nonnegative integers. This case of integer-valued random variables, which satisfy Franken's condition, is more general than 2-runs statistic. For example, all results hold also for sums of independent random variables. On the other hand, results are not so explicit. When proving results of this Section we seek to get such results, that, one one hand, for 2-runs statistic are comparable to the results from previous section and, on the other hand, for independent summands are comparable to known results, such as (1.12) and (1.14).

For the sake of convenience we repeat the main notation. Let X be random variable concentrated on non-negative integers. We then denote its kth factorial moment by

$$\nu_k = \mathrm{E}X(X-1)\cdots(X-k+1).$$

Let $X_1, X_2, \dots, X_n$ be identically distributed 1-dependent random variables concentrated on non-negative integers. Let, for $m = 1, 2, \dots;\ i_j = 1, 2, \dots;\ j = 1, \dots, m$,

$$a(i_1, i_2, \dots, i_m) = \frac{\mathrm{E}X_1(X_1-1)\cdots(X_1-i_1+1)\cdots X_m(X_m-1)\cdots(X_m-i_m+1)}{i_1! i_2! \cdots i_m!}.$$

For formulation of our results we need the following notation. Let

$$\Gamma_1 = n\nu_1, \quad \Gamma_2 = \frac{n(\nu_2 - \nu_1^2)}{2} + (n-1)(a(1,1) - \nu_1^2),$$

$$\begin{aligned}
\Gamma_3 &= n\Big(\frac{\nu_3}{6} - \frac{\nu_1\nu_2}{2} + \frac{\nu_1^3}{3}\Big) + (n-1)\Big(a(1,2) + a(2,1) - \nu_1\nu_2 + 2\nu_1(\nu_1^2 - a(1,1))\Big) \\
&\quad + (n-2)(a(1,1,1) - 2\nu_1 a(1,1) + \nu_1^3), \\
r_1 &= \nu_3 + \nu_1\nu_2 + \nu_1^3 + a(1,2) + a(2,1) + \nu_1 a(1,1) + a(1,1,1), &(2.18)\\
r &= a(3,1) + a(2,2) + a(1,3) + a(1,1,1,1) + a^2(1,1) + a(2,1,1) + a(1,2,1) \\
&\quad + a(1,1,2) + \nu_1 a(2,1) + \nu_1 a(1,2) + \nu_1 a(1,1,1) + \nu_4 + \nu_1\nu_3 + \nu_2^2 + \nu_1^4. &(2.19)
\end{aligned}$$

The distribution of $X_1 + X_2 + \cdots + X_n$ we denote by F_n.

Recalling the definition of factorial moments, we see that

$$a(1) = \nu_1, \quad a(2) = \frac{\nu_2}{2}, \quad a(3) = \frac{\nu_3}{3!}, \quad a(1,1) = \mathrm{E}X_1X_2.$$

Further on we use notation $a(i,j)$ for mixed moments only, since then, it is easier to compare our results with (1.14).

We assume that, for $n \to \infty$,

$$\nu_1 = o(1), \quad \nu_2 = o(\nu_1), \quad a(1,1) = o(\nu_1), \quad |X_1| \leqslant C_{15}, \quad n\nu_1 \to \infty. \tag{2.20}$$

Further on we assume $C_{15} \geqslant 1$. It is easy to check that (2.20) is stronger than Franken's condition (1.11). On the other hand, there are many variables which satisfy condition (2.20). As can be checked, 2-runs statistic satisfies (2.20) if $p \to 0$. However, we think that investigation of 1-dependent random variables under (2.20) is more natural in the following context. Let us consider the sum of m-dependent indicator variables. Redefining partial sums consisting of m subsequent variables as new random variables we switch from m-dependent case to 1-dependent case. The new variables now are not the indicator variables. However, one can expect that under quite mild assumptions on the initial variables the analogue of Franken's condition will hold.

For example, let us consider 3-runs, that is $\xi_1\xi_2\xi_3 + \xi_2\xi_3\xi_4 + \xi_3\xi_4\xi_5 + \cdots$, where ξ_j are i.i.d. Bernoulli variables, $P(\xi_1 = 1) = p = 1 - P(\xi_1 = 0)$. Let $\tilde\eta_1 = \xi_1\xi_2\xi_3 + \xi_2\xi_3\xi_4$, $\tilde\eta_2 = \xi_3\xi_4\xi_5 + \xi_4\xi_5\xi_6$, etc. It is easy to check that $\tilde{\eta}_1, \tilde{\eta}_2, \tilde{\eta}_3, \dots$ form the series of 1-dependent random variables satisfying (2.20), provided that $p \to 0$.

It also should be mentioned, that, in principle, (2.20) can be replaced by weaker condition: $\nu_2 < \tilde{C}\nu_1$, $|a(1,1)| \leqslant \tilde{C}\nu_1$, where $\tilde{C}$ is very small absolute constant. However, the proofs then become extremely long.

Distributions satisfying (2.20) have dominating probability mass at zero. This condition is natural for the so-called aggregate claim distribution in the individual model. More precisely, if we assume that $X_i = \bar{\xi}_i \bar{\eta}_i$ where $\bar{\xi}_i$ and $\bar{\eta}_i$ are independent, $\bar{\xi}_i$ is Bernoulli random variable and $\bar{\eta}_i$ is a positive random variable concentrated on integers, then one can give the following interpretation: $\bar{\xi}_i$ reflects possibility of occurrence of claim with the small probability and $\bar{\eta}_i$ has the distribution of the claim amount. Then, though $\bar{\eta}_i$ can have large factorial moments, the factorial moments of X_i are small.

Let us define measures for approximations of F_n:

$$D_1 = \exp\{\Gamma_1 U\}, \quad D_2 = \exp\{\Gamma_1 U + \Gamma_2 U^2\}, \quad D_3 = \exp\{\Gamma_1 U + \Gamma_2 U^2 + \Gamma_3 U^3\}.$$

Note that, in general, we deal with signed measures, since Γ_2, Γ_3 can be negative. For the completeness of results we begin from the Poisson approximation.

Theorem 2.2.1 *Let assumptions (2.20) be satisfied. Then*

$$\|F_n - D_1\| = O\Big(\frac{\nu_2 + a(1,1) + \nu_1^2}{\nu_1}\Big), \qquad (2.21)$$

$$\|F_n - D_1\|_\infty = O\Big(\frac{\nu_2 + a(1,1) + \nu_1^2}{\nu_1\sqrt{n\nu_1}}\Big). \qquad (2.22)$$

In (2.21) the accuracy is no better than $O(\nu_1)$. Though we know that due to assumption (2.20) $\nu_1 = o(1)$, its convergence to zero can be very slow. The next theorem shows that situation can not be much improved by the standard Poisson asymptotic expansion.

Theorem 2.2.2 *Let assumptions (2.20) be satisfied. Then*

$$\|F_n - D_1(I + \Gamma_2 U^2)\| = O\Big(\frac{(\nu_2 + a(1,1) + \nu_1^2)^2}{\nu_1^2} + \frac{r_1}{\nu_1\sqrt{n\nu_1}}\Big), \qquad (2.23)$$

$$\|F_n - D_1(I + \Gamma_2 U^2)\|_\infty = O\Big(\frac{(\nu_2 + a(1,1) + \nu_1^2)^2}{\nu_1^2\sqrt{n\nu_1}} + \frac{r_1}{n\nu_1^2}\Big). \qquad (2.24)$$

We see that the second-order Poisson approximation improved the accuracy of approximation. However, it is no better than $O(\nu_1^2)$, which can mean a very poor accuracy. Let us check how approximation improves when the signed compound Poisson approximations are applied.

Theorem 2.2.3 *Let assumptions (2.20) be satisfied. Then*

$$\|F_n - D_2\| = O\Big(\frac{r_1}{\nu_1\sqrt{n\nu_1}}\Big), \qquad (2.25)$$

$$\|F_n - D_2\|_\infty = O\Big(\frac{r_1}{n\nu_1^2}\Big). \qquad (2.26)$$

The accuracy in (2.25) is at least of the order $O((n\nu_1)^{-1/2})$. Moreover, if, in addition, we assume that all X_i are independent, then the order of accuracy coincides with the right hand-side of (1.14). It is natural to expect that asymptotic expansion to D_2 will improve the accuracy even more.

Theorem 2.2.4 *Let assumptions (2.20) be satisfied. Then*

$$\|F_n - D_3\| = O\Big(\frac{r}{n\nu_1^2}\Big), \qquad (2.27)$$

$$\|F_n - D_3\|_\infty = O\Big(\frac{r}{n\nu_1^2\sqrt{n\nu_1}}\Big). \qquad (2.28)$$

In Theorem 2.2.4, we used longer expansion in the exponent. One can also apply an expansion of a more standard form.

Theorem 2.2.5 *Let assumptions (2.20) be satisfied. Then*

$$\|F_n - D_2(I+\Gamma_3 U^3)\| = O\Big(\frac{r}{n\nu_1^2} + \frac{r_1^2}{n\nu_1^3}\Big), \tag{2.29}$$

$$\|F_n - D_2(I+\Gamma_3 U^3)\|_\infty = O\Big(\frac{r}{n\nu_1^2\sqrt{n\nu_1}} + \frac{r_1^2}{n\nu_1^3\sqrt{n\nu_1}}\Big). \tag{2.30}$$

From the point of practical calculations, $D_2(I+\Gamma_3 U^3)$ is simpler than D_3, because convolution with U^3 just means the third backward difference for the 'probabilities' of D_2.

It was mentioned in above, that if $p = o(1)$ and $np^2 \to \infty$, then 2-runs statistic considered in the first part of this paper satisfies assumptions (2.20). For 2-runs statistic $\nu_1 = p^2$, $a(1,1) = p^3$, $a(1,1,1) = p^4$, $a(1,1,1,1) = p^5$ and all other quantities ($\nu_2, \nu_3, a(2,1)\dots$) are equal to zero. Therefore, it is not difficult to check that (2.21) – (2.30) have the same order of accuracy as corresponding estimates from the first part of this paper with absolute constants replaced by the symbol $O(\cdot)$. If we consider the case of independent summands, then (2.21) and (2.25) have the same order of accuracy as (1.12) and (1.14), respectively.

2.3 Poisson type approximations for sums of 1-dependent symmetric three point distributions

In this section, we prove one analogue of (1.10). As far as we know symmetry of distribution so far was not taken into account, when compound Poisson approximations were used for sums of weakly dependent random variables.

Let $\hat{X}_j$, $j = 1, 2, \dots, n$ be a triangular array of 1-dependent identically distributed three-point random variables, $P(\hat{X}_j = 1) = p_1$, $P(\hat{X}_j = -1) = p_{-1}$, $P(\hat{X}_j = 0) = 1 - p_1 - p_{-1}$. We denote the distribution and characteristic function of $S_n = \hat{X}_1 + \hat{X}_2 + \dots + \hat{X}_n$ by P_n and $\widehat{P}_n(t)$, respectively. Let $z = \mathrm{e}^{\mathrm{i}t} - 1$, $z_{-1} = \bar{z} = \mathrm{e}^{-\mathrm{i}t} - 1$, $\bar{p} = p_{-1} + p_1$,

$$\begin{aligned}
&h(j_1, j_2) = P(\hat{X}_1 = j_1, \hat{X}_2 = j_2) - P(\hat{X}_1 = j_1)P(\hat{X}_2 = j_2),\\
&h(j_1, j_2, j_3) = P(\hat{X}_1 = j_1, \hat{X}_2 = j_2, \hat{X}_3 = j_3) - P(\hat{X}_1 = j_1)P(\hat{X}_2 = j_2)P(\hat{X}_3 = j_3),\\
&b_j = \mathrm{E}(\mathrm{e}^{\mathrm{i}t\hat{X}_1} - 1)\cdots(\mathrm{e}^{\mathrm{i}t\hat{X}_j} - 1),\\
&H_j = \widehat{\mathrm{E}}(\mathrm{e}^{\mathrm{i}t\hat{X}_1} - 1)\cdots(\mathrm{e}^{\mathrm{i}t\hat{X}_j} - 1) := b_j - \sum_{k=1}^{j-1} H_k b_{j-k}, \quad H_1 = p_1 x + p_{-1}\bar{x},\\
&K_1 = |h(-1,-1) - h(-1,1) - h(1,-1) + h(1,1)| + \bar{p}|p_1 - p_{-1}|,\\
&K_2 = |h(-1,1) - 2h(1,1) + h(1,-1)| + \sum_{j,k\in\{-1,1\}} |h(j,k,-1) - h(j,k,1)|,\\
&K_3 = \sum_{j,k\in\{-1,1\}} |h(j,k)| + \bar{p}^2.
\end{aligned}$$

Our goal is to investigate the closeness of P_n to its accompanying compound Poisson law. More precisely, let B^n be a compound Poisson distribution with the following characteristic function:

$$\widehat{B}^n(t) = \exp\{nH_1\} = \exp\{np_1 z + np_{-1}\bar{z}\}.$$

The closeness is estimated in the uniform Kolmogorov and local metrics.

Theorem 2.3.1 *Let*

$$\sum_{j,k\in\{-1,1\}} |h(j,k)|/\bar{p} + 90\sqrt{\bar{p}} \leqslant 1/3. \tag{2.31}$$

Then, for all $n = 1, 2, \ldots,$

$$\sup_x |P_n\{(-\infty, x]\} - B^n\{(-\infty, x]\}| \leqslant C_{16} n K_1 \min\left(1, \frac{1}{n\bar{p}}\right) + \\ + C_{17} n K_2 \min\left(1, \frac{1}{n\bar{p}\sqrt{n\bar{p}}}\right) + C_{18} n K_3 \min\left(1, \frac{1}{(n\bar{p})^2}\right) \tag{2.32}$$

and

$$\sup_x |P_n\{x\} - B^n\{x\}| \leqslant C_{19} n K_1 \min\left(1, \frac{1}{n\bar{p}\sqrt{n\bar{p}}}\right) + \\ + C_{20} n K_2 \min\left(1, \frac{1}{(n\bar{p})^2}\right) + C_{21} n K_3 \min\left(1, \frac{1}{(n\bar{p})^2\sqrt{n\bar{p}}}\right). \tag{2.33}$$

Condition (2.31) is a technical one and quite probably can be improved. It is only marginally better than $p = o(1)$, $a(j,k) = o(p)$. Formally, it allows for p to be a (very) small absolute constant. If $\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_n$ are symmetric independent random variables, then the right-hand-side in (2.32) becomes $C_{22} n^{-1}$, which is consistent with (1.10). It is not difficult to construct an example of dependent array, which satisfies (2.31). Let $\xi_1, \xi_2, \ldots$ be symmetric i.i.d. r.v., having distribution $P(\xi_1 = 1) = P(\xi_1 = -1) = \alpha$, $P(\xi_1 = 0) = 1 - 2\alpha$. Let $X_1 = \xi_1\xi_2$, $X_2 = \xi_2\xi_3$, etc. If $\alpha = o(1)$, then (2.31) holds and the accuracy of approximation in (2.32) is $O(n\alpha^3 \wedge (n\alpha)^{-1})$.

2.4 Poisson type approximations for sums of 1-dependent non-identically distributed Bernoulli variables

In this section, we obtain some generalization of (1.8). Let $\tilde{X}_j$, $j = 1, 2, \ldots, n$ be a sequence of 1-dependent not identically distributed Bernoulli variables, $P(\tilde{X}_j = 1) = p_j = 1 - P(\tilde{X}_j = 0)$. We denote the distribution and characteristic function of $\tilde{S} = \tilde{X}_j + \tilde{X}_{j+1} + \cdots + \tilde{X}_{j+n-1}$ by $\mathcal{L}(\tilde{S})$ and $\widehat{M}_n(t)$ respectively. Let $\lambda_i = \sum_{k=1}^n p_k^i$, $(i = 1, 2)$, for the sake of brevity, denote by $\lambda = \lambda_1$.

Further we need the following notation. Let

$$\tilde{p}_{i,j} = P(\tilde{X}_j = 1, \tilde{X}_{j+1} = 1, \ldots, \tilde{X}_{j+i-1} = 1),$$

and let

$$a_{i,j} = \widehat{\mathrm{E}}\tilde{X}_j\tilde{X}_{j+1}\ldots\tilde{X}_{j+i-1} = \\ = \sum_{l=1}^{i}(-1)^{l-1} \sum_{\substack{i_1+\cdots+i_l=i \\ i_m \geqslant 1}} E\tilde{X}_j \ldots \tilde{X}_{j+i_1-1} E\tilde{X}_{j+i_1} \ldots \tilde{X}_{j+i_1+i_2-1} \ldots E\tilde{X}_{j+i_1+\cdots+i_{l-1}} \ldots \tilde{X}_{j+i-1}.$$

Note that due to 1-dependence and Holder's inequality

$$\tilde{p}_{i,j} = \begin{cases} \mathrm{E}\tilde{X}_1 \ldots \tilde{X}_j \leqslant \sqrt{\mathrm{E}\tilde{X}_1^2\tilde{X}_3^2 \ldots \tilde{X}_{j+i-2}^2 \mathrm{E}\tilde{X}_2^2 \ldots \tilde{X}_{j+i-1}^2} \\ \quad = \sqrt{p_j p_{j+1} \ldots p_{j+i-1}} & \text{if } j+i-1 \text{ - even number}, \\ \mathrm{E}\tilde{X}_1 \ldots \tilde{X}_j \leqslant \sqrt{\mathrm{E}\tilde{X}_1^2\tilde{X}_3^2 \ldots \tilde{X}_{j+i-1}^2 \mathrm{E}\tilde{X}_2^2 \ldots \tilde{X}_{j+i-2}^2} \\ \quad = \sqrt{p_j p_{j+1} \ldots p_{j+i-1}} & \text{if } j+i-1 \text{ - odd number}. \end{cases} \tag{2.34}$$

Consequently,

$$a_{i,j} \leqslant C\sqrt{p_j p_{j+1} \ldots p_{j+i-1}}. \tag{2.35}$$

If r.v. are independent then $a_{i,j} = 0$. For any real t and $k \geqslant j$ we have

$$a_{i,j}(\mathrm{e}^{\mathrm{i}t} - 1)^i = \widehat{\mathrm{E}}(\mathrm{e}^{\mathrm{i}t\tilde{X}_{j-i+1}} - 1)(\mathrm{e}^{\mathrm{i}t\tilde{X}_{j-i+2}} - 1)\ldots(\mathrm{e}^{\mathrm{i}t\tilde{X}_j} - 1).$$

For approximation of $\mathcal{L}(\tilde{S})$ we use signed compound Poisson measures M_i, $i = 1, 2, 3$:

$$M_1 = \mathrm{e}^{\lambda U}, \quad M_2 = \exp\Big\{\lambda U + \Big(\sum_{k=2}^{n} a_{2,k} - \frac{1}{2}\lambda_2\Big)U^2\Big\},$$

$$\begin{aligned} M_3 &= \exp\Big\{\lambda U + \Big(\sum_{k=2}^{n} a_{2,k} - \frac{1}{2}\lambda_2\Big)U^2 \\ &\quad + \Big(\sum_{k=3}^{n} a_{3,k} - \sum_{k=2}^{n} a_{2,k}(p_k + p_{k-1}) + \frac{1}{3}\lambda_3\Big)U^3\Big\}. \end{aligned}$$

Moreover, let

$$\begin{aligned} M_{11} &= \Big(\sum_{k=2}^{n} a_{2,k} - \frac{1}{2}\lambda_2\Big), \\ M_{21} &= \sum_{k=3}^{n} a_{3,k} - \sum_{k=2}^{n} a_{2,k}(p_k + p_{k-1}) + \frac{1}{3}\lambda_3. \end{aligned}$$

The results of this section will be obtained under the following assumption

$$\begin{cases} \max_{1\le j\le n} p_j = o(1), \\ \sum_{j=1}^{n} a_{2,j} = o(\lambda_1). \end{cases} \tag{2.36}$$

In principle, the first condition in (2.36) can be replaced by the weaker one, requiring $\max p_j$ to be smaller than some absolute constant. We assume that p_j is small and the dependence of variables is weak. Unfortunately, due to the estimates used in the proofs, the constant is very small.

We can formulate our results.

Theorem 2.4.1 *Let (2.36) hold. Then, for all $n = 1, 2, \ldots,$*

$$\begin{aligned} \|\mathcal{L}(\tilde{S}) - M_1\|_\infty &= O\Big\{\sum_{k=1}^{n} \big(p_k^2 + |2a_{2,k}|\big) \min\Big(1, \frac{1}{\lambda^{3/2}}\Big) \\ &\quad + \sum_{k=1}^{n} \big(|a_{3,k}|\big) \min\Big(1, \frac{1}{\lambda^2}\Big)\Big\}, \end{aligned} \tag{2.37}$$

$$\begin{aligned} \|\mathcal{L}(\tilde{S}) - M_1\| &= O\Big\{\sum_{k=1}^{n} \big(p_k^2 + |2a_{2,k}|\big) \min\Big(1, \frac{1}{\lambda}\Big) \\ &\quad + \sum_{k=1}^{n} \big(|a_{3,k}|\big) \min\Big(1, \frac{1}{\lambda^{3/2}}\Big)\Big\}, \end{aligned} \tag{2.38}$$

$$\begin{aligned} \|\mathcal{L}(\tilde{S}) - M_1\|_{\mathrm{W}} &= O\Big\{\sum_{k=1}^{n} \big(p_k^2 + |2a_{2,k}|\big) \min\Big(1, \frac{1}{\lambda^{1/2}}\Big) \\ &\quad + \sum_{k=1}^{n} \big(|a_{3,k}|\big) \min\Big(1, \frac{1}{\lambda}\Big)\Big\}. \end{aligned} \tag{2.39}$$

Theorem 2.4.2 *Let (2.36) hold. Then, for all* $n = 1, 2, \ldots,$

$$\begin{aligned}\|\mathcal{L}(\tilde{S}) - M_2\|_\infty &= O\Big\{\sum_{k=1}^{n}(p_k^3 + |a_{3,k}| + |a_{2,k}|p_k)\min\Big(1, \frac{1}{\lambda^2}\Big) \\ &+ \sum_{k=1}^{n}\big(|a_{4,k}| + a_{2,k}^2 + |a_{3,k}|p_k\big)\min\Big(1, \frac{1}{\lambda^{5/2}}\Big) \\ &+ \sum_{k=1}^{n}\big(|a_{5,k}| + |a_{2,k}a_{3,k}|\big)\min\Big(1, \frac{1}{\lambda^3}\Big)\Big\}, \qquad (2.40)\end{aligned}$$

$$\begin{aligned}\|\mathcal{L}(\tilde{S}) - M_2\| &= O\Big\{\sum_{k=1}^{n}(p_k^3 + |a_{3,k}| + |a_{2,k}|p_k)\min\Big(1, \frac{1}{\lambda^{3/2}}\Big) \\ &+ \sum_{k=1}^{n}\big(|a_{4,k}| + a_{2,k}^2 + |a_{3,k}|p_k\big)\min\Big(1, \frac{1}{\lambda^2}\Big) \\ &+ \sum_{k=1}^{n}\big(|a_{5,k}| + |a_{2,k}a_{3,k}|\big)\min\Big(1, \frac{1}{\lambda^{5/2}}\Big)\Big\}, \qquad (2.41)\end{aligned}$$

$$\begin{aligned}\|\mathcal{L}(\tilde{S}) - M_2\|_{\mathrm{W}} &= O\Big\{\sum_{k=1}^{n}(p_k^3 + |a_{3,k}| + |a_{2,k}|p_k)\min\Big(1, \frac{1}{\lambda}\Big) \\ &+ \sum_{k=1}^{n}\big(|a_{4,k}| + a_{2,k}^2 + |a_{3,k}|p_k\big)\min\Big(1, \frac{1}{\lambda^{3/2}}\Big) \\ &+ \sum_{k=1}^{n}\big(|a_{5,k}| + |a_{2,k}a_{3,k}|\big)\min\Big(1, \frac{1}{\lambda^2}\Big)\Big\}. \qquad (2.42)\end{aligned}$$

Theorem 2.4.3 *Let (2.36) hold. Then, for all* $n = 1, 2, \ldots,$

$$\begin{aligned}\|\mathcal{L}(\tilde{S}) - M_3\|_\infty &= O\Big\{R_4\min\Big(1, \frac{1}{\lambda_1^{5/2}}\Big) + R_5\min\Big(1, \frac{1}{\lambda^3}\Big) \\ &+ R_6\min\Big(1, \frac{1}{\lambda^{7/2}}\Big) + R_7\min\Big(1, \frac{1}{\lambda^4}\Big)\Big\}, \qquad (2.43)\end{aligned}$$

$$\begin{aligned}\|\mathcal{L}(\tilde{S}) - M_3\| &= O\Big\{R_4\min\Big(1, \frac{1}{\lambda^2}\Big) + R_5\min\Big(1, \frac{1}{\lambda^{5/2}}\Big) \\ &+ R_6\min\Big(1, \frac{1}{\lambda^3}\Big) + R_7\min\Big(1, \frac{1}{\lambda^{7/2}}\Big)\Big\}, \qquad (2.44)\end{aligned}$$

$$\begin{aligned}\|\mathcal{L}(\tilde{S}) - M_3\|_{\mathrm{W}} &= O\Big\{R_4\min\Big(1, \frac{1}{\lambda^{3/2}}\Big) + R_5\min\Big(1, \frac{1}{\lambda^2}\Big) \\ &+ R_6\min\Big(1, \frac{1}{\lambda^{5/2}}\Big) + R_7\min\Big(1, \frac{1}{\lambda^3}\Big)\Big\}, \qquad (2.45)\end{aligned}$$

where

$$
\begin{aligned}
R_4 &= \sum_{k=1}^{n}\big(p_k^4+|a_{2,k}|(p_k^2+|a_{2,k}|)+|a_{3,k}|p_k+|a_{4,k}|\big),\\
R_5 &= \sum_{k=1}^{n}\big(p_k a_{2,k}^2+|a_{2,k}a_{3,k}|+|a_{3,k}|p_k^2+|a_{4,k}|p_k+|a_{5,k}|\big),\\
R_6 &= \sum_{k=1}^{n}\big(|a_{2,k}a_{3,k}|p_k+a_{3,k}^2+|a_{2,k}a_{4,k}|+|a_{2,k}|^3+|a_{5,k}|p_k+|a_{6,k}|\big),\\
R_7 &= \sum_{k=1}^{n}\big(a_{2,k}^2|a_{3,k}|+|a_{3,k}a_{4,k}|+|a_{2,k}a_{5,k}|+|a_{7,k}|\big).
\end{aligned}
$$

Theorem 2.4.4 *Let (2.36) hold. Then, for all* $n = 1, 2, \dots,$

$$\|\mathcal{L}(\tilde{S}) - M_1(I + M_{11})\| = O\bigg\{\bigg(\sum_{k=1}^{n}(p_k^2 + |2a_{2,k}|)\bigg)^2 \min\Big(1, \frac{1}{\lambda^2}\Big)\bigg\}. \tag{2.46}$$

Theorem 2.4.5 *Let (2.36) hold. Then, for all* $n = 1, 2, \dots,$

$$\|\mathcal{L}(\tilde{S}) - M_2(1 + M_{21})\| = O\bigg\{\bigg(\sum_{k=1}^{n}(p_k^3 + |a_{3,k}| + |a_{2,k}|p_k)\bigg)^2 \min\Big(1, \frac{1}{\lambda^3}\Big)\bigg\}. \tag{2.47}$$

Examples

It is easy to check that if, in Theorem 2.4.2 and Theorem 2.4.3, $\tilde{X}_1, \tilde{X}_2, \dots, \tilde{X}_n$ are independent r.v., then

1. Local estimates (2.37), (2.40) and (2.43) become:

$$\|\mathcal{L}(\tilde{S}) - M_1\|_\infty = O\Big(\lambda_2 \min\Big(1, \frac{1}{\lambda^{3/2}}\Big)\Big),$$

$$\|\mathcal{L}(\tilde{S}) - M_2\|_\infty = O\Big(\lambda_3 \min\Big(1, \frac{1}{\lambda^2}\Big)\Big),$$

$$\|\mathcal{L}(\tilde{S}) - M_3\|_\infty = O\Big(\lambda_4 \min\Big(1, \frac{1}{\lambda^{5/2}}\Big)\Big).$$

2. Total variations (2.38), (2.41) and (2.44) becomes:

$$\|\mathcal{L}(\tilde{S}) - M_1\| = O\Big(\lambda_2 \min\Big(1, \frac{1}{\lambda}\Big)\Big),$$

$$\|\mathcal{L}(\tilde{S}) - M_2\| = O\Big(\lambda_3 \min\Big(1, \frac{1}{\lambda^{3/2}}\Big)\Big),$$

$$\|\mathcal{L}(\tilde{S}) - M_3\| = O\Big(\lambda_4 \min\Big(1, \frac{1}{\lambda^2}\Big)\Big),$$

   which up to constant coincides with (1.8) and similar results for other metrics in [27].

3. Let us consider 2-runs as defined in the first section of this thesis. Then from (2.40) and (2.41) we obtain

$$\|\mathcal{L}(\tilde{S}) - M_2\|_\infty = O\Big(\frac{1}{n}\Big)$$

   and

$$\|\mathcal{L}(\tilde{S}) - M_2\| = O\Big(\frac{p}{\sqrt{n}}\Big).$$

Both estimates are of the right order, see Theorem 2.1.2. Note that, in this case, $M_2 = G_2$.

Chapter 3

Proofs

3.1 General auxiliary results

Further on we denote by C positive absolute constants. The letter Θ stands for any finite signed measure on $\mathbb{Z}$ satisfying $\|\Theta\| \leqslant 1$. The values of C and Θ can vary from line to line, or even within the same line. Sometimes to avoid possible ambiguity, the C are supplied with indices. Throughout this paper, we set $0^0 = 1$.

If $\|M - I\| < 1$, then we set the logarithm of measure M, to be

$$lnM = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (M - I)^k.$$

For the proof of theorems we need general auxiliary results.

Lemma 3.1.1 *Let M be a finite variation measure concentrated on integers. For all $v \in \mathbb{R}$ and $u > 0$, we then have*

$$\|M\| \leqslant \big(1 + u\pi\big)^{1/2} \bigg(\frac{1}{2\pi} \int\limits_{-\pi}^{\pi} |\widehat{M}(t)|^2 + \frac{1}{u^2} |\big(\mathrm{e}^{-itv} \widehat{M}(t)\big)'|^2 \mathrm{d}t \bigg)^{1/2}, \tag{3.1}$$

and

$$\|M\|_\infty \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} |\widehat{M}(t)| \mathrm{d}t, \tag{3.2}$$

$$|M| \leqslant \frac{1}{4} \int_{-\pi}^{\pi} \frac{|\widehat{M}(t)|}{|t|} \mathrm{d}t. \tag{3.3}$$

The estimate (3.1) is well-known; see, for example, [33]. The estimate (3.2) follows from the formula of inversion, the estimate (3.3) is the well-known Tsaregradskii's inequality.

In the following four lemmas, $C(k)$ denotes an absolute positive constant depending on k.

Lemma 3.1.2 *Let M be concentrated on $\mathbb{Z}$, $\alpha \in \mathbb{R}$, $b \geqslant 1$. Then,*

$$|M| \geqslant C \bigg| \int_{-\infty}^{\infty} \mathrm{e}^{-t^2/2} \widehat{M}\Big(\frac{t}{b}\Big) \mathrm{e}^{-\mathrm{i}t\alpha} \mathrm{d}t \bigg|, \tag{3.4}$$

$$\|M\|_\infty \geqslant \frac{C}{b} \bigg| \int_{-\infty}^{\infty} \mathrm{e}^{-t^2/2} \widehat{M}\Big(\frac{t}{b}\Big) \mathrm{e}^{-\mathrm{i}t\alpha} \mathrm{d}t \bigg|. \tag{3.5}$$

The estimates (3.4) and (3.5) remain valid if $\mathrm{e}^{-t^2/2}$ is replaced by $t\mathrm{e}^{-t^2/2}$.

Lemma's proof can be found in [47].

Lemma 3.1.3 *Let $t \in (0,\infty)$ and $k = 0,1,2,\dots$. We then have*

$$\|U^2 \mathrm{e}^{tU}\| \leqslant \frac{3}{t\mathrm{e}}, \qquad \|U^k \mathrm{e}^{tU}\| \leqslant \Big(\frac{2k}{t\mathrm{e}}\Big)^{k/2}, \qquad \|U^k \mathrm{e}^{tU}\|_\infty \leqslant \frac{C(k)}{t^{(k+1)/2}}.$$

The first inequality was proven in [37]. The second bound follows from formula (3.8) in [19] and the properties of the total variation norm. The third relation follows from the formula of inversion.

For our asymptotically sharp estimates we need the following lemma. Set

$$\varphi_k(x) = \frac{1}{\sqrt{2\pi}} \frac{\mathrm{d}^k}{\mathrm{d}x^k} \mathrm{e}^{-x^2/2}, \quad \|\varphi_k\|_1 = \int_{\mathbb{R}} |\varphi_k(x)|\,\mathrm{d}x, \quad \|\varphi_k\|_\infty = \sup_{x\in\mathbb{R}} |\varphi_k(x)| \quad (k = 0,1,\dots).$$

Lemma 3.1.4 *Let $t > 0$ and $k = 0,1,2\dots$. Then*

$$\begin{aligned} \Big| \|U^k \mathrm{e}^{tU}\| - \frac{\|\varphi_k\|_1}{t^{k/2}} \Big| &\leqslant \frac{C(k)}{t^{(k+1)/2}}, \\ \Big| \|U^k \mathrm{e}^{tU}\|_\infty - \frac{\|\varphi_k\|_\infty}{t^{(k+1)/2}} \Big| &\leqslant \frac{C(k)}{t^{k/2+1}}. \end{aligned}$$

The proof follows from a more general Proposition 4 in [39]. Note that $\|\varphi_2\|_1 = \tilde{C}_1$, $\|\varphi_2\|_\infty = \tilde{C}_2$, $\|\varphi_3\|_1 = \tilde{C}_3$, $\|\varphi_3\|_\infty = \tilde{C}_4$, see [17].

Lemma 3.1.5 *Let $\lambda > 0$, $k = 0,1,2,\dots$. Then*

$$\begin{aligned} |\sin(t/2)|^k \mathrm{e}^{-\lambda \sin^2(t/2)} &\leqslant \frac{C(k)}{\lambda^{k/2}}, \\ \int_{-\pi}^{\pi} |\sin(t/2)|^k \mathrm{e}^{-\lambda \sin^2(t/2)} \mathrm{d}t &\leqslant \frac{C(k)}{\max(1, \lambda^{(k+1)/2})}. \end{aligned} \tag{3.6}$$

The first estimate is trivial. For possible constants in the second estimate one can consult [27], p.47.

Lemma 3.1.6 *If A, B are complex numbers and, for some $V > 0$,*

$$|\mathrm{e}^A| \leqslant \mathrm{e}^V, \qquad |\mathrm{e}^B| \leqslant \mathrm{e}^V,$$

Then

$$|\mathrm{e}^A - \mathrm{e}^B| \leqslant \mathrm{e}^V |A - B|.$$

Proof. Let $Re(A-B) \leqslant 0$. Then

$$\begin{aligned} |\mathrm{e}^A - \mathrm{e}^B| &\leqslant |\mathrm{e}^B(\mathrm{e}^{A-B} - 1)| \leqslant \mathrm{e}^V \Big| \int_0^2 (\mathrm{e}^{(A-B)\tau})' \mathrm{d}\tau \Big| \\ &\leqslant \mathrm{e}^V |A-B| \int_0^1 |\mathrm{e}^{(A-B)\tau}| \mathrm{d}\tau \leqslant \mathrm{e}^V |A-B|. \end{aligned}$$

The proof for the case when $Re(B-A) \leqslant 0$ is absolutely symmetric.

Let $Z_1, Z_2, \dots$ be a sequence of arbitrary complex-valued random variables. Then $\widehat{\mathrm{E}} Z_1 = \mathrm{E} Z_1$ and, for a product of $k \geqslant 2$ random variables $Z_1 Z_2 \cdots Z_k$, the symbol $\widehat{\mathrm{E}} Z_1 Z_2 \cdots Z_k$ is defined recursively by

$$\widehat{\mathrm{E}} Z_1 Z_2 \cdots Z_k = \mathrm{E} Z_1 Z_2 \cdots Z_k - \sum_{j=1}^{k-1} \widehat{\mathrm{E}} Z_1 \cdots Z_j \mathrm{E} Z_{j+1} \cdots Z_k. \tag{3.7}$$

This symbol was introduced by V. Statulevičius [42]; see also, [25], and the references therein.

Lemma 3.1.7 *([25]) Let $Z_1, \dots, Z_N$ be a sequence of 1-dependent rv's. Then the product representation*

$$\mathrm{E}e^{zS_N} = \varphi_1(z)\varphi_2(z)\dots\varphi_N(z)$$

holds for each $z \in K_N$, where $\varphi_1(z) = \mathrm{E}e^{zZ_1}$ and for $k = 2, \dots, N$

$$\varphi_k(z) = \frac{\mathrm{E}e^{zS_k}}{\mathrm{E}e^{zS_{k-1}}} = \mathrm{E}e^{zZ_k} + \sum_{j=1}^{k-1} \frac{\hat{\mathrm{E}}(e^{zZ_j}-1)(e^{zZ_{j+1}}-1)\dots(e^{zZ_k}-1)}{\varphi_j(z)\varphi_{j+1}(z)\dots\varphi_{k-1}(z)},$$

where

$$w_N = \max_{1\leqslant k\leqslant N}(\mathrm{E}|e^{zZ_k}-1|^2)^{1/2}, \quad K_N = \{z \in C^1 : w_N(z) \leqslant 1/6\}.$$

Furthermore, the following estimates are true for each $z \in K_N$ and $k = 1.2. \dots, N$:

$$|\varphi_k(z)-1| \underset{(>)}{\leq} |\mathrm{E}e^{zZ_k}-1| \underset{(-)}{+} \frac{2(\mathrm{E}|e^{zZ_{k-1}}-1|^2\mathrm{E}|e^{zZ_k}-1|^2)^{1/2}}{1-4w_n(z)}$$

or

$$|\varphi_k(z)-1| \underset{(>)}{\leq} |\mathrm{E}e^{zZ_k}-1| \underset{(-)}{+} 6(w_N(z))^2.$$

Lemma 3.1.8 *Let $Z_1, \dots, Z_N$ be a sequence of 1-dependent rv's. Then the estimate*

$$\begin{aligned}\Big|\ln \mathrm{E}e^{zS_N} \quad &- \quad \sum_{k=1}^{N}\mathrm{E}(e^{zZ_k}-1) - \sum_{k=2}^{N}\hat{\mathrm{E}}(e^{zZX_{k-1}}-1)(e^{zZ_k}-1)\Big| \\ &\leqslant \quad 2w_N(z)\Big(\sum_{k=1}^{N}|\mathrm{E}e^{zZ_k}-1| + 22\sum_{k=1}^{N}\mathrm{E}|e^{zZ_k}-1|^2\Big)\end{aligned}$$

holds for each $z \in K_N$

Lemma 3.1.9 *([25]) Let $Z_1, Z_2, \dots Z_k$ be 1-dependent random variables with* $\mathrm{E}|Z_j|^2 < \infty, \quad j = 1, \dots, k$. *Then*

$$|\widehat{\mathrm{E}}Z_1Z_2\dots Z_j| \leqslant 2^{j-1}\prod_{k=1}^{j}\sqrt{\mathrm{E}|Z_k|^2}.$$

Lemma 3.1.10 *([25]) Let $Z_1, Z_2, \dots, Z_n$ be a sequence of 1-dependent rv's with $M_{pn} = E|Z_k|^p < \infty$. If $w_n(it) \leqslant 1/6$ then*

$$\left|\frac{\mathrm{d}^p\hat{\varphi}_k}{\mathrm{d}t^p}\right| \leqslant C(p)M_{pn},$$

where $C(p)$ is a constant only depending on p.

3.2 Auxiliary results for compound Poisson approximations for runs statistic

We recall that $\eta_j = \xi_{j-1}\xi_j$, where all ξ_j are independent Bernoulli (indicator) variables, $P(\xi_j = 1) = p = 1 - P(\xi_1)$, $(j = 0, 1, 2, \dots)$. Let $S_n = \eta_1 + \dots + \eta_n$, $\varphi_1(t) = \mathrm{E}\exp\{it\eta_1\}$ and

$$\varphi_k(t) = \frac{\mathrm{E}\exp\{\mathrm{i}tS_k\}}{\mathrm{E}\exp\{\mathrm{i}tS_{k-1}\}} \qquad (k = 2, 3, \dots, n).$$

For the sake of brevity further on we write φ_k instead of $\varphi_k(t)$. We recall that $z = \mathrm{e}^{\mathrm{i}t} - 1$. All derivatives are taken with respect to t.

Lemma 3.2.1 *Let $0 < p < 1$, $k = 1, 2, \ldots$. Then*

$$\widehat{\mathrm{E}}(\exp\{\mathrm{i}t\eta_1\} - 1)(\exp\{\mathrm{i}t\eta_2\} - 1) \cdots (\exp\{\mathrm{i}t\eta_k\} - 1) = z^k p^{k+1}(1-p)^{k-1}$$

Proof. The proof easily follows by induction from the definition (3.7).
Let $k = 1$, then

$$\widehat{\mathrm{E}}(\exp\{\mathrm{i}t\eta_1\} - 1) = zp^2.$$

Let $k = 2$, then

$$\widehat{\mathrm{E}}(\exp\{\mathrm{i}t\eta_1\} - 1)(\exp\{\mathrm{i}t\eta_2\} - 1) = z^2 p^3 (1-p).$$

Say, that for k proof is standing. Let $k = k+1$, then

$$\begin{aligned}
\widehat{\mathrm{E}}\exp\{\mathrm{i}t\eta_1\} &- 1)(\exp\{\mathrm{i}t\eta_2\} - 1) \cdots (\exp\{\mathrm{i}t\eta_k + 1\} - 1) \\
&= E(\exp\{\mathrm{i}t\eta_1\} - 1)(\exp\{\mathrm{i}t\eta_2\} - 1) \cdots (\exp\{\mathrm{i}t\eta_k + 1\} - 1) \\
&- \sum_{j=1}^{k} \widehat{E}(\exp\{\mathrm{i}t\eta_1\} - 1) \cdots (\exp\{\mathrm{i}t\eta_j\} - 1)\mathrm{E}(\exp\{\mathrm{i}t\eta_{j+1}\} - 1) \cdots (\exp\{\mathrm{i}t\eta_{k+1}\} - 1) \\
&= z^{k+1}p^{k+2} - \sum_{j=1}^{k} z^j p^{j+1}(1-p)^{j-1} p^{k-j+2} z^{k-j+1} = z^{k+1}[p^{k+2} - \sum_{j=1}^{k} p^{k+3}(1-p)^{j-1}] \\
&= z^{k+1}p^{k+2}[1 - p\sum_{j=1}^{k}(1-p)^{j-1}] = z^{k+1}p^{k+2}(1-p)^k.
\end{aligned}$$

□

Lemma 3.2.2 *Let $0 < p < 1$, $k = 1, 2, \ldots, n$. Then*

$$\mathrm{E}\exp\{\mathrm{i}tS_n\} = \prod_{k=1}^{n} \varphi_k \tag{3.8}$$

and, for $k = 2, 3, \ldots, n$,

$$\varphi_k = p^2 z + \sum_{j=1}^{k-1} \frac{z^{k-j+1}p^{k-j+2}(1-p)^{k-j}}{\varphi_j \varphi_{j+1} \cdots \varphi_{k-1}}. \tag{3.9}$$

Proof. The proof follows from Lemma 3.1 in 3.1.7 and Lemma 3.2.1. Note that we use only the first part of Lemma 3.1 and, as can be seen from its proof, the additional assumption $\mathrm{E}|\exp\{\mathrm{i}tX_k\} - 1|^2)^{1/2} \leqslant 1/6$ is not needed . □

Lemma 3.2.3 *Let $p \leqslant 1/5$. Then, for all $k = 1, 2, \ldots,$*

$$\begin{aligned}
|\varphi_k - 1| &\leqslant \frac{16p^2}{5}, \qquad |\varphi_k - 1 - p^2 z| \leqslant 2|z|^2 p^3(1-p), \qquad \frac{1}{|\varphi_k|} \leqslant \frac{125}{109}, && (3.10) \\
|\varphi_k'| &\leqslant 4p^2, \qquad |\varphi_k' - p^2 z'| \leqslant C|z|p^3, \quad |\varphi_k''| \leqslant 10p^2. && (3.11)
\end{aligned}$$

Proof. Note that the last estimate in (3.10) follows from the first estimate. Indeed, we have

$$||\varphi_k| - 1| \leqslant |\varphi_k - 1| \leqslant \frac{16}{5}p^2 \leqslant \frac{16}{5} \cdot \frac{1}{25} = \frac{16}{125}.$$

Therefore,

$$|\varphi_k| - 1 \geqslant -\frac{16}{125}, \quad |\varphi_k| \geqslant \frac{109}{125}, \quad \frac{1}{|\varphi_k|} \leqslant \frac{125}{109}.$$

The first estimate of (3.10) is proved by induction. For $k = 1$ the proof follows easily from the definition of φ_1. Let us assume that it holds for $\varphi_1, \dots, \varphi_{k-1}$. Taking into account trivial estimate $|z| \leqslant 2$ and (3.9), we then obtain

$$\begin{aligned}|\varphi_k - 1| &\leqslant 2p^2 + \sum_{j=1}^{k-1} \frac{|z|^{k-j+1} p^{k-j+2}(1-p)^{k-j}}{|\varphi_j| \cdots |\varphi_{k-1}|} \leqslant 2p^2 + \sum_{j=1}^{k-1} 2^{k-j+1} p^{k-j+2}(1-p)^{k-j} \Big(\frac{125}{109}\Big)^{k-j} \\ &\leqslant 2p^2 \Big\{ 1 + \sum_{j=1}^{\infty} \Big(2p(1-p)\frac{125}{109}\Big)^j \Big\} \leqslant 2p^2 \sum_{j=0}^{\infty} \Big(\frac{40}{109}\Big)^j \leqslant \frac{16p^2}{5}.\end{aligned}$$

For the proof of the second estimate in (3.10) we once more apply (3.9):

$$\begin{aligned}|\varphi_k - 1 - p^2 z| &\leqslant \sum_{j=1}^{k-1} |z|^{k-j+1} p^{k-j+2}(1-p)^{k-j}\Big(\frac{125}{109}\Big)^{k-j} \\ &\leqslant |z|^2 p^3 (1-p)\frac{125}{109} \sum_{j=0}^{\infty}\Big(\frac{40}{109}\Big)^j \leqslant 2|z|^2 p^3(1-p).\end{aligned}$$

The estimate (3.11) is proved arguing similarly. First, we calculate derivative of (3.9), then prove the first estimate by induction and, finally, obtain the second estimate. Note that for the proof of (3.11) one must apply (3.10). We prove the last estimate of (3.11). From Lemma 3.2.2 it follows that

$$\begin{aligned}\varphi_k'' &= -\mathrm{e}^{\mathrm{i}t} p^2 + \sum_{j=1}^{k-1} \frac{\Psi''}{\varphi_j \cdots \varphi_{k-1}} - 2\sum_{j=1}^{k-1} \frac{\Psi'}{\varphi_j \cdots \varphi_{k-1}} \sum_{i=j}^{k-1} \frac{\varphi_i'}{\varphi_i} \\ &\quad + \sum_{j=1}^{k-1} \frac{\Psi}{\varphi_j \cdots \varphi_{k-1}} \Big(\sum_{i=j}^{k-1} \frac{\varphi_i'}{\varphi_i}\Big)^2 - \sum_{j=1}^{k-1} \frac{\Psi}{\varphi_j \cdots \varphi_{k-1}} \sum_{i=j}^{k-1} \Big[\frac{\varphi_i''}{\varphi_i} - \Big(\frac{\varphi_i}{\varphi_i}\Big)^2\Big]. \qquad (3.12)\end{aligned}$$

Here

$$\Psi = z^{k-j+1} p^{k-j+2}(1-p)^{k-j}.$$

Applying (3.10) and obvious estimate $p(1-p) \leqslant 4/25$ we get

$$\begin{aligned}|\varphi_k''| &\leqslant p^2 + p^2 \sum_{j=1}^{k-1} \Big(\frac{40}{109}\Big)^{k-j} \Big[\frac{(k-j+1)(k-j)}{2} + (k-j+1)\Big] \\ &\quad + \frac{40}{109} p^2 \sum_{j=1}^{k-1} \Big(\frac{40}{109}\Big)^{k-j} (k-j+1)(k-j) \\ &\quad + 2p^2 \sum_{j=1}^{k-1} \Big(\frac{40}{109}\Big)^{k-j} \Big[\Big(\frac{20}{109}\Big)^2 (k-j)^2 + \Big(\frac{50}{109} + \frac{16 \cdot 25}{109^2}\Big)(k-j)\Big] \leqslant 10p^2.\end{aligned}$$

□

Lemma 3.2.4 *Let $p \leqslant 1/5$. For all t, $k = 1, 2, 3, \dots$, we then have*

$$|\varphi_k| \leqslant 1 - \frac{16}{25} p^2 \sin^2 \frac{t}{2} \leqslant \exp\Big\{ -\frac{16}{25} p^2 \sin^2 \frac{t}{2} \Big\}. \qquad (3.13)$$

Proof. It is easy to check, that

$$|1 + p^2 z|^2 = (1 - p^2 + p^2 \cos t)^2 + p^4 \sin^2 t = 1 - 4p^2(1-p^2)\sin^2(t/2).$$

Consequently, $|1 + p^2 z| \leqslant 1 - 2p^2(1-p^2)\sin^2(t/2)$. Therefore, applying the second estimate from (3.10), we obtain

$$|\varphi_k| \leqslant |1 + p^2 z| + |\varphi_k - 1 - pz^2| \leqslant 1 - 2p^2(1 - p^2 - 4p(1-p))\sin^2 \frac{t}{2} \leqslant 1 - \frac{16}{25} p^2 \sin^2 \frac{t}{2}.$$

The second estimate in (3.13) trivially follows from the first one . □

In Lemmas 3.2.5 – 3.2.9, the values of $\theta = \theta(p,t)$ can vary from line to line, but θ always satisfies inequality $|\theta| \leqslant 1$.

Lemma 3.2.5 *Let $p \leqslant 1/5$, $k = 3, 4, \ldots$. Then*

$$\begin{aligned}
\varphi_1 &= 1 + p^2 z, \\
\varphi_2 &= 1 + p^2 z + p^3(1-p)z^2 - p^5(1-p)z^3 + C\theta p^5|z|^4, \\
\varphi_k &= 1 + p^2 z + p^3(1-p)z^2 + p^4(1-p)(1-2p)z^3 + C\theta p^5|z|^4.
\end{aligned}$$

Proof. We give the proof for $k = 3, 4, \ldots$ only. Applying (3.9) we obtain

$$\varphi_k - 1 = p^2 z + \frac{z^2 p^3(1-p)}{\varphi_{k-1}} + \frac{z^3 p^4(1-p)^2}{\varphi_{k-2}\varphi_{k-1}} + \sum_{j=1}^{k-3} \frac{z^{k-j+1} p^{k-j+2}(1-p)^{k-j}}{\varphi_j \cdots \varphi_{k-1}}. \tag{3.14}$$

Applying the last estimate of (3.10) to the sum in (3.14) we get the following estimate

$$|z|^4 p^5 \sum_{j=1}^{k-3} (2p)^{k-3-j}(1-p)^{k-j}\left(\frac{125}{109}\right)^{k-j} \leqslant Cp^5|z|^4 \sum_{j=1}^{k-3}\left(\frac{40}{109}\right)^{k-j-3} \leqslant Cp^5|z|^4. \tag{3.15}$$

Once more applying (3.10) we obtain

$$\begin{aligned}
\frac{1}{\varphi_k} &= \frac{1}{1+(\varphi_k - 1)} = 1 - (\varphi_k - 1) + C\theta|\varphi_k - 1|^2 \sum_{j=0}^{\infty}\left(\frac{16p^2}{5}\right)^j \\
&= 1 - p^z + C\theta|\varphi_k - 1 - pz^2| + C\theta p^4|z|^4 \sum_{j=0}^{\infty}\left(\frac{16}{125}\right)^j = 1 - p^2 z + C\theta p^3|z|^2. \qquad (3.16)
\end{aligned}$$

The proof of Lemma now follows from (3.16) , (3.15) and (3.14) . □

Lemma 3.2.6 *Let $p \leqslant 1/5$, $k = 3, 4, \ldots$. Then*

$$\begin{aligned}
\varphi_1' &= p^2 z', \\
\varphi_2' &= p^2 z' + p^3(1-p)(z^2)' - p^5(1-p)(z^3)' + C\theta p^5|z|^3, \\
\varphi_k' &= p^2 z' + p^3(1-p)(z^2)' + p^4(1-p)(1-2p)(z^3)' + C\theta p^5|z|^3.
\end{aligned}$$

Proof. For $k = 1$ and $k = 2$ the required estimates follow directly from the definition of φ_1 and (3.9). For $k = 3, 4, \ldots$ we calculate derivative of (3.14)

$$\begin{aligned}
(\varphi_k - 1)' &= p^2 z' + \frac{(z^3)' p^4 (1-p)^2}{\varphi_{k-2}\varphi_{k-1}} - \frac{z^3 p^4(1-p)^2(\varphi_{k-2}'\varphi_{k-1} + \varphi_{k-2}\varphi_{k-1}')}{(\varphi_{k-2}\varphi_{k-1})^2} \\
&\quad + \frac{(z^2)' p^3(1-p)}{\varphi_{k-1}} - \frac{z^2 p^3(1-p)\varphi_{k-1}'}{\varphi_{k-1}^2} + \left(\sum_{j=1}^{k-3} \frac{z^{k-j+1}p^{k-j+2}(1-p)^{k-j}}{\varphi_j \cdots \varphi_{k-1}}\right)'.
\end{aligned}$$

We treat various summands differently. Applying Lemma 3.2.3 to the third summand we prove that it is $C\theta p^5|z|^3$. We use (3.16) for estimation of the second, the fourth and the fifth summands. Moreover, from (3.11) it follows that $\varphi' = p^2 z' + C\theta p^3|z|$. We use this short expansion for the fifth summand. It remains to estimate the last, sixth summand. Note that

$$\begin{aligned}
\left(\sum_{j=1}^{k-3} \frac{z^{k-j+1}p^{k-j+2}(1-p)^{k-j}}{\varphi_j \cdots \varphi_{k-1}}\right)' &= \sum_{j=1}^{k-3} \frac{(k-j+1)z^{k-j}z' p^{k-j+2}(1-p)^{k-j}}{\varphi_j \cdots \varphi_{k-1}} \\
&\quad - \sum_{j-1}^{k-3} \frac{z^{k-j+1}p^{k-j+2}(1-p)^{k-j}}{\varphi_j \cdots \varphi_{k-1}} \sum_{m=j}^{k-3} \frac{\varphi_m'}{\varphi_m}.
\end{aligned}$$

Applying Lemma 3.2.3 we obtain that each sum is less than

$$Cp^5|z|^4 \sum_{j=1}^{k-3}(k-j+1)\left(\frac{40}{109}\right)^{k-j-3} \leqslant Cp^5|z|^4.$$

The last estimate completes Lemma's proof . □

Lemma 3.2.7 *Let $p \leqslant 1/5$, $k = 3, 4, \ldots$. Then*

$$\begin{aligned}
\varphi_1'' &= p^2 z'', \\
\varphi_2'' &= p^2 z'' + p^3(1-p)(z^2)'' - p^5(1-p)(z^3)'' + C\theta p^5|z|^2, \\
\varphi_k'' &= p^2 z'' + p^3(1-p)(z^2)'' + p^4(1-p)(1-2p)(z^3)'' + C\theta p^5|z|^2.
\end{aligned}$$

Proof. We prove Lemma for $k \geqslant 3$ only. Applying (3.12) and estimating the last two sums just as in the proof of Lemma 3.2.3 and applying estimate $|\Psi| \leqslant p^3(4/25)^{k-j-1}$ we prove that they are less than $Cp^5|z|^2$. Similarly, taking into account Lemmas 3.2.3–3.2.6 we obtain

$$\sum_{j=1}^{k-1}\frac{\Psi'}{\varphi_j\cdots\varphi_{k-1}}\sum_{i=j}^{k-1}\frac{\varphi_i'}{\varphi_i} = \sum_{j=k-2}^{k-1} + \sum_{j=1}^{k-3} = \sum_{j=k-2}^{k-1} + C\theta p^5|z|^2 = \frac{(z)'p^3(1-p)\varphi_{k-1}'}{\varphi_{k-1}^2} + C\theta p^5|z|^2$$

and

$$\sum_{j=1}^{k-1}\frac{\Psi''}{\varphi_j\cdots\varphi_{k-1}} = \frac{p^3(1-p)(z^2)''}{\varphi_{k-1}} + \frac{p^4(1-p)^2(z^3)''}{\varphi_{k-2}\varphi_{k-1}} + C\theta p^5|z|^2.$$

To complete the proof one should apply (3.10). □

Lemma 3.2.8 *Let $p \leqslant 1/5$, $k = 3, 4, \ldots$. Then*

$$\begin{aligned}
\ln\varphi_1 &= p^2 z - \frac{p^4}{2}z^2 + \frac{p^6}{3}z^3 + C\theta p^5|z|^4, \\
\ln\varphi_2 &= p^2 z + \frac{p^3(2-3p)}{2}z^2 + \frac{p^5(7p-6)}{3}z^3 + C\theta p^5|z|^4, \\
\ln\varphi_k &= p^2 z - \frac{p^3(2-3p)}{2}z^2 + \frac{p^4(3-12p+10p^2)}{3}z^3 + C\theta p^5|z|^4.
\end{aligned}$$

Proof. Note that due to (3.10) $|\varphi_k - 1| \leqslant 16p^2/5 \leqslant 16/125$. Therefore,

$$\ln\varphi_k = \sum_{j=1}^{\infty}\frac{(-1)^{j+1}(\varphi_k-1)^j}{j} = \sum_{j=1}^{3}\frac{(-1)^{j+1}(\varphi_k-1)^j}{j} + C\theta|\varphi_k-1|^4.$$

Now, it remains to apply Lemma 3.2.5 . □

The proof of the following Lemmas is almost identical to the proof of Lemma 3.2.8 and, therefore, is omitted

Lemma 3.2.9 *Let $p \leqslant 1/5$, $k = 3, 4, \ldots$. We then have*

$$\begin{aligned}
(\ln\varphi_1)' &= p^2 z' - \frac{p^4}{2}(z^2)' + \frac{p^6}{3}(z^3)' + C\theta p^5|z|^3, \\
(\ln\varphi_2)' &= p^2 z' + \frac{p^3(2-3p)}{2}(z^2)' + \frac{p^5(7p-6)}{3}(z^3)' + C\theta p^5|z|^3, \\
(\ln\varphi_k)' &= p^2 z' - \frac{p^3(2-3p)}{2}(z^2)' + \frac{p^4(3-12p+10p^2)}{3}(z^3)' + C\theta p^5|z|^3.
\end{aligned}$$

Lemma 3.2.10 *Let $p \leqslant 1/5$, $k = 3,4,\ldots$. We then have*

$$\begin{aligned}(\ln\varphi_1)'' &= p^2z'' - \frac{p^4}{2}(z^2)'' + \frac{p^6}{3}(z^3)'' + C\theta p^5|z|^2,\\ (\ln\varphi_2)' &= p^2z'' + \frac{p^3(2-3p)}{2}(z^2)'' + \frac{p^5(7p-6)}{3}(z^3)'' + C\theta p^5|z|^2,\\ (\ln\varphi_k)' &= p^2z'' - \frac{p^3(2-3p)}{2}(z^2)'' + \frac{p^4(3-12p+10p^2)}{3}(z^3)'' + C\theta p^5|z|^2.\end{aligned}$$

Now, we investigate some properties of approximating measures.

Lemma 3.2.11 *Let $p \leqslant 1/5$. Then, for all t,*

$$|\widehat{G}_2(t)| \leqslant C\exp\Big\{-\frac{6np^2}{5}\sin^2\frac{t}{2}\Big\}, \quad |\widehat{G}_3(t)| \leqslant C\exp\Big\{-np^2\sin^2\frac{t}{2}\Big\}. \tag{3.17}$$

Proof. It is easy to check that

$$|\widehat{G}_2(t)| \leqslant C\exp\{np^2 Rez + |z|^2np^3(2-3p)/2\}.$$

Here $Rez = -2\sin^2(t/2)$ denotes the real part of z. Since $|z|^2 = 4\sin^2(t/2)$ and $p \leqslant 1/5$ the desired result easily follows. The estimate for $\widehat{G}_3(t)$ is proved similarly applying the estimate $|z|^3 \leqslant 2|z|^2$. □

Lemma 3.2.12 *Let M be finite variation signed measure concentrated on integers, $p \leqslant 1/5$, $\tau \in [0,1]$, $\alpha \in [0,1]$. Then*

$$\begin{aligned}\|M\exp\{np^2U + \alpha\gamma_2U^2 + \tau\gamma_3U^3\}\| &\leqslant C\|M\exp\{0.4np^2U\}\|,\\ \|M\exp\{np^2U + \alpha\gamma_2U^2 + \tau\gamma_3U^3\}\|_\infty &\leqslant C\|M\exp\{0.4np^2U\}\|_\infty.\end{aligned}$$

Proof. For $p \leqslant 1/5$ the following estimate holds

$$p\frac{(2-3p)}{2} + p^3(3-12p+10p^2)\frac{2}{3} \leqslant \frac{1}{6}.$$

Therefore, taking into account that $\|I\| = 1$ and $\|U\| \leqslant 2$, we obtain

$$\begin{aligned}&np^2U + \alpha\gamma_2U^2 + \tau\gamma_3U^3\\ &\quad= np^2U + np^2U^2\left(\alpha\frac{p(2-3p)}{2}I + \tau p^2(3-12p+10p^2)\frac{U}{3}\right) + \Theta C\\ &\quad= np^2U + \frac{np^2}{6}U^2\Theta + \Theta C.\end{aligned}$$

Consequently,

$$\begin{aligned}\|M\exp\{np^2U + \alpha\gamma_2U^2 + \tau\gamma_3U^3\}\| &\leqslant C\Big\|M\exp\Big\{np^2U + \frac{np^2}{6}U^2\Theta\Big\}\Big\|\\ &\leqslant C\Big\|M\exp\Big\{0.4np^2U\Big\}\Big\|\,\Big\|\exp\Big\{0.6np^2U + \frac{np^2}{6}U^2\Theta\Big\}\Big\|.\end{aligned}$$

To complete the proof one needs to show that the second norm is bounded by absolute constant. This can be proved applying the definition of exponential measure and Lemma 3.1.3:

$$\begin{aligned}\Big\|\exp\Big\{0.6np^2U + \frac{np^2}{6}U^2\Theta\Big\}\Big\| &\leqslant 1 + \sum_{m=1}^{\infty}\frac{1}{m!}\Big\|\frac{np^2}{6}U^2\exp\Big\{\frac{0.6np^2}{m}U\Big\}\Big\|^m\\ &\leqslant 1 + \sum_{m=1}^{\infty}\frac{\mathrm{e}^m}{m^m\sqrt{2\pi m}}\left(\frac{m}{2\mathrm{e}0.6}\right)^m \leqslant C.\end{aligned}$$

We used the fact that the total variation norm of any distribution equals unity. Therefore, $\|\exp\{0.6np^2U\}\| = 1$. Estimate for the local norm follows from

$$\begin{aligned}&\|M\exp\{np^2U+\alpha\gamma_2U^2+\tau\gamma_3U^3\}\|_\infty\\ &\qquad\leqslant C\Big\|\exp\Big\{0.6np^2U+\frac{np^2}{6}U^2\Theta\Big\}\Big\|\,\Big\|M\exp\Big\{0.4np^2U\Big\}\Big\|_\infty.\end{aligned}$$

□

3.3 Proof of Theorems 2.1.1 – 2.1.9

Proof of Theorem 2.1.4. To make expressions shorter we write φ_k and $\widehat{G}_3$ instead $\varphi_k(t)$ and $\widehat{G}_3(t)$, respectively. Taking into account (3.8), (3.13), (3.17) and Lemma 3.2.8 we obtain

$$\begin{aligned}\Big|\prod_{k=1}^n\varphi_k-\widehat{G}_3\Big| &\leqslant Ce^{-Cnp^2\sin^2(t/2)}\Big|\sum_{k=1}^n\ln\varphi_k-\ln\widehat{G}_3\Big|\\ &\leqslant Cnp^5|\sin(t/2)|^4e^{-Cnp^2\sin^2(t/2)}.\end{aligned}\tag{3.18}$$

From Lemma 3.2.9 it follows that

$$\Big|\sum_{k=1}^n(\ln\varphi_k)'-inp^2\Big|\leqslant np^2|e^{it}-1|+Cnp^3|z|^2\leqslant Cnp^2|z|.$$

Therefore, applying Lemma 3.2.9, (3.18) and (3.6), we obtain

$$\begin{aligned}&\Big|\Big(e^{-itnp^2}\prod_{k=1}^n\varphi_k-e^{-itnp^2}\widehat{G}_3\Big)'\Big|\\ &\quad=\Big|\Big(\sum_{k=1}^n\ln\varphi_k-itnp^2\Big)'\prod_{k=1}^n\varphi_ke^{-itnp^2}-(\ln\widehat{G}_3-itnp^2)'\widehat{G}_3e^{-itnp^2}\Big|\\ &\quad\leqslant\Big|\Big(\sum_{k=1}^n\ln\varphi_k-itnp^2\Big)'\Big(\prod_{k=1}^n\varphi_k-\widehat{G}_3\Big)e^{-itnp^2}+e^{-itnp^2}\widehat{G}_3\Big(\sum_{k=1}^n(\ln\varphi_k)'-(\ln\widehat{G}_3)'\Big)\Big|\\ &\quad\leqslant Ce^{-Cnp^2\sin^2(t/2)}\Big\{\Big|\sum_{k=1}^n(\ln\varphi_k)'-inp^2\Big|np^5|\sin(t/2)|^4+np^5|\sin(t/2)|^3\Big\}\\ &\quad\leqslant Ce^{-Cnp^2\sin^2(t/2)}np^5|\sin(t/2)|^3.\end{aligned}\tag{3.19}$$

Let us take in Lemma 3.1.1 $v=np^2$ and $u=\max(1,np^2)$ and $M=\mathcal{L}(S)-G_3$. The estimates (2.5) and (2.6) follow from (3.18), (3.19) and (3.6) . □

Proof of Theorem 2.1.2. Taking into account Lemma 3.2.12 we obtain

$$\begin{aligned}\|G_3-G_2\| &= \|G_2(\exp\{\ln G_3-\ln G_2\}-I)\|=\Big\|G_2\int_0^1(\exp\{\tau(\ln G_3-\ln G_2)\})'\mathrm{d}\tau\Big\|\\ &= \Big\|G_2(\ln G_2-\ln G_3)\int_0^1\exp\{\tau(\ln G_3-\ln G_2)\}\mathrm{d}\tau\Big\|\\ &= \Big\|(\ln G_2-\ln G_3)\int_0^1\exp\{\tau\ln G_3+(1-\tau)\ln G_2\}\mathrm{d}\tau\Big\|\\ &\leqslant C\|(\ln G_3-\ln G_2)\exp\{0.4np^2U\}\|\leqslant C\|np^4U^3\exp\{0.4np^2U\}\|\\ &\leqslant Cnp^4\min(1,(np^2)^{-3/2}).\end{aligned}$$

For the last estimate we used Lemma 3.1.3. Similar estimate holds for local norm. To complete the proof of Theorem 2.1.3 one needs to use Theorem 2.1.4 and the triangle inequality. □

Proof of Theorem 2.1.5. We have

$$G_3 - G_2(I+\gamma_3 U^3) = G_2(\exp\{\gamma_3 U^3\} - I - \gamma_3 U^3) = G_2(\gamma_3 U^3)^2 \int_0^1 (1-\tau)\exp\{\tau\gamma_3 U^3\}\mathrm{d}\tau.$$

From Lemma 3.2.12 and Lemma 3.1.3 it follows that

$$\|G_3 - G_2(I+\gamma_3 U^3)\| \leqslant C\|\gamma_3^2 U^6 \exp\{0.4np^2 U\}\| \leqslant C(np^4)^2 \min(1,(np^2)^{-3}).$$

The triangle inequality and Theorem 2.1.4 complete the proof of (2.7). The proof of (2.8) is absolutely analogous. One simply needs to replace the total variation norm by the local one. □

Proof of Theorem 2.1.1. We have

$$G_2 - G_1(I+\gamma_2 U^2) = G_1(\exp\{\gamma_2 U^2\} - I - \gamma_2 U^2) = G_1(\gamma_2 U^2)^2 \int_0^1 (1-\tau)\exp\{\tau\gamma_2 U^2\}\mathrm{d}\tau.$$

From Lemma 3.2.12 and Lemma 3.1.3 it follows that

$$\|G_2 - G_1(I+\gamma_2 U^2)\| \leqslant C\|\gamma_2^2 U^4 \exp\{0.4np^2 U\}\| \leqslant C(np^3)^2 \min(1,(np^2)^{-2}).$$

The triangle inequality and Theorem 2.1.2 complete the proof of (2.1). The proof of (2.2) is absolutely analogous. One simply needs to replace the total variation norm by the local one. □

Proof of Theorem 2.1.3. All proofs are very similar. Therefore, we prove (2.4) only. Arguing as in the proof of Theorem 2.1.2 and applying Lemma 3.1.3 and Lemma 3.2.12 we obtain

$$\begin{aligned}\|(G_2-G_1)U^3\| &= \Big\|U^3 G_1 \int_0^1 \gamma_2 U^2 \exp\{\tau\gamma_2 U^2\}\mathrm{d}\tau\Big\| \\ &\leqslant Cnp^3 \int_0^1 \Big\|G_1 \mathrm{e}^{\tau\gamma_2 U} U^5 \mathrm{d}\tau\Big\| \leqslant Cnp^3 \Big\|U^5 \mathrm{e}^{0.4np^2 U}\Big\| \leqslant \frac{Cp}{\sqrt{n}}. \qquad (3.20)\end{aligned}$$

It is not difficult to check that, for any finite measures M, V and constant C_0, the following inequality holds

$$\Big|\|M\| - C_0\Big| \leqslant \|M-V\| + \Big|\|V\| - C_0\Big|.$$

Consequently applying Theorem 2.1.5, Lemma 3.1.3, (3.20) and Lemma 3.1.4 we obtain

$$\begin{aligned}\Big|\|\mathcal{L}(S) - G_2\| - \frac{\tilde{C}_3 p}{\sqrt{n}}\Big| &\leqslant \|\mathcal{L}(S) - G_2(I+\gamma_3 U^3)\| + \Big|\|G_2\gamma_3 U^3\| - \frac{\tilde{C}_3 p}{\sqrt{n}}\Big| \\ &\leqslant \frac{Cp}{n} + \|G_2 U^3(\gamma_3 - np^4)\| + \Big|np^4\|G_2 U^3\| - \frac{\tilde{C}_3 p}{\sqrt{n}}\Big| \\ &\leqslant \frac{Cp^2}{\sqrt{n}} + np^4\|(G_2-G_1)U^3\| + \Big|np^4\|G_1 U^3\| - \frac{\tilde{C}_3 p}{\sqrt{n}}\Big| \\ &\leqslant \frac{Cp^2}{\sqrt{n}} + np^4\Big|\|G_1 U^3\| - \frac{\tilde{C}_3}{(np^2)^{3/2}}\Big| \\ &\leqslant \frac{Cp^2}{\sqrt{n}} + \frac{C}{n}.\end{aligned}$$

For the local estimates one should use the local metric. □

Proof of Theorem 2.1.8. We assumed that $p \leqslant 1/5$ and $n \geqslant 3$. Therefore,

$$|\gamma_3| = \gamma_3 \geqslant \frac{np^4(3 - 12p + 10p^2)}{3} \geqslant \frac{np^4}{3}. \tag{3.21}$$

Applying Lemma 3.2.8 and Lemma 3.2.11 we obtain

$$|\widehat{F}(t) - \widehat{G}_3(t)| \leqslant |\ln \widehat{F}(t) - \ln \widehat{G}_3(t)| \leqslant Cnp^5|z|^4 \leqslant Cnp^5|t|^4, \tag{3.22}$$

We have

$$\begin{aligned} |\widehat{G}_3(t) - \widehat{G}_2(t)(1 + \gamma_3 z^3)| &= |\widehat{G}_2(t)(\exp\{\gamma_3 z^3\} - 1 - \gamma_3 z^3)| \\ &= |\widehat{G}_2(t)\gamma_3^2 z^6 \int_0^1 (1-\tau)\exp\{\tau\gamma_3 z^3\}\mathrm{d}\tau| \\ &\leqslant C\gamma_3^2|z|^6 \leqslant Cn^2p^8t^6, \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} &\widehat{F}(t) - \widehat{G}_2(t) = \widehat{G}_2(t)\gamma_3(\mathrm{i}t)^3 + \widehat{G}_2(t)\gamma_3(z^3 - (\mathrm{i}t)^3) \\ &+(\widehat{G}_3(t) - \widehat{G}_2(t)(1 + \gamma_3 z^3)) + (\widehat{F}(t) - \widehat{G}_3(t)). \end{aligned} \tag{3.24}$$

Let $b = h\max(1, \sqrt{n}p)$, $h \geqslant 1$. Then applying (3.21) and (3.24) we obtain

$$\begin{aligned} \Big|\int_{-\infty}^{\infty} t\mathrm{e}^{-t^2/2}(\widehat{F}(t/b) &- \widehat{G}_2(t/b))\mathrm{e}^{-\mathrm{i}t\alpha}\mathrm{d}t\Big| \geqslant \Big|\int_{-\infty}^{\infty} t\mathrm{e}^{-t^2/2}\widehat{G}_2(t/b)\mathrm{e}^{-\mathrm{i}t\alpha}\frac{\gamma_3 t^3}{b^3}\mathrm{d}t\Big| \\ &- C_1\Big|\int_{-\infty}^{\infty} t\mathrm{e}^{-t^2/2}\Big(\frac{|\gamma_3||t|^4}{b^4} + \frac{n^2p^8t^6}{b^6} + \frac{np^5|t|^4}{b^4}\Big)\mathrm{d}t\Big| \\ &\geqslant \frac{|\gamma_3|}{b^3}\Big|\int_{-\infty}^{\infty} t^4\mathrm{e}^{-t^2/2}\mathrm{d}t\Big| - C_2\Big|\int_{-\infty}^{\infty} t^2\frac{|\gamma_3|np^3t^6}{b^6}\mathrm{e}^{-t^2/2}\mathrm{d}t\Big| \\ &- C_3\Big(\frac{np^4}{b^4} + \frac{n^2p^8}{b^6} + \frac{np^5}{b^4}\Big) \\ &\geqslant C_4\frac{np^4}{b^3} - C_5\frac{n^2p^6}{b^6} - C_6\frac{np^4}{b^4} \\ &\geqslant C_4\frac{np^4}{b^3} - \frac{C_7np^4}{h^4\max(1, \sqrt{n}p)^3} \\ &\geqslant C_4\frac{np^4}{h^3\max(1, n^{3/2}p^3)}\Big(1 - \frac{C_7}{h}\Big) \\ &\geqslant \frac{C_4}{h^3}\min(\frac{p}{\sqrt{n}}, np^4)\Big(1 - \frac{C_7}{h}\Big). \end{aligned}$$

It suffices to take $h = 2C_7$ an apply Lemma 3.1.2 with $\alpha = np^2$. □

Proof of Theorem 2.1.6 We assumed that $p \leqslant 1/5$ and $n \geqslant 3$. Therefore,

$$|\gamma_2| = \gamma_2 \geqslant \frac{np^3(2 - 3p)}{2} \geqslant \frac{7np^3}{10}. \tag{3.25}$$

Applying Lemma 3.2.8 and Lemma 3.2.11 we obtain

$$|\widehat{F}(t) - \widehat{G}_2(t)| \leqslant |\ln \widehat{F}(t) - \ln \widehat{G}_2(t)| \leqslant Cnp^4|z|^3 \leqslant Cnp^4|t|^3, \tag{3.26}$$

We have

$$\begin{aligned} |\widehat{G}_2(t) - \widehat{G}_1(t)(1 + \gamma_2 z^2)| &= |\widehat{G}_1(t)(\exp\{\gamma_2 z^2\} - 1 - \gamma_2 z^2)| \\ &= |\widehat{G}_1(t)\gamma_2^2 z^4 \int_0^1 (1-\tau)\exp\{\tau\gamma_2 z^2\}\mathrm{d}\tau| \\ &\leqslant C\gamma_2^2|z|^4 \leqslant Cn^2p^6t^4, \end{aligned} \tag{3.27}$$

and

$$\widehat{F}(t)-\widehat{G}_1(t)=\widehat{G}_1(t)\gamma_2(\mathrm{i}t)^2+\widehat{G}_1(t)\gamma_2(z^2-(\mathrm{i}t)^2)$$
$$+(\widehat{G}_2(t)-\widehat{G}_1(t)(1+\gamma_2 z^2))+(\widehat{F}(t)-\widehat{G}_2(t)). \tag{3.28}$$

Let $b=h\max(1,\sqrt{n}p)$, $h\geqslant 1$. Then applying (3.25) and (3.28) we obtain

$$\begin{aligned}
\Big|\int_{-\infty}^{\infty}\mathrm{e}^{-t^2/2}(\widehat{F}(t/b) &- \widehat{G}_1(t/b))\mathrm{e}^{-\mathrm{i}t\alpha}\mathrm{d}t\Big| \geqslant \Big|\int_{-\infty}^{\infty}\mathrm{e}^{-t^2/2}\widehat{G}_1(t/b)\mathrm{e}^{-\mathrm{i}t\alpha}\frac{\gamma_2 t^2}{b^2}\mathrm{d}t\Big|\\
&- C_8\Big|\int_{-\infty}^{\infty}\mathrm{e}^{-t^2/2}\Big(\frac{|\gamma_2||t|^3}{b^3}+\frac{n^2p^6t^4}{b^4}+\frac{np^4|t|^3}{b^3}\Big)\mathrm{d}t\Big|\\
&\geqslant \frac{|\gamma_2|}{b^2}\Big|\int_{-\infty}^{\infty}t^2\mathrm{e}^{-t^2/2}\mathrm{d}t\Big|-C_9\Big|\int_{-\infty}^{\infty}t^2\frac{|\gamma_2|np^2t^4}{b^4}\mathrm{e}^{-t^2/2}\mathrm{d}t\Big|\\
&- C_{10}\Big(\frac{np^3}{b^3}+\frac{n^2p^6}{b^4}+\frac{np^4}{b^3}\Big)\\
&\geqslant C_{11}\frac{np^3}{b^2}-C_{12}\frac{n^2p^5}{b^4}-C_{13}\frac{np^3}{b^3}\\
&\geqslant C_{11}\frac{np^3}{b^2}-\frac{C_{14}np^3}{h^3\max(1,\sqrt{n}p)^2}\\
&\geqslant C_{11}\frac{np^3}{h^2\max(1,np^2)}\Big(1-\frac{C_{14}}{h}\Big)\\
&\geqslant \frac{C_{11}}{h^2}\min(p,np^3)\Big(1-\frac{C_{14}}{h}\Big).
\end{aligned}$$

It suffices to take $h=2C_{14}$ an apply Lemma 3.1.2 with $\alpha=np^2$. □

Proof of Theorem 2.1.7. We have

$$\begin{aligned}
\Big|\hat{G}_2(t)-\hat{G}_1(t)\Big(1+\gamma_2 z^2+\frac{(\gamma_2 z^2)^2}{2}\Big)\Big| &= \Big|G_1(t)\Big(\mathrm{e}^{(\gamma_2 z^2)^2/2}-1-\gamma_2 z^2-\frac{(\gamma_2 z^2)^2}{2}\Big)\Big|\\
&= \Big|G_1(t)(\gamma_2 z^2)^3/2\int_0^1(1-\tau)^2\mathrm{e}^{\tau\gamma_2 z^2}\Big| \qquad (3.29)\\
&\leqslant C\gamma_2^3|z^6|\leqslant Cn^3p^9t^6,
\end{aligned}$$

and

$$\Big(\hat{F}(t)-\hat{G}_1(t)(1+\gamma_2 z^2)\Big)=\Big(\hat{G}_1(t)\frac{(\gamma_2 z^2)^2}{2}\Big) \tag{3.30}$$
$$+\quad\Big(\hat{G}_2(t)-\hat{G}_1(t)\Big(1+\gamma_2 z^2+\frac{(\gamma_2 z^2)^2}{2}\Big)\Big)+(\hat{F}(t)-\hat{G}_2(t)).$$

Let $b = h\max(1, \sqrt{n}p)$, $h \geqslant 1$. Then applying (3.25) and (3.30) we obtain

$$\begin{aligned}
\Big|\int_{-\infty}^{\infty} \mathrm{e}^{-t^2/2}(\hat{F}(t/b) - \hat{G}_1(t/b)(1 &+ \gamma_2 z^2))\mathrm{e}^{-\mathrm{i}t\alpha}\mathrm{d}t\Big| \geqslant \Big|\int_{-\infty}^{\infty} \mathrm{e}^{-t^2/2}\widehat{G}_1(t/b)\mathrm{e}^{-\mathrm{i}t\alpha}\frac{\gamma_2^2 t^4}{2b^4}\mathrm{d}t\Big| \\
&- C_{15}\Big|\int_{-\infty}^{\infty} \mathrm{e}^{-t^2/2}\Big(\frac{n^3p^9t^6}{b^6} + \frac{np^4|t|^3}{b^3}\Big)\mathrm{d}t\Big| \\
&\geqslant \frac{|\gamma_2^2|}{2b^4}\Big|\int_{-\infty}^{\infty} t^2\mathrm{e}^{-t^2/2}\mathrm{d}t\Big| - C_{16}\Big|\int_{-\infty}^{\infty} t^2\frac{|\gamma_2^2|np^2t^6}{b^6}\mathrm{e}^{-t^2/2}\mathrm{d}t\Big| \\
&-C_{17}\Big(\frac{n^3p^9}{b^6} + \frac{np^4}{b^3}\Big) \\
&\geqslant C_{18}\frac{n^2p^6}{b^4} - C_{19}\frac{n^3p^8}{b^6} - C_{20}\frac{np^4}{b^3} \\
&\geqslant C_{18}\frac{np^4}{b^3} - \frac{C_{21}np^4}{h^3\max(1,\sqrt{n}p)^2} \\
&\geqslant C_{18}\frac{np^4}{h^2\max(1,np^2)}\Big(1 - \frac{C_{21}}{h}\Big) \\
&\geqslant \frac{C_{18}}{h^2}\min(p^2, np^4)\Big(1 - \frac{C_{21}}{h}\Big).
\end{aligned}$$

It suffices to take $h = 2C_{21}$ an apply Lemma 3.1.2 with $\alpha = np^2$. □

Proof of Theorem 2.1.9. All proofs are similar. Therefore, we prove (2.15) and (2.16) only. Let $a = np^2$, $M = \mathcal{L}(S) - G_2$. Summing up the formula of inversion we obtain:

$$M\{[m,h]\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\widehat{M}(t)(\mathrm{e}^{-\mathrm{i}tm} - \mathrm{e}^{-\mathrm{i}t(h+1)})}{1-\mathrm{e}^{-\mathrm{i}t}}\,\mathrm{d}t.$$

Considering limit as $h \to \infty$, by Lebesgue theorem we get

$$M\{[m,\infty)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\widehat{M}(t)\mathrm{e}^{-\mathrm{i}tm}}{1-\mathrm{e}^{-\mathrm{i}t}}\,\mathrm{d}t.$$

Let

$$u(t) = \frac{\widehat{M}(t)\mathrm{e}^{-\mathrm{i}ta}}{1-\mathrm{e}^{-\mathrm{i}t}}.$$

Let us assume that $m - a \neq 0$. Integrating by parts and taking into account that $\mathrm{e}^{\mathrm{i}\pi} = \mathrm{e}^{-\mathrm{i}\pi}$ and, consequently, $u(\pi) = u(-\pi)$, leads us to the following relation

$$M\{[m,\infty)\} = -\frac{1}{2\pi(a-m)^2}\int_{-\pi}^{\pi} u(t)''\mathrm{e}^{\mathrm{i}t(a-m)}\,\mathrm{d}t.$$

Consequently,

$$|M\{[m,\infty)\}| \leqslant \frac{1}{2\pi(a-m)^2}\int_{-\pi}^{\pi} |u(t)''|\,\mathrm{d}t. \tag{3.31}$$

On the other hand,

$$|M\{[m,\infty)\}| \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} |u(t)|\,\mathrm{d}t. \tag{3.32}$$

Taking into account Lemmas 3.1.4–3.1.5 we obtain

$$\begin{aligned}
|u(t)| \quad &\leqslant \quad C\exp\Big\{-\frac{16}{25}np^2\sin^2\frac{t}{2}\Big\}\frac{\Big|\ln\sum_1^n\varphi_k - \widehat{G}_2\Big|}{|z|} \\
&\leqslant C \quad \exp\Big\{-\frac{4}{25}np^2\sin^2\frac{t}{2}\Big\}np^4|\sin(t/2)|^2.
\end{aligned}$$

Similarly we prove

$$|u'(t)| \leqslant C\frac{p}{\sqrt{n}}np^2\exp\Big\{-\frac{4}{5}np^2\sin^2\frac{t}{2}\Big\}, \tag{3.33}$$

$$|u''(t)| \leqslant C\frac{p}{\sqrt{n}}np^2\sqrt{np^2}\exp\Big\{-\frac{4}{5}np^2\sin^2\frac{t}{2}\Big\}. \tag{3.34}$$

Substituting these estimates into (3.31) and (3.32) we complete the proof. Note that $M\{[0,\infty\} = 0$. Therefore,

$$M\{[m,\infty)\} = -M\{[0,m)\}.$$

The estimates for Wasserestein metric follow from

$$\sum_{k=0}^{\infty}\frac{1}{1+(k-a)^2/a^2} \leqslant C\Big(1+a\int_{-\infty}^{\infty}\frac{1}{1+y^2}\mathrm{d}y\Big).$$

□

3.4 Auxiliary results for approximations of sums under analogue of Franken's condition

Let $X_1, X_2, \dots, X_n$ be identically distributed 1-dependent random variables concentrated on non-negative integers, $t \in \mathbb{R}$ and let

$$z = \mathrm{e}^{\mathrm{i}t} - 1, \quad S_n = X_1 + X_2 + \cdots + X_n, \quad \widehat{F}_n(t) = \mathrm{E}\mathrm{e}^{\mathrm{i}tS_n},$$

$$Y_k = \mathrm{e}^{\mathrm{i}tX_k} - 1, \quad \psi_1(t) = \mathrm{E}\mathrm{e}^{\mathrm{i}tX_1}, \quad \psi_k(t) = \frac{\mathrm{E}\mathrm{e}^{\mathrm{i}tS_k}}{\mathrm{E}\mathrm{e}^{\mathrm{i}tS_{k-1}}} \quad (k = 2, 3, \dots, n),$$

$$w_n(t) := (\mathrm{E}|\mathrm{e}^{\mathrm{i}tX_k} - 1|^2)^{1/2}, \quad \mathcal{K}(t) := \{t : w_n(t) \leqslant 1/6\}.$$

Lemma 3.4.1 *If $t \in \mathcal{K}(t)$, then*

$$\widehat{F}_n(t) = \prod_{k=1}^{n}\psi_k(t),$$

$$\psi_k(t) = \mathrm{E}\mathrm{e}^{\mathrm{i}tX_k} + \sum_{j=1}^{k-1}\frac{\widehat{\mathrm{E}}(\mathrm{e}^{\mathrm{i}tX_j}-1)(\mathrm{e}^{\mathrm{i}tX_{j+1}}-1)\cdots(\mathrm{e}^{\mathrm{i}tX_k}-1)}{\psi_j(t)\psi_{j+1}(t)\cdots\psi_{k-1}(t)} \tag{3.35}$$

and

$$|\psi_k(t) - 1| \leqslant |\mathrm{E}\mathrm{e}^{\mathrm{i}tX_k} - 1| + 6(w_n(t))^2.$$

Lemma 3.4.1 is a partial case of Lemma 3.1.7.

Lemma 3.4.2 *If $t \in \mathcal{K}(t)$, then*

$$\Big|\ln\widehat{F}_n(t) - n\mathrm{E}Y_1 - \sum_{k=2}^{n}\widehat{\mathrm{E}}Y_{k-1}Y_k\Big| \leqslant 2w_n(t)\Big(n|\mathrm{E}Y_1| + 22n\mathrm{E}|Y_k|^2\Big).$$

Lemma 3.4.2 is a partial case of Lemma 3.1.8.

Lemma 3.4.3 *If $1 \leqslant j \leqslant k$, then*

$$|\widehat{\mathrm{E}}Y_jY_{j+1}\cdots Y_k| \leqslant 2^{k-j}(C_0\nu_1)^{(k-j+1)/2}|z|^{k-j+1}, \tag{3.36}$$

$$|(\widehat{\mathrm{E}}Y_jY_{j+1}\cdots Y_k)'| \leqslant (k-j+1)2^{k-j}(C_0\nu_1)^{(k-j+1)/2}|z|^{k-j}. \tag{3.37}$$

Proof. We have

$$|e^{itX_m}-1| \leqslant X_m|z|, \quad E|Y_m|^2 \leqslant |z|^2 EX_m^2 \leqslant C_0\nu_1|z|^2, \quad (m=j,\dots,k). \tag{3.38}$$

From (3.38) and Lemma 3.1.9 follows (3.36). As follows from the formula in [25] p. 502

$$(\widehat{E}Y_jY_{j+1}\cdots Y_k)' = \sum_{i=j}^{k} \widehat{E}Y_j\dots Y_i'\dots Y_k.$$

We have $E|Y_i'|^2 = EX_i^2 \leqslant C_0\nu_1$. Therefore, for the proof of (3.37) it remains to apply Lemma 3.1.9.
□

Lemma 3.4.4 *If (2.20) is satisfied and n is sufficiently large, then, for all t,*

$$|\widehat{F}_n(t)| \leqslant \exp\{-Cn\nu_1\sin^2(t/2)\}, \quad |\widehat{D}_j(t)| \leqslant \exp\{-Cn\nu_1\sin^2(t/2)\}, \quad (j=1,2,3). \tag{3.39}$$

Proof. For sufficiently large n, $w_n(t) \in \mathcal{K}(t)$. Indeed, $|Y_1| \leqslant X_1|z|$, $|z| = 2|\sin(t/2)|$. Therefore,

$$w_n(t) \leqslant |z|\sqrt{EX_1^2} \leqslant 2\sqrt{C_0}\sqrt{\nu_1}|\sin(t/2)| \leqslant 2\sqrt{C_0}\sqrt{\nu_1}. \tag{3.40}$$

For sufficiently large n, the last estimate is less than 1/6. Therefore, applying Lemma 3.4.2, we obtain

$$\begin{aligned}
\Big|\ln\widehat{F}_n(t) - nEY_1 - \sum_{k=2}^{n}\widehat{E}Y_{k-1}Y_k\Big| &\leqslant 2w_n(t)\Big(n|EY_1| + 22nE|Y_k|^2\Big)\\
&\leqslant 2\sqrt{C_0}\sqrt{\nu_1}|\sin(t/2)|\Big(n\nu_1|z| + 22n|z|^2EX_1^2\Big)\\
&\leqslant 2\sqrt{C_0}\sqrt{\nu_1}|\sin(t/2)|\Big(n\nu_1|z| + 44n|z|C_0\nu_1\Big)\\
&= 4\sqrt{C_0}(1+44C_0)n\nu_1\sqrt{\nu_1}\sin^2(t/2).
\end{aligned} \tag{3.41}$$

Note that, for non-negative integers s,k,

$$e^{itk} - 1 = \sum_{j=1}^{s-1}\binom{k}{j}z^j + z^s\sum_{j=s}^{k}\binom{j-1}{s-1}e^{(k-j)it} = \sum_{j=1}^{s-1}\binom{k}{j}z^j + \theta\binom{k}{s}|z|^s. \tag{3.42}$$

Here, as usual, we assume $\binom{k}{j} = 0$, if $k < j$. Therefore,

$$EY_1 = \nu_1 z + \theta\frac{\nu_2}{2}|z|^2 = \nu_1 z + \theta 2\nu_2\sin^2(t/2). \tag{3.43}$$

Similarly,

$$|\widehat{E}Y_1Y_2| = |EY_1Y_2 - EY_1EY_2| \leqslant |EY_1Y_2| + |EY_1EY_2| \leqslant |EY_1Y_2| + 4\nu_1^2\sin^2(t/2)$$

and

$$|EY_1Y_2| \leqslant a(1,1)|z|^2 = 4a(1,1)\sin^2(t/2).$$

Combining the last two estimates with (3.43) and (3.41), we obtain

$$|\ln\widehat{F}_n(t) - n\nu_1 z| \leqslant 2n\left(\nu_2 + 2\nu_1^2 + 2a(1,1) + 42\sqrt{C_0}(1+44C_0)n\nu_1\sqrt{\nu_1}\right)\sin^2(t/2).$$

Therefore, for sufficiently large n,

$$\begin{aligned}
|\widehat{F}_n(t)| &\leqslant |\exp\{n\nu_1 z\}|\exp\{|\ln\widehat{F}_n(t) - n\nu_1 z|\}\\
&\leqslant \exp\{-2n\sin^2(t/2)\nu_1(1-o(1))\} \leqslant \exp\{-C\nu_1\sin^2(t/2)\}.
\end{aligned}$$

If n sufficiently large, then

$$|\Gamma_2| = o(n\nu_1), \qquad |\Gamma_3| = o(n\nu_1). \tag{3.44}$$

Indeed, due to (2.20), we have

$$\nu_3 = \mathrm{E}X_1(X_1-1)(X_1-2) \leqslant C_0\nu_2 = o(\nu_1), \quad a(1,2) \leqslant C_0 a(1,1) = o(\nu_1).$$

Consequently,

$$|\Gamma_2| \leqslant n\bigg(\frac{|\nu_2-\nu_1^2|}{2} + |a(1,1)-\nu_1^2|\bigg) = o(n\nu_1)$$

and

$$|\Gamma_3| \leqslant n\bigg|\frac{\nu_3}{6} + \frac{\nu_1\nu_2}{2} + \frac{\nu_1^3}{3}\bigg| + n\bigg|\frac{a(1,2)+a(2,1)}{2} - \nu_1\nu_2 + 2\nu_1(\nu_1^2 - a(1,1))\bigg| = o(n\nu_1).$$

Taking into account (3.44) we obtain

$$|\widehat{D}_3(t)| \leqslant \exp\{-2n\nu_1\sin^2(t/2) + 4\sin^2(t/2)(|\Gamma_2| + |z||\Gamma_3|)\} = \exp\{-2n\nu_1\sin^2(t/2)(1-o(1))\}.$$

If n is sufficiently large, then $|\widehat{D}_3(t)| \leqslant \exp\{-Cn\nu_1\sin^2(t/2)\}$. Estimates for $\widehat{D}_1(t)$ and $\widehat{D}_2(t)$ are proved similarly. $\square$

To shorten our expressions let

$$b_1 = \nu_1, \quad b_2 = \frac{\nu_2}{2} + a(1,1) - \nu_1^2, \quad b_3 = \tilde{b}_3 + a(1,1,1) - 2\nu_1 a(1,1) + \nu_1^3,$$

$$\tilde{b}_3 = \frac{\nu_3}{6} + [a(1,2) + a(2,1) - \nu_1\nu_2] + \nu_1(\nu_1^2 - a(1,1))$$

and let r be defined as in (2.19). For the sake of brevity we write ψ_j instead of $\psi_j(t)$.

Lemma 3.4.5 *If (2.20) is satisfied, and n is sufficiently large, then*

$$\begin{aligned}
\psi_1(t) &= 1 + \nu_1 z + \frac{\nu_2}{2}z^2 + \frac{\nu_3}{6}z^3 + C\theta\nu_4|z|^4,\\
\psi_2(t) &= 1 + b_1 z + b_2 z^2 + \tilde{b}_3 z^3 + C\theta r|z|^4,\\
\psi_k(t) &= 1 + b_1 z + b_2 z^2 + b_3 z^3 + C\theta r|z|^4, \qquad (k \geqslant 3).
\end{aligned}$$

Proof. Let $k \geqslant 8$. From Lemma 3.4.1 it follows that

$$\psi_k = 1 + \mathrm{E}Y_k + \sum_{j=1}^{k-1}\frac{\widehat{\mathrm{E}}Y_jY_{j+1}\cdots Y_k}{\psi_j\psi_{j+1}\cdots\psi_{k-1}} = 1 + \mathrm{E}Y_k + \sum_{j=k-2}^{k-1} + \sum_{j=k-6}^{k-3} + \sum_{j=1}^{k-7}. \tag{3.45}$$

Applying (3.42) to the second term in (3.45) we obtain

$$\mathrm{E}Y_k = \nu_1 z + \frac{\nu_2}{2}z^2 + \frac{\nu_3}{6}z^3 + C\theta\nu_4|z|^4. \tag{3.46}$$

Due to Lemma 3.4.1 and (3.40) we have

$$|\psi_k - 1| \leqslant \nu_1|z| + 6C_0\nu_1|z|^2 \leqslant 2(1+12C_0)\nu_1|\sin(t/2)|. \tag{3.47}$$

We have $\nu_1 = o(1)$. Therefore, for sufficiently large n,

$$C_0\nu_1 \leqslant \frac{1}{400}, \quad |\psi_k - 1| \leqslant \frac{1}{10}, \qquad \frac{1}{|\psi_k|} \leqslant \frac{10}{9}. \tag{3.48}$$

Taking into account (3.48) and (3.36) we can estimate the last term in (3.45):

$$\Big|\sum_{j=1}^{k-7}\frac{\widehat{\mathrm{E}}Y_jY_{j+1}\cdots Y_k}{\psi_j\psi_{j+1}\cdots\psi_{k-1}}\Big| \leqslant \sum_{j=1}^{k-7}\Big(\frac{10}{9}\Big)^{k-j}2^{k-j}|z|^{k-j+1}(C_0\nu_1)^{(k-j+1)/2}$$
$$\leqslant (C_0\nu_1)^4|z|^8\sum_{j=1}^{k-7}\Big(\frac{10}{9}\Big)^{k-j}2^{k-j}2^{k-j-7}\Big(\frac{1}{20}\Big)^{k-j-7} \leqslant C\nu_1^4|z|^4. \qquad (3.49)$$

From (3.48) it follows that

$$\Big|\sum_{j=k-6}^{k-3}\frac{\widehat{\mathrm{E}}Y_jY_{j+1}\cdots Y_k}{\psi_j\psi_{j+1}\cdots\psi_{k-1}}\Big| \leqslant \sum_{j=k-6}^{k-3}|\widehat{\mathrm{E}}Y_jY_{j+1}\cdots Y_k|\Big(\frac{10}{9}\Big)^{k-j-1} \leqslant C\sum_{j=k-6}^{k-3}|\widehat{\mathrm{E}}Y_jY_{j+1}\cdots Y_k|.$$

Applying inequality $|\mathrm{e}^{\mathrm{i}X}-1|\leqslant X|z|$ it is easy to check that

$$\mathrm{E}|Y_1Y_2\cdots Y_m|\leqslant a(1,1,\dots,1)|z|^m, \quad \mathrm{E}|Y_1(Y_1-1)Y_2|\leqslant 2a(2,1)|z|^3 \qquad (3.50)$$

and so on. Thus, from (3.7) we get

$$|\widehat{\mathrm{E}}Y_1Y_2| \leqslant \mathrm{E}|Y_1Y_2|+\mathrm{E}|Y_1|\mathrm{E}|Y_2|\leqslant(a(1,1)+\nu_1^2)|z|^2, \qquad (3.51)$$
$$|\widehat{\mathrm{E}}Y_1Y_2Y_3| \leqslant \mathrm{E}|Y_1Y_2Y_3|+\mathrm{E}|Y_1|\mathrm{E}|Y_2Y_3|+|\widehat{\mathrm{E}}Y_1Y_2|\mathrm{E}|Y_3|$$
$$\leqslant \big(a(1,1,1)+2\nu_1a(1,1)+\nu_1^2\big)|z|^3. \qquad (3.52)$$

and

$$|\widehat{\mathrm{E}}Y_1Y_2Y_3Y_4| \leqslant \mathrm{E}|Y_1Y_2Y_3Y_4|+\mathrm{E}|Y_1|\mathrm{E}|Y_2Y_3Y_4|+|\widehat{\mathrm{E}}Y_1Y_2|\mathrm{E}|Y_3Y_4|+|\widehat{\mathrm{E}}Y_1Y_2Y_3|\mathrm{E}|Y_4|$$
$$\leqslant a(1,1,1,1)|z|^4+\nu_1a(1,1,1)|z|^4+|\widehat{\mathrm{E}}Y_1Y_2|a(1,1)|z|^2+|\widehat{\mathrm{E}}Y_1Y_2Y_3|\nu_1|z|$$
$$\leqslant a(1,1,1,1)|z|^4+\nu_1a(1,1,1)|z|^4+(a(1,1)+\nu_1^2)a(1,1)|z|^4$$
$$+[a(1,1,1)+2\nu_1a(1,1)+\nu_1^2]\nu_1|z|$$
$$\leqslant [a(1,1,1,1)+2\nu_1a(1,1,1)+3a(1,1)\nu_1^2+a^2(1,1)+\nu_1^4]|z|^4$$
$$\leqslant [a(1,1,1,1)+2\nu_1a(1,1,1)+4a^2(1,1)+4\nu_1^4]|z|^4\leqslant Cr|z|^4.$$

Similarly estimating all remaining terms, we finally obtain

$$\Big|\sum_{j=k-6}^{k-3}\frac{\widehat{\mathrm{E}}Y_jY_{j+1}\cdots Y_k}{\psi_j\psi_{j+1}\cdots\psi_{k-1}}\Big|\leqslant Cr|z|^4. \qquad (3.53)$$

Collecting the last estimate, (3.46) and (3.49) and substituting them into (3.45) we have

$$\psi_k = 1+\sum_{j=1}^{3}\nu_j\frac{z^j}{j}+\frac{\widehat{\mathrm{E}}Y_{k-2}Y_{k-1}Y_k}{\psi_{k-2}\psi_{k-1}}+\frac{\widehat{\mathrm{E}}Y_{k-1}Y_k}{\psi_{k-1}}+C\theta r|z|^4. \qquad (3.54)$$

It is easy to see that $\nu_4\leqslant C_0^2\nu_2$, $\nu_4\leqslant C_0^2\nu_2$, $a(1,1,1)\leqslant C_0a(1,1)$, $a(k,1)\leqslant C(k)a(1,1)$, since all X_j are bounded by C_0. Therefore, from (3.54), arguing as in above, we obtain the following rough estimate

$$\psi_k = 1+\nu_1z+C\theta(\nu_2+a(1,1)+\nu_1^2)|z|^2. \qquad (3.55)$$

Taking into account (3.48) and (3.55), for a sufficiently large n, we prove

$$\frac{1}{\psi_k} = \frac{1}{1-(1-\psi_k)} = 1+(1-\psi_k)+C\theta|1-\psi_k|^2\sum_{j=0}^{\infty}|1-\psi_k|^j$$
$$+ \quad 1+(1-\psi_k)+C\theta\nu_1^2|z|^2\sum_{j=0}^{\infty}\frac{1}{10^j} = 1-\nu_1z+C\theta[\nu_2+a(1,1)+\nu_1^2]|z|^2. \qquad (3.56)$$

From (3.48) it follows that

$$\left|\frac{1}{\psi_{k-2}\psi_{k-1}}-1\right|\leqslant C|1-\psi_{k-2}\psi_{k-1}|\leqslant C\big(|1-\psi_{k-2}|+|\psi_{k-2}|\,|1-\psi_{k-1}|\big)\leqslant C\nu_1|z|. \tag{3.57}$$

Combining estimates (3.56), (3.57), (3.51) and (3.52) with (3.54) we obtain

$$\psi_k=1+\sum_{j=1}^{3}\frac{\nu_j z^j}{j}+\widehat{\mathrm{E}}Y_{k-2}Y_{k-1}Y_k+\widehat{\mathrm{E}}Y_{k-1}Y_k(1-\nu_1 z)+C\theta r|z|^4. \tag{3.58}$$

Applying (3.42) with $s=4$ and $s=3$, we get

$$\begin{aligned}\mathrm{E}Y_1Y_2 &= \mathrm{E}\Big\{X_1z+\frac{X_1(X_1-1)z^2}{2}+\frac{X_1(X_1-1)(X_1-2)z^3}{6}\Big\}Y_2+X\theta a(3,1)|z|^4\\ &= \mathrm{E}X_1z\Big\{X_2z+\frac{X_2(X_2-1)z^2}{2}+C\theta X_2(X_2-1)(X_2-2)|z|^3\Big\}\\ &\quad +\mathrm{E}\frac{X_1(X_1-1)z^2}{2}\big(X_2z+C\theta X_2(X_2-1)|z|^2\big)+C\theta a(3,1)|z|^4\\ &= a(1,1)z^2+\big(a(1,2)+a(2,1)\big)z^3+C\theta\big(a(3,1)+a(2,2)+a(1,3)\big)|z|^4. \end{aligned}\tag{3.59}$$

Similarly

$$\begin{aligned}\mathrm{E}Y_1\mathrm{E}Y_2 &= \nu_1z\left(\nu_1z+\frac{\nu_2z^2}{2}+C\theta\nu_3|z|^3\right)+\frac{\nu_2z^2}{2}(\nu_1z+C\theta\nu_2|z|^2)+C\theta\nu_1\nu_3|z|^4\\ &= \nu_1^2z^2+\nu_1\nu_2z^3+C\theta(\nu_1\nu_3+\nu_2^2)|z|^4. \end{aligned}\tag{3.60}$$

Combining the last two expressions with the definition of $\widehat{\mathrm{E}}Y_1Y_2$ we obtain

$$\widehat{\mathrm{E}}Y_1Y_2(1-\nu_1z)=[a(1,1)-\nu_1^2]z^2+\Big\{[a(1,2)+a(2,1)-\nu_1\nu_2]-\nu_1[a(1,1)-\nu_1^2]\Big\}z^3+C\theta r|z|^4. \tag{3.61}$$

Arguing in the similar way we prove

$$\widehat{\mathrm{E}}Y_1Y_2Y_3=[a(1,1,1)-2\nu_1a(1,1)+\nu_1^3]z^3+C\theta r|z|^4.$$

Putting the last expression and (3.61) into (3.58) we get $\psi_k=1+b_1z+b_2z^2+b_3z^3+C\theta r|z|^4$. If $k=3,4,\dots,7$ then arguing is exactly the same if not simpler, since the last term in (3.45) is absent. The case $k=2$ is proved similarly. The case $k=1$ follows from (3.42) with $s=4$. $\square$

Lemma 3.4.6 *If (2.20) is satisfied and n is sufficiently large, then*

$$|\psi_k'|\leqslant 6C_0\nu_1,\qquad (k=1,2,...). \tag{3.62}$$

Proof. We prove (3.62) by induction. It is easy to check that

$$|\psi_1'|\leqslant|\mathrm{iE}X_1\mathrm{e}^{\mathrm{i}tX_1}|\leqslant\mathrm{E}|X_1\mathrm{e}^{\mathrm{i}tX_1}|=\nu_1\leqslant C_0\nu_1. \tag{3.63}$$

Let us assume that (3.62) holds for $m=1,2,\dots,k-1$. From (3.35) it follows that

$$|\psi_k'|\leqslant\nu_1+\sum_{j=1}^{k-1}\frac{|(\widehat{\mathrm{E}}Y_j\cdots Y_k)'|}{|\psi_j\cdots\psi_{k-1}|}+\sum_{j=1}^{k-1}\frac{|\widehat{\mathrm{E}}Y_j\cdots Y_k|}{|\psi_j\cdots\psi_{k-1}|}\sum_{m=j}^{k-1}\left|\frac{\psi_m'}{\psi_m}\right|. \tag{3.64}$$

Taking into account (3.37) and (3.48) we obtain

$$\begin{aligned}\sum_{j=1}^{k-1}\frac{|(\widehat{\mathrm{E}}Y_j\cdots Y_k)'|}{|\psi_j\cdots\psi_{k-1}|} &\leqslant \sum_{j=1}^{k-1}\left(\frac{10}{9}\right)^{k-j}(k-j+1)2^{k-j}|z|^{k-j}(C_0\nu_1)^{(k-j+1)/2}\\ &\leqslant C_0\nu_1\sum_{j=1}^{k-1}\left(\frac{10}{9}\right)^{k-j}(k-j+1)4^{k-j}\left(\frac{1}{400}\right)^{(k-j-1)/2}\\ &= 20C_0\nu_1\sum_{j=1}^{k-1}(k-j+1)\left(\frac{2}{9}\right)^{k-j}\leqslant 4.5C_0\nu_1. \end{aligned}\tag{3.65}$$

For estimation of the last sum, we used a standard combinatorial approach by taking $x = 2/9$ and applying the following formula

$$\sum_{j=1}^{k-2}(k-j+1)x^{k-j} \leqslant 3x^2 + 4x^3 + \cdots = \Big(\sum_{m=3}^{\infty} x^m\Big)' = \Big(\frac{x^3}{1-x}\Big)' = \frac{3x^2}{1-x} + \frac{x^3}{(1-x)^2}.$$

Taking into account inductions assumption, (3.48) and (3.36) we obtain

$$\begin{aligned}
\sum_{j=1}^{k-1}\frac{|\widehat{\mathrm{E}}Y_j\cdots Y_k|}{|\psi_j\cdots\psi_{k-1}|}\sum_{m=j}^{k-1}\Big|\frac{\psi_m'}{\psi_m}\Big| &\leqslant 6C_0\nu_1\sum_{j=1}^{k-1}\Big(\frac{10}{9}\Big)^{k-j+1}(k-j)2^{k-j}|z|^{k-j+1}(C_0\nu_1)^{(k-j+1)/2}\\
&\leqslant 6C_0\nu_1\sum_{j=1}^{k-1}(k-j)\Big(\frac{10}{9}\Big)^{k-j+1}2^{k-j}2^{k-j+1}(20)^{-k+j-1}\\
&= 3C_0\nu_1\sum_{j=1}^{k-1}(k-j)\Big(\frac{2}{9}\Big)^{k-j+1} \leqslant \frac{12C_0\nu_1}{49}. \qquad (3.66)
\end{aligned}$$

Combining (3.66) and (3.65) with (3.64) we complete Lemma's proof. □

Lemma 3.4.7 *If (2.20) is satisfied and n is sufficiently large,then*

$$\begin{aligned}
\psi_1'(t) &= \nu_1 z' + \frac{\nu_2}{2}(z^2)' + \frac{\nu_3}{6}(z^3)' + C\theta\nu_4|z|^3,\\
\psi_2'(t) &= b_1 z' + b_2(z^2)' + \tilde{b}_3(z^3)' + C\theta r|z|^3,\\
\psi_k'(t) &= b_1 z' + b_2(z^2)' + b_3(z^3)' + C\theta r|z|^3, \qquad (k \geqslant 3).
\end{aligned}$$

Proof. Lemma's proof is a combination of the proofs of Lemma 3.4.5 and Lemma 3.4.6 and is therefore omitted. □

Lemma 3.4.8 *If (2.20) is satisfied and n is sufficiently large, then*

$$\begin{aligned}
\ln\psi_1(t) &= \nu_1 z + \frac{\nu_2-\nu_1^2}{2}z^2 + \Big(\frac{\nu_3}{6} - \frac{\nu_1\nu_2}{2} + \frac{\nu_1^3}{3}\Big)z^3 + C\theta r|z|^4,\\
\ln\psi_2(t) &= b_1 z + \Big(b_2 - \frac{b_1^2}{2}\Big)z^2 + \Big(\tilde{b}_3 - b_1 b_2 + \frac{b_1^3}{3}\Big)z^3 + C\theta r|z|^4,\\
\ln\psi_k(t) &= b_1 z + \Big(b_2 - \frac{b_1^2}{2}\Big)z^2 + \Big(b_3 - b_1 b_2 + \frac{b_1^3}{3}\Big)z^3 + C\theta r|z|^4, \qquad (k \geqslant 3).
\end{aligned}$$

Proof. From (3.47) and (3.48) it follows that

$$\begin{aligned}
\ln\psi_k &= (\psi_k - 1) - \frac{(\psi_k-1)^2}{2} + \frac{(\psi_k-1)^3}{3} + C\theta|\psi_k - 1|^4\sum_{j=0}^{\infty}\frac{|\psi_k-1|^j}{j}\\
&= (\psi_k - 1) - \frac{(\psi_k-1)^2}{2} + \frac{(\psi_k-1)^3}{3} + C\theta\nu_1^4|z|^4\sum_{j=0}^{\infty}\Big(\frac{1}{10}\Big)^j.
\end{aligned}$$

Applying Lemma 3.4.5 we complete the proof. □

Lemma 3.4.9 *If (2.20) is satisfied and n is sufficiently large, then*

$$\begin{aligned}
(\ln\psi_1(t))' &= \nu_1 z' + \frac{\nu_2-\nu_1^2}{2}(z^2)' + \Big(\frac{\nu_3}{6} - \frac{\nu_1\nu_2}{2} + \frac{\nu_1^3}{3}\Big)(z^3)' + C\theta r|z|^3,\\
(\ln\psi_2(t))' &= b_1 z' + \Big(b_2 - \frac{b_1^2}{2}\Big)(z^2)' + \Big(\tilde{b}_3 - b_1 b_2 + \frac{b_1^3}{3}\Big)(z^3)' + C\theta r|z|^3,\\
(\ln\psi_k(t))' &= b_1 z' + \Big(b_2 - \frac{b_1^2}{2}\Big)(z^2)' + \Big(b_3 - b_1 b_2 + \frac{b_1^3}{3}\Big)(z^3)' + C\theta r|z|^3, \qquad (k \geqslant 3).
\end{aligned}$$

Proof. We have

$$(\ln\psi_k)' = \frac{\psi_k'}{\psi_k}.$$

Arguing as in (3.56) we prove

$$\frac{1}{\psi_k} = 1 + (1-\psi_k) + (1-\psi_k)^2 + C\theta\nu_1^3|z|^5.$$

The proof now follows from Lemmas 3.4.7 and 3.4.5. □

Lemma 3.4.10 *Let M be finite variation signed measure concentrated on integers, $\tau\in[0,1]$, $\alpha\in[0,1]$. If (2.20) is satisfied and n is sufficiently large, then*

$$\begin{aligned}\|M\exp\{n\nu_1 U + \alpha\Gamma_2U^2 + \tau\Gamma_3U^3\}\| &\leqslant \|M\exp\{0.5n\nu_1U\}\|,\\ \|M\exp\{n\nu_1 U + \alpha\Gamma_2U^2 + \tau\Gamma_3U^3\}\|_\infty &\leqslant \|M\exp\{0.5n\nu_1U\}\|_\infty.\end{aligned}$$

Proof. Due to (3.44), if n is sufficiently large, then

$$\|\Gamma_2+\Gamma_3U\| = \Theta(|\Gamma_2| + 2|\Gamma_3|) = \Theta\frac{n\nu_1}{10}.$$

Total variation norm of a distribution is always equal to unity. Consequently, $\|\exp\{0.5\gamma_1U\}\| = 1$. If n is sufficiently large, then, taking into account Lemma 3.1.3 below, we obtain

$$\begin{aligned}&\|\exp\{0.5\Gamma_1U + \alpha\Gamma_2U^2 + \tau\Gamma_3U^3\}\| = \|\exp\{0.5n\nu_1U + 0.1n\nu_1\Theta\}\|\\ &\quad\leqslant\ 1 + \sum_{m=1}^\infty \frac{1}{m!}\Big\|\frac{n\nu_1}{10}U^2\exp\Big\{\frac{n\nu_1}{2m}U\Big\}\Big\|^m \leqslant 1 + \sum_{m=1}^\infty \frac{\mathrm{e}^m}{m^m\sqrt{2\pi m}}\Big(\frac{3m}{5\mathrm{e}}\Big)^m \leqslant C.\end{aligned}$$

The proof of Lemma's assertion now follows from the properties of the total variation and local norms. Indeed, we have

$$\begin{aligned}&\|M\exp\{\Gamma_1U + \alpha\Gamma_2U^2 + \tau\Gamma_3U^3\}\|_\infty\\ &\qquad\leqslant\ \|M\exp\{0.5\Gamma_1U\}\|_\infty\|M\exp\{0.5\Gamma_1U + \alpha\Gamma_2U^2 + \tau\Gamma_3U^3\}\|.\end{aligned}$$

Similar relation also holds for the total variation norm. □

3.5 Proof of Theorems 2.2.1–2.2.5

Proof of Theorem 2.2.4. To make expressions shorter we write $\widehat{F}_n$ and $\widehat{D}_3$ instead of $\widehat{F}_n(t)$ and $\widehat{D}_3(t)$, respectively. We assume that n is sufficiently large and all auxiliary results hold. Taking into account (3.39) and Lemma 3.4.8 we obtain

$$\begin{aligned}|\widehat{F}_n - \widehat{D}_3| &\leqslant C\mathrm{e}^{-Cn\nu_1\sin^2(t/2)}|\ln\widehat{F}_n(t) - \ln\widehat{D}_3|\\ &\leqslant Cnr|\sin(t/2)|^4\mathrm{e}^{-C\nu_1\sin^2(t/2)}.\end{aligned}\tag{3.67}$$

Applying formula of inversion (3.2) and (3.6) we easily prove (2.28). From Lemma 3.4.9 it follows that

$$|(\ln\widehat{F}_n)' - n\nu_1\mathrm{i}| \leqslant |(\ln\widehat{F}_n)' - n\nu_1z'| + n\nu_1|z'-\mathrm{i}| \leqslant Cn\nu_1|z|.$$

Therefore, applying Lemma 3.4.9, (3.67) and (3.6), we obtain

$$\begin{aligned}&\Big|\big(\mathrm{e}^{-\mathrm{i}tn\nu_1}\widehat{F}_n - \mathrm{e}^{-\mathrm{i}tn\nu_1}\widehat{D}_3\big)'\Big|\\ &\quad=\ \Big|\big(\ln\widehat{F}_n - \mathrm{i}tn\nu_1\big)'\widehat{F}_n\mathrm{e}^{-\mathrm{i}tn\nu_1} - (\ln\widehat{D}_3 - \mathrm{i}tn\nu_1)'\widehat{D}_3\mathrm{e}^{-\mathrm{i}tn\nu_1}\Big|\\ &\quad\leqslant\ \Big|\big(\ln\widehat{F}_n - \mathrm{i}tn\nu_1\big)'(\widehat{F}_n - \widehat{D}_3)\mathrm{e}^{-\mathrm{i}tn\nu_1} + \mathrm{e}^{-\mathrm{i}tn\nu_1}\widehat{D}_3\big((\ln\widehat{F}_n)' - (\ln\widehat{D}_3)'\big)\Big|\\ &\quad\leqslant\ C\mathrm{e}^{-Cn\nu_1\sin^2(t/2)}\Big\{|(\ln\widehat{F}_n)' - \mathrm{i}n\nu_1|nr|\sin(t/2)|^4 + nr|\sin(t/2)|^3\Big\}\\ &\quad\leqslant\ C\mathrm{e}^{-Cn\nu_1\sin^2(t/2)}nr|\sin(t/2)|^3.\end{aligned}\tag{3.68}$$

Let us take in Lemma 3.1.1 $v = u = n\nu_1$ and $M = F_n - D_3$. If n is sufficiently large and $n\nu_1 \geqslant 1$, then the estimate (2.27) follow from (3.67), (3.68) and (3.6) . □

Proof of Theorem 2.2.5. We have

$$D_3 - D_2(I + \Gamma_3 U^3) = D_2(\exp\{\Gamma_3 U^3\} - I - \Gamma_3 U^3) = D_2(\Gamma_3 U^3)^2 \int_0^1 (1-\tau)\exp\{\tau\Gamma_3 U^3\}\mathrm{d}\tau.$$

From Lemma 3.4.10 and Lemma 3.1.3 it follows that

$$\begin{aligned} \|D_3 - D_2(I + \Gamma_3 U^3)\| &\leqslant \int_0^1 \|D_2 \exp\{\tau\Gamma_3 U^3\}\Gamma_3^2 U^6\|\mathrm{d}\tau \\ &\leqslant C|\Gamma_3|^2 \|U^6 \exp\{0.5 n\nu_1 U\}\| \leqslant C\frac{|\Gamma_3|^2}{(n\nu_1)^3}. \end{aligned}$$

Estimate (2.29) follows from the triangle inequality and Theorem 2.2.4. For the proof of (2.30) one must replace the total variation norm by the local one. □

Proof of Theorem 2.2.3. If n is sufficiently large, then $n\nu_1 > 1$. Therefore, applying Lemmas 3.4.10 and 3.1.3 and Theorem 2.2.5, we obtain

$$\begin{aligned} \|F_n - D_2\| &\leqslant \|F_n - D_2(I + \Gamma_3 U^3)\| + \|\Gamma_3 U^3 D_2\| \\ &\leqslant \frac{Cr}{n\nu_1^2} + \frac{Cr_1^2}{n\nu_1^3} + Cnr_1\|U^3 \exp\{0.5n\nu_1 U\}\| \\ &\leqslant \frac{Cr}{\nu_1\sqrt{n\nu_1}} + \frac{Cr_1^2}{n\nu_1^3} + \frac{Cr_1}{\nu_1\sqrt{n\nu_1}} \leqslant \frac{Cr_1}{\nu_1\sqrt{n\nu_1}}. \end{aligned}$$

□

Proof of Theorem 2.2.2. We have

$$D_2 - D_1(I + \Gamma_2 U^2) = D_1(\exp\{\Gamma_2 U^2\} - I - \Gamma_2 U^2) = D_1(\Gamma_2 U^2)^2 \int_0^1 (1-\tau)\exp\{\tau\Gamma_2 U^2\}\mathrm{d}\tau.$$

From Lemma 3.4.10 and Lemma 3.1.3 it follows that

$$\begin{aligned} \|D_2 - D_1(I + \Gamma_2 U^2)\| &\leqslant \int_0^1 \|D_1 \exp\{\tau\Gamma_2 U^2\}\Gamma_2^2 U^4\|\mathrm{d}\tau \\ &\leqslant C|\Gamma_2|^2 \|U^4 \exp\{0.5 n\nu_1 U\}\| \leqslant C\frac{|\Gamma_2|^2}{(n\nu_1)^2}. \end{aligned}$$

Estimate (2.23) follows from the triangle inequality and Theorem 2.2.3. For the proof of (2.24) one must replace the total variation norm by the local one. □

Proof of Theorem 2.2.1. If n is sufficiently large, then $n\nu_1 > 1$. Therefore, applying Lemmas 3.4.10 and 3.1.3 and Theorem 2.2.2, we obtain

$$\begin{aligned} \|F_n - D_1\| &\leqslant \|F_n - D_1(I + \Gamma_2 U^2)\| + \|\Gamma_2 U^2 D_1\| \\ &\leqslant \frac{(\nu_2 + a(1,1) + \nu_1^2)^2}{\nu_1^2} + \frac{r_1}{\nu_1\sqrt{n\nu_1}} \\ &+ Cn(\nu_2 + a(1,1) + \nu_1^2)\|U^2 \exp\{0.5n\nu_1 U\}\| \\ &\leqslant \frac{(\nu_2 + a(1,1) + \nu_1^2)^2}{\nu_1^2} + \frac{r_1}{\nu_1\sqrt{n\nu_1}} \\ &+ \frac{(\nu_2 + a(1,1) + \nu_1^2)}{\nu_1} \leqslant \frac{(\nu_2 + a(1,1) + \nu_1^2)}{\nu_1}. \end{aligned}$$

□

3.6 Auxiliary results for Poisson type approximations for sums of 1-dependent symmetric three point distributions

Let $s = \mathrm{i}t$ and $\upsilon_1 = \upsilon_1(s) = \mathrm{E}\mathrm{e}^{s\hat{X}_1}$,

$$\upsilon_k = \frac{\mathrm{E}\mathrm{e}^{sS_k}}{\mathrm{E}\mathrm{e}^{sS_{k-1}}}, \quad w_n(\mathrm{i}t) := \sqrt{\mathrm{E}|\mathrm{e}^{\mathrm{i}tX_j} - 1|^2},\ k = 2, 3, \dots, n.$$

The following lemma follows from Lemma 3.1 and Lemma 3.2 [25].

Lemma 3.6.1 *Let (2.31) hold. Then, for $k = 1, 2, \dots n$, and all real t*

$$\begin{aligned}
\upsilon_k &= \mathrm{E}\mathrm{e}^{s\hat{X}_k} + \sum_{j=1}^{k-1} \frac{\widehat{\mathrm{E}}(\mathrm{e}^{s\hat{X}_j} - 1)(\mathrm{e}^{s\hat{X}_{j+1}} - 1)\dots(\mathrm{e}^{s\hat{X}_k} - 1)}{\upsilon_j \upsilon_{j+1} \dots \upsilon_{k-1}}, \\
|\upsilon_k - 1| &\leqslant |\mathrm{E}\mathrm{e}^{s\hat{X}_k} - 1| + 2\sqrt{\mathrm{E}|\mathrm{e}^{s\hat{X}_{k-1}} - 1|^2 \mathrm{E}|\mathrm{e}^{s\hat{X}_k} - 1|^2}/(1 - 4w_n(z)) \\
&\leqslant 13\bar{p}|\mathrm{e}^s - 1| \le 1/5, \\
|\ln \widehat{B}_n(t) - nH_1| &\leqslant n\bar{p}\left(\frac{\sum_{j,k\in\{-1,1\}} |h(j,k)|}{\bar{p}} + 90\sqrt{\bar{p}}\right)|z|^2.
\end{aligned}$$

Lemma 3.6.2 *Let (2.31) be satisfied. Then, for all $|t| \leqslant \pi$,*

$$\max\{|\widehat{P}_n(t)|, |\widehat{B}^n(t)|\} \leqslant \exp\{-C_8 n\bar{p}\sin^2(t/2)\}.$$

Proof. Note that

$$|\widehat{P}_n(t)| \leqslant \Big|\exp\{nH_1\}\Big| \exp\{|\ln \widehat{P}_n(t) - nH_1|\}$$

and apply Lemma 3.6.1. The estimate for $\widehat{B}^n(t)$ follows directly from its definition and (2.31). □

Lemma 3.6.3 *Let condition (2.31) be satisfied. Then, for $k \geqslant 7$, we have*

$$\begin{aligned}
\upsilon_k - 1 &= H_1 + H_2 + H_3 - H_2H_1 + H_4 - 2H_3H_1 + H_2(H_1^2 - H_2) \\
&+ H_5 - 3H_4H_1 + H_3(3H_1^2 - 3H_2) + 3H_2^2H_1 \\
&+ H_6 - 4H_5H_1 + H_3(12H_2H_1 - 2H_3) + H_2(2H_2^2 - 4H_4) \\
&+ H_7 + H_3(10H_2^2 - 5H_4) - 5H_2H_5 + C_9\theta\bar{p}^4|z|^4. \qquad (3.69)
\end{aligned}$$

Moreover, for $k = 2, 3, 4, 5, 6$ the estimate (3.69) holds with $H_3 = H_4 = H_5 = H_6 = H_7 = 0$, $H_4 = H_5 = H_6 = H_7 = 0$, $H_5 = H_6 = H_7 = 0$, $H_6 = H_7 = 0$, $H_7 = 0$ respectively.

Proof. Applying Lemma 3.6.1 we obtain

$$\begin{aligned}
\upsilon_k - 1 &= H_1 + \frac{H_2}{\upsilon_{k-1}} + \frac{H_3}{\upsilon_{k-2}\upsilon_{k-1}} + \frac{H_4}{\upsilon_{k-3}\upsilon_{k-2}\upsilon_{k-1}} + \frac{H_5}{\upsilon_{k-4}\upsilon_{k-3}\upsilon_{k-2}\upsilon_{k-1}} \\
&+ \frac{H_6}{\upsilon_{k-5}\upsilon_{k-4}\upsilon_{k-3}\upsilon_{k-2}\upsilon_{k-1}} + \frac{H_7}{\upsilon_{k-6}\upsilon_{k-5}\upsilon_{k-4}\upsilon_{k-3}\upsilon_{k-2}\upsilon_{k-1}} \\
&+ \sum_{j=1}^{k-7} \frac{\widehat{\mathrm{E}}(\mathrm{e}^{s\hat{X}_j} - 1)(\mathrm{e}^{s\hat{X}_{j+1}} - 1)\dots(\mathrm{e}^{s\hat{X}_k} - 1)}{\upsilon_j\upsilon_{j+1}\dots\upsilon_{k-1}}. \qquad (3.70)
\end{aligned}$$

and

$$\frac{1}{|\upsilon_k|} \leqslant \frac{1}{1 - |1 - \upsilon_k|} \leqslant \frac{5}{4}.$$

From Lemma 3.1.9 we obtain

$$\begin{aligned}
|\widehat{\mathrm{E}}(\mathrm{e}^{s\hat{X}_j} - 1)(\mathrm{e}^{s\hat{X}_{j+1}} - 1)\dots(\mathrm{e}^{s\hat{X}_k} - 1)| &\leqslant 2^{k-j} \prod_{m=j}^{k} \sqrt{\mathrm{E}|\mathrm{e}^{s\hat{X}_m} - 1|^2} \\
&= 2^{k-j}|z|^{k-j+1}\bar{p}^{(k-j+1)/2} \leqslant C_{10}\bar{p}^4|z|^4(4\sqrt{\bar{p}})^{k-j-7}.
\end{aligned}$$

Noting that, due to (2.31), we can assume $\bar{p}$ to be small. Therefore,

$$\sum_{j=1}^{k-7} \frac{\widehat{\mathrm{E}}(\mathrm{e}^{s\hat{X}_j}-1)(\mathrm{e}^{s\hat{X}_{j+1}}-1)\dots(\mathrm{e}^{s\hat{X}_k}-1)}{\upsilon_j\upsilon_{j+1}\dots\upsilon_{k-1}}$$
$$\leqslant C_{11}\bar{p}^4|z|^4\sum_{j=1}^{k-7}(5\sqrt{\bar{p}})^{k-j-7}\leqslant C_{12}\bar{p}^4|z|^4. \qquad (3.71)$$

From (3.70) and (3.71) we get

$$\begin{aligned}
\frac{1}{\upsilon_k} &= 1+(1-\upsilon_k)+(1-\upsilon_k)^2+C_{13}\theta\bar{p}^3|z|^4 = 1-H_1-\frac{H_2}{\upsilon_{k-1}}-\frac{H_3}{\upsilon_{k-1}\upsilon_{k-2}}\\
&- \frac{H_4}{\upsilon_{k-1}\upsilon_{k-2}\upsilon_{k-3}}-\frac{H_5x^5}{\upsilon_{k-1}\upsilon_{k-2}\upsilon_{k-3}\upsilon_{k-4}}+(1-\upsilon_k)^2+C_{14}\theta\bar{p}^3|z|^4\\
&= 1-H_1+H_1^2-H_2+3H_2H_1-H_3+2H_2^2+4H_3H_1-H_4+5H_2H_3-H_5+C_{15}\theta\bar{p}^3|z|^4.
\end{aligned}$$

Putting the last expression into (3.70) we complete the proof of Lemma 3.6.3, for $k\geqslant 7$. The cases $k=2,3,4,5,6$ are proved similarly. □

Lemma 3.6.4 *Let condition (2.31) be satisfied. Then, for $k\geqslant 7$,*

$$\ln\upsilon_k = H_1+C_{16}\theta(K_1|z|^2+K_2|z|^3+K_3|z|^4).$$

Proof. Applying Lemma 3.6.1 and Lemma 3.6.3, it is not difficult to show that

$$\ln\upsilon = H_1+C_{17}\theta\bigg(\sum_{j=2}^{7}|H_j|+|H_1|^2+\bar{p}^2|z|^4\bigg).$$

Since $z=-\bar{z}-|z|^2$ one can easily obtain the following estimates

$$\begin{aligned}
H_1 &= (p_1-p_{-1})z-p_{-1}|z|^2,\\
H_1^2 &= C_{17}\theta(\bar{p}|p_1-p_{-1}||z|^2+\bar{p}^2|z|^4),\\
|H_2| &\leqslant |h(-1,-1)-h(-1,1)-h(1,-1)+h(1,1)|\,|z|^2+\\
&\quad |h(-1,1)+h(1,-1)-2h(1,1)|\,|z|^3+|h(1,1)|\,|z|^4,\\
|H_3| &\leqslant \sum_{j,k\in\{-1,1\}}|h(j,k,-1)-h(j,k,1)|\,|z|^3+\sum_{j,k\in\{-1,1\}}|h(j,k,-1)|\,|z|^4+|H_2|.
\end{aligned}$$

Let $h(j_1,j_2,\dots,j_k)=P(\hat{X}_1=j_1,\hat{X}_2=j_2,\dots,\hat{X}_k=j_k)-P(\hat{X}_1=j_1)P(\hat{X}_2=j_2)\cdots P(\hat{X}_k=j_k)$ and let $\sum_k^*$ denote the sum over all $j_1,\dots,j_k\in\{-1,1\}$. Then, for $k=4,5,6,7$,

$$|H_k|\leqslant {\sum}_k^*|h(j_1,j_2,\dots,j_k)|\,|z|^k+C_{18}\sum_{m=2}^{k-1}|H_m|.$$

Lemma's statement now follows from the following estimate

$$|h(j_1,j_2,\dots,j_k)|\leqslant C_{19}|a(j_1,j_2)|+C_{20}\bar{p}^2.$$

□

Lemma 3.6.5 *Let (2.31) be satisfied and let $|t|\leqslant\pi$. Then*

$$|\widehat{P}_n(t)-\widehat{B}^n(t)| \leqslant C_{21}\exp\{-C_{22}npt^2\}[K_1|t|^2+K_2|t|^3+K_3|t|^4].$$

3.7 Proof of Theorem 2.3.1

Proof. Applying Lemma 3.6.2 we get

$$|\widehat{P}_n(t)-\widehat{B}^n(t)| \leqslant C_{23}\exp\{-C_{22}n\bar{p}t^2\}|\ln\widehat{P}_n(t)-\ln\widehat{B}^n(t)|.$$

Note that $\ln\widehat{P}_n(t)=\sum_{k=1}^n \ln\upsilon_k$. Consequently, from Lemma 3.6.4 we get the required estimate.

Theorems proof now follows from Lemmas 3.6.5 and 3.1.1. □

3.8 Auxiliary results for sums of 1-dependent non-identically distributed Bernoulli variables

For the sake of brevity set $s=\mathrm{i}t$, $z=\mathrm{e}^{\mathrm{i}t}-1$. Moreover, we denote by θ all quantities satisfying $|\theta|\leqslant 1$ and use C for all absolute constants, which may vary from line to line. Let $\varphi_1=\varphi_1(z)=\mathrm{E}\mathrm{e}^{zX_1}$,

$$\varphi_k=\frac{\mathrm{E}\mathrm{e}^{zS_k}}{\mathrm{E}\mathrm{e}^{zS_{k-1}}},\quad w_n(\mathrm{i}t):=\sqrt{\mathrm{E}|\mathrm{e}^{\mathrm{i}tX_j}-1|^2},\ k=2,3,\dots,n.$$

More precisely, let

$$\frac{2\sum_{j=1}^n a_{2,j}}{\sum_{j=1}^n p_j}+180\max_{1\le j\le n}\sqrt{p_j}\le C<1. \tag{3.72}$$

The following lemma follows from Lemma 3.1 and Lemma 3.2 in Heinrich [25].

Lemma 3.8.1 *Let condition (3.72) hold. Then, for $k=1,2,\dots n$, and all real t*

$$\hat{\varphi}_k=\mathrm{E}\mathrm{e}^{s\tilde{X}_k}+\sum_{j=1}^{k-1}\frac{\widehat{\mathrm{E}}(\mathrm{e}^{s\tilde{X}_j}-1)(\mathrm{e}^{s\tilde{X}_{j+1}}-1)\dots(\mathrm{e}^{s\tilde{X}_k}-1)}{\hat{\varphi}_j\hat{\varphi}_{j+1}\cdots\hat{\varphi}_{k-1}} \tag{3.73}$$

and

$$\begin{aligned}|\hat{\varphi}_k-1| &\leqslant |\mathrm{E}\mathrm{e}^{s\tilde{X}_k}-1|+2\frac{\sqrt{\mathrm{E}|\mathrm{e}^{s\tilde{X}_{k-1}}-1|^2\mathrm{E}|\mathrm{e}^{s\tilde{X}_k}-1|^2}}{1-4w_n(z)}\\ &\leqslant p_k|\mathrm{e}^s-1|+12\sqrt{p_kp_{k-1}}|\mathrm{e}^s-1|\\ &\leqslant C(p_k+p_{k-1})|\mathrm{e}^s-1|, \qquad (3.74)\\ |\ln\mathrm{E}\mathrm{e}^{\mathrm{i}t\tilde{S}}-\lambda_1(\mathrm{e}^s-1)| &\leqslant \lambda_1\left(\frac{|\sum_{k=1}^n a_{2,k}|}{\lambda_1}+90\max_{1\le k\le n}\sqrt{p_k}\right)|\mathrm{e}^s-1|^2. \qquad (3.75)\end{aligned}$$

Lemma 3.8.2 *Let condition (3.72) be satisfied. Then, for all $|t|\leqslant\pi$,*

$$\max\{|\mathrm{E}\mathrm{e}^{\mathrm{i}t\tilde{S}}|,|\widehat{D}(t)|\}\leqslant\exp\{-C\lambda_1(\mathrm{e}^s-1)\}.$$

Proof. Note that

$$|\mathrm{E}\mathrm{e}^{\mathrm{i}t\tilde{S}}|\leqslant\left|\exp\Big\{\lambda_1(\mathrm{e}^s-1)\Big\}\right|\exp\Big\{|\ln\mathrm{E}\mathrm{e}^{\mathrm{i}t\tilde{S}}-\lambda_1(\mathrm{e}^s-1)|\Big\}$$

and apply (3.75). The estimate for $\widehat{M}(t)$ follows directly from its definition and (3.72). □

Lemma 3.8.3 *Let condition (3.72) be satisfied. Then, for $k \geqslant 7$, we have*

$$\begin{aligned}
\hat{\varphi}_k - 1 &= p_k z + a_{2,k} z^2 + (a_{3,k} - a_{2,k} p_{k-1}) z^3 \\
&+ (a_{4,k} - a_{3,k}(p_{k-2} + p_{k-1}) + a_{2,k}(p_{k-1}^2 - a_{2,k-1})) z^4 \\
&+ (a_{5,k} - a_{4,k}(p_{k-3} + p_{k-2} + p_{k-1}) + a_{3,k}(p_{k-1}^2 + p_{k-2}^2 + p_{k-1}p_{k-2} - a_{2,k-1} - a_{2,k-2}) \\
&+ a_{2,k}(2a_{2,k-1}p_{k-1} + a_{2,k-1}p_{k-2} - a_{3,k-1})) z^5 \\
&+ (a_{6,k} - a_{5,k}(p_{k-4} + p_{k-3} + p_{k-2} + p_{k-1}) - a_{4,k}(a_{2,k-1} + a_{2,k-2} + a_{2,k-3}) \\
&+ a_{3,k}(2a_{2,k-1}p_{k-1} + 2a_{2,k-2}p_{k-2} + 2a_{2,k-1}p_{k-2} + a_{2,k-2}p_{k-3} + a_{2,k-2}p_{k-1} - a_{3,k-1} \\
&- a_{3,k-2}) + a_{2,k}(a_{2,k-1}a_{2,k-2} + a_{2,k-1}^2 + 2a_{3,k-1}p_{k-1} + a_{3,k-1}p_{k-2} + a_{3,k-1}p_{k-3} \\
&- a_{4,k-1})) z^6 + (a_{7,k} - a_{5,k}(a_{2,k-1} + a_{2,k-2} + a_{2,k-3} + a_{2,k-4}) \\
&- a_{4,k}(a_{3,k-1} + a_{3,k-2} + a_{3,k-3}) + a_{3,k}(a_{2,k-2}a_{2,k-3} + a_{2,k-2}^2 \\
&+ 2a_{2,k-1}a_{2,k-2} + a_{2,k-1}^2 - a_{4,k-1} - a_{4,k-2}) \\
&+ a_{2,k}(2a_{2,k-1}a_{3,k-1} + a_{2,k-2}a_{3,k-1} + a_{2,k-3}a_{3,k-1} + a_{2,k-1}a_{3,k-2} - a_{5,k-1})) z^7 \\
&+ C\theta(p_k^4 + p_{k-1}^4 + p_{k-2}^4 + p_{k-3}^4 + p_{k-4}^4 + p_{k-5}^4 + p_{k-6}^4 + p_{k-7}^4)|z|^4. \qquad (3.76)
\end{aligned}$$

Moreover, for $k = 2, 3, 4, 5, 6$ the estimate (3.76) holds with $a_{3,k} = a_{4,k} = a_{5,k} = a_{6,k} = = a_{7,k} = 0$, $a_{4,k} = a_{5,k} = a_{6,k} = a_{7,k} = 0$, $a_{5,k} = a_{6,k} = a_{7,k} = 0$, $a_{6,k} = a_{7,k} = 0$, $a_{7,k} = 0$, respectively.

Proof. Let $k \geqslant 7$. Then from (3.73) we obtain

$$\begin{aligned}
\hat{\varphi}_k - 1 &= p_k z + \frac{a_{2,k} z^2}{\hat{\varphi}_{k-1}} + \frac{a_{3,k} z^3}{\hat{\varphi}_{k-2}\hat{\varphi}_{k-1}} \frac{a_{4,k} z^4}{\hat{\varphi}_{k-3}\hat{\varphi}_{k-2}\hat{\varphi}_{k-1}} \\
&+ \frac{a_{5,k} z^5}{\hat{\varphi}_{k-4}\hat{\varphi}_{k-3}\hat{\varphi}_{k-2}\hat{\varphi}_{k-1}} \frac{a_{6,k} z^6}{\hat{\varphi}_{k-5}\hat{\varphi}_{k-4}\hat{\varphi}_{k-3}\hat{\varphi}_{k-2}\hat{\varphi}_{k-1}} \\
&+ \frac{a_{7,k} z^7}{\hat{\varphi}_{k-6}\hat{\varphi}_{k-5}\hat{\varphi}_{k-4}\hat{\varphi}_{k-3}\hat{\varphi}_{k-2}\hat{\varphi}_{k-1}} \\
&+ \sum_{j=1}^{k-7} \frac{\widehat{\mathrm{E}}(\mathrm{e}^{s\tilde{X}_j} - 1)(\mathrm{e}^{s\tilde{X}_{j+1}} - 1)\dots(\mathrm{e}^{s\tilde{X}_k} - 1)}{\hat{\varphi}_j \hat{\varphi}_{j+1} \cdots \hat{\varphi}_{k-1}}. \qquad (3.77)
\end{aligned}$$

From Lemma 3.1.9 we obtain

$$\begin{aligned}
&|\widehat{\mathrm{E}}(\mathrm{e}^{s\tilde{X}_j} - 1)(\mathrm{e}^{s\tilde{X}_{j+1}} - 1)\dots(\mathrm{e}^{s\tilde{X}_k} - 1)| \\
&\qquad \leqslant 2^{k-j} \prod_{m=j}^{k} \sqrt{\mathrm{E}|\mathrm{e}^{s\tilde{X}_m} - 1|^2} \\
&\qquad = 2^{k-j} |z|^{k-j+1} \sqrt{p_j p_{j+1} \cdots p_k}. \qquad (3.78)
\end{aligned}$$

From (3.74) it follows that

$$\frac{1}{|\hat{\varphi}_k|} \leqslant \frac{1}{1 - |1 - \hat{\varphi}_k|} \leqslant \frac{5}{4}. \qquad (3.79)$$

Consequently, noting that due to (3.72) all p_j are very small, from (3.78), (3.79) we get

$$\begin{aligned}
&\sum_{j=1}^{k-7} \frac{\widehat{\mathrm{E}}(\mathrm{e}^{s\tilde{X}_j} - 1)(\mathrm{e}^{s\tilde{X}_{j+1}} - 1)\dots(\mathrm{e}^{s\tilde{X}_k} - 1)}{\hat{\varphi}_j \hat{\varphi}_{j+1} \cdots \hat{\varphi}_{k-1}} \\
&\leqslant \sqrt{p_{k-7} \cdots p_k} |z|^4 \sum_{j=1}^{k-7} 5^{k-j} \left(\frac{1}{180}\right)^{k-j-8+1} \\
&\qquad \leqslant C \sqrt{p_{k-7} \cdots p_k} |z|^4 \sum_{j=1}^{k-7} \left(\frac{5}{180}\right)^{k-j} \\
&\qquad\qquad \leqslant C \sqrt{p_{k-7} \cdots p_k} |z|^4 \qquad (3.80)
\end{aligned}$$

From (2.35), (3.74) and (3.80) we get

$$
\begin{aligned}
\frac{1}{\hat{\varphi}_k} &= 1 + (1-\hat{\varphi}_k) + (1-\hat{\varphi}_k)^2 + C\theta(p_k^3 + p_{k-1}^3)|z|^4 \\
&= 1 - p_k z - \frac{a_{2,k} z^2}{\hat{\varphi}_{k-1}} - \frac{a_{3,k} z^3}{\hat{\varphi}_{k-1}\hat{\varphi}_{k-2}} - \frac{a_{4,k} z^4}{\hat{\varphi}_{k-1}\hat{\varphi}_{k-2}\hat{\varphi}_{k-3}} - \frac{a_{5,k} z^5}{\hat{\varphi}_{k-1}\hat{\varphi}_{k-2}\hat{\varphi}_{k-3}\hat{\varphi}_{k-4}} + (1-\hat{\varphi}_k)^2 \\
&+ C\theta(p_k^3 + p_{k-1}^3)|z|^4 = 1 - p_k z + (p_k^2 - a_{2,k}) z^2 + (2a_{2,k} p_k + a_{2,k} p_{k-1} - a_{3,k}) z^3 \\
&+ (a_{2,k}^2 + a_{2,k} a_{2,k-1} + 2a_{3,k} p_k + a_{3,k} p_{k-1} + a_{3,k} p_{k-2} - a_{4,k}) z^4 \\
&+ (2a_{2,k} a_{3,k} + a_{2,k} a_{3,k-1} + a_{3,k} a_{2,k-1} + a_{3,k} a_{2,k-2} - a_{5,k}) z^5 \\
&+ C\theta(p_k^3 + p_{k-1}^3 + p_{k-2}^3 + p_{k-3}^3 + p_{k-4}^3 + p_{k-5}^3)|z|^4. \qquad (3.81)
\end{aligned}
$$

Putting the last expression into (3.77) we complete the proof of Lemma 3.8.3, for $k \geqslant 7$. The cases $k = 2, 3, 4, 5, 6$ are proved similarly. □

Lemma 3.8.4 *Let condition (3.72) be satisfied. Then, for $k \geqslant 7$,*

$$
\begin{aligned}
\frac{d}{dt}(\hat{\varphi}_k) &= i e^{it} \big[p_k + 2a_{2,k} z + 3(a_{3,k} - a_{2,k} p_{k-1}) z^2 \\
&+ 4(a_{4,k} - a_{3,k}(p_{k-2} + p_{k-1}) + a_{2,k}(p_{k-1}^2 - a_{2,k-1})) z^3 \\
&+ 5(a_{5,k} - a_{4,k}(p_{k-3} + p_{k-2} + p_{k-1}) + a_{3,k}(p_{k-1}^2 + p_{k-2}^2 + p_{k-1} p_{k-2} - a_{2,k-1} - a_{2,k-2}) \\
&+ a_{2,k}(2a_{2,k-1} p_{k-1} + a_{2,k-1} p_{k-2} - a_{3,k-1})) z^4 \\
&+ 6(a_{6,k} - a_{5,k}(p_{k-4} + p_{k-3} + p_{k-2} + p_{k-1}) - a_{4,k}(a_{2,k-1} + a_{2,k-2} + a_{2,k-3}) \\
&+ a_{3,k}(2a_{2,k-1} p_{k-1} + 2a_{2,k-2} p_{k-2} + 2a_{2,k-1} p_{k-2} + a_{2,k-2} p_{k-3} + a_{2,k-2} p_{k-1} - a_{3,k-1} \\
&- a_{3,k-2}) + a_{2,k}(a_{2,k-1} a_{2,k-2} + a_{2,k-1}^2 + 2a_{3,k-1} p_{k-1} + a_{3,k-1} p_{k-2} + a_{3,k-1} p_{k-3} \\
&- a_{4,k-1})) z^5 + 7(a_{7,k} - a_{5,k}(a_{2,k-1} + a_{2,k-2} + a_{2,k-3} + a_{2,k-4}) \\
&- a_{4,k}(a_{3,k-1} + a_{3,k-2} + a_{3,k-3}) + a_{3,k}(a_{2,k-2} a_{2,k-3} + a_{2,k-2}^2 \\
&+ 2a_{2,k-1} a_{2,k-2} + a_{2,k-1}^2 - a_{4,k-1} - a_{4,k-2}) \\
&+ a_{2,k}(2a_{2,k-1} a_{3,k-1} + a_{2,k-2} a_{3,k-1} + a_{2,k-3} a_{3,k-1} + a_{2,k-1} a_{3,k-2} - a_{5,k-1})) z^6 \big] \\
&+ C\theta(p_k^4 + p_{k-1}^4 + p_{k-2}^4 + p_{k-3}^4 + p_{k-4}^4 + p_{k-5}^4 + p_{k-6}^4 + p_{k-7}^4)|z|^5. \qquad (3.82)
\end{aligned}
$$

Proof. Let $k \geqslant 7$. Then from (3.73) we obtain

$$
\begin{aligned}
\frac{d}{dt}(\hat{\varphi}_k) = \frac{d}{dt}\Bigg(p_k z &+ \frac{a_{2,k} z^2}{\hat{\varphi}_{k-1}} + \frac{a_{3,k} z^3}{\hat{\varphi}_{k-2}\hat{\varphi}_{k-1}} + \frac{a_{4,k} z^4}{\hat{\varphi}_{k-3}\hat{\varphi}_{k-2}\hat{\varphi}_{k-1}} + \frac{a_{5,k} z^5}{\hat{\varphi}_{k-4}\hat{\varphi}_{k-3}\hat{\varphi}_{k-2}\hat{\varphi}_{k-1}} \\
&+ \frac{a_{6,k} z^6}{\hat{\varphi}_{k-5}\hat{\varphi}_{k-4}\hat{\varphi}_{k-3}\hat{\varphi}_{k-2}\hat{\varphi}_{k-1}} + \frac{a_{7,k} z^7}{\hat{\varphi}_{k-6}\hat{\varphi}_{k-5}\hat{\varphi}_{k-4}\hat{\varphi}_{k-3}\hat{\varphi}_{k-2}\hat{\varphi}_{k-1}} \\
&+ \sum_{j=1}^{k-7} \frac{\widehat{\mathrm{E}}(e^{s\tilde{X}_j} - 1)(e^{s\tilde{X}_{j+1}} - 1)\dots(e^{s\tilde{X}_k} - 1)}{\hat{\varphi}_j \hat{\varphi}_{j+1} \cdots \hat{\varphi}_{k-1}} \Bigg). \qquad (3.83)
\end{aligned}
$$

Noting that due to (3.72) all p_j are very small, we get

$$
\begin{aligned}
&\sum_{j=1}^{k-7} \frac{\widehat{\mathrm{E}}(e^{s\tilde{X}_j} - 1)(e^{s\tilde{X}_{j+1}} - 1)\dots(e^{s\tilde{X}_k} - 1)}{\hat{\varphi}_j \hat{\varphi}_{j+1} \cdots \hat{\varphi}_{k-1}} \\
&\leqslant \sqrt{p_{k-7}\cdots p_k}\,|z|^3 \sum_{j=1}^{k-7} (k-j+1) 5^{k-j} \left(\frac{1}{180}\right)^{k-j-8} \\
&\leqslant C\sqrt{p_{k-7}\cdots p_k}\,|z|^3 \sum_{j=1}^{k-7} (k-j+1) \left(\frac{5}{180}\right)^{k-j} \\
&\leqslant C\sqrt{p_{k-7}\cdots p_k}\,|z|^3 \leqslant C(p_{k-7}^4 + p_{k-6}^4 + p_{k-5}^4 + p_{k-4}^4 \\
&+ p_{k-3}^4 + p_{k-2}^4 + p_{k-1}^4 + p_k^4)|z|^3. \qquad (3.84)
\end{aligned}
$$

From (3.74), (3.81), (3.84) and Lemma 3.1.10 we get

$$\begin{aligned}\frac{d}{dt}\left(\frac{a_{2,k}z^2}{\hat{\varphi}_{k-1}}\right) &= ie^{it}\big(2a_{2,k}z - 3a_{2,k}p_{k-1}z^2 + 4a_{2,k}(p_{k-1}^2 - a_{2,k-1})z^3 + 5a_{2,k}(2a_{2,k-1}p_{k-1}\\ &+ a_{2,k-1}p_{k-2} - a_{3,k-1})z^4 + 6a_{2,k}(a_{2,k-1}^2 + a_{2,k-1}a_{2,k-2} + 2a_{3,k-1}p_{k-1} + a_{3,k-1}p_{k-2}\\ &+ a_{3,k-1}p_{k-3} - a_{4,k-1})z^5 + 7a_{2,k}(2a_{2,k-1}a_{3,k-1} + a_{2,k-1}a_{3,k-2} + a_{3,k-1}a_{2,k-2}\big)\\ &+ a_{3,k-1}a_{2,k-3} - a_{5,k-1})z^6 + C\theta(p_k^4 + p_{k-1}^4 + p_{k-2}^4 + p_{k-3}^4 + p_{k-4}^4 + p_{k-5}^4 + p_{k-6}^4\\ &+ p_{k-7}^4)|z|^5, \end{aligned} \tag{3.85}$$

$$\begin{aligned}\frac{d}{dt}\left(\frac{a_{3,k}z^3}{\hat{\varphi}_{k-1}\hat{\varphi}_{k-2}}\right) &= ie^{it}\big(3a_{3,k}z^2 - 4(a_{3,k}p_{k-1} - a_{3,k}p_{k-2})z^3 + 5a_{3,k}(p_{k-1}^2 + p_{k-2}^2\\ &+ p_{k-2}^2 + p_{k-1}p_{k-2} - a_{2,k-1} - a_{2,k-2})z^4 + 6a_{3,k}(2a_{2,k-2}p_{k-2} + a_{2,k-2}p_{k-3}\\ &+ a_{2,k-2}p_{k-1} + 2a_{2,k-1}p_{k-2} + 2a_{2,k-1}p_{k-1} - a_{3,k-1} - a_{3,k-2})z^5\\ &+ 7a_{3,k}(a_{2,k-2}^2 + 2a_{2,k-1}a_{2,k-2} + a_{2,k-2}a_{2,k-3} + a_{2,k-1}^2 - a_{4,k-1} - a_{4,k-2})z^6\big)\\ &+ C\theta(p_k^4 + p_{k-1}^4 + p_{k-2}^4 + p_{k-3}^4 + p_{k-4}^4 + p_{k-5}^4 + p_{k-6}^4 + p_{k-7}^4)|z|^5, \end{aligned} \tag{3.86}$$

$$\begin{aligned}\frac{d}{dt}\left(\frac{a_{4,k}z^4}{\hat{\varphi}_{k-1}\hat{\varphi}_{k-2}\hat{\varphi}_{k-3}}\right) &= ie^{it}\big(4a_{4,k}z^3 - 5a_{4,k}(p_{k-1} + p_{k-2} + p_{k-3})z^4 - 6a_{4,k}(a_{2,k-1}\\ &+ a_{2,k-2} + a_{2,k-3}z^5) - 7a_{4,k}(a_{3,k-1} + a_{3,k-2} + a_{3,k-3})z^6\big) + C\theta(p_k^4 + p_{k-1}^4 + p_{k-2}^4\\ &+ p_{k-3}^4 + p_{k-4}^4 + p_{k-5}^4 + p_{k-6}^4 + p_{k-7}^4)|z|^5, \end{aligned} \tag{3.87}$$

$$\begin{aligned}\frac{d}{dt}\left(\frac{a_{5,k}z^5}{\hat{\varphi}_{k-1}\hat{\varphi}_{k-2}\hat{\varphi}_{k-3}\hat{\varphi}_{k-4}}\right) &= ie^{it}\big(5a_{5,k}z^4 - 6ie^{it}a_{5,k}z^5(p_{k-1} + p_{k-2} + p_{k-3} + p_{k-4})|z|\big)\\ &- 7a_{5,k}z^6(a_{2,k-1} + a_{2,k-2} + a_{2,k-3} + a_{2,k-4})\big) + C\theta(p_k^4 + p_{k-1}^4 + p_{k-2}^4 + p_{k-3}^4\\ &+ p_{k-4}^4 + p_{k-5}^4 + p_{k-6}^4 + p_{k-7}^4)|z|^5, \end{aligned} \tag{3.88}$$

$$\begin{aligned}\frac{d}{dt}\left(\frac{a_{6,k}z^6}{\hat{\varphi}_{k-1}\hat{\varphi}_{k-2}\hat{\varphi}_{k-3}\hat{\varphi}_{k-4}\hat{\varphi}_{k-5}}\right) &= 6ie^{it}a_{6,k}z^5 + C\theta(p_k^4 + p_{k-1}^4\\ &+ p_{k-2}^4 + p_{k-3}^4 + p_{k-4}^4 + p_{k-5}^4 + p_{k-6}^4 + p_{k-7}^4)|z|^5, \end{aligned} \tag{3.89}$$

$$\begin{aligned}\frac{d}{dt}\left(\frac{a_{7,k}z^7}{\hat{\varphi}_{k-1}\hat{\varphi}_{k-2}\hat{\varphi}_{k-3}\hat{\varphi}_{k-4}\hat{\varphi}_{k-5}\hat{\varphi}_{k-6}}\right) &= 7ie^{it}a_{7,k}z^6 + C\theta(p_k^4 + p_{k-1}^4\\ &+ p_{k-2}^4 + p_{k-3}^4 + p_{k-4}^4 + p_{k-5}^4 + p_{k-6}^4 + p_{k-7}^4)|z|^5. \end{aligned} \tag{3.90}$$

Putting the (3.84)-(3.90) into (3.83)we complete the proof Lemma 3.8.4 for $k \geqslant 7$. The cases $k = 2, 3, 4, 5, 6$ are proved similarly . □

Lemma 3.8.5 *Let condition (3.72) be satisfied. Then, for $k \geqslant 7$,*

$$
\begin{aligned}
\ln \varphi_k &= p_k z + \big(a_{2,k} - \frac{1}{2}p_k^2\big)z^2 + \big(a_{3,k} - a_{2,k}(p_k + p_{k-1}) + \frac{1}{3}p_k^3\big)z^3 \\
&+ \big(a_{4,k} - a_{3,k}(p_k + p_{k-1} + p_{k-2}) + a_{2,k}(p_k^2 + p_{k-1}^2 + p_k p_{k-1} - \frac{1}{2}a_{2,k} - a_{2,k-1})\big)z^4 \\
&+ \big(a_{5,k} - a_{4,k}(p_{k-3} + p_{k-2} + p_{k-1} + p_k) + a_{3,k}(p_k p_{k-1} + p_k p_{k-2} + p_{k-1}p_{k-2} + p_k^2 \\
&+ p_{k-1}^2 + p_{k-2}^2 - a_{2,k} - a_{2,k-1} - a_{2,k-2}) + a_{2,k}(a_{2,k}p_k + a_{2,k}p_{k-1} + a_{2,k-1}p_k \\
&+ 2a_{2,k-1}p_{k-1} + a_{2,k-1}p_{k-2} - a_{3,k-1})\big)z^5 \\
&+ \big(a_{6,k} - a_{5,k}(p_{k-4} + p_{k-3} + p_{k-2} + p_{k-1} + p_k) - a_{4,k}(a_{2,k-3} + a_{2,k-2} + a_{2,k-1} + a_{2,k}) \\
&+ a_{3,k}(a_{2,k-1}p_k + a_{2,k-2}p_k + 2a_{2,k}p_k + 2a_{2,k}p_{k-1} + a_{2,k}p_{k-2} + 2a_{2,k-1}p_{k-1} \\
&+ 2a_{2,k-2}p_{k-2} + 2a_{2,k-1}p_{k-2} + a_{2,k-2}p_{k-3} + a_{2,k-2}p_{k-1} - \frac{1}{2}a_{3,k} - a_{3,k-1} - a_{3,k-2}) \\
&+ a_{2,k}(a_{2,k}a_{2,k-1} + a_{2,k-1}a_{2,k-2} + a_{2,k-1}^2 + a_{3,k-1}p_k + 2a_{3,k-1}p_{k-1} + a_{3,k-1}p_{k-2} \\
&+ a_{3,k-1}p_{k-3} - a_{4,k-1} + \frac{1}{3}a_{2,k}^2)\big)z^6 \\
&+ \big(a_{7,k} - a_{5,k}(a_{2,k} + a_{2,k-1} + a_{2,k-2} + a_{2,k-3} + a_{2,k-4}) - a_{4,k}(a_{3,k} + a_{3,k-1} + a_{3,k-2} \\
&+ a_{3,k-3}) + a_{3,k}(2a_{2,k-1}a_{2,k-2} + a_{2,k-2}a_{2,k-3} + a_{2,k-1}^2 + a_{2,k-2}^2 - a_{4,k-1} - a_{4,k-2}) \\
&+ a_{2,k}(a_{2,k}a_{3,k-1} + 2a_{2,k-1}a_{3,k-1} + a_{2,k-2}a_{3,k-1} + a_{2,k-3}a_{3,k-1} + a_{2,k-1}a_{3,k-2} - a_{5,k-1}) \\
&+ a_{2,k}a_{3,k}(a_{2,k} + 2a_{2,k-1} + a_{2,k-2})\big)z^7 \\
&+ C\theta(p_k^4 + p_{k-1}^4 + p_{k-2}^4 + p_{k-3}^4 + p_{k-4}^4 + p_{k-5}^4 + p_{k-6}^4 + p_{k-7}^4)|z|^4.
\end{aligned}
$$

Proof. From (3.74) we obtain

$$\ln \hat{\varphi}_k = (\hat{\varphi}_k - 1) - \frac{(\hat{\varphi}_k - 1)^2}{2} + \frac{(\hat{\varphi}_k - 1)^3}{3} + C\theta(p_k^4 + p_{k-1}^4)|e^z - 1|^4.$$

To complete the proof one needs to apply Lemma 3.8.5 . □

Lemma 3.8.6 *Let condition (3.72) be satisfied. Then, for $k \geqslant 7$,*

$$
\begin{aligned}
\frac{d}{dt}\Big(\ln \varphi_k\Big) &= i\mathrm{e}^{it}\big[p_k + \big(2a_{2,k} - p_k^2\big)z + \big(3a_{3,k} - 3a_{2,k}(p_k + p_{k-1}) + p_k^3\big)z^2 \\
&+ 4\big(a_{4,k} - a_{3,k}(p_k + p_{k-1} + p_{k-2}) + a_{2,k}(p_k^2 + p_{k-1}^2 + p_k p_{k-1} - \frac{1}{2}a_{2,k} - a_{2,k-1})\big)z^3 \\
&+ 5\big(a_{5,k} - a_{4,k}(p_{k-3} + p_{k-2} + p_{k-1} + p_k) + a_{3,k}(p_k p_{k-1} + p_k p_{k-2} + p_{k-1}p_{k-2} + p_k^2 \\
&+ p_{k-1}^2 + p_{k-2}^2 - a_{2,k} - a_{2,k-1} - a_{2,k-2}) + a_{2,k}(a_{2,k}p_k + a_{2,k}p_{k-1} + a_{2,k-1}p_k \\
&+ 2a_{2,k-1}p_{k-1} + a_{2,k-1}p_{k-2} - a_{3,k-1})\big)z^4 \\
&+ 6\big(a_{6,k} - a_{5,k}(p_{k-4} + p_{k-3} + p_{k-2} + p_{k-1} + p_k) - a_{4,k}(a_{2,k-3} + a_{2,k-2} + a_{2,k-1} + a_{2,k}) \\
&+ a_{3,k}(a_{2,k-1}p_k + a_{2,k-2}p_k + 2a_{2,k}p_k + 2a_{2,k}p_{k-1} + a_{2,k}p_{k-2} + 2a_{2,k-1}p_{k-1} \\
&+ 2a_{2,k-2}p_{k-2} + 2a_{2,k-1}p_{k-2} + a_{2,k-2}p_{k-3} + a_{2,k-2}p_{k-1} - \frac{1}{2}a_{3,k} - a_{3,k-1} - a_{3,k-2}) \\
&+ a_{2,k}(a_{2,k}a_{2,k-1} + a_{2,k-1}a_{2,k-2} + a_{2,k-1}^2 + a_{3,k-1}p_k + 2a_{3,k-1}p_{k-1} + a_{3,k-1}p_{k-2} \\
&+ a_{3,k-1}p_{k-3} - a_{4,k-1} + \frac{1}{3}a_{2,k}^2)\big)z^5 \\
&+ 7\big(a_{7,k} - a_{5,k}(a_{2,k} + a_{2,k-1} + a_{2,k-2} + a_{2,k-3} + a_{2,k-4}) - a_{4,k}(a_{3,k} + a_{3,k-1} + a_{3,k-2} \\
&+ a_{3,k-3}) + a_{3,k}(2a_{2,k-1}a_{2,k-2} + a_{2,k-2}a_{2,k-3} + a_{2,k-1}^2 + a_{2,k-2}^2 - a_{4,k-1} - a_{4,k-2}) \\
&+ a_{2,k}(a_{2,k}a_{3,k-1} + 2a_{2,k-1}a_{3,k-1} + a_{2,k-2}a_{3,k-1} + a_{2,k-3}a_{3,k-1} + a_{2,k-1}a_{3,k-2} - a_{5,k-1}) \\
&+ a_{2,k}a_{3,k}(a_{2,k} + 2a_{2,k-1} + a_{2,k-2})\big)z^6\big] \\
&+ C\theta(p_k^4 + p_{k-1}^4 + p_{k-2}^4 + p_{k-3}^4 + p_{k-4}^4 + p_{k-5}^4 + p_{k-6}^4 + p_{k-7}^4)|z|^3.
\end{aligned}
$$

Using (3.81) and (3.82) and Lemma 3.8.6 we get the stated of Lemma. □

Lemma 3.8.7 *Let condition (3.72) be satisfied. Let* $A_3 = \sum_{k=3}^n a_{3,k} - \sum_{k=2}^n a_{2,k}(p_k + p_{k-1}) + \frac{1}{3}\lambda_3$ *and* $A_2(t) = \Big(\sum_{k=2}^n a_{2,k} - \frac{1}{2}\lambda_2\Big)$, *then*

$$\|\mathrm{e}^{0.9\lambda U + tA_2U^2}\| \leqslant C \tag{3.91}$$

and

$$\|\mathrm{e}^{0.9\lambda U + A_2U^2 + tA_3U^3}\| \leqslant C. \tag{3.92}$$

Proof. The estimates are provided similarly, so we give the details of (3.92). In view of Lemma 3.1.4 and (3.72), we see that

$$\begin{aligned}
\|\mathrm{e}^{0.9\lambda U + A_2U^2 + A_3tU^3}\| &= \Big\|\mathrm{e}^{0.9\lambda U}\sum_{r=0}^{\infty}\frac{(A_2U^2 + A_3tU^3)^r}{r!}\Big\| \\
&= \Big\|\mathrm{e}^{0.9\lambda U} + \sum_{r=1}^{\infty} e^{0.9\lambda U}\frac{(A_2U^2 + A_3tU^3)^r}{r!}\Big\| \\
&\leqslant 1 + \sum_{r=1}^{\infty}\frac{1}{r!}\Big\|\mathrm{e}^{0.9\lambda U r}(A_2U^2 + A_3tU^3)\Big\|^r \\
&\leqslant 1 + \sum_{r=1}^{\infty}\frac{1}{r!}\Big\|\mathrm{e}^{0.9\lambda U r}U^2\Big\|(|A_2| + 2|A_3|)^r \\
&\leqslant 1 + \sum_{r=1}^{\infty}\frac{1}{r!}(|A_2| + 2|A_3|)^r\Big(\frac{3r}{0.9\mathrm{e}\lambda}\Big)^r \\
&\leqslant 1 + \sum_{r=1}^{\infty}\frac{\mathrm{e}^r}{r^r\sqrt{2\pi r}}\Big(\frac{3(|A_2| + 2|A_3|)}{0.9\lambda}\Big)^r\frac{r^r}{e^r}.
\end{aligned} \tag{3.93}$$

We used the facts $\|\exp 0,9\lambda U\| = 1$ and $\|U\| \leqslant 2$. Applying condition (3.72), we obtain

$$\Big(\frac{3(|A_2| + 2|A_3|)}{0.9\lambda}\Big) < 1. \tag{3.94}$$

The proof is completed. □

3.9 Proof of Theorems 2.4.1–2.4.5

Proof of Theorem 2.4.3. Applying Lemma's 3.8.2, 3.8.5 and 3.1.1 we obtain

$$\begin{aligned}
\|\mathcal{L}(\tilde{S}) - M_3\|_\infty &\leqslant C\int_{-\pi}^{\pi}|\mathrm{E}\mathrm{e}^{\mathrm{i}t\tilde{S}} - \widehat{M}_3(t)|\mathrm{d}t \\
&\leqslant C\int_{-\pi}^{\pi}\exp\{-C\lambda t^2\}|\ln \mathrm{E}\mathrm{e}^{\mathrm{i}t\tilde{S}} - \ln\widehat{M}_3(t)|\mathrm{d}t \\
&\leqslant C\int_0^{\pi}\mathrm{e}^{-C\lambda t^2}[R_4|t|^4 + R_5|t|^5 + R_6|t|^6 + R_7|t|^7]\mathrm{d}t.
\end{aligned}$$

Let $\lambda \geqslant 1$ and $\beta = \max(1, \sqrt{\lambda})$. Applying Lemma's 3.8.2, 3.1.1, 3.8.5 and 3.8.6 we obtain

$$\begin{aligned}\|\mathcal{L}(\tilde{S}) - M_3\|^2 \leqslant C \int_{-\pi}^{\pi} \Big(\beta|\mathrm{E}\mathrm{e}^{\mathrm{i}t\tilde{S}} - \widehat{M}_3(t)|^2 \\ + \quad \frac{1}{\beta}\Big|\Big(\mathrm{e}^{-it\lambda}|\mathrm{E}\mathrm{e}^{\mathrm{i}t\tilde{S}} - \widehat{M}_3(t)|\Big)'\Big|^2 \mathrm{d}t\Big) \\ \leqslant \quad C\int_0^{\pi} \Big(\mathrm{e}^{-C\lambda t^2}\beta\Big[R_4^2|t|^8 + R_5^2|t|^{10} + R_6^2|t|^{12} + R_7^2|t|^{14}\Big] \\ + \quad \mathrm{e}^{-C\lambda t^2}\frac{1}{\beta}\Big[R_4^2|t|^6 + R_5^2|t|^8 + R_6^2|t|^{10} + R_7^2|t|^{12}\Big]\Big)\mathrm{d}t.\end{aligned}$$

Let $\lambda \geqslant 1$ and $\beta = \max(1, \sqrt{\lambda})$. Applying Lemma's 3.8.2, 3.1.1 and Lemma's 3.8.5, 3.8.6 where exiting expressions are divided by x, also using simple 1.1 equality we obtain

$$\begin{aligned}\|\mathcal{L}(\tilde{S}) - M_3\|_{\mathrm{W}}^2 \leqslant C \int_{-\pi}^{\pi} \Big(\beta|\mathrm{E}\mathrm{e}^{\mathrm{i}t\tilde{S}} - \widehat{M}_3(t)|^2 \\ + \quad \frac{1}{\beta}\Big|\Big(\mathrm{e}^{-it\lambda}|\mathrm{E}\mathrm{e}^{\mathrm{i}t\tilde{S}} - \widehat{M}_3(t)|\Big)'\Big|^2 \mathrm{d}t\Big) \\ \leqslant \quad C\int_0^{\pi} \Big(\mathrm{e}^{-C\lambda t^2}\beta\Big[R_4^2|t|^6 + R_5^2|t|^8 + R_6^2|t|^{10} + R_7^2|t|^{12}\Big] \\ + \quad \mathrm{e}^{-C\lambda t^2}\frac{1}{\beta}\Big[R_4^2|t|^4 + R_5^2|t|^6 + R_6^2|t|^8 + R_7^2|t|^{10}\Big]\Big)\mathrm{d}t.\end{aligned}$$

where

$$\begin{aligned} R_4 &= \sum_{k=1}^{n}\big(p_k^4 + |a_{2,k}|(p_k^2 + |a_{2,k}|) + |a_{3,k}|p_k + |a_{4,k}|\big), \\ R_5 &= \sum_{k=1}^{n}\big(p_k a_{2,k}^2 + |a_{2,k}a_{3,k}| + |a_{3,k}|p_k^2 + |a_{4,k}|p_k + |a_{5,k}|\big), \\ R_6 &= \sum_{k=1}^{n}\big(|a_{2,k}a_{3,k}|p_k + a_{3,k}^2 + |a_{2,k}a_{4,k}| + |a_{2,k}|^3 + |a_{5,k}|p_k + |a_{6,k}|\big), \\ R_7 &= \sum_{k=1}^{n}\big(a_{2,k}^2|a_{3,k}| + |a_{3,k}a_{4,k}| + |a_{2,k}a_{5,k}| + |a_{7,k}|\big). \end{aligned}$$

□

Proof of Theorem 2.4.1 and Theorem 2.4.2. The proof of Theorem 2.4.1 and Theorem 2.4.2 are similar to the proof of Theorem 2.4.3, the only difference is that one should use shorter asymptotic expansion. □

Proof of Theorems 2.4.4 and 2.4.5. All estimates are proved similarly, therefore we give the details of the proof of (2.47) only. We have

$$\|\mathcal{L}(\tilde{S}) - M_2(I + M_{21}U^3)\| \quad \leqslant \quad \|\mathcal{L}(\tilde{S}) - M_3\| + \|M_3 - M_2(I + M_{21}U^3)\|.$$

Applying Lemma 3.8.7 and the properties of the total variation norm (see Introduction), we get

$$\begin{aligned}
\|M_3 - M_2(I + M_{21}U^3)\| &= \|\mathrm{e}^{\lambda U + M_{11}U^2}\big(\mathrm{e}^{M_{21}U^3} - I - M_{21}U^3\big)\| \\
&= \|M_{21}^2 U^6 \int_0^1 \mathrm{e}^{\lambda U + M_{11}U^2 + tM_{21}U^3}(1-t)\mathrm{d}t\| \\
&\leqslant \|M_{21}^2 U^6 \mathrm{e}^{0,1\lambda x}\| \int_0^1 \|\mathrm{e}^{\lambda U + M_{11}U^2 + tM_{21}U^3}\|(1-t)\mathrm{d}t \\
&\leqslant C\|M_{21}^2 U^6 \mathrm{e}^{0,1\lambda U}\|.
\end{aligned}$$

The proof is completed by applying Lemma 3.1.4. □

Bibliography

[1] R. Arratia, L. Goldstein, and L. Gordon, Poisson approximation and the Chen-Stein method, *Statist. Sci.,* **5:** 403–434, 1990.

[2] A.D. Barbour, Asymptotic expansions in the Poisson limit theorem, *Ann. Probab.,* **15:** 748–766, 1987.

[3] A.D. Barbour and V. Čekanavičius, Total variation asymptotics for sums of independent integer random variables, *Ann. Probab.,* **30:** 509–545, 2002.

[4] A.D. Barbour and O. Chryssaphino, Compound Poisson approximation: a user's guide. *Ann. Appl. Probab.,* **11:** 964–1002, 2001.

[5] A.D. Barbour and J.L. Jensen, Local and tail approximations near the Poisson limit, *Scand. J. Statist.,* **16:** 75–87, 1989.

[6] A.D. Barbour and A. Xia, Poisson perturbations, *ESAIM: Probab. Statist.,* **3:** 131–150, 1999.

[7] A.D. Barbour and P. Hall, On the rate of Poisson convergence, *Math. Proc. Camb. Phil. Soc.,* **95:** 473–480, 1984.

[8] A.D. Barbour, L. Holst, and S. Janson, *Poisson Approximation,* Oxford Clarendon Press, 1992.

[9] A. Bikelis, Asymptotic expansions for distributions of statistics, in *Proc. 36th Conf. Lith. Math. Soc., Vilnius, June 22–23, 1995*, R. Kudžma, V. Mackevičius (Eds.), Vilnius University Press, (1996), pp. 5–28, (Russian).

[10] A. Bikelis and A. Žemaitis, Asymptotic expansions for probabilities of large deviations, *Application of Probab. Theory and Math. Statist.*, **3:** 9–39, 1980, (Russian).

[11] L. Bortkewitsch, *Das Gesetz der Kleinen Zahlen.* Leipzig, Teubner, 1898.

[12] T.C. Brown and A. Xia, Stein's method and birth-death processes, *Ann. Probab.,* **29:** 1373–1403, 2001.

[13] V. Čekanavičius, Non-uniform theorems for discrete measures, *Lithuanian Math. J.,***33:** 114–126, 1993.

[14] V. Čekanavičius and M. Mikalauskas, Signed Poisson approximations for Markov chains, *Stochastic. Proc. Appl.,* **82:** 205–227, 1999.

[15] V. Čekanavičius and M. Mikalauskas, Local theorems for the Markov binomial distribution, *Lithuanian Math. J.* **41**, 219–231, 2001.

[16] V. Čekanavičius and B. Roos, Two-parametric compound binomial approximations, *Lith. Math. J.*, **44**, 354–373, 2004.

[17] V. Čekanavičius and B. Roos, An Expansion in the Exponent for Compound Binomial Approximations, *Liet. Mat. Rink.*, **46**(1): 67–110, 2006.

[18] L.H.Y. Chen, Poisson approximation for dependent trials, *Ann. Probab.*, **3**(3): 534–545.

[19] P. Deheuvels and D. Pfeifer, On a relationship between Uspensky's theorem and Poisson approximations, *Ann. Inst. Statist. Math.,* **40**, 671–681.

[20] J. Dhaene and N. De Pril, On a class of approximative computation methods in individual risk model, *Insurance Math. Econom.,* **14**: 181–196, 1994.

[21] P. Eichelsbacher and M. Roos, Compound Poisson approximation for dissociated random variables via Stein's method, *Combin. Probab. Comput.,* **8**: 335–346, 1999.

[22] P. Franken, Approximation der Verteilungen von Summen unabhangiger nichtnegativer ganzzahliger Zufallsgrossen durch Poissonsche Verteilungen, *Math. Nachr.,* **6**: 303–340, 1964.

[23] B. Grigelionis, On asymptotic expansion of the reminder term in the case of convergence to a Poisson law, *Litovsk. Matem. Sb.,* **2**: 35–48, 1962, (Russian).

[24] C. Hipp, Improved approximations for the aggregate claims distribution in the individual model, *Astin Bull.,* **16**: 89–100, 1986.

[25] L. Heinrich, A Method for the Derivation of Limit Theorems for Sums of m-dependent Random Variables, *Z. Wahrscheinlichkeitstheorie verw. Gebiete,* **60:** 501–515, 1982.

[26] L. Heinrich, A method for the derivation of limit theorems for sums of weakly dependent random viriables: a survey. *Optimization,***18:** 715-735, 1987.

[27] J. Kruopis, Precision of approximation of the generalized binomial distribution by convolutions of Poisson measures, *Lithuanian Math. J.,* **26**: 37–49, 1986.

[28] J. Kruopis, Approximation of distributions of sums of lattice random variables I, *Lithuanian Math. J.,* **26**: 234–244, 1986.

[29] L. Le Cam, An approximation theorem for Poisson binomial distribution, *Pacific J. Math.,* **10**: 1181–1197, 1960.

[30] A. M. Mood, The distribution theory of runs, *Ann. of Math. Stat.,* **11**: 367–392, 1940.

[31] J. Petrauskienė and V. Čekanavičius, Compound Poisson approximations for sums of 1-dependent random variables I. *Lith. Math. J.,* 2010.

[32] E.L. Presman, Approximation of binomial distributions by infinitely divisible ones, *Teor. Veroyatn. i Primen.,* **28**: 372–382, (1983), (Russian).

[33] E.L. Presman, Approximation in variation of the distribution of a sum of independent Bernoulli variables with a Poisson law, *Teor. Veroyatn. i Primen.,* **30**: 391–396, 1985.

[34] Yu.V. Prokhorov, Asymptotic behavior of the binomial distribution, *Uspekhi Mat. Nauk.,* **8**: 135–142, 1953, (Russian).

[35] S.T. Rachev, *Probability metrics and the stability of stochastic models,* Willey, 494, 1991.

[36] A. Röllin, Approximation of sums of conditionally independent variables by the translated Poisson distribution, *Bernoulli,* **11**: 1115–1128, 2005.

[37] B. Roos, Sharp constants in the Poisson approximation, *Statist. Probab. Lett.,* **52**: 155–168, 2001.

[38] B. Roos, Poisson approximation via the convolution with Kornya-Presman signed measures, *Theory Probab. Appl.,* **48**: 555–560, 2003.

[39] B. Roos, Asymptotics and sharp bounds in the Poisson approximation to the Poisson-binomial distributions, *Bernoulli,* **5**: 1021–1034, 1999.

[40] R.J. Serfling, A general Poisson approximation theorem, *Ann. Probab.,* **3**: 726–731, 1975.

[41] V.V. Shergin, On the convergence rate in the Central Limit Theorem for m-dependent dandom variables, *Teor. Veroyatn. i Primen.,* **24**: 781–794, 1979, Russian.

[42] V. Statulevičius, Large-deviation theorems for sums of dependent random variables I, *Lith. Math. J.,* **19**: 289-296, 1979.

[43] J. Sunklodas, Rate of convergence in the central limit theorem for random variables with strong mixing, *Lith. Math. J.,* **24**: 182–190, 1984.

[44] J. Sunklodas, On a lower bound of the rate of convergence in the central limit theorem for m -dependent random variables, *Lith. Math. J.,* **37**: 291–299, 1997.

[45] J. Šiaulys and V. Čekanavičius, Approximation of distributions of integer additive functions by discrete charges I, *Lith. Math. J.,* **28**: 392–401, 1988.

[46] A.N. Tichomirov, On the convergence rate in the central limit theorem for weakly dependent random variables, *Teor. Veroyatn. i Primen.,* **25**: 800–818, 1980, Russian.

[47] Wang Xiaoxin and A. Xia, On negative binomial approximation to k-runs, *J. Appl. Probab.,* **45**: 456-471, 2008.

[48] A. Xia and Mei Zhang, On approximation of Markov binomial distributions, *Bernoulli*, **15**: 1335-1350, 2009.