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# Joint Discrete Universality in the Selberg-Steuding Class and Non-Trivial Zeroes of the Riemann Zeta-Function

**Selbergo-Štoidingo klasės diskretus jungtinis universalumas ir netrivialūs Rymano dzeta funkcijos nuliai**

Master's Thesis

Author: Benjaminas Togobickij
VU email: benjaminas.togobickij@mif.stud.vu.lt

Supervisor: prof. dr. Roma Kačinskaitė

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# Joint Discrete Universality in the Selberg-Steuding Class and Non-Trivial Zeroes of the Riemann Zeta-Function


### Abstract

In this thesis, the approximation property of a certain class of zeta-functions is studied. It is shown that $L$-functions from the Selberg-Steuding class $\mathscr{S}$ are jointly universal in the Voronin sense concerning discrete shifts involving the non-trivial zeroes of the Riemann zeta-function $\zeta(s)$, or, more precisely, we approximate simultaneously any collection of non-vanishing analytic functions on compact subsets by using shifts $L(s+i\gamma_k h_j)$ with accuracy $\varepsilon > 0$. Here $\gamma_k$ are the imaginary parts of the non-trivial zeroes of the function $\zeta(s)$. Also, a modification of this theorem is obtained, i.e., we extend the result to positive density and show that the limit exists for all but at most countably many $\varepsilon > 0$. These theorems under the weak Montgomery pair correlation conjecture and certain linear independence for the fixed $h_j$'s are shown. The proof involves the application of Mergelyan's approximation theorem and a limit theorem in the space of analytic functions.



**Key words:** approximation, discreteness, joint universality, non-trivial zeroes, Riemann zeta-function, Selberg-Steuding class, weak convergence, universality.


# Selbergo-Štoidingo klasės diskretus jungtinis universalumas ir netrivialūs Rymano dzeta funkcijos nuliai


### Santrauka

Šiame darbe nagrinėjama aproksimavimo tam tikra dzeta funkcijų klase savybė. Įrodoma, kad $L$ funkcijos iš Selbergo-Štoidingo klasės $\mathscr{S}$ yra universalios Voronino prasme, kai postūmių aibė sudaroma panaudojant Rymano dzeta funkcijos $\zeta(s)$ netrivialiuosius nulius. Tiksliau pasakius, darbe yra nagrinėjamas vienalaikis analizinių, nelygių nuliui funkcijų rinkinių aproksimavimas $L(s+i\gamma_k h_j)$ postūmiais $\varepsilon > 0$ tikslumu; čia $\gamma_k$ – Rymano dzeta funkcijos $\zeta(s)$ netrivialių nulių menamosios dalys. Taip pat įrodoma šios teoremos modifikacija išplečiant rezultatą teigiamam tankiui, t. y. parodome, kad egzistuoja riba visiems $\varepsilon > 0$, išskyrus daugiausia skaičią jų aibę. Šios teoremos yra įrodomos pareikalaujant, kad būtų išpildytos dvi sąlygos: teisinga silpnoji Montgomerio porų koreliacijos hipotezė ir tam tikra fiksuotų $h_j$ tiesine nepriklausomybė. Įrodymui naudojama Mergeliano aproksimacijos teorema ir ribinė teorema analizinių funkcijų erdvėje.



**Raktiniai žodžiai:** aproksimavimas, diskretumas, jungtinis universalumas, netrivialūs nuliai, Rymano dzeta funkcija, Selbergo-Štoidingo klasė, silpnasis konvergavimas, universalumas.

# Notation

| | |
|---|---|
| $\mathbb{N}$ | the set of positive integer numbers |
| $\mathbb{N}_0$ | the set of non-negative integer numbers |
| $\mathbb{Z}$ | the set of integer numbers |
| $\mathbb{P}$ | the set of prime numbers |
| $\mathbb{Q}$ | the set of rational numbers |
| $\mathbb{R}$ | the set of real numbers |
| $\mathbb{R}^+$ | the set of positive real numbers |
| $\mathbb{C}$ | the set of complex numbers |
| $n$, $m$ | natural numbers |
| $s = \sigma + it$ | a complex number, $i$ – imaginary unit |
| $\Re(s) = \sigma$ | real part of $s$ |
| $\Im(s) = t$ | imaginary part of $s$ |
| $\gcd(a, b)$ | greatest common divisor of $a$ and $b$ |
| $\liminf\limits_{T\to\infty} x_T$ | lower density of $x_T$ |
| $\limsup\limits_{T\to\infty} x_T$ | upper density of $x_T$ |
| $\mathbf{1}_A(x)$ | indicator function of the set $A$ |
| $\text{meas}\{A\}$ | Lebesgue measure of a set $A \subset \mathbb{R}$ |
| $\#\{A\}$ | cardinality of a set $A$ |
| $A \times B$ | Cartetisian product of sets $A$ and $B$ |
| $f(x) \sim g(x)$, $x \to \infty$ | asymptotic equivalence of functions $f(x)$ and $g(x)$ as $x \to \infty$, i.e., $\lim\limits_{x\to\infty} f(x)/g(x) = 1$. |
| $f(x) \ll_a g(x)$ or $f = O(g(x))$ | $\lvert f(x)\rvert \leq C(a)g(x)$ |
| $\log a$ | natural logarithm of $a$ |
| $D(a, b)$ | the strip $\{s \in \mathbb{C} : a < \sigma < b\}$ for $a < b$, $a, b \in \mathbb{R}$ |
| $\mathcal{K}(a, b)$ | the set of compact subsets of the strip $D(a, b)$ with connected complements |
| $H_{a,b}(K)$ | the set of continuous functions on $K$ and analytic in its interior |
| $H^0_{a,b}(K)$ | the subclass of non-vanishing functions of $H_{a,b}(K)$ |

# Introduction

P.G.L. Dirichlet is considered the pioneer of analytic number theory. In 1837, he proved [24] what is now commonly known as Dirichlet's prime number theorem. It states that arithmetic progressions of the form $an+b$, $a, b \in \mathbb{N}$, $\gcd(a,b)=1$, contain infinitely many prime numbers. To prove the theorem, Dirichlet went "beyond" the bounds of integers and employed mathematical analysis, limits, and continuity. This gave rise to a branch of mathematics called analytic number theory, which investigates problems of integers using real and complex numbers. More precisely, Dirichlet first defined a complex valued function, now known as the Dirichlet $L$-function,

$$L(s,\chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}, \quad s = \sigma + it, \quad \sigma > 1, \tag{1}$$

where $\chi(m) : \mathbb{Z} \to \mathbb{C}$ is an arithmetic function called the Dirichlet character, and proved that, for a non-trivial Dirichlet character, it is nonzero at $s = 1$ (the full proof can be found in [6]).

Since the XIX century, many similar functions to (1), named zeta- or $L$-functions, have been defined [21, Chapter 450]. All of these functions can be expressed by the generalized Dirichlet series

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \tag{2}$$

converging on some half-plane of the complex numbers. Here $\{a_n\}$ is some sequence of complex numbers, and $\{\lambda_n\}$ is a strictly increasing non-negative sequence of real numbers. The simplest zeta-function, originally discovered by L. Euler, who only analysed the function with real argument, is called the Riemann zeta-function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \sigma > 1.$$

It is named after B. Riemann, who proved many properties of the function with complex argument in his paper [8] in an attempt to prove the prime number theorem.

One of the most important areas of study in analytic number theory is the non-trivial zeroes of the Riemann zeta-function $\zeta(s)$. Zeroes of the form $s = -2n$, $n \in \mathbb{N}$, are called the trivial zeroes of function $\zeta(s)$. All other zeroes are called non-trivial, and it is known that they all lie in the so called critical strip $\{s \in \mathbb{C} : 0 < \sigma < 1\}$. One importance of the zeroes comes from the derivation made by Riemann on the explicit expression of the prime counting function $\pi(s)$ over the zeroes of $\zeta(s)$. Also, Riemann conjectured that all of the non-trivial zeroes lie on the line $\sigma = 1/2$, which is called the critical line. Therefore, the difficulty in finding a proof for the Riemann hypothesis and its relation with the prime counting function $\pi(x)$ has attracted much attention to the analysis of the non-trivial zeroes of $\zeta(s)$.

One such analysis of the zeroes is the number of zeroes up to some imaginary part $T$ in the critical strip $\{s \in \mathbb{C} : 0 < \sigma < 1, 0 < t < T\}$ (denote it by $N(T)$). Then the Riemann-von Mangoldt formula is

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T). \tag{3}$$

It was first discovered by Riemann and later proved by H.C.F. von Mangoldt (see [10, Theorem 9.4]). An important consequence follows from this formula, which will be needed later in this work. If we order the non-trivial zeroes $\rho_k = \beta_k + i\gamma_k$ of $\zeta(s)$, such that $\gamma_{k+1} \geq \gamma_k$, then it follows that (see [10, (9.4.4)])

$$\gamma_k \sim \frac{2\pi k}{\log k}, \quad k \to \infty. \tag{4}$$

In 1975, S.M. Voronin first proved [32] that any non-vanishing analytic function can be approximated uniformly by shifts of the Riemann zeta-function $\zeta(s+i\tau)$ in the critical strip, this property is now called the universality property of a zeta- or $L$-functions. The modern version of Voronin's theorem can be formulated in terms of positive lower density. For this purpose, some preliminary notation is needed. Let $D(a,b) = \{s \in \mathbb{C} : a < \sigma < b\}$, and $\text{meas}\{A\}$ denotes the Lebesgue measure of the set $A \subset \mathbb{R}$. Also, let $\mathcal{K}(a,b)$ be the class of compact subsets of the strip $D(a,b)$ with connected complements. Let, for $K \in \mathcal{K}(a,b)$, $H_{a,b}(K)$ be the class of continuous functions on $K$ that are analytic in the interior of $K$, and let $H^0_{a,b}(K)$ be the subclass of non-vanishing functions of $H_{a,b}(K)$.

**Theorem A.** *Let $K \subset D(1/2,1)$ is a compact set with connected complement, and $f(s)$ be a continuous non-vanishing continuous function on $K$ and analytic in the interior of $K$. Then,* *for every $\varepsilon > 0$, it holds an inequality*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{|s|\le r} |\zeta(s+i\tau) - f(s)| < \varepsilon\right\} > 0.$$

The proof of the theorem can be found in [16, Theorem 1.7] or [2, Theorem 6.5.2]. A key role in the proof plays the fact that the function $\zeta(s)$ can be written as the Euler product, i.e.,

$$\zeta(s) = \prod_{p\in\mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \sigma > 1,$$

where $\mathbb{P}$ is the set of prime numbers. Later B. Bagchi improved [9] Voronin's result to arbitrary compact sets in the right-half of the critical strip with connected complement. He also proved the result in terms of probability theory, which relies heavily on the weak convergence in limit theorems.

After Voronin and Bagchi the property of universality has been proven for many other zeta- and $L$-functions. In Voronin's original paper [32] he also proved that the universality property holds for the Dirichlet $L$-function $L(s,\chi)$. Voronin in his doctoral thesis [31] was also the first to prove the so-called joint universality property for $L(s,\chi)$'s. More precisely, that for a collection of non-equivalent characters their corresponding Dirichlet $L$-functions can simultaneously approximate a collection of non-vanishing analytic functions [16]. Again, Bagchi proved this result in a different form and the strongest form of this result is as follow.

**Theorem B.** *Let $\chi_1 \bmod q_1, \ldots, \chi_r \bmod q_r$ be pairwise non-equivalent Dirichlet characters, and, for each $1 \le j \le r$, let $K_j \in \mathcal{K}\left(\frac{1}{2},1\right)$ and $f_j(s) \in H^0_{\frac{1}{2},1}(K_j)$. Then, for all $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \max_{1\le j\le r} \max_{s\in K_j} |L(s+i\tau,\chi_j) - f_j(s)| < \varepsilon\right\} > 0.$$

Also, it turns out, that one can change "lim inf" into "lim" for all but at most countably many $\varepsilon > 0$. This was first shown for the Riemann zeta-function $\zeta(s)$ by J.-L. Mauclaire [19], and independently by A. Laurinčikas and L. Meška [3]. More precisely, they proved the following theorem.

**Theorem C.** *Let $K \in \mathcal{K}\left(\frac{1}{2},1\right)$ and $f(s) \in H^0_{\frac{1}{2},1}(K)$. Then, for all but at most countably many $\varepsilon > 0$, it holds*

$$\lim_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau) - f(s)| < \varepsilon\right\} > 0.$$

The universality described above is of the continuous kind. In 1980, A. Reich proved [5] universality for a type of zeta-function generalising $\zeta(s)$, called the Dedekind zeta-function. Discrete universality deals with the shifts restricted to an arithmetic progression $\{hk\}$ for fixed real $h \in \mathbb{R}^+$. Bagchi in his thesis also proved from the results of Reich an analogous discrete version of the continuous joint universality theorem for the Dirichlet $L$-function.

**Theorem D.** *Let* $\chi_1 \mod q_1, \ldots, \chi_r \mod q_r$ *be pairwise non-equivalent Dirichlet characters, and, for each* $1 \le j \le r$, *let* $K_j \in \mathcal{K}\left(\frac{1}{2}, 1\right)$ *and* $f_j(s) \in H^0_{\frac{1}{2},1}(K_j)$. *Then, for all* $h \in \mathbb{R}^+$ *and all* $\varepsilon > 0$,

$$\liminf_{N\to\infty} \frac{1}{N} \# \left\{ 1 \le k \le N : \max_{1\le j\le r} \max_{s\in K_j} |L(s + ihk, \chi_j) - f_j(s)| < \varepsilon \right\} > 0.$$

Similarly, one can change the arithmetic progressions $\{hk\}$ into more complex sequences, and it turns out that this is not trivial. The first progress in this direction was made in [1] for the Riemann zeta-function by A. Dubickas and A. Laurinčikas. They looked at sequences of the form $\{k^\alpha h\}$ with fixed $0 < \alpha < 1$ and $h > 0$. More precisely, they proved the following statement.

**Theorem E.** *Let* $K \in \mathcal{K}\left(\frac{1}{2}, 1\right)$ *and* $f(s) \in H^0_{\frac{1}{2},1}(K)$, *and suppose that* $0 < \alpha < 1$ *and* $h > 0$. *Then, for all* $\varepsilon > 0$,

$$\liminf_{N\to\infty} \frac{1}{N} \# \left\{ 1 \le k \le N : \sup_{s\in K} |\zeta(s + ik^\alpha h) - f(s)| < \varepsilon \right\} > 0.$$

An important extension of this was later proven by Ł. Pańkowski [20] for the Dirichlet $L$-functions with joint shifts of the form $L(s + i\alpha_j k^{a_j} \log^{b_j} k, \chi_j)$, where, for each $1 \le j \le r$, $\alpha_j \in \mathbb{R}$, $a_j \in \mathbb{R}^+$, $b_j \in \mathbb{R}$ if $a_j \notin \mathbb{Z}$, and $b_j \in \mathbb{R} \setminus (0, 1]$ if $a_j \in \mathbb{N}$, and $a_j \ne a_k$ and $b_j \ne b_k$ if $k \ne j$.

Not too long after Bagchi's doctoral thesis, in 1989, A. Selberg defined [7] a general class of zeta- and $L$-functions satisfying certain conditions to study the general properties that these functions share. One of them is the property of universality, which will be discussed later in the work.

In this thesis, the **aim** is to prove a certain joint discrete universality theorem for functions belonging to a certain extension of the Selberg class, called the Selberg-Steuding class, when the sequence of shifts is taken to be the the imaginary parts of the non-trivial zeroes of the

Riemann zeta-function. Also, it is shown that this result can be modified as in Theorem C. The **tasks** of this thesis are therefore to

- become familiar with properties of the Selberg-Steuding class;
- analyse current literature on the discrete joint universality property for the Selberg-Steuding class;
- become familiar with the method used in proving discrete universality;
- formulate the new joint discrete universality theorem;
- prove the required auxiliary probabilistic results needed to prove the formulated theorems;
- by application of the main limit theorem, Mergelyan's theorem, and properties of weak convergence prove the formulated theorem.

In the first chapter of the thesis, some elements of the theory regarding general classes of ordinary Dirichlet series are presented. In the second chapter, the required auxiliary and relevant results from the literature together with the new formulated theorems are given. In the third chapter, some needed known properties regarding weak convergence of probability measures is presented from the literature. Also, in the same chapter, auxiliary lemmas and the main limit theorem are proved. Finally, in the last chapter, the main universality theorems are proved.

# Chapter 1

# The Selberg-Steuding class

As was stated in the introduction, all zeta- and $L$-functions can be written as a general Dirichlet series, but there is no known exact definition for what is a zeta-function, as Huxley puts it "we know one when we see one" [16]. But one can analyse classes of Dirichlet series satisfying certain conditions. One such general class was defined by Selberg in 1989 [7], which became a great deal of interest. This chapter is primarily written following J. Steudings monograph [16]. To talk about general classes of Dirichlet series we first recall a few results on ordinary Dirichlet series.

If in (2) we take the sequence $\{\lambda_n\}$ to be simply the sequence $\{\log n\}$, we obtain a complex function defined by the Dirichlet series, known as the ordinary Dirichlet series,

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

We will assume that the function $L(s)$ has a representation as an ordinary Dirichlet series on some half-plane of the complex numbers $\mathbb{C}$, i.e.,

$$L(s) = \sum_{m=1}^{\infty} \frac{a(m)}{m^s}.$$

Then we say that $L(s)$ belongs to the Selberg class $\mathcal{S}$ if it satisfies the following hypotheses:

$1^0$ *Ramanujan hypothesis.* Coefficients of $L(s)$ satisfy $a(m) \ll m^{\varepsilon}$ for every $\varepsilon > 0$.

$2^0$ *Analytic continuation.*There exists a non-negative integer $\alpha \in \mathbb{N}_0$ such that $(s-1)^{\alpha} L(s)$ is an entire function of finite order.

$3^0$ *Functional equation.* $L(s)$ satisfies the equation

$$\Lambda_L(s) = w\overline{\Lambda_L(1-\overline{s})},$$

where

$$\Lambda_L(s) := L(s)Q^s \prod_{j=1}^{f} \Gamma(\lambda_j s + \mu_j)$$

with positive real numbers $Q$ and $\lambda_j$, and with complex numbers $\mu_j$ and $w$, such that $\Re\mu_j \geq 0$ and $|w| = 1$.

$4^0$ *Euler product.* $L(s)$ can be written as the product

$$L(s) = \prod_{p\in\mathbb{P}} L_p(s),$$

where

$$\log L_p(s) = \sum_{l=1}^{\infty} \frac{b(p^l)}{p^{ls}}$$

with coefficients $b(p^l)$ such that $b(p^l) \ll p^{\theta l}$ for some $\theta < 1/2$.

Steuding in his monograph extensively analysed the Selberg class and extended it in order to satisfy certain conjectured properties about $L$-functions in $\mathcal{S}$. For example, it is conjectured that all $L$-functions in $\mathcal{S}$ are automorphic. He defined a class, now called the Steuding class $\tilde{\mathcal{S}}$ as the set of functions $L(s)$ satisfying the following conditions.

$\mathrm{i}^0$ *Ramanujan hypothesis.* The same hypothesis as $1^0$.

$\mathrm{ii}^0$ *Analytic continuation.* There exists a real number $\sigma_L < 1$ such that $L(s)$ has an analytic continuation on the half-plane $\sigma > \sigma_L$ except for at most a pole at $s = 1$.

$\mathrm{iii}^0$ *Finite order.* There exists a non-negative constant $\mu_L$, such that, for all fixed $\sigma > \sigma_L$ and positive $\varepsilon$,

$$L(\sigma + it) \ll_\varepsilon |t|^{\mu_L+\varepsilon}$$

as $|t| \to \infty$.

$\mathrm{iv}^0$ *Polynomial Euler product.* There exists $m \in \mathbb{N}_0$ and, for every prime $p$, there exists a sequence of complex numbers $\alpha_j(p)$, $1 \leq j \leq m$, such that

$$L(s) = \prod_{p\in\mathbb{P}} \prod_{j=1}^{m} \left(1 - \frac{\alpha_j(p)}{p^s}\right)^{-1}.$$

$\mathrm{v}^0$ *Prime mean-square.* There exists a positive constant $\kappa$ such that

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\leq x} |a(p)|^2 = \kappa.$$

Steuding then showed that the axioms ii$^0$ and iii$^0$ can be deduced from $\mathcal{S}$ axiom $2^0$. Only the two axioms iv$^0$ and v$^0$ cannot be deduced from the axioms of $\mathcal{S}$. However, as he mentions, they are expected to hold since for all known examples of functions in the Selberg class their Euler product has the form iv$^0$ and the axiom v$^0$ is a sort of prime number theorem for the coefficients of the polynomial Euler product. He then proved universality for a subclass of Selberg class $\mathcal{S} \cap \widetilde{\mathcal{S}}$, i.e., functions satisfying $2^0$, $3^0$, iv$^0$ and v$^0$. Later in [11] a stronger result was obtained removing the condition iv$^0$. So, the Selberg-Steuding class, denoted here as $\mathscr{S}$, is referred to the class of Dirichlet series satisfying conditions $1^0$ to $4^0$ and v$^0$.

An important parameter in the analysis of the structure of $\mathscr{S}$ is the degree of $L(s) \in \mathscr{S}$, which is defined as

$$d_L = 2\sum_{j}^{f} \lambda_j,$$

where $\lambda_j$ and $f$ are as in the $3^0$ condition of $\mathcal{S}$. For example, an equivalent Riemann-von Mangoldt formula as in (3) can be obtained. If $N_L(T)$ is the number of zeroes up to some imaginary part $T$ in the critical strip, one can obtain the following estimate

$$N_L(T) \sim \frac{d_L}{\pi} T \log T.$$

For more details, see [16, Theorem 7.7].

# Chapter 2

# Statement of main results

In this chapter, some results necessary in the proof of the main theorems, together with the related known propositions are presented. Also, the main propositions of the thesis are formulated.

## 2.1 Auxiliary propositions

As in the introduction, let $\{\gamma_k : k \in \mathbb{N}\}$ be the sequence of imaginary parts of the non-trivial zeros of the Riemann zeta-function $\zeta(s)$. Assuming the truth of the Riemann hypothesis, H. Montgomery studied [13] the distribution of consecutive zeroes of the function $\zeta(s)$ and conjectured the asymptotic pair relation

$$\sum_{\substack{0<\gamma_k,\gamma_l<T \\ 2\pi\alpha_1/\log T \le \gamma_k-\gamma_l \le 2\pi\alpha_2/\log T}} 1 \sim \left( \int\limits_{\alpha_1}^{\alpha_2} \left(1 - \frac{\sin \pi u}{\pi u}^2\right) du + \delta(\alpha_1, \alpha_2) \right) \frac{T}{2\pi} \log T$$

as $T \to \infty$, where $\alpha_1 < \alpha_2$ are fixed, and $\delta(\alpha_1, \alpha_2) = 1$ if $0 \in [\alpha_1, \alpha_2]$, and $\delta(\alpha_1, \alpha_2) = 0$ otherwise. In [27], a weaker form of the Montgomery conjecture was defined by R. Garunkštis, A. Laurinčikas and R. Macaitienė, and now it is called the weak Montgomery pair correlation conjecture. More precisely, for some constant $c$, it can be shown that the estimate

$$\sum_{\substack{0<\gamma_k,\gamma_l<T \\ |\gamma_k-\gamma_l|<c/\log T}} 1 \ll T \log T, \quad T \to \infty, \tag{5}$$

follows from Montgomery's pair correlation conjecture.

Now we may state the result obtained in [27].

**Theorem F.** *Assume that* (5) *is true, and let* $K \in \mathcal{K}\left(\frac{1}{2}, 1\right)$ *and* $f \in H^0_{\frac{1}{2},1}(K)$*. Then, for every* $\varepsilon > 0$ *and* $h > 0$,

$$\liminf_{N\to\infty} \frac{1}{N} \# \left\{ 1 \le k \le N : \sup_{s\in K} |\zeta(s + i\gamma_k h) - f(s)| < \varepsilon \right\} > 0.$$

To prove this theorem one important initial step was to show that the sequence $\{a\gamma_k\}$ is distributed modulo 1. The importance of this arise in the proof of the limit theorem, which we will see later. Recall that a sequence $\{x_k : k \in \mathbb{N}\} \subset \mathbb{R}$ is called distributed modulo 1 if, for each interval $[a, b) \subset [0, 1)$, the equality

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbf{1}_{[a,b)}(\{x_k\}) = b - a$$

holds, where $\mathbf{1}_{[a,b)}$ is the indicator function of the interval $[a, b)$ and $\{x_k\}$ denotes the fractional part of $x_k$. Intuitively, it means that the fractional parts of the sequence converge to a uniform distribution on the interval. An important role in the analysis of uniform distribution modulo 1 is the Weyl criterion (see [12]), which allows such questions about the uniform distribution of fractional parts to be reduced to bounds of exponential sums.

**Theorem 1** (Weyl criterion)**.** *A sequence* $\{x_k : k \in \mathbb{N}\} \subset \mathbb{R}$ *is uniformly disitributed modulo 1 if and only if, for all non zero integers* $m$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} e^{2\pi i m x_k} = 0.$$

We state the lemma obtained in [27], where this criterion was used.

**Lemma 1.** *The sequence* $\{a\gamma_k\}$ *with* $a \ne 0$ *is uniformly distributed modulo 1.*

Another important result, which connects discrete and continuous mean squares of certain continuous functions is Gallagher's lemma.

**Lemma 2** (Gallagher)**.** *Let* $T_0$ *and* $T \ge \delta > 0$ *be real numbers and* $\mathcal{T}$ *be finite subset of the interval* $\{T_0 - \delta/2, T_0 + T - \delta/2\}$*. Define the counting function*

$$N_\delta(x) = \sum_{\substack{t\in\mathcal{T} \\ |t-x|<\delta}} 1$$

*and let* $S$ *a complex-valued valued continuous function on* $[T_0, T + T_0]$ *having a continuous derivative on* $(T_0, T + T_0)$*. Then*

$$\sum_{t\in\mathcal{T}} N_\delta^{-1}(t) |S(t)|^2 \le \frac{1}{\delta} \int_{T_0}^{T_0+T} |S(x)|^2 \, dx + \left( \int_{T_0}^{T_0+T} |S(x)|^2 \, dx \int_{T_0}^{T_0+T} |S'(x)|^2 \, dx \right)^{1/2}.$$

The proof of this lemma can be found in [15, Lemma 1.4].

Also, for the proof of universality property a key role is played by Mergelyan's theorem on the approximation of continuous functions by polynomials.

**Theorem 2** (Mergelyan)**.** *Let $K$ be a compact subset of $\mathbb{C}$ with connected complements. Then any continuous function $f(s)$ on $K$, which is analytic in its interior, can be uniformly approximated on $K$ by the polynomials of $s$, i.e., for every $\varepsilon > 0$, there exists a polynomial $p(s)$ such that*

$$\sup_{s\in K} |f(s) - p(s)| < \varepsilon.$$

For the proof of the theorem, see [17].

## 2.2 Main results

Now we state some helpful results regarding the universality in the Selberg-Steuding class $\mathscr{S}$. Denote $\sigma_L = \max(1/2, 1 - 1/d_L)$ and, for brevity, $D_L = D(\sigma_L, 1)$

The first proof of universality for the Selberg-Steuding class $\mathscr{S}$, as is defined in this thesis, was obtained by H. Nagoshi and Steuding in [11]. There it was shown that invoking the condition $\mathrm{v}^0$ to the Selberg class $\mathcal{S}$ was enough to prove universality of continuous kind for the function $L(s) \in \mathcal{S}$.

**Theorem G.** *Let $L(s) \in \mathscr{S}$, $K \in \mathcal{K}(\sigma_L, 1)$ and $f(s) \in H^0_{\sigma_L,1}(K)$. Then, for all $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0, T] : \sup_{s\in K} |L(s + i\tau) - f(s)| < \varepsilon\right\} > 0.$$

A general continuous kind joint universality theorem was shown by R. Kačinskaitė, Laurinčikas and B. Žemaitienė in [26].

**Theorem H.** *Let $L(s) \in \mathscr{S}$, and real algebraic numbers $a_1, \dots, a_r$ are linearly independent over the field of rational numbers. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}(\sigma_L, 1)$ and $f_j(s) \in H^0_{\sigma_L,1}(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0, T] : \sup_{1\le j\le r} \sup_{s\in K_j} |L(s + ia_j\tau) - f_j(s)| < \varepsilon\right\} > 0.$$

In [4], discrete universality for $L(s) \in \mathscr{S}$ with shifts of the form $L(s+ikh)$ and the same conditions as in Theorem G was obtained by Laurinčikas and Macaitienė. An important part in the proof of the discrete case was the multiset

$$L(\mathbb{P}, h, \pi) = \{(\log p : p \in \mathbb{P}), 2\pi/h\}.$$

Two separate cases needed to be analysed: when $L(\mathbb{P}, h, \pi)$ is linearly independent over the rational numbers $\mathbb{Q}$ and when they are linearly dependant over $\mathbb{Q}$. In the case of the discrete joint universality, when many different positive fixed $h_1, \ldots, h_r$ are taken, denote

$$L(\mathbb{P}, \underline{h}, 2\pi) = \{(h_j \log p : p \in \mathbb{P}), j = 1, \ldots, r; 2\pi\},$$

where $\underline{h} = (h_1, \ldots, h_r)$. With this in mind, an analogous joint discrete version of the discrete universality for $L(s) \in \mathscr{S}$ was obtained in [29] by Kačinskaitė, Laurinčikas and Žemaitienė.

The main result for which the generalized version in this thesis is obtained was proved by Kačinskaitė in [25]. There the discrete approximation by the shifts of $L$-functions from the Selberg-Steuding class $\mathscr{S}$ was studied when the shifting parameter involves the set $\{\gamma_k\}$ with using of the weak Montgomery conjecture (5).

**Theorem I.** *Suppose that $L(s) \in \mathscr{S}$ and estimate (5) holds. Let $K \in \mathcal{K}(\sigma_L, 1)$ and $f(s) \in H^0_{\sigma_L,1}(K)$. Then, for fixed $h > 0$ and any $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N} \# \left\{1 \le k \le N : \sup_{s\in K} |L(s + i\gamma_j h) - f(s)| < \varepsilon \right\} > 0.$$

As we have noted, for the majority of these universality theorems, it is also shown that $\liminf$ can be replaced by $\lim$ for all but at most countably many $\varepsilon > 0$.

Now we state two main universality theorems of this thesis.

**Theorem 3.** *Suppose that $L(s) \in \mathscr{S}$, the estimate (5) holds, and the set $L(\mathbb{P}, \underline{h}, 2\pi)$ is linearly independent over the field of rational numbers $\mathbb{Q}$. For $j = 1, \ldots, r$, let $K_j \in \mathcal{K}(\sigma_L, 1)$ and $f_j(s) \in H^0_{\sigma_L,1}(K_j)$. Then, for every $\underline{h} \in (\mathbb{R}^+)^r$ and every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N} \# \left\{1 \le k \le N : \sup_{1\le j\le r} \sup_{s\in K_j} |L(s + i\gamma_k h_j) - f_j(s)| < \varepsilon \right\} > 0.$$

**Theorem 4.** *Under the same conditions as in Theorem 3 the limit*

$$\lim_{N\to\infty} \frac{1}{N} \# \left\{ 1 \le k \le N : \sup_{1\le j\le r} \sup_{s\in K_j} |L(s+i\gamma_k h_j) - f_j(s)| < \varepsilon \right\} > 0$$

*exists for all but at most countably many $\varepsilon > 0$.*

For the proof of these theorems, a joint discrete limit theorem in the space of analytic functions is used and will be discussed in the next chapter of the thesis.

# Chapter 3

# Limit theorem

In this chapter, the main bulk of the work needed to prove the Theorem 3 is presented since the proof relies heavily on a limit theorem of weakly convergent probability measures.

## 3.1 Some elements from the theory of weak convergence of probability measures

This part of the thesis is primarily written following Laurinčikas' monograph [2] since we need some preliminaries from classical probability and measure theories.

Firstly, we say that a sequence of probability measures $\{P_n : n \in \mathbb{N}\}$ defined on $(S, \mathcal{B}(S))$ (ussually abbreviated as probability measures $P_n$ on $(S, \mathcal{B}(S))$, omitting that it is a sequence) converges weakly to a probability measure $P$ on $(S, \mathcal{B}(S))$ as $n \to \infty$ if for all real bounded continuous functions $f$ on $S$,

$$\int_S f dP_n \to \int_S f dP, \quad n \to \infty.$$

Sometimes weak convergence is denoted as $P_n \Longrightarrow P$. A useful result in proving weak convergence of measures is due to Billingsley [22, Theorem 2.1].

**Theorem 5.** *Let $P_n$ and $P$ be probability measures on $(S, \mathcal{B}(S))$. Then the three assertions are equivalent:*

$i^0$ $P_n \Longrightarrow P$;

$ii^0$ $\lim_{n\to\infty} P_n(A) = P(A)$ *for all continuity sets of $P$, i. e., sets $A \in \mathcal{B}(S)$, such that $P(\partial A) = 0$;*

*iii*$^0$ $\liminf_{n\to\infty} P_n(G) \geq P(G)$ *for all open sets* $G$.

Another well known result is Tikhonov's theorem for compact topological spaces (for the proof, see [18, Theorem 5.13]).

**Theorem 6** (Tikhonov)**.** *A cartesian product of an arbitrary family of compact topological spaces is compact with respect to the product topology.*

Here the product topology or sometimes called Tikhonov topology is the weakest topology with respect of which all the projections are continuous, i.e., the intersection of all topologies, such that the projections are continuous, defined on the Cartesian product of compact topological spaces.

A very important type of the measure is the Haar measure, which is an invariant Borel measure on a compact topological group. The question of existence of such a measure is answered in the following theorem (see [30, Theorem 5.14] for the proof).

**Theorem 7.** *On every compact topological group exists a unique probability Haar measure.*

The next theorem is Lévys continuity theorem, allowing the study of limit distributions by looking at their characteristic functions (for the proof see [23, Theorem 26.3]).

**Theorem 8.** *Let* $P_n$ *and* $P$ *be probability measures with characteristic functions respectively* $\varphi_n(t)$ *and* $\varphi(t)$*. Then* $P_n \Longrightarrow P$*, if and only if* $\varphi_n(t) \to \varphi(t)$ *for all* $t$*.*

**Theorem 9** ([22, Theorem 5.1])**.** *Let* $h$ *be a mapping between metric spaces* $S$ *and* $S'$*, and* $P_n$ *and* $P$ *probability measures defined on* $(S, \mathcal{B}(S))$*. If* $P_n \Longrightarrow P$ *and* $P(D_h) = 0$*, where* $D_h$ *is the set of discontinuities of* $h$*, then* $P_n h^{-1} \Longrightarrow Ph^{-1}$*.*

Here $Ph^{-1}(A) = P\left(h^{-1}(A)\right)$.

Now denote $\xrightarrow{\mathcal{D}}$ convergence by distribution. The following result will be used in order to prove the main limit theorem.

**Theorem 10** ([22, Theorem 4.2])**.** *Let, for each* $N \in \mathbb{N}$, $Y_N$, $X_{N,1}$, $X_{N,2}$, $\ldots$ *be a sequence of random elements on* $S$*. Suppose, that, for each* $n$, $X_{N,n} \xrightarrow{\mathcal{D}} X_n$ *as* $N \to \infty$*, and that* $X_n \xrightarrow{\mathcal{D}} X$ *as* $n \to \infty$*. Also, let, for every* $\varepsilon > 0$,

$$\lim_{n\to\infty} \limsup_{N\to\infty} P\left\{\varrho\left(X_{N,n}, Y_n\right) \geq \varepsilon\right\} = 0.$$

*Then* $Y_n \xrightarrow{\mathcal{D}} X$ *as* $N \to \infty$*.*

## 3.2 Auxiliary results on limit theorems

In this section, limit lemmas on a torus are proven. Let

$$\Omega = \prod_{p\in\mathbb{P}} \mathbb{X}_p$$

be the infinite-dimensional torus, where $\mathbb{X}_p = \{s \in \mathbb{C} : |s| = 1\}$ for all primes $p \in \mathbb{P}$.

Since each $\mathbb{X}_p$ is a compact set then,by the Tikhonov theorem, $\Omega$ with the product topology and pointwise multiplication is a compact topological Abelian group. Now we construct the set $\Omega^r = \Omega_1 \times \cdots \times \Omega_r$, where $\Omega_j = \Omega$, $j = 1, \ldots, r$. Then, by the Tikhonov theorem again, we have that $\Omega^r$ is a compact topological group. Denote by $\underline{\omega} = (\omega_1, \ldots, \omega_r)$, $\omega_j \in \Omega_j$, $\omega_j = (\omega_j(p) : p \in \mathbb{P})$, $j = 1, \ldots, r$, where $\omega_j(p)$ is the projection of $\omega_j \in \Omega_j$ to the coordinate space $\mathbb{X}_p$, the elements of $\Omega^r$. So by Theorem 7 there is a unique probability Haar measure $m_{Hj}$ on each $(\Omega_j, \mathcal{B}(\Omega_j))$, $j = 1, \ldots, r$, which are products of Haar measures on the coordinate spaces

$$m_{Hj}\{\omega : \omega \in A\} = \prod_{p\in\mathbb{P}} m_{Hjp}\{\omega : \omega_j(p) \in A_{jp}\},$$

where $A_{jp}$ is the projection of $A \in \mathcal{B}(\Omega_j)$ onto $\mathbb{X}_p$. Therefore, each $\{\omega_j(p) : p \in \mathbb{P}\}$ is a sequence of independent complex-valued random elements on the probability space $(\Omega_j, \mathcal{B}(\Omega_j), m_{Hj})$. Moreover, by Theorem 7 there is a unique probability Haar measure $m_H$ on $(\Omega^r, \mathcal{B}(\Omega^r))$, which is the product of Haar measures $m_{Hj}$:

$$m_H(A) = m_{H1}(A_1) \cdot \cdots \cdot m_{Hr}(A_r), \quad \forall A = A_1 \times \ldots \times A_r \in \mathcal{B}(\Omega^r).$$

Further, let

$$\omega_j(m) = \prod_{p^l \| m} \omega_j^l(p), \quad m \in \mathbb{N},$$

where $p^l \| m$ denotes that $p^l | m$ and $p^{l+1} \nmid m$. This extends the function $\omega_j(p)$ to the set of positive integers.

Recall that the Fourier transform of a certain measure $Q$ on $(\Omega, \mathcal{B}(\Omega))$ is defined by

$$g(\underline{k}) = \int_{\Omega} \prod_{p\in\mathbb{P}}{}^{*} \omega(p)^{k_p} dQ,$$

where the star on the product indicates that only a finite number of $k_p$ are non-zeroes and $\underline{k} = (k_p : p \in \mathbb{P})$, and $\omega(p)$ denotes the projection of $\omega \in \Omega$ onto $\mathbb{X}_p$ as before. We state a special case of the Theorem 8 from [2, Theorem 1.3.21]. The proof can be found in [14, Theorem 1.4.2] as a special case of the continuity theorem for compact Abelian groups.

**Theorem 11.** *Let $\{Q_n\}$ be a sequence of probability measures on $(\Omega, \mathcal{B}(\Omega))$ and let $\{g_n(k)\}$ be the sequence of their corresponding Fourier transforms. Then, if for every $\underline{k}$ vector the limit $\lim_{n\to\infty} g_n(\underline{k}) = g(\underline{k})$ exists then there exists a probability measure $Q$ on $(\Omega, \mathcal{B}(\Omega))$ such that $Q_n \Longrightarrow Q$ and $g(\underline{k})$ is its Fourier transform.*

Moreover, let $H(D_L)$ be the space of analytic functions on $D_L$ endowed with the topology of uniform convergence on compact sets, and denote

$$H^r(D_L) = \prod_{j=1}^{r} H(D_L).$$

Now, for a set $A \in \mathcal{B}(\Omega^r)$, define the probability measure

$$Q_N(A) = \frac{1}{N}\#\left\{1 \le k \le N : \left(\left(p^{-i\gamma_k h_1} : p \in \mathbb{P}\right), \dots, \left(p^{-i\gamma_k h_r} : p \in \mathbb{P}\right)\right) \in A\right\}. \tag{6}$$

We prove a a limit lemma for the measure $Q_N$.

**Lemma 3.** *Suppose that the estimate (5) holds and the set $L(\mathbb{P}, \underline{h}, 2\pi)$ is linearly independent over $\mathbb{Q}$. Then, $Q_N(A)$ converges weakly to the Haar measure $m_H$ as $N \to \infty$.*

*Proof.* Since $\Omega^r$ is an Abelian topological group we can define its characters as continuous homomorphisms $\chi(\underline{\omega}) : \Omega^r \to \mathbb{C}$ as

$$\chi(\underline{\omega}) = \prod_{j=1}^{r} \prod_{p\in\mathbb{P}}{}^{*} \omega_j^{k_{jp}}(p)$$

with integers $k_{jp}$, where the star indicates that only a finite number of $k_{jp}$ are non-zeroes. Therefore, the Fourier transform $g_N(\underline{k}_1, \dots, \underline{k}_r)$, $\underline{k}_j = (k_{jp} : k_{jp} \in \mathbb{Z}, p \in \mathbb{P})$, $j = 1, \dots, r$, of measure $Q_N$ is following

$$\begin{aligned} g_N(\underline{k}_1, \dots, \underline{k}_r) &= \int_{\Omega^r} \prod_{j=1}^{r} \prod_{p\in\mathbb{P}}{}^{*} \omega_j^{k_{jp}}(p) dQ_N(\underline{\omega}) \\ &= \frac{1}{N} \sum_{k=1}^{N} \prod_{j=1}^{r} \prod_{p\in\mathbb{P}}{}^{*} p^{-ih_j k_{jp} \gamma_k} \\ &= \frac{1}{N} \sum_{k=1}^{N} \exp\left\{-i\gamma_k \sum_{j=1}^{r} h_j \sum_{p\in\mathbb{P}}{}^{*} k_{jp} \log p\right\}. \end{aligned} \tag{7}$$

By Theorem 11, it is sufficient to show that the Fourier transform $g_N(\underline{k}_1, \ldots, \underline{k}_r)$ letting $N \to \infty$ gives

$$\lim_{N\to\infty} g_N(\underline{k}_1, \ldots, \underline{k}_r) = \begin{cases} 1, & \text{if } (\underline{k}_1, \ldots, \underline{k}_r) = (\underline{0}, \ldots, \underline{0}), \\ 0, & \text{otherwise}, \end{cases} \tag{8}$$

i.e, that the measure $Q_N$ converges weakly to the Haar measure $m_H$.

From (7) it is obvious that $g_N(\underline{0}, \ldots, \underline{0}) = 1$. Now suppose that $(\underline{k}_1, \ldots, \underline{k}_r) \neq (\underline{0}, \ldots, \underline{0})$. So, there exists $j \in \{1, \ldots, r\}$ such that $\underline{k}_j \neq \underline{0}$. Therefore, there exists a prime number $p$ such that $k_{jp} \neq 0$. Then, since the set $L(\mathbb{P}, \underline{h}, 2\pi)$ is linearly independent over $\mathbb{Q}$, we have that

$$\sum_{j=1}^{r} h_j {\sum_{p\in\mathbb{P}}}^{*} k_{jp} \log p \neq 0.$$

If this is not true, then we would have from (7) that

$$\sum_{j=1}^{r} h_j {\sum_{p\in\mathbb{P}}}^{*} k_{jp} \log p = 2\pi n$$

for some $n \in \mathbb{Z}$, which contradicts the linear independence of $L(\mathbb{P}, \underline{h}, 2\pi)$. Therefore, due to Lemma 1 we have that the sequence

$$\left\{ \frac{1}{2\pi} \gamma_k \sum_{j=1}^{r} h_j {\sum_{p\in\mathbb{P}}}^{*} k_{jp} \log p : k \in \mathbb{N} \right\}$$

is uniformly distributed modulo 1. From this, together with (7) and the Weyl criterion, we have that, for $(\underline{k}_1, \ldots, \underline{k}_r) \neq (\underline{0}, \ldots, \underline{0})$,

$$\lim_{N\to\infty} g_N(\underline{k}_1, \ldots, \underline{k}_r) = 0.$$

The proof follows since the limit measure is uniquely determined by its Fourier transform, therefore the right hand side of (8) must be the Haar measure $m_H$ on $(\Omega^r, \mathcal{B}(\Omega^r))$. ■

Now let $\theta > 1/2$ be a fixed number and put

$$v_n(m;\theta) = \exp\left\{ -\left(\frac{m}{n}\right)^{\theta} \right\}, \quad m, n \in \mathbb{N}.$$

Define the series

$$L_n(s) = \sum_{m=1}^{\infty} \frac{a(m) v_n(m;\theta)}{m^s}$$

and functions

$$L_n(s, \omega_j) = \sum_{m=1}^{\infty} \frac{a(m)\omega_j(m) v_n(m;\theta)}{m^s}, \quad j = 1, \ldots, r.$$

If $L(s) \in \mathscr{S}$, then $a(m) \ll m^\varepsilon$ for all $\varepsilon > 0$. Since $v_n(m;\theta)$ decreases exponentially with respect to $m$, therefore, $L_n(s)$ and $L_n(s,\omega_j)$ are absolutely convergent for $\sigma > \sigma_a$ with some $\sigma_a$ and fixed $n \in \mathbb{N}$. Let

$$\underline{L}_n(s + i\gamma_k \underline{h}) = (L_n(s + i\gamma_k h_1), \ldots, L_n(s + i\gamma_k h_r)),$$

and

$$\underline{L}_n(s, \underline{\omega}) = (L_n(s, \omega_1), \ldots, L_n(s, \omega_r)).$$

Define the probability measure

$$P_{n,N}(A) = \frac{1}{N} \# \{1 \le k \le N : \underline{L}_n(s + i\gamma_k \underline{h}) \in A\}, \quad A \in \mathcal{B}(H^r(D_L)).$$

**Lemma 4.** *There exists a probability measure $P_n$ on $(H^r(D_L), \mathcal{B}(H^r(D_L)))$ such that $P_{n,N}$ converges weakly to $P_n$ as $N \to \infty$.*

*Proof.* Define mappings $u_n(\underline{\omega}) : \Omega^r \to H^r(D_L)$ given by $u_n(\underline{\omega}) = \underline{L}_n(s, \underline{\omega})$. Each series $L_n(s, \omega_j)$, $j = 1, \ldots, r$, converges absolutely. This implies that $u_n$ is a continuous mapping. Therefore $u_n$ is measurable in the space $(H^r(D_L), \mathcal{B}(H^r(D_L)))$, i.e., $u_n(\underline{\omega})$ are $H^r(D_L)$-valued random elements. Therefore, every probability measure $P$ on $(\Omega^r, \mathcal{B}(\Omega^r))$ induces a unique probability measure

$$Pu_n^{-1}(A) = P(u_n^{-1}A), \quad A \in \mathcal{B}(H^r(D_L)),$$

on $(H^r(D_L), \mathcal{B}(H^r(D_L)))$. So, taking the measure (6) we get, that for every $A \in \mathcal{B}(H^r(D_L))$

$$P_{n,N}(A) = \frac{1}{N} \# \left\{1 \le k \le N : \left(\left(p^{-i\gamma_k h_j} : p \in \mathbb{P}\right), j = 1, \ldots, r\right) \in u_n^{-1} A\right\} = Q_N u_n^{-1}(A).$$

Therefore, by Lemma 3, the continuity of $u_n$ and Theorem 9, it follows that $P_{n,N}$ converges weakly to $P_n = m_H u_n^{-1}$. ■

Let

$$L(s, \omega_j) = \sum_{m=1}^{\infty} \frac{a(m)\omega_j(m)}{m^s}, \quad j = 1, \ldots, r.$$

It is known (see Lemma 4.1 in [16]) that, for almost all $\omega_j$, the Dirichlet series $L(s, \omega_j)$ is uniformly convergent on compact subsets of the strip $D_L$. Therefore, $L(s, \omega_j)$, $j = 1, \ldots, r$, are $H(D_L)$-random elements. Also, since the Haar measure $m_H$ on $(\Omega^r, \mathcal{B}(\Omega^r))$ is the product of the Haar measures $m_{Hj}$ on respectively spaces $(\Omega_j, \mathcal{B}(\Omega_j))$, therefore $\underline{L}(s, \underline{\omega}) = (L(s, \omega_1), \ldots, L(s, \omega_r))$

is an $H^r(D_L)$-random element defined on $(\Omega^r, \mathcal{B}(\Omega^r))$. Let $P_{\underline{L}}$ be the disitribution of $\underline{L}(s, \underline{\omega})$, i.e.,

$$P_{\underline{L}}(A) = m_H \left\{ \underline{\omega} \in \Omega^r : \underline{L}(s, \underline{\omega}) \in A \right\}, \quad A \in \mathcal{B}\left(H^r(D_L)\right).$$

**Lemma 5.** *The measure $P_n = m_H u_n^{-1}$ converges weakly to the measure $P_{\underline{L}}$ as $n \to \infty$. Moreover, the support of $P_{\underline{L}}$ is*

$$S_{\underline{L}} := \left( \{ g \in H(D_L) : g(s) \neq 0 \ \text{or}\ g(s) \equiv 0 \} \right)^r. \tag{9}$$

*Proof.* The measure $P_n$ coincides with same as in the continuous case (see [26]). Therefore, the proof is given in Lemma 8 in said paper. While the second part is proven in [26, Lemma 9].

The first proof uses weak convergence in terms of continuity sets by Theorem 5. Firstly, a certain group is proved to be ergodic, i.e., that the $\sigma$-algebra formed by the invariant sets with respect to $m_H$ consists only of sets with measure $m_H$ equal 1 or 0. Then a certain random variable on $(\Omega^r, \mathcal{B}(\Omega^r), m_H)$ is defined for which ergodicity is implied. Then, by application of the Birkhoff–Khintchine ergodic theorem (see [2, Theorem 1.6.6]) and Theorem 5, the proof of first part follows.

The second result we get applying Lemma 5.12 from [16]. ∎

Next we prove an approximation lemma on $H^r(D_L)$. Firstly, let

$$\underline{L}(s + i\gamma_k \underline{h}) = \left( L(s + i\gamma_k h_1), \ldots, L(s + i\gamma_k h_r) \right).$$

The topology of uniform convergence on compact sets of $D_L$ for $H_j(D_L)$ is induced by the metric

$$\varrho(f, g) = \sum_{j=1}^{\infty} 2^{-j} \frac{\sup_{s \in K_j} |f(s) - g(s)|}{1 + \sup_{s \in K_j} |f(s) - g(s)|}, \tag{10}$$

where $f, g \in H(D_L)$ and $\{K_j : j \in \mathbb{N}\}$ is a sequence of compact subsets of $D_L$. Then, for $\underline{g}_l = (g_{l1}, \ldots, g_{lr}) \in H^r(D_L)$, $l = 1, 2$, define the metric $\underline{\varrho}$ on $H^r(D_L)$,

$$\underline{\varrho}(\underline{g}_1, \underline{g}_2) = \max_{1 \le m \le r} \varrho(g_{1m}, g_{2m}).$$

**Lemma 6.** *Suppose that $L(s) \in \mathscr{S}$ and the estimate (5) is true. Then, for arbitrary fixed numbers $h_1, \ldots, h_r$,*

$$\lim_{n \to \infty} \limsup_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \underline{\varrho}(\underline{L}(s + i\gamma_k \underline{h}), \underline{L}_n(s + i\gamma_k \underline{h})) = 0.$$

*Proof.* From the definition of the metric (10) on $H(D_L)$ it suffices to show that, for every compact set $K \subset D_L$,

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \sup_{s\in K} |L(s+i\gamma_k h_j) - L_n(s+i\gamma_k h_j)| = 0, \quad j = 1, \dots, r. \tag{11}$$

We fix a compact set $K \subset D_L$ and positive number $h$. It is known (see [16, Section 4.4]) that $L_n(s)$ has the following integral representation

$$L_n(s) = \frac{1}{2\pi i} \int\limits_{\theta - i\infty}^{\theta + i\infty} L(s+z) l_n(z;\theta) dz, \tag{12}$$

where

$$l_n(s;\theta) = \frac{1}{\theta} \Gamma\left(\frac{s}{\theta}\right) n^s,$$

and $\theta$ is the same as in the definition of $v_n(m;\theta)$. There exists $\delta = \delta(K)$ such that $\sigma_L + 2\delta \leq \sigma \leq 1 - \delta$ for all $\sigma + it \in K$. Thus, let $\theta_1 = \sigma - \sigma_L - \delta > 0$ and $\theta = \sigma_L + \delta > 1/2$. Due to the poles of the gamma function and axiom $2^0$ of $\mathscr{S}$ the integrand in (12) has a simple pole at $z = 0$ and a possible simple pole at $z = 1 - s$. Therefore, by the calculus of residues shifting the integration path to the left we get

$$L_n(s) - L(s) = \frac{1}{2\pi i} \int\limits_{\sigma - \theta_1 - i\infty}^{\sigma - \theta_1 + i\infty} L(s+z) l_n(z;\theta) dz + R(s),$$

where

$$R(s) = \operatorname*{Res}_{z=1-s} L(s+z) l_n(z;\theta) = a l_n(1-s;\theta),$$

and $a = \operatorname*{Res}_{s=1} L(s)$, and $L(s)$ comes from the residue of the integrand at $z = 0$. If in axiom $2^0$ $\alpha = 0$, then $R(s) = 0$. Hence, for all $s = \sigma + it \in K$,

$$\begin{aligned}
&L_n(s+i\gamma_k h) - L(s+i\gamma_k h) \\
&\quad = \frac{1}{2\pi i} \int\limits_{-\infty}^{\infty} L(s+i\gamma_k h + \sigma_L - \sigma + \delta + i\tau) l_n(\sigma_L - \sigma + \delta + i\tau;\theta) d\tau + R(s+i\gamma_k h) \\
&\quad = \frac{1}{2\pi i} \int\limits_{-\infty}^{\infty} L(\sigma_L + \delta + i\gamma_k h + i\tau) l_n(\sigma_L + \delta - s + i\tau;\theta) d\tau + R(s+i\gamma_k h) \\
&\quad \ll \int\limits_{-\infty}^{\infty} |L(\sigma_L + \delta + i\gamma_k h + i\tau)| \sup_{s\in K} |l_n(\sigma_L + \delta - s + i\tau;\theta)| \, d\tau + \sup_{s\in K} |R(s+i\gamma_k h)| \, .
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{N}\sum_{k=1}^{N}\sup_{s\in K}|L(s+i\gamma_k h_j)-L_n(s+i\gamma_k h_j)|\\
&\ll \int_{-\infty}^{\infty}\left(\frac{1}{N}\sum_{k=1}^{N}|L(\sigma_L+\delta+i\gamma_k h+i\tau)|\right)\sup_{s\in K}|l_n(\sigma_L+\delta-s+i\tau;\theta)|\,d\tau\\
&\quad+\frac{1}{N}\sum_{k=1}^{N}\sup_{s\in K}|R(s+i\gamma_k h)| =: I_N+Z_N. \qquad (13)
\end{aligned}$$

It is known (see [16, (2.14)]) that, for a fixed $\sigma\in(\sigma_L,1)$,

$$\int_{-T}^{T}|L(\sigma+it)|^2\,dt \ll_{\sigma,L} T. \qquad (14)$$

From the Cauchy integral formula for a point $s_0$ we have

$$L'(s_0)=\frac{1}{2\pi i}\oint_{\mathscr{C}}\frac{L(z)}{(z-s_0)^2}dz.$$

Taking the contour $\mathscr{C}$ to be a circle centered at $\sigma+it$ with radius $\delta$ we find

$$|L'(\sigma+it)|\le\frac{1}{2\pi\delta}\int_{0}^{2\pi}\left|L(\sigma+it+\delta e^{i\varphi})\right|d\varphi.$$

Taking the maximum of the modulus and by application of the maximum modulus principle we obtain the estimate

$$|L'(\sigma+it)|\le\frac{1}{\delta}|L(\sigma+it)|.$$

From here, integrating over $[-T,T]$ and applying (14), it follows that

$$\int_{-T}^{T}|L'(\sigma+it)|^2\,dt \ll_{\sigma,L} T.$$

So, we obtain that, for all $\tau\in\mathbb{R}$ and $\sigma\in(\sigma_L,1)$,

$$\int_{0}^{T}|L(\sigma+i\tau+it)|^2\,dt \ll_{\sigma,L} T+|\tau| \quad\text{and}\quad \int_{0}^{T}|L'(\sigma+i\tau+it)|^2\,dt \ll_{\sigma,L} T+|\tau|. \qquad (15)$$

Now let $\delta=c/\log\frac{c_1 N}{\log N}$, where $c,c_1>0$, and define

$$N_\delta(\gamma_k)=\sum_{\substack{l=1\\|\gamma_l-\gamma_k|<\delta}}^{N}1.$$

Then, from the estimate (4) for the sequence $\{\gamma_k\}$ and Mongomery's weak conjecture (5), we obtain the estimate

$$\sum_{k=1}^{N} N_\delta(\gamma_k) = \sum_{\substack{k=1 \\ |\gamma_l-\gamma_k|<\delta}}^{N} \sum_{l=1}^{N} 1 \ll \sum_{\substack{0<\gamma_k,\gamma_l\le c_1 N/\log N \\ |\gamma_k-\gamma_l|<c/\log\frac{c_1 N}{\log N}}} 1 \ll N. \tag{16}$$

Next we find

$$\sum_{k=1}^{N} |L(\sigma_L+\delta+i\gamma_k h+i\tau)| = \sum_{k=1}^{N} \sqrt{N_\delta(\gamma_k h) N_\delta^{-1}(\gamma_k h)}\, |L(\sigma_L+\delta+i\gamma_k h+i\tau)|\,.$$

Applying Cauchy-Schwarz inequality we obtain

$$\sum_{k=1}^{N} \sqrt{N_\delta(\gamma_k h) N_\delta^{-1}(\gamma_k h)}\, |L(\sigma_L+\delta+i\gamma_k h+i\tau)|$$
$$\ll \left(\sum_{k=1}^{N} N_\delta(\gamma_k h) \sum_{k=1}^{N} N_\delta^{-1}(\gamma_k h)\, |L(\sigma_L+\delta+i\gamma_k h+i\tau)|^2\right)^{1/2}.$$

Then, from Gallagher's lemma and the estimate (16), it follows

$$\left(\sum_{k=1}^{N} N_\delta(\gamma_k h) \sum_{k=1}^{N} N_\delta^{-1}(\gamma_k h)\, |L(\sigma_L+\delta+i\gamma_k h+i\tau)|^2\right)^{1/2}$$
$$\ll_{h,L} \sqrt{N} \left(\log N \int_{\gamma_1}^{c(h)N/\log N} |L(\sigma_L+\delta+it+i\tau)|^2\, dt\right.$$
$$\left. + \left(\int_{\gamma_1}^{c(h)N/\log N} |L(\sigma_L+\delta+it+i\tau)|^2\, dt \int_{\gamma_1}^{c(h)N/\log N} |L'(\sigma_L+\delta+it+i\tau)|^2\, dt\right)^{1/2}\right)^{1/2}.$$

Finally, from this and the estimates (15), we find

$$\sum_{k=1}^{N} |L(\sigma_L+\delta+i\gamma_k h+i\tau)| \ll_{h,\delta,L} N(1+|\tau|) \tag{17}$$

Furthermore, it is known that, by application of Stirlings formulas (see [28, Theorem 8.18]), for the gamma function in some strip $\sigma_1 \le \sigma \le \sigma_2$ of the complex plane, the following estimate holds

$$|\Gamma(\sigma+it)| \sim \sqrt{\frac{2\pi}{|t|}}\, |t|^\sigma\, e^{-\pi|t|/2} \left(1+O(|t|^{-1})\right).$$

From the last, it is not difficult to obtain the estimate

$$\Gamma(\sigma+it) \ll e^{-c_2|t|}, \quad c_2>0. \tag{18}$$

(18) and the definition of $l_n(s;\theta)$ yield the following estimate for all $s \in K$:

$$l_n(\sigma_L+\delta-s+i\tau;\theta) \ll_\theta n^{\sigma_L+\delta-\sigma}\left|\Gamma\left(\frac{1}{\theta}\left(\sigma_L+\delta-s+i\tau\right)\right)\right|$$
$$\ll_\theta n^{-\delta}e^{-\frac{c_1}{\theta}|\tau-t|} \ll_{\theta,K} n^{-\delta}e^{-c_3|\tau|}, \quad c_3>0.$$

Moreover, the latter estimate and (17) show that

$$I_N \ll_{\delta,L,h,\theta,K} n^{-\delta}\int\limits_{-\infty}^{\infty}\left(1+|\tau|\right)e^{-c_3|\tau|}d\tau \ll_{\delta,L,h,\theta,K} n^{-\delta}. \tag{19}$$

Applying estimate (18) once more for $R(s+i\gamma_k h)$, we get similarly, for all $s\in K$,

$$R(s+i\gamma_k h) \ll_\theta n^{1-\sigma}e^{-c_4|\gamma_k h-t|} \ll_{\theta,K} n^{1-\sigma_L-2\delta}e^{-c_5\gamma_k h}, \quad c_4,c_5>0.$$

Again, due to (4), we obtain that

$$Z_N \ll_{\theta,K,a} n^{1-\sigma_L-2\delta}\frac{1}{N}\sum_{k=1}^{N}e^{-c_5\gamma_k h} \ll_{\theta,K,a,h} n^{1-\sigma_L-2\delta}\left(\frac{\log N}{N}+\frac{1}{N}\sum_{k\ge N}e^{-c_5\gamma_k h}\right)$$
$$\ll_{\theta,K,a,h} n^{1-\sigma_L-2\delta}\frac{\log N}{N}.$$

This, and (19) and (13) lead to the estimate

$$\frac{1}{N}\sum_{k=1}^{N}\sup_{s\in K}|L(s+i\gamma_k h_j)-L_n(s+i\gamma_k h_j)| \ll_{\delta,L,h,\theta,K,a} n^{-\delta}+n^{1-\sigma_L-2\delta}\frac{\log N}{N}.$$

Thus, taking $N\to\infty$ and then $n\to\infty$, we obtain (11), and complete the proof of the lemma.
■

Now we shall prove the main joint limit theorem. Firstly, for $A\in\mathcal{B}(H^r(D_L))$, we set

$$P_N(A)=\frac{1}{N}\#\left\{1\le k\le N : \underline{L}(s+i\gamma_k\underline{h})\in A\right\}.$$

**Theorem 12.** *Suppose that $L(s)\in\mathscr{S}$, the estimate (5) holds and the set $L(\mathbb{P},\underline{h},2\pi)$ is linearly independent over $\mathbb{Q}$. Then $P_N$ converges weakly to $P_{\underline{L}}$ as $N\to\infty$.*

*Proof.* Due to Lemma 5, it is sufficient to show that $P_n$ and $P_N$ converges to same limit measure as $n\to\infty$ and $N\to\infty$, respectively. Denote by $\xrightarrow{\mathcal{D}}$ the convergence in distribution.

On certain probability space $(\Omega,\mathcal{R},\nu)$ with measure $\nu$, define a random variable $\xi_N$ by the formula

$$\nu(\xi_N=h\gamma_k)=\frac{1}{N}, \quad k=1,\dots,N.$$

Let $X_n = X_n(s)$ and $X = X(s)$ be $H^r(D_L)$-valued random elements with distributions $P_n$ and $P_{\underline{L}}$, respectively. Define the two more $H^r(D_L)$-valued random elements

$$X_{n,N} = X_{n,N}(s) = \underline{L}_n(s + i\underline{h}\xi_N) \quad \text{and} \quad Y_N = Y_N(s) = \underline{L}(s + i\underline{h}\xi_N).$$

The defined random elements $X_{n,N}$ and $Y_N$ have distributions $P_{n,N}$ and $P_N$, respectively. Then, by Lemmas 4 and 5, we have

$$X_{n,N} \xrightarrow[N\to\infty]{\mathcal{D}} X_n \quad \text{and} \quad X_n \xrightarrow[n\to\infty]{\mathcal{D}} X, \tag{20}$$

respectively. Moreover, applying Lemma 6, we get that, for every $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} \limsup_{N\to\infty} \nu \left\{ \underline{\varrho}\left(Y_N, X_{N,n}\right) \geq \varepsilon \right\} \\
&\quad = \lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N} \# \left\{ 1 \leq k \leq N : \underline{\varrho}\left(\underline{L}(s + i\gamma_k\underline{h}), \underline{L}_n(s + i\gamma_k\underline{h})\right) \geq \varepsilon \right\} \\
&\quad \leq \lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{\varepsilon N} \sum_{k=1}^{N} \underline{\varrho}\left(\underline{L}(s + i\gamma_k\underline{h}), \underline{L}_n(s + i\gamma_k\underline{h})\right) = 0.
\end{aligned}$$

From this equality and (20), the hypotheses of Theorem 10 are satisfied. Therefore,

$$Y_N \xrightarrow[N\to\infty]{\mathcal{D}} X,$$

which proves the assertion of the theorem. ■

# Chapter 4

# Proof of main theorems

In this chapter, the main theorems, Theorems 3 and 4, of the thesis are proved. The proofs rely on an application of Mergelyan's theorem and the main limit theorem, Theorem 12.

*Proof of Theorem 3.* Because $f_j(s)$ are continuous non-vanishing functions, analytic in the interior of $K_j$, by application of Mergelyan's theorem, there exists non-vanishing in $K_j$ polynomials $p_1(s), \ldots, p_r(s)$ such that

$$\sup_{1\leq j\leq r} \sup_{s\in K_j} |f_j(s) - p_j(s)| < \frac{\varepsilon}{4}. \tag{21}$$

Since the polynomials $p_j(s)$'s have only finitely many zeroes, we may find a compact subset of $D_L$ with connected complement such that $K_j \subset \widehat{K}_j$ and $p_j(s) \neq 0$ on $\widehat{K}_j$. Therefore, $\log p_j(s)$ is continuous in $\widehat{K}_j$ and analytic in its interior.

Applying Mergelyan's theorem again we find polynomials $q_1(s), \ldots, q_r(s)$ such that

$$\sup_{1\leq j\leq r} \sup_{s\in K_j} \left|p_j(s) - e^{q_j(s)}\right| < \frac{\varepsilon}{4}. \tag{22}$$

From (21) and (22), we obtain

$$\sup_{1\leq j\leq r} \sup_{s\in K_j} \left|f_j(s) - e^{q_j(s)}\right| < \frac{\varepsilon}{2}. \tag{23}$$

The tuple $\left(e^{q_1(s)}, \ldots, e^{q_r(s)}\right)$ is an element of the support $S_{\underline{L}}$ of $P_{\underline{L}}$ defined in (9). Then, by Lemma 5, the set

$$G_\varepsilon := \left\{(g_1, \ldots, g_r) \in H^r(D_L) : \sup_{1\leq j\leq r} \sup_{s\in K_j} \left|g_j(s) - e^{q_j(s)}\right| < \frac{\varepsilon}{2}\right\}$$

is an open neighbourhood of the element of the support $S_{\underline{L}}$. This, in view of the definition of a support, means that $P_{\underline{L}}(G_\varepsilon) > 0$.

Now let

$$\hat{G}_\varepsilon := \left\{(g_1, \ldots, g_r) \in H^r(D_L) : \sup_{1\le j\le r} \sup_{s\in K_j} |g_j(s) - f_j(s)| < \varepsilon\right\}. \tag{24}$$

Then from (23) we have that $G_\varepsilon \subset \hat{G}_\varepsilon$. Therefore, from Theorems 12 and 5 and assertion iii$^0$, we have

$$\liminf_{N\to\infty} P_N(\hat{G}_\varepsilon) \ge P_{\underline{L}}(\hat{G}_\varepsilon) > P_{\underline{L}}(G_\varepsilon) > 0.$$

Finally, the rest of the proof of the theorem follows from the definition of the measure $P_N$. More precisely,

$$\liminf_{N\to\infty} \frac{1}{N} \#\left\{1 \le k \le N : \underline{L}(s + i\gamma_k \underline{h}) \in \hat{G}_\varepsilon\right\} > 0$$

or

$$\liminf_{N\to\infty} \frac{1}{N} \#\left\{1 \le k \le N : \sup_{1\le j\le r} \sup_{s\in K_j} |L(s + i\gamma_k h_j) - f_j(s)| < \varepsilon\right\} > 0.$$

■

*Proof of Theorem 4.* The beginning of the proof follows the same steps as in the proof of Theorem 3. Therefore, we preserve the same the notation here also.

The boundary of the set $\hat{G}_\varepsilon$ lies in

$$\left\{(g_1, \ldots, g_r) \in H^r(D_L) : \sup_{1\le j\le r} \sup_{s\in K_j} |g_j(s) - f_j(s)| = \varepsilon\right\}.$$

Observe that the boundaries, for different $\varepsilon_1 \ne \varepsilon_2$, are disjoint, i.e., $\partial\hat{G}_{\varepsilon_1} \cap \partial\hat{G}_{\varepsilon_2} = \emptyset$. Therefore, $P_{\underline{L}}\left(\partial\hat{G}_\varepsilon\right) > 0$ for at most countably many $\varepsilon > 0$, i.e., $\hat{G}_\varepsilon$ is a continuity set of all but at most countably many $\varepsilon > 0$. Again, from Theorems 12 and 5 and assertion ii$^0$, we have

$$\lim_{N\to\infty} P_N(\hat{G}_\varepsilon) = P_{\underline{L}}(\hat{G}_\varepsilon) > P_{\underline{L}}(G_\varepsilon) > 0$$

for all but at most countably many $\varepsilon > 0$. Finally, the proof of the theorem follows from the definition of the measure $P_N$, same as in the previous given proof. ■

# Conclusions

In this thesis a certain joint discrete approximation for the Selberg-Steuding class $\mathscr{S}$ has been analysed. A short introduction regarding the class $\mathscr{S}$ was given, and two universality results are presented. Then the two main joint discrete universality theorems were formulated. Afterwards, relevant auxiliary results on limit theorems were proved. Finally, the two main theorems were proved by application of Mergelyan's approximation theorem and the main limit theorem. Therefore, from the thesis, the following conclusions follow.

1. Assuming the Montgomery pair correlation conjecture for the imaginary parts of the non-trivial zeroes of $\zeta(s)$, $\gamma_k$, collections of discrete shifts $(L(s+i\gamma_k h_1), \ldots, L(s+i\gamma_k h_r))$ of functions from the Selberg-Steuding class $\mathscr{S}$ have the property of joint discrete universality, when the set $L(\mathbb{P}, \underline{h}, \pi)$ is linearly independent over $\mathbb{Q}$.
2. The latter result is also true for all but at most countably $\varepsilon > 0$, when it is modified so that regular density is taken instead of lower density in the universality inequality.

In the future, this result could be generalised to composite universality, i.e., for compositions of operators defined on the space of analytic functions. Further it would be of interest to go to one more general class of zeta-functions, for example, the class of Matsumoto zeta-functions or the Pánkowski class.

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