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EIGENVALUE PROBLEM FOR DIFFERENTIAL EQUATIONS WITH NON-LOCAL AND TRANSMISSION BOUNDARY CONDITIONS

Tikrinių reikšmių uždavinys diferencialinėms lygtims su nelokaliosiomis ir perkelties kraštinėmis sąlygomis

Master's thesis

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Summary

In this work we analyze the eigenvalues and eigenfunctions of a second-order ordinary differential equation with classical, non-local and transmission conditions. We present a solution method for the problem as well as the conditions that need to be satisfied by the eigenvalues. We provide both analytic and geometric methods to analyze the eigenvalues. In the literature review we give a few examples of problems with non-local boundary conditions encountered in the natural sciences and engineering, as well as present classical, non-local, transmission conditions and their problems. The novelty of the work is in the combination of both transmission and non-local boundary conditions in the problem, as well as a geometric method to analyze the eigenvalues.

Keywords: Eigenvalue problem, non-local boundary conditions, transmission conditions

Santrauka

Šiame darbe tiriame antros eilės paprastosios diferencialinės lygties su klasikine, nelokalia ir pernašos kraštinėmis sąlygomis tikrines reikšmes ir tikrines funkcijas. Pristatome sprendimo metodą; taip pat sąlygas, kurias turi tenkinti tikrinės reikšmės. Tikrinėms reikšmėms ištirti naudojame analizinius ir geometrinius metodus. Literatūros apžvalgoje pateikiame kelis uždavinių su nelokaliomis kraštinėmis sąlygomis taikymų gamtos moksluose ir inžinerijoje pavyzdžius, taip pat pristatome klasikines, nelokalias ir pernašos kraštines sąlygas ir jų uždavinius. Darbo naujumas susideda iš to, jog nelokaliosios ir pernašos kraštinės sąlygos yra nagrinėjamos drauge tame pačiame uždavinyje, bei geometrinio metodo tirti tikrinėms reikšmėms.

Raktiniai žodžiai: tikrinių reikšmių uždavinys, nelokaliosios kraštinės sąlygos, perkelties sąlygos

Notation

- $\mathbb{N}$ denotes positive integers $\{1, 2, 3, ...\}$.
- $\mathbb{N}_0$ denotes positive integers with 0: $\{0, 1, 2, 3, ...\}$.
- $\mathring{\mathbb{R}}$ denotes the real numbers without zero: $\mathbb{R} \setminus \{0\}$.
- u', u'' denote the first and second derivative of the function u, respectively.
- $< l, u >$ denotes a linear functional l acting on a function u.

1 Introduction

In this literature review we will first present some examples of applications of models with non-local boundary conditions in the natural sciences. Next, we will remind the reader of the classical boundary value and Cauchy initial-value problems for ordinary differential equations. Then we will consider non-local boundary conditions and give extra attention to the corresponding Sturm-Liouville problem. Lastly, we will introduce transmission conditions.

1.1 Examples of Models with Non-Local Boundary Conditions

Here we will introduce a few examples of problems with non-local boundary conditions in the natural sciences and engineering: an equation for the free surface of a droplet, diffusion on a semi-conductive material, thermoelasticity in mechanics, control of a thermostat and a circulating bioreactor. Most of these examples can be found in M. Sapagovas' textbook [10].

1.1.1 The Equation for the Free Surface of a Droplet

The equation for the free surface height $u(r)$ of a symmetric droplet is found by minimizing the potential energy $E_1 + E_2$, where

$$E_1 = \pi\rho g \int_0^a r u^2 dr,$$
$$E_2 = 2\pi\sigma \int_0^a r\sqrt{1 + (du/dr)^2} dr$$

under a fixed known volume

$$V = 2\pi \int_0^a r u dr. \tag{1.1}$$

Here ρ is the density of the fluid, g is the gravitational constant, σ is the coefficient of surface tension, a is the radius of the circle where the droplet touches the surface, r is the polar coordinate, $u(r)$ is the height of the droplet above the surface.

According to the principle of calculus of variations we can derive the Euler-Lagrange equation for the free surface [1]:

$$\frac{1}{r}\frac{d}{dr}\left(\frac{r}{\sqrt{1 + (du/dr)^2}}\frac{du}{dr}\right) - Ku + \lambda = 0,$$

where $K = \rho g/\sigma$ and λ is the Lagrange multiplier (an unknown constant).

In order to uniquely determine λ and a, as well as the two constants of integration for $u(r)$, we need to impose a total of four boundary conditions. One such condition is (1.1), which is a non-local integral boundary condition that ensures the conservation of the total volume of the droplet. The other three conditions, all of which are classical, are:

$$u'(0) = 0, \tag{1.2}$$

$$u(a) = 0, \tag{1.3}$$

$$u'(a) = \cos\gamma. \tag{1.4}$$

Condition (1.2) ensures that the surface at the very top of the droplet is tangent to the surface below the droplet. Condition (1.3) means that the radius of the droplet is a, at which point the height of the droplet becomes 0. Lastly, the condition (1.4) fixes the angle that the droplet makes with the surface at the droplet's boundary.

1.1.2 Diffusion of a Mixture from a Limited Source

The implantation-diffusion process in a semi-conductive material happens in two repeated steps: implantation and diffusion. The one-dimensional thermodiffusion process is modeled by the following nonlinear diffusion equation [2]:

$$\frac{\partial n}{\partial \eta} = \frac{\partial}{\partial x}\left(D_0 n^\sigma \frac{\partial n}{\partial x}\right),$$

together with the following boundary conditions:

$$n(x, 0) = 0, \tag{1.5}$$

$$n(l, t) = 0, \tag{1.6}$$

$$\int_0^l n(x, t)dx = m. \tag{1.7}$$

Here $n(x, t)$ stands for the concentration of an ion species on the surface of a semi-conductor at the point $0 < x < l$ and time moment $t > 0$. m is the total amount of the diffusing material and $D_0 > 0, \sigma > 0$ are diffusion constants.

The boundary conditions (1.5)–(1.6) are classical and imply that, initially, there are no ions on the surface, and that the ion concentration is zero at the right end-point $x = l$ of the semi-conductor throughout the experiment. The non-local integral boundary condition (1.7) ensures the conservation of mass, that a fixed amount of material diffuses.

1.1.3 Quasistatic Thermoelasticity

W. A. Day [3] considers the problem of quasistatic thermoelasticity. In the article it is proved that the entropy density $u(x, t)$ satisfies the heat equation

$$(1 + \delta^2)\frac{\partial \eta}{\partial t} = \frac{\partial^2 \eta}{\partial x^2} \tag{1.8}$$

with non-local boundary conditions

$$\eta(0,t) = -\delta^2 \int_0^1 \eta(x,t)dx,$$
$$\eta(1,t) = -\delta^2 \int_0^1 \eta(x,t)dx$$

and the initial condition

$$\eta(x,0) = \eta_0(x), \tag{1.9}$$

where

$$\delta = (3\lambda + 2\mu)\alpha\Big(\frac{\Theta_0}{(\lambda+2\mu)c}\Big)^{1/2}$$

λ, μ are the elastic moduli, α is the thermal expansion coefficient, Θ_0 is the initial temperature, c is the specific heat capacity.

W. A. Day also considered another deformation model of a quasistatic thermoelastic material of unit length [4]. The entropy satisfies the same equation (1.8) together with non-local integral boundary conditions:

$$\eta(0,t) = -2\delta^2 \int_0^1 (2-3x)\eta(x,t)dx,$$
$$\eta(1,t) = 2\delta^2 \int_0^1 (1-3x)\eta(x,t)dx,$$

and the initial condition (1.9), where

$$\delta^2 = \frac{\Theta_0 B^2}{cA},$$

A is the bending stiffness of a rod, B measures the interaction between thermal and mechanic effects.

1.1.4 Thermostat Problem

Consider the problem of an air-conditioning system where the thermostat and the sensor are placed at the opposite sides of a room [7]:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \tag{1.10}$$
$$u(x,0) = \phi(x), \tag{1.11}$$
$$\frac{\partial u(0,t)}{\partial x} = \beta(u(0,t)+1), \tag{1.12}$$
$$\frac{\partial u(1,t)}{\partial x} = H(u(0,t)). \tag{1.13}$$

(1.10) is the partial differential equation for the propagation of heat in a room and it requires three conditions. (1.11) represents the initial condition - the initial distribution of heat in the room. (1.12) is a classical boundary condition signifying that the rate of change of heat at the location of a sensor, that is situated at the boundary, is proportional to the heat at the sensor. The last condition, the non-local boundary condition (1.13), represents the action of the thermostat, which is a function H of the heat at the sensor that sits at the opposite side of the room with respect to the thermostat. In other words, the last condition (1.13) connects the values of heat at opposite sides of a room.

1.1.5 Cyclical Bioreactors

The article [11] considers a model of a cultivation process of bacteria or yeast in a cylindrical bioreactor. We consider a system of diffusion-convection-reaction equations for the concentration of mass $X(z,t)$ at height z and time t of a living culture, subject to the concentration of a substrate $S(z,t)$ that the culture consumes:

$$\frac{\partial X}{\partial t} = \frac{1}{B}\frac{\partial^2 X}{\partial z^2} - \frac{\partial X}{\partial z} + D\frac{XS}{K+S}, \tag{1.14}$$

$$\frac{\partial S}{\partial t} = \frac{1}{B}\frac{\partial^2 S}{\partial z^2} - \frac{\partial S}{\partial z} - D\frac{XS}{K+S} \tag{1.15}$$

with boundary conditions:

$$\frac{\partial X(1,t)}{\partial z} = 0, \tag{1.16}$$

$$\frac{\partial S(1,t)}{\partial z} = 0, \tag{1.17}$$

$$X(0,t) - \frac{1}{B}\frac{\partial X(0,t)}{\partial z} = \frac{1}{1+\gamma} + \frac{\gamma}{1+\gamma}X(1,t) \tag{1.18}$$

$$S(0,t) - \frac{1}{B}\frac{\partial S(0,t)}{\partial z} = \frac{c}{1+\gamma} + \frac{\gamma}{1+\gamma}S(1,t) \tag{1.19}$$

and initial conditions:

$$X(z,0) = X_0(z),$$
$$S(z,0) = S_0(z).$$

Here B, D are, respectively, the Bodenstein and Damkohler numbers, γ, K, c are constants.

The terms

$$\frac{\partial X}{\partial t} = \frac{1}{B}\frac{\partial^2 X}{\partial z^2}$$

and

$$\frac{\partial S}{\partial t} = \frac{1}{B}\frac{\partial^2 S}{\partial z^2}$$

in (1.14)–(1.15) represent the diffusion of the culture and substrate in the bioreactor. The terms

$$\frac{\partial X}{\partial t} = -\frac{\partial X}{\partial z}$$

and

$$\frac{\partial S}{\partial t} = -\frac{\partial S}{\partial z}$$

in (1.14)–(1.15) represent convection due to water circulation. Whereas

$$\frac{\partial X}{\partial t} = D\frac{XS}{K+S},$$

$$\frac{\partial S}{\partial t} = -D\frac{XS}{K+S}$$

in (1.14)–(1.15) represent the consumption of the substrate, and hence the growth of the culture, where K controls for the saturation of substrate consumption: the consumption is at half-maximal rate when $S = K$.

The conditions (1.16)–(1.17) are classical Neumann boundary conditions at the top of the reactor for the culture and substrate. The conditions (1.18)–(1.19) are non-local and imply that there is circulation: a fraction $\frac{\gamma}{1+\gamma}$ of the culture and substrate at the top of the bioreactor is reintroduced to the bottom and another fraction is supplied anew. c is the relative strength of new supply of substrate compared to culture.

1.2 Classical Problems in Ordinary Differential Equations

A differential equation is an equation that involves an unknown function along with its derivatives. Consider the linear second-order ordinary differential equation

$$a(t)\frac{d^2u}{dt^2} + b(t)\frac{du}{dt} + c(t)u = f(t), \text{ for } t \in (0,1), \tag{1.20}$$

where $a(t), b(t), c(t), f(t)$ are some real-valued functions and $u(t)$ is a twice-differentiable unknown function that satisfies the equation above.

We know from the general theory of ordinary differential equations (take any introductory textbook on ordinary differential equations, for example [5]) that the solution to (1.20) is a function of one variable

$$u(t) = c_1u_1(t) + c_2u_2(t) + u_0(t),$$

where u_1 ir u_2 are two linearly-independent solutions of the homogeneous equation (equation (1.20) with $f(t) = 0$), whereas u_0 is the particular solution. c_1 and c_2 are some constants to be determined.

If we want to uniquely determine the solution (i.e., solve for the constants), then we need to specify further conditions upon the system. In the classical case such conditions can be:

$$u(0) = \mu_0, \quad \frac{du(0)}{dx} = \mu_1, \tag{1.21}$$

that specify the values of the function $u(t)$ and its derivative $u'(t)$ at the initial value $t = 0$, which together with (1.20) formulates Cauchy's initial-value problem, or

$$u(0) = \mu_0, \quad u(1) = \mu_1, \tag{1.22}$$

called Dirichlet boundary conditions that define the values of the function $u(t)$ at the two end-points of the interval $(0,1)$ under consideration, which together with (1.20) forms a classical boundary-value problem.

Another example of classical boundary conditions are Neumann boundary conditions:

$$u'(0) = \mu_0, \quad u'(1) = \mu_1, \tag{1.23}$$

that enforce the slopes of the curve $u(t)$ at the boundary points.

Note that these conditions are local in the sense that the values of the function $u(t)$ at which the conditions are specified are independent of each other.

1.3 Non-Local Boundary Conditions

In practice sometimes classical boundary conditions such as (1.21)–(1.23) are insufficient; sometimes the value of the function at one point depends on its value at other points. In such a case we will have a so-called non-local boundary-value problem. For example [10],

1. Periodic conditions:

$$u(a) = u(b), \tag{1.24}$$

$$u'(a) = u'(b) \tag{1.25}$$

connects the values of a function and its derivative at the two end-points of an interval $[a, b]$ under consideration.

2. Connected boundary-value conditions:

$$a_0 u(a) + a_1 u'(a) + b_0 u(b) + b_1 u'(b) = \mu_0,$$
$$c_0 u(a) + c_1 u'(a) + d_0 u(b) + d_1 u'(b) = \mu_1,$$

where the periodic conditions (1.24)–(1.25) are a particular case.

3. Non-local integral conditions:

$$u(a) = \int_a^b \alpha(x) u(x) dx + \mu_1,$$
$$u(b) = \int_a^b \beta(x) u(x) dx + \mu_1;$$

or

$$\int_a^b \gamma_1(x) u(x) dx = \mu_1$$
$$\int_a^b \gamma_2(x) u(x) dx = \mu_2.$$

4. Non-local Bitsadze-Samarskii conditions:

$$u(a) = \gamma_0 u(\xi_0) + \mu_0, \quad a < \xi_0 < b,$$
$$u(b) = \gamma_1 u(\xi_1) + \mu_1, \quad a < \xi_1 < b.$$

More generally, we can consider second-order ordinary differential equations together with the condition $< l, u(t) >= 0$, where l is a functional, which connects the values of the function $u(t)$ and/or its derivative in at least two different points of the interval. A good introduction to non-local boundary problems is provided in the textbook [10].

1.4 Sturm–Liouville Problem with Non-Local Boundary Conditions

We can also consider the Sturm–Liouville eigenvalue problem for a second-order differential operator together with non-local boundary conditions. For example, consider the problem:

$$-u'' = \lambda u, \text{ for } t \in (0,1), \tag{1.26}$$

$$u(0) = 0, \tag{1.27}$$

$$u(1) = \gamma u(\xi), \tag{1.28}$$

$\xi \in (0,1)$, eigenvalue $\lambda \in \mathbb{C}$, $\gamma \in \mathbb{R}$ - some constant, (1.28) is a Bisadze–Simarskii type non-local boundary condition. The problem is to find the values λ for which there exists a non-zero solution to (1.26)–(1.28). Such solutions are known as the eigenfunctions.

There is already a substantial amount of results known for this problem in the literature, for example [12, 13, 14]. We will provide a few here.

1.4.1 The cases with classical boundary conditions

If $\gamma = 0$, then we have a classical Sturm–Liouville problem. In this case, all eigenvalues of (1.26)–(1.28) are positive and simple [14]:

$$\lambda = s^2, \ s = \pi n, \ n \in \mathbb{N}$$

and the eigenfunctions are

$$u(t) = \sin(st). \tag{1.29}$$

If $\gamma \neq 0$, but $\xi = 0$, then we have the classical case as above. Likewise, if $\gamma \neq 1$ and $\xi = 1$, then we have the classical case also.

If we formally allow $\gamma = \infty$ by turning the non-local Bisadze–Simarskii condition (1.28) into

$$\lim_{\gamma \to \infty} \frac{1}{\gamma} u(1) = u(\xi),$$

then we get the condition $u(\xi) = 0$. For $\xi > 0$ the result is similar to the classical case:

$$\lambda = s^2, \ s = \frac{\pi}{\xi} n, \ n \in \mathbb{N}$$

and the eigenfunctions have the same form as in (1.29).

1.4.2 The case with one classical boundary condition

If $\gamma = \infty$ and $\xi = 0$, then the second boundary condition (1.28) becomes the same as the first boundary condition (1.27). Whereas if $\gamma = 1$ and $\xi = 1$, then (1.28) becomes an identity $0 = 0$. In both cases we have the problem with a single classical boundary condition [14]:

$$-u'' = \lambda u, \tag{1.30}$$

$$u(0) = 0, \tag{1.31}$$

for $t \in (0,1)$.

If $\lambda = 0$, then the solution $u(t) = Ct$ satisfies the problem (1.30)–(1.31) for an arbitrary constant $C \in \mathbb{R}$.

Let us define a bijection $\lambda = s^2$ such that $s \in \mathbb{C}_s = \{z \in \mathbb{C} : -\pi/2 < arg(z) \leq \pi/2 \text{ or } z = 0\}$ and $\lambda \in \mathbb{C}$. If we have an $s \in \mathbb{C}_s$, then the corresponding eigenvalue of (1.30)–(1.31) is λ. If $s \neq 0$, then the eigenfunction for (1.30)–(1.31) is

$$u(t) = C\sin(st). \tag{1.32}$$

We can combine the eigenfunctions for the cases $s = 0$ and $s \neq 0$ into a single eigenfunction:

$$u(t) = C\frac{\sin(st)}{s}, \ \ s \in \mathbb{C}_s. \tag{1.33}$$

This is because the function (1.33) is an analytic function with a removable singularity at the point $s = 0$:

$$\lim_{s\to 0} C\frac{\sin(st)}{s} = Ct. \tag{1.34}$$

This is clear if we write down the Maclaurin series:

$$\frac{\sin(st)}{s} = \sum_{n=0}^{\infty}(-1)^n\frac{s^{2n}}{(2n+1)!}t^{2n+1}. \tag{1.35}$$

If $s \neq 0$, then the s in the denominator of (1.33) can be absorbed by the arbitrary constant C and we retrieve (1.32). Therefore, without loss of generality, we can work with the function (1.33). Since eigenfunctions are defined up to an arbitrary multiplicative constant C, we can work with $C = 1$.

1.4.3 The case when $\lambda = 0$

Let us return to the problem (1.26)–(1.28) and consider the case when $\gamma \neq 0$ and $0 < \xi < 1$. If we substitute the function (1.33) into the non-local boundary condition (1.28), we get

$$C\Big(\frac{\sin s}{s} - \gamma\frac{\sin(s\xi)}{s}\Big) = 0.$$

There exists a non-zero solution (an eigenfunction) if s satisfies

$$\frac{\sin s}{s} - \gamma\frac{\sin(s\xi)}{s} = 0. \tag{1.36}$$

When $\lambda = s = 0$, we have the equality $\gamma\xi = 1$ [14]. Therefore, the eigenvalue $\lambda = 0$ exists if and only if $\gamma = \frac{1}{\xi}$. The corresponding eigenfunction is $u(t) = t$.

1.4.4 The case when $\lambda > 0$

For convenience, we will define

$$s = \sigma + i\omega \tag{1.37}$$

that satisfies

$$s^2 = \lambda, \tag{1.38}$$

such that $s \in \mathbb{C}_s$. Then we will have a bijection between $\mathbb{C}_s$ and $\mathbb{C}_\lambda = \mathbb{C}$. From the (1.37)–(1.38) above we get

$$\lambda = \sigma^2 - \omega^2 + 2\sigma\omega i. \tag{1.39}$$

It is clear that $\lambda > 0$ when $\sigma > 0$ and $\omega = 0$. There exists an eigenfunction

$$u(t) = \sin(\sigma t)$$

whenever $\sigma > 0$ satisfies

$$\sin\sigma - \gamma\sin(\sigma\xi) = 0. \tag{1.40}$$

If $\sin s = 0$ and $\sin(s\xi) = 0$, then (1.36) is valid for all $\gamma \in \mathbb{R}$. In this case we have eigenvalues $\lambda = s^2$ that do not depend on the parameter γ. We will say that λ is a constant eigenvalue if, for a given fixed ξ, it does not depend on the value of γ. Suppose that m and n are positive co-prime integers. A countable infinity of positive constant eigenvalues exist only for rational $\xi = \frac{m}{n} \in (0,1)$, and those eigenvalues are $\lambda = s^2,\ s = \pi k,\ k \in n\mathbb{N} = \{n, 2n, 3n, ...\}$. The corresponding eigenfunctions are (1.32). All other eigenvalues are non-constant [14].

Furthermore, if $|\gamma| \notin [1, \frac{1}{\xi}]$, then the real roots of (1.36) are simple. In particular, the positive real roots σ are simple. Whereas if $|\gamma| = \frac{1}{\xi}$, then for non-simple roots σ of (1.40) both sides are equal to zero and $\sigma > 0$ [6].

In general, the analysis of (1.40) is very difficult. We will provide an example when $\xi = \frac{1}{2}$. In this case we have

$$\sin\sigma = \gamma\sin\left(\frac{\sigma}{2}\right).$$

By the double-angle formula,

$$2\sin\left(\frac{\sigma}{2}\right)\cos\left(\frac{\sigma}{2}\right) = \gamma\sin\left(\frac{\sigma}{2}\right).$$

When σ is a root of $\sin\left(\frac{\sigma}{2}\right) = 0$, then $\lambda = \sigma^2$ is a constant eigenvalue. The roots corresponding to constant eigenvalues are

$$\sigma = 2\pi n,\ n \in \mathbb{N}.$$

When the roots σ satisfy

$$2\cos\left(\frac{\sigma}{2}\right) = \gamma,$$

that is, when

$$\sigma = 2\arccos\left(\frac{\gamma}{2}\right) + 2\pi n,\ n \in \mathbb{N},$$

provided that $|\gamma| \le 2$, then we have non-constant eigenvalues.

1.4.5 The case when $\lambda < 0$

We will have that $\lambda < 0$ if and only if $\sigma = 0$ and $\omega > 0$. Then the eigenfunction is

$$u(t) = \sinh(\omega t)$$

and the condition (1.36), for the eigenfunction to exist becomes

$$\sinh(\omega) - \gamma \sinh(\omega\xi) = 0. \tag{1.41}$$

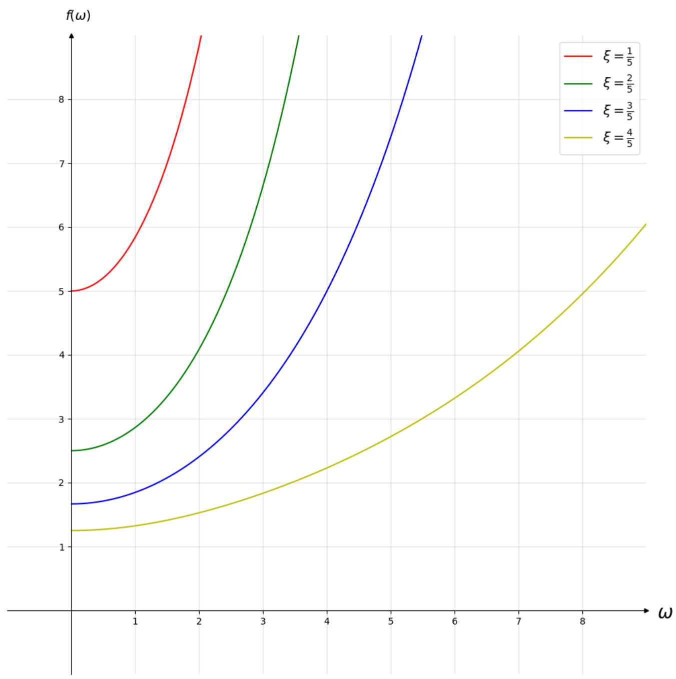

Figure 1: Plots of the function $f_\xi(\omega)$ for several values of ξ: $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$. The plot shows that $f_\xi(\omega)$ is a positive, monotonically increasing function, where smaller values of ξ correspond to larger values of $\lim_{\omega\to 0} f_\xi(\omega) = \frac{1}{\xi}$ and larger $f'_\xi(\omega)$ for all $\omega > 0$.

Let us consider the function $f_\xi(\omega) = \frac{\sinh(\omega)}{\sinh(\xi\omega)}$ for $\omega > 0$ and fixed $0 < \xi < 1$ [10]. We have that

$$\lim_{\omega\to 0} f_\xi(\omega) = \frac{1}{\xi}, \tag{1.42}$$

$$\lim_{\omega\to \infty} f_\xi(\omega) = \infty. \tag{1.43}$$

From the Figure 1 we can see that $f_\xi(\omega)$ is a monotonically increasing function and it is steeper for smaller values of ξ. From the monotonicity of the function and (1.41)–(1.43) we can see that the equation (1.41) has a unique root $\omega > 0$ if and only if $\gamma > \frac{1}{\xi}$.

1.4.6 Complex eigenvalues

If $\lambda = s^2,\ s = \sigma + i\omega$ is a complex eigenvalue, then the corresponding eigenfunction is

$$u(t) = \sin\sigma\cosh\omega + i\cos\sigma\sinh\omega,$$

for $\sigma > 0,\ \omega > 0$.

We will provide a negative result: if $|\gamma| \le 1$, then the problem (1.26)–(1.28) does not have complex eigenvalues [10].

Let us separate the equation

$$\sin s - \gamma\sin(s\xi) = 0 \tag{1.44}$$

into real and imaginary parts by substituting $s = \sigma + i\omega$. Then we get

$$\sin\sigma\cosh\omega - \gamma\sin(\sigma\xi)\cosh(\omega\xi) = 0,$$
$$\sinh\omega\cos\sigma - \gamma\sinh(\omega\xi)\cos(\sigma\xi) = 0.$$

By expressing $\sin\sigma$ from the first equation and $\cos\sigma$ from the second, we can rewrite the equation $\sin^2\sigma + \cos^2\sigma = 1$ as:

$$\Big(\gamma\frac{\cosh(\omega\xi)}{\cosh\omega}\sin(\sigma\xi)\Big)^2 + \Big(\gamma\frac{\sinh(\omega\xi)}{\sinh\omega}\cos(\sigma\xi)\Big)^2 = 1. \tag{1.45}$$

But since $0 < \xi < 1$, then for all $\omega > 0$ we have that

$$0 < \frac{\cosh(\omega\xi)}{\cosh\omega} < 1,\ 0 < \frac{\sinh(\omega\xi)}{\sinh\omega} < 1.$$

If we assume that $|\gamma| \le 1$, then

$$\Big(\gamma\frac{\cosh(\omega\xi)}{\cosh\omega}\Big)^2 < 1,\ \Big(\gamma\frac{\sinh(\omega\xi)}{\sinh\omega}\Big)^2 < 1.$$

Therefore, the equation (1.45) cannot be satisfied. Thus, if $|\gamma| \le 1$ and $0 < \xi < 1$, then the problem (1.26)–(1.28) does not have complex eigenvalues.

A further result is that all complex roots of (1.44) are simple [6].

Considering the results from the previous subsections 1.4.3–1.4.5, we can conclude that if $\gamma \in (\infty, \frac{1}{\xi})$ and $0 < \xi < 1$, then the problem (1.26)–(1.28) can only have real positive eigenvalues [10].

1.5 Differential Equations with Transmission Conditions

A further complication of ordinary differential equation problems is when we impose a discontinuity in the solution of the differential equation (1.20). This can be achieved by specifying special conditions, called transmission [9], or sometimes also interface [8], conditions. An example of such transmission conditions are

$$u(t_d + 0) = du(t_d - 0), \tag{1.46}$$
$$u'(t_d + 0) = du'(t_d - 0), \tag{1.47}$$

where $d \in \mathring{\mathbb{R}}$ and $0 < t_d < 1$. In practice, these conditions enforce a discontinuity at the point $t = t_d$. The size of the discontinuity is proportional to d.

Another, more general, example of transmission conditions is

$$u(t_d + 0) = a_{11}u(t_d - 0) + a_{12}u'(t_d - 0),$$
$$u'(t_d + 0) = a_{21}u(t_d - 0) + a_{22}u'(t_d - 0),$$

where $a_{11},\ a_{12},\ a_{21},\ a_{22} \in \mathbb{R}$, not all zero.

We can also have multiple points of discontinuity at $t = t_{d_1}, t_{d_2}, ..., t_{d_n}$. Then the transmission conditions may take the form of

$$\begin{aligned}
u(t_{d_1} + 0) &= d_1 u(t_{d_1} - 0),\\
u'(t_{d_1} + 0) &= d_1 u'(t_{d_1} - 0),\\
u(t_{d_2} + 0) &= d_2 u(t_{d_2} - 0),\\
u'(t_{d_2} + 0) &= d_2 u'(t_{d_2} - 0),\\
&\ldots\\
u(t_{d_n} + 0) &= d_n u(t_{d_n} - 0),\\
u'(t_{d_n} + 0) &= d_n u'(t_{d_n} - 0)
\end{aligned}$$

for $d_i \in \mathring{\mathbb{R}}$.

The solution to

$$a(t)\frac{d^2u}{dt^2} + b(t)\frac{du}{dt} + c(t)u = 0, \quad t \in (0,1), \tag{1.48}$$

along with (1.46)–(1.47), will now be composed of solutions at two intervals, separated by the discontinuity:

$$u(t) = \begin{cases} u_1(t) = C_1\psi_1(t) + C_2\psi_2(t), & \text{for } 0 \le t \le t_d - 0,\\ u_2(t) = C_3\psi_1(t) + C_4\psi_2(t), & \text{for } t_d + 0 \le t \le 1, \end{cases} \tag{1.49}$$

where $\psi_1(t)$ and $\psi_2(t)$ are two linearly-independent solutions of (1.48).

Notice that the solution (1.49) involves four unknown constants, therefore the two transmission conditions (1.46)–(1.47) are insufficient in order to determine the solution uniquely. Hence, we need to enforce further constraints. These can be classical or they can also be non-local.

2 The Sturm–Liouville Problem with Transmission and Non-Local Boundary Conditions

In this section we will consider the eigenvalue problem of a linear second-order differential operator together with non-local boundary and transmission conditions:

$$-u'' = \lambda u, \quad t \in (0, t_d) \cup (t_d, 1), \tag{2.1}$$

$$u(0) = 0, \tag{2.2}$$

$$u(t_d + 0) = du(t_d - 0), \tag{2.3}$$

$$u'(t_d + 0) = du'(t_d - 0), \tag{2.4}$$

$$u(1) = \gamma_1 u(\xi_1) + \gamma_2 u(\xi_2) \tag{2.5}$$

where $0 < \xi_1 \leq t_d \leq \xi_2 < 1,\ d \in \mathring{\mathbb{R}}$ and $\gamma_1, \gamma_2 \in \mathbb{R}$, $\lambda \in \mathbb{C}$ is the eigenvalue of the problem.

We can interpret the two transmission conditions (2.3)–(2.4) as imposing a discontinuity of the first kind upon the solution at the point $t = t_d$. However, here we will take the interpretation that we have two domains $I_1 = (0, t_d) \cup \{t_d - 0\}$ and $I_2 = \{t_d + 0\} \cup (t_d, 1)$, where the the solution set $u(t)$ will be composed of two real-valued functions for the two different intervals:

$$u(t) = \begin{cases} u_1(t), \text{ for } t \in I_1, \\ u_2(t), \text{ for } t \in I_2. \end{cases} \tag{2.6}$$

The two solutions are not independent, but are connected through the boundary conditions (2.3)–(2.5).

The condition (2.5) is a non-local Bitsadze-Samarskii type boundary condition, where $\gamma_1, \gamma_2 \in \mathbb{R}$ are some constants. ξ_1, ξ_2 are defined such that $0 < \xi_1 \leq t_d \leq \xi_2 < 1$. In other words, $\xi_1 \in I_1$ and $\xi_2 \in I_2$. If $\gamma_1 \neq 0$, then the two solutions $u_1(t), u_2(t)$ are connected not just through the point $t = t_d$ via the transmission conditions (2.3)–(2.4), but also through the condition (2.5). Condition (2.1) is classical.

Obviously, $u(t) = 0$ is a solution to (2.1)–(2.5) for any $\lambda \in \mathbb{C}$. Therefore, the goal will be to look for non-zero solutions (i.e., the eigenfunctions) of the problem and retrieve the conditions that σ, ω in (1.37)–(1.39) must satisfy in order for λ to be an eigenvalue.

Remark 1. Before going further, we will consider the case when $d = 0$. We will conclude that for the eigenvalue problem to be well-posed, we must assume that $d \neq 0$.

Consider the problem (2.1)–(2.5) when $d = 0$. Then (2.1)–(2.5) becomes

$$-u'' = \lambda u, \qquad \text{for } t \in I_1 \cup I_2, \tag{2.7}$$
$$u(0) = 0, \tag{2.8}$$
$$u(t_d + 0) = 0, \tag{2.9}$$
$$u'(t_d + 0) = 0, \tag{2.10}$$
$$u(1) = \gamma_1 u(\xi_1) + \gamma_2 u(\xi_2). \tag{2.11}$$

The conditions (2.9)–(2.10) are simply Cauchy's initial value conditions for the solution in the interval I_2. They imply that that $u_2(t) = 0$, for $t \in I_2$. Thus $u(1) = u(\xi_2) = 0$, which turns (2.11) into

$$\gamma_1 u(\xi_1) = 0, \tag{2.12}$$

that is a local condition. The solution to (2.7)–(2.8) is

$$u_1(t) = C\frac{\sin(st)}{s}, \text{ for } t \in I_1, \tag{2.13}$$

where the remarks about (1.32)–(1.35) apply. If $\gamma_1 = 0$, then from (2.12) we cannot determine the constant C or the value of s in (2.13), which makes the problem ill-posed due to non-uniqueness. Then (2.13) is an eigenfunction for any $\lambda \in \mathbb{C}$.

If $\gamma_1 \neq 0$, then (2.12) implies that $u(\xi_1) = 0$, or, equivalently, that

$$C\frac{\sin(s\xi_1)}{s} = 0. \tag{2.14}$$

We are looking for non-zero solutions, therefore we assume that $C \neq 0$. If $s = 0$, then by (2.14) and (1.34) we have $\xi_1 = 0$. But we have assumed that $0 < \xi_1 \leq t_d$. Thus, $s \neq 0$.

Therefore, by assuming that $C \neq 0$ and $s \neq 0$, s in the denominator of (2.14) can be absorbed by the constant C, and we have the condition that

$$\sin(s\xi_1) = 0.$$

This has solutions

$$s = \frac{\pi n}{\xi_1}, \text{ for } n \in \mathbb{Z}^+.$$

Thus, if $d = 0$ and $\gamma_1 \neq 0$, we have a family of solutions

$$u_1(t) = \sin\left(\frac{\pi n}{\xi_1}t\right), \text{ for } t \in I_1, n \in \mathbb{Z}^+,$$
$$u_2(t) = 0, \qquad \text{for } t \in I_2.$$

In what follows we will assume that $d \neq 0$.

Remark 2. Another special case is when $d = 1$. Since from $-u'' = \lambda u$ it follows that u'', u' are continuous whenever u is continuous, the case $d = 1$ corresponds to when we have a removable discontinuity at the point $t = t_d$. We can eliminate this discontinuity by defining $u(t_d) = u(t_d - 0) =$

$u(t_d + 0)$. Then the transmission conditions (2.3)–(2.4) can essentially be ignored and we are left with a Sturm–Liouville problem with a non-local three-point Bisadze–Samarskii boundary condition (2.1)–(2.2), (2.5).

Remark 3. If $\xi_1 = t_d$ or $\xi_2 = t_d$, then we have the limiting cases, where $u(\xi_1) = u(t_d - 0)$ or $u(\xi_2) = u(t_d + 0)$.

Remark 4. If $\gamma_1 = \gamma_2 = 0$, then the boundary condition (2.5) becomes local and we have a case that was already presented in 1.4.2.

Remark 5. The reason we are interested in the equation (2.1) is because it represents the zeroth order term in asymptotic expansion or perturbation series of more general second-order differential operators.

2.1 Solution Development

In this section we will provide a solution method for the problem (2.1)–(2.5).

The solutions in both intervals will be linear combinations of two linearly independent solutions to equation (2.1), whenever λ is an eigenvalue. Luckily, we can easily find such solutions: $\psi_1(t) = \cos(st)$ and $\psi_2(t) = \frac{\sin(st)}{s}$, where $s \in \mathbb{C}_s$ and $s^2 = \lambda$. Hence, we will search for the solution $u(t)$ in the form

$$u(t) = \begin{cases} u_1(t) = C_1 \cos(st) + C_2 \frac{\sin(st)}{s}, \text{ for } t \in I_1, \\ u_2(t) = C_3 \cos(st) + C_4 \frac{\sin(st)}{s}, \text{ for } t \in I_2. \end{cases} \tag{2.15}$$

Note that we have four unknown constants. Correspondingly, we have four conditions (2.2)–(2.5). We can retrieve the constants by substituting the solution (2.15) into the conditions (2.1)–(2.5). Then we get a system of equations:

$$\begin{aligned} C_1 &= 0, \\ C_3 \cos(st_d) + C_4 \frac{\sin(st_d)}{s} &= d\Big(C_1 \cos(st_d) + C_2 \frac{\sin(st_d)}{s}\Big), \\ -sC_3 \sin(st_d) + C_4 \cos(st_d) &= d\Big(-sC_1 \sin(st_d) + C_2 \cos(st_d)\Big), \\ C_3 \cos s + C_4 \frac{\sin s}{s} &= \gamma_1 \Big(C_1 \cos(s\xi_1) + C_2 \frac{\sin(s\xi_1)}{s}\Big) + \gamma_2 \Big(C_3 \cos(s\xi_2) + C_4 \frac{\sin(s\xi_2)}{s}\Big). \end{aligned}$$

In matrix-vector form this becomes

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -d\cos(st_d) & -d\frac{\sin(st_d)}{s} & \cos(st_d) & \frac{\sin(st_d)}{s} \\ -sd\sin(st_d) & -d\cos(st_d) & -s\sin(st_d) & \cos(st_d) \\ -\gamma_1 \cos(s\xi_1) & -\gamma_1 \frac{\sin(s\xi_1)}{x} & \cos s - \gamma_2 \cos(s\xi_2) & \frac{\sin s}{s} - \gamma_2 \frac{\sin(s\xi_2)}{s} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

We get non-zero solutions if and only if the determinant of the matrix is equal to zero:

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ -d\cos(st_d) & -d\frac{\sin(st_d)}{s} & \cos(st_d) & \frac{\sin(st_d)}{s} \\ -sd\sin(st_d) & -d\cos(st_d) & -s\sin(st_d) & \cos(st_d) \\ -\gamma_1\cos(s\xi_1) & -\gamma_1\frac{\sin(s\xi_1)}{x} & \cos s - \gamma_2\cos(s\xi_2) & \frac{\sin s}{s} - \gamma_2\frac{\sin(s\xi_2)}{s} \end{vmatrix} = 0.$$

Expanding the determinant by the first row we get

$$\begin{vmatrix} -d\frac{\sin(st_d)}{s} & \cos(st_d) & \frac{\sin(st_d)}{s} \\ -d\cos(st_d) & -s\sin(st_d) & \cos(st_d) \\ -\gamma_1\frac{\sin(s\xi_1)}{x} & \cos s - \gamma_2\cos(s\xi_2) & \frac{\sin s}{s} - \gamma_2\frac{\sin(s\xi_2)}{s} \end{vmatrix} = 0.$$

By multiplying through the diagonal entries and summing them up we get

$$\begin{aligned} & d\sin^2(st_d)\Big(\frac{\sin s}{s} - \gamma_2\frac{\sin(s\xi_2)}{s}\Big) \\ & -\gamma_1\cos^2(st_d)\frac{\sin(s\xi_1)}{s} \\ & -d\frac{\sin(st_d)\cos(st_d)}{s}\Big(\cos s - \gamma_2\cos(s\xi_2)\Big) \\ & -\gamma_1\sin^2(st_d)\frac{\sin(s\xi_1)}{s} \\ & +d\frac{\sin(st_d)\cos(st_d)}{s}\Big(\cos s - \gamma_2\cos(s\xi_2)\Big) \\ & +d\cos^2(st_d)\Big(\frac{\sin s}{s} - \gamma_2\frac{\sin(s\xi_2)}{s}\Big) = 0, \end{aligned}$$

which, upon simplification, yields

$$d\frac{\sin s}{s} - \gamma_1\frac{\sin(s\xi_1)}{s} - \gamma_2 d\frac{\sin(s\xi_2)}{s} = 0. \tag{2.16}$$

This is the condition that $s \in \mathbb{C}_s$ must satisfy in order for λ to be an eigenvalue, and thus for us to have non-zero solutions to (2.1)–(2.5).

However, we still need to find the eigenfunctions. For this, we will offer an alternative solution method to the problem (2.1)–(2.5).

Substituting the solution (2.15) into the classical condition (2.2) we find that

$$0 = u(0) = C_1\cos(0) + C_2\sin(0) = C_1,$$

thus

$$u_1(t) = C_2\frac{\sin(st)}{s}.$$

Because eigenfunctions are non-zero and only differ by a constant multiple, then without loss of generality we will pick $C_2 = 1$. Therefore,

$$u_1(t) = \frac{\sin(st)}{s}, \text{ for } t \in I_1$$

is the eigenfunction for the first interval.

From the transmission conditions (2.3)–(2.4) we get the relations

$$C_3 \cos(st_d) + C_4 \frac{\sin(st_d)}{s} = d\frac{\sin(st_d)}{s},$$
$$-C_3 s \sin(st_d) + C_4 \cos(st_d) = d\cos(st_d).$$

These allow us to easily conclude that $C_3 = 0$ and $C_4 = d$. Therefore, our eigenfunctions are:

$$u_1(t) = \frac{\sin(st)}{s}, \quad \text{for } t \in I_1, \tag{2.17}$$
$$u_2(t) = d\frac{\sin(st)}{s}, \ \text{for } t \in I_2. \tag{2.18}$$

Inserting these into the non-local boundary condition (2.5) we get

$$\gamma_1 \frac{\sin(s\xi_1)}{s} + \gamma_2 d \frac{\sin(s\xi_2)}{s} = d\frac{\sin s}{s}, \tag{2.19}$$

which agrees with (2.16).

If $s \in \mathbb{C}_s$ satisfies the equation above, then $\lambda = s^2$ is an eigenvalue of the problem. In conclusion, the problem (2.1)–(2.5) has a non-zero solution (2.6), (2.17)–(2.18) whenever s satisfies (2.19).

2.2 Analysis of Eigenvalues

In this subsection we will analyze the roots of the equation (2.19). These roots will correspond to the eigenvalues of the problem (2.1)–(2.5).

2.2.1 The case when $\lambda = 0$

The case when $\lambda = 0$ is very important theoretically, because it determines when the operator $\mathcal{L} = -\frac{d^2}{dt^2}$ in (2.1) together with the boundary conditions (2.2)–(2.5) is singular. This, in turn, determines when we have none or infinitely many solutions.

$\lambda = 0$ corresponds to the case when $s = 0$. Then the condition (2.19) becomes

$$\gamma_1 \xi_1 + d\gamma_2 \xi_2 = d \tag{2.20}$$

and the corresponding eigenfunctions are

$$u_1(t) = t, \quad \text{for } t \in I_1,$$
$$u_2(t) = dt, \ \text{for } t \in I_2,$$

Note that the condition (2.20) does not involve t_d directly, but only indirectly through the relation $0 < \xi_1 \leq t_d \leq \xi_2 < 1$.

We can rewrite (2.20) as

$$\frac{\gamma_1}{\frac{d}{\xi_1}} + \frac{\gamma_2}{\frac{1}{\xi_2}} = 1, \tag{2.21}$$

which is an equation of a line for the variables γ_1, γ_2, where we assume that $d, \ \xi_1, \ \xi_2$ are given. Hence,

$\lambda = 0$ will be an eigenvalue if and only if γ_1 and γ_2 belong to the line (2.21). The line (2.21) has a γ_1-intercept at the point $\gamma_1 = \frac{d}{\xi_1}$ and a γ_2-intercept at the point $\gamma_2 = \frac{1}{\xi_2}$.

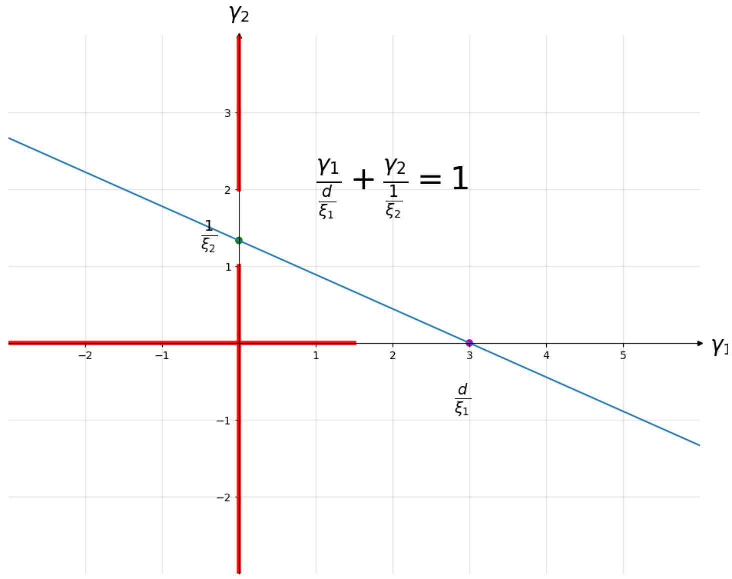

Figure 2: The line (2.21) in blue, where $\xi_1 = \frac{1}{4}, \xi_2 = \frac{3}{4}, d = \frac{3}{4}$ and $t_d = \frac{1}{2}$. The red lines represent the values that the γ_1-intercept and γ_2-intercept cannot occupy when we fix $d = \frac{3}{4},\ t_d = \frac{1}{2}$ and vary ξ_1, ξ_2.

We shall investigate the allowed regions for the line (2.21) when we change ξ_1, ξ_2 while keeping $d \in \mathring{\mathbb{R}}$ and $0 < t_d < 1$ fixed.

From the assumption that $0 < \xi_1 \leq t_d \leq \xi_2 < 1$, we find that

$$-\infty < -\frac{|d|}{\xi_1} \leq -\frac{|d|}{t_d}, \text{ for } d < 0, \tag{2.22}$$

$$\frac{|d|}{t_d} \leq \frac{|d|}{\xi_1} < \infty, \quad \text{for } d > 0 \tag{2.23}$$

$$1 < \frac{1}{\xi_2} \leq \frac{1}{t_d} < \infty, \tag{2.24}$$

which implies that the γ_1-intercept belongs to $\big(-\infty, -\frac{|d|}{t_d}\big]$ if $d < 0$ and γ_1-intercept belongs to $\big[\frac{|d|}{t_d}, \infty\big)$ if $d > 0$, and the γ_2-intercept belongs to $(1, \frac{1}{t_d}]$. This allows us to determine some theoretical conditions upon the line (2.21) if we know the values of d and t_d, as can be seen from Figure 2.

If we fix ξ_1, ξ_2, t_d and vary $d \in \mathring{\mathbb{R}}$, then we can see that the γ_1-intercept $\frac{d}{\xi_1}$ can achieve arbitrary non-zero values, as can be seen from the Figure 3. Thus, for arbitrary non-zero values of d and fixed ξ_1, ξ_2, t_d, the absolute restriction is that γ_2-intercept $\notin (-\infty, 1] \cup [\frac{1}{t_d}, \infty)$.

If we now consider the case when we fix ξ_1, ξ_2, d, but vary $0 < t_d < 1$, then we see that

$$1 < \frac{1}{t_d} < \infty,$$

$$|d| < \frac{|d|}{t_d} < \infty, \quad \text{for } d > 0,$$

$$-\infty < -\frac{|d|}{t_d} < -|d|, \text{ for } d < 0$$

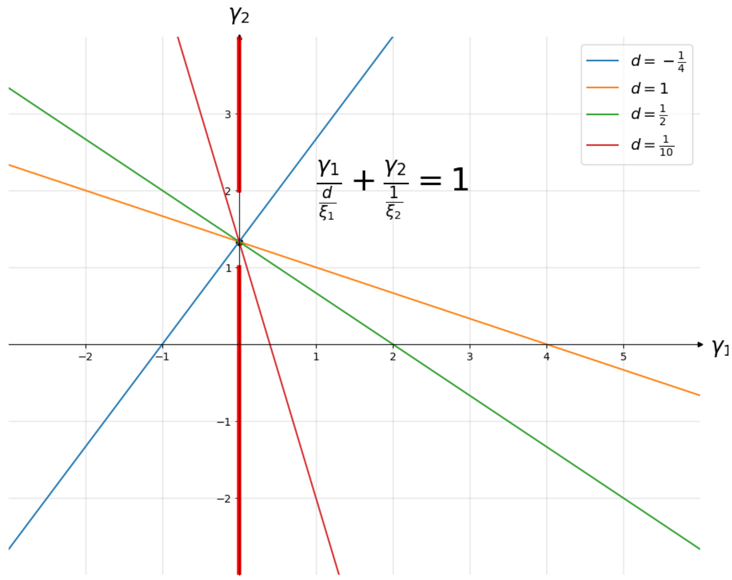

Figure 3: The lines (2.21) for varying values of $d : -\frac{1}{4}, 1, \frac{1}{2}, \frac{1}{10}$, where $\xi_1 = \frac{1}{4}, \xi_2 = \frac{3}{4}$ and $t_d = \frac{1}{2}$ are fixed. The thick red lines on the γ_2-axis represent the restrictions for γ_1-intercept and γ_2-intercept as we vary d, but keep ξ_1, ξ_2, t_d fixed.

from which we conclude that for fixed ξ_1, ξ_2, d and varying t_d, the γ_1-intercept $\in (-\infty, -|d|)$ for $d < 0$ and (d, ∞) for $d > 0$, while γ_2-intercept > 1.

When $d = 1$, we have that (2.20) becomes

$$\gamma_1\xi_1 + \gamma_2\xi_2 = 1$$

and the solution is

$$u(t) = t, \text{ for } t \in I_1 \cup I_2.$$

We can define $u(t_d) = t_d$ and obtain a solution for the entire interval $t \in (0, 1)$.

2.2.2 The case when $\lambda > 0$

$\lambda > 0$ whenever $\sigma > 0$ and $\omega = 0$. Then the condition (2.19) becomes

$$\gamma_1 \sin(\sigma\xi_1) + \gamma_2 d \sin(\sigma\xi_2) = d \sin\sigma. \tag{2.25}$$

This equation is notoriously difficult to study and therefore it is left for future studies. Here we will consider only a few special cases.

If $\gamma_1 = 0$, then (2.25) becomes

$$\gamma_2 \sin(\sigma\xi_2) = \sin\sigma, \tag{2.26}$$

which is (1.36) with $\gamma_2 := \gamma$ and $\xi_2 := \xi$. The results for this situation have already been investigated and presented in 1.4.4.

If $\gamma_2 = 0$, then (2.25) becomes

$$\frac{\gamma_1}{d} \sin(\sigma\xi_1) = \sin\sigma, \tag{2.27}$$

which is (1.36) with $\frac{\gamma_1}{d} := \gamma$ and $\xi_1 := \xi$.

If $\xi_1 = \xi_2 := \xi$, then (1.36) becomes

$$\big(\frac{\gamma_1}{d} + \gamma_2\big)\sin(\sigma\xi) = \sin\sigma, \tag{2.28}$$

which is, again, (1.36) with $\frac{\gamma_1}{d} + \gamma_2 := \gamma$. An example when $\xi_1 = \xi_2 = \frac{1}{2}$ was shown in 1.4.4.

A more interesting case is when $\frac{\gamma_1}{d} = \gamma_2 := \gamma$. Then we have

$$\gamma\big(\sin(\sigma\xi_1) + \sin(\sigma\xi_2)\big) = \sin\sigma.$$

We can rewrite this as

$$2\gamma\sin\Big(\frac{(\xi_1+\xi_2)}{2}\sigma\Big)\cos\Big(\frac{(\xi_1-\xi_2)}{2}\sigma\Big) = 2\sin\big(\frac{\sigma}{2}\big)\cos\big(\frac{\sigma}{2}\big).$$

If we further assume that $\xi_1 + \xi_2 = 1$, then we will have constant eigenvalues $\sigma = 2\pi n$ for $n \in \mathbb{N}$. The non-constant eigenvalues will then satisfy

$$\gamma\cos\Big(\frac{(2\xi_1 - 1)}{2}\sigma\Big) = \cos\big(\frac{\sigma}{2}\big).$$

2.2.3 The case when $\lambda < 0$

The case of negative eigenvalues has been investigated thoroughly and it is one of the main focus of this work. From (1.39) $\lambda < 0$ corresponds to the case when $\sigma = 0$ and $\omega > 0$. The eigenfunctions will be in the form of

$$u_1(t) = \sinh(\omega), \quad \text{for } t \in I_1,$$
$$u_2(t) = d\sinh(\omega), \ \text{for } t \in I_2$$

and the condition (2.19) will be

$$\gamma_1\sinh(\omega\xi_1) + \gamma_2 d\sinh(\omega\xi_2) = d\sinh(\omega). \tag{2.29}$$

Since $\sinh(\beta\omega)$ is a monotonically increasing function that intersects 0 at $\omega = 0$, when $\beta > 0$, the function $\sinh(\beta\omega)$ will be positive for $\omega > 0$. Therefore, after rearranging, we can turn the condition (2.29) into

$$\frac{\gamma_1}{d\frac{\sinh(\omega)}{\sinh(\xi_1\omega)}} + \frac{\gamma_2}{\frac{\sinh(\omega)}{\sinh(\xi_2\omega)}} = 1, \tag{2.30}$$

which is an equation of a line in γ_1, γ_2 for fixed $d,\ \xi_1,\ \xi_2,\ \omega$. Hence, $\lambda < 0$ will be an eigenvalue if and only if γ_1 and γ_2 belong to the line (2.30). The γ_1-intercept is at the point $\gamma_1 = d\frac{\sinh(\omega)}{\sinh(\xi_1\omega)}$ and the γ_2-intercept is at the point $\gamma_2 = \frac{\sinh(\omega)}{\sinh(\xi_2\omega)}$.

From Figure 1 we can see that $\frac{\sinh\omega}{\sinh(\omega\beta)}$ is an increasing function for $0 < \beta < 1$. Therefore, in analogy with (2.22)–(2.24), we can conclude that

$$\begin{aligned} \frac{|d|}{\xi_1} < \ |d|\frac{\sinh(\omega)}{\sinh(\xi_1\omega)} &< \infty, \\ -\infty < -|d|\frac{\sinh(\omega)}{\sinh(\xi_1\omega)} &< -\frac{|d|}{\xi_1}, \\ 1 < \frac{1}{\xi_2} < \qquad \frac{\sinh(\omega)}{\sinh(\xi_2\omega)} &< \infty. \end{aligned}$$

This means that the γ_1-intercept $> \frac{d}{\xi_1}$ for $d > 0$ and $< -\frac{|d|}{\xi_1}$ for $d < 0$, while the γ_2-intercept $> \frac{1}{\xi_2}$.

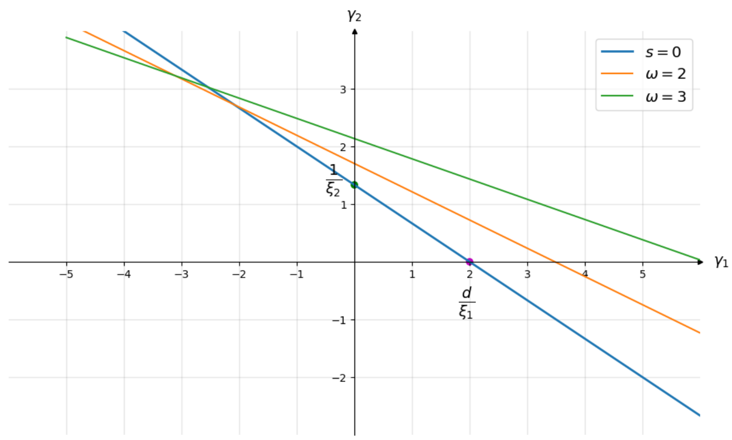

Figure 4: The plot of the lines (2.30) for $\omega = 2, 3$ in yellow and green. The blue line corresponds to (2.20). We can see that for some values of $\gamma_1,\ \gamma_2$, the green and yellow lines cross. This means that for those values of $\gamma_1,\ \gamma_2$, we have that $\omega = 2$ and $\omega = 3$ are the two solutions to (2.30). In this figure $\xi_1 = \frac{1}{4},\ \xi_2 = \frac{3}{4},\ d = \frac{1}{2},\ t_d = \frac{2}{5}$.

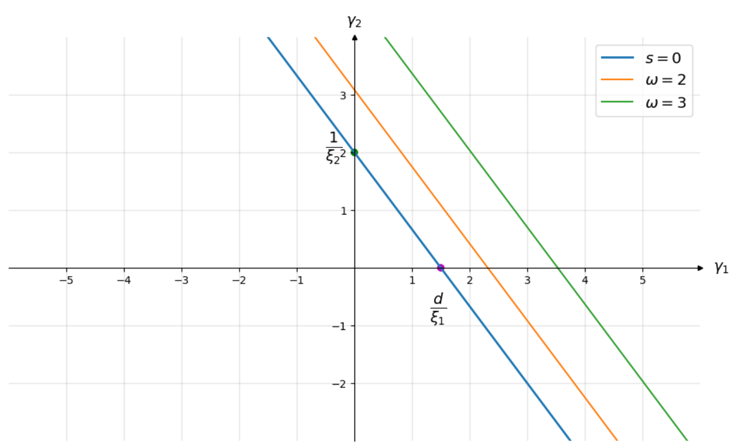

Figure 5: The plot of the lines (2.30) for $\omega = 2, 3$ in yellow and green when $\xi_1 = \xi_2$. The blue line corresponds to (2.20). We see that in this case two lines do not cross. In this figure $\xi_1 = t_d = \xi_2 = \frac{1}{2},\ d = \frac{3}{4}$.

Figures 4 and 5 show a few lines (2.30) for several values of $\omega > 0$. Looking at the lines in Figure 4 we can notice an interesting observation: for some fixed values of $d,\ \xi_1,\ \xi_2,\ \gamma_1,\ \gamma_2$ there will be two lines that intersect at the same point. In other words, under certain conditions, there will be two values of ω that satisfy (2.30) for fixed $d,\ \xi_1,\ \xi_2,\ \gamma_1,\ \gamma_2$, and therefore, we will have two negative eigenvalues. On the other hand, in the limiting case when $\xi_1 = \xi_2$ that is depicted in Figure 5, we can see that the lines do not cross.

We shall investigate this more closely. Let us define a parametric curve in the $(x, y) \in \mathbb{R}^2$ plane

$$g(\omega; \xi_1, \xi_2) = \Big(\frac{\sinh(\xi_1\omega)}{\sinh(\omega)}, \frac{\sinh(\xi_2\omega)}{\sinh(\omega)}\Big), \text{ for } \omega > 0. \tag{2.31}$$

We will denote the two coordinate functions as

$$x = g_1(\omega) = \frac{\sinh(\xi_1\omega)}{\sinh(\omega)},$$
$$y = g_2(\omega) = \frac{\sinh(\xi_2\omega)}{\sinh(\omega)}.$$

We can rewrite (2.30) as a line in the (x, y) plane:

$$l : \frac{\gamma_1}{d}x + \gamma_2 y = 1. \tag{2.32}$$

These definitions allow us to incur a separation of variables: the curve g depends only on $\xi_1,\ \xi_2$, whereas the line l depends on $\gamma_1,\ \gamma_2,\ d$. The equation (2.30) will be satisfied when the curve g crosses the line l. By studying g and l we will be able to understand the conditions that must be satisfied for there to be two, one or no negative eigenvalues.

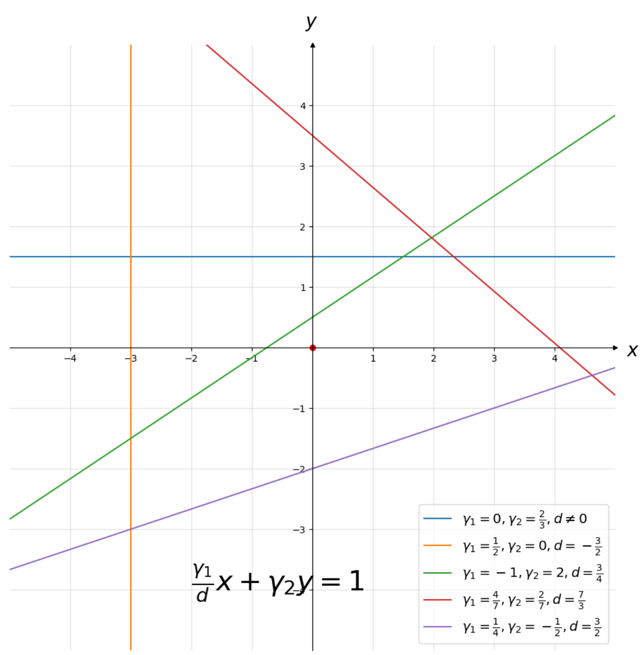

Figure 6: The line l (2.32) for various values of γ_1, γ_2, d. The red line has a negative slope, the green and purple lines have a positive slope. When $\gamma_1 = 0$, the line is horizontal and $y = \frac{1}{\gamma_2}$, which is demonstrated by the blue line. When $\gamma_2 = 0$, the line is vertical and $x = \frac{d}{\gamma_1}$, which is demonstrated by the yellow line. The x- and y-intercepts can take both positive and negative values. The red point at the origin corresponds to the only restriction upon the line l: it does not cross the point $(x, y) = (0, 0)$.

First, we will investigate the line l. The x-intercept $= \frac{d}{\gamma_1}$ when $\gamma_1 \neq 0$, whereas the y-intercept $= \frac{1}{\gamma_2}$ when $\gamma_2 \neq 0$. It is clear that the point $(x, y) = (0, 0)$ can never satisfy the line (2.32) because we would get $0 = 1$. Since we assume that $\gamma_1, \gamma_2 \neq \infty$, this can never happen. We have also assumed that $\gamma_1,\ \gamma_2$ can never be both zero and that $d \neq 0$. If $\gamma_1 = 0$, but $\gamma_2 \neq 0$, then we have a horizontal line $y = \frac{1}{\gamma_2}$. If $\gamma_2 = 0$, but $\gamma_1 \neq 0$, then we have a vertical line $x = \frac{d}{\gamma_1}$. If $\gamma_2 \neq 0$, then we can rewrite the equation (2.32) as

$$y = mx + b, \tag{2.33}$$

where we have defined

$$m = -\frac{\gamma_1}{d\gamma_2}, \tag{2.34}$$

$$b = \frac{1}{\gamma_2}. \tag{2.35}$$

As $|\gamma_2| \to \infty$, the y-intercept $b \to 0$. The slope m depends on the values of γ_1, γ_2, d and when they are all positive, the slope is negative. If one variable or all three are negative, then the slope is positive. In general, the only restriction placed upon the line l is that it cannot cross the point $(x, y) = (0, 0)$, otherwise it is free. Some examples of lines (2.32) are shown in Figure 6.

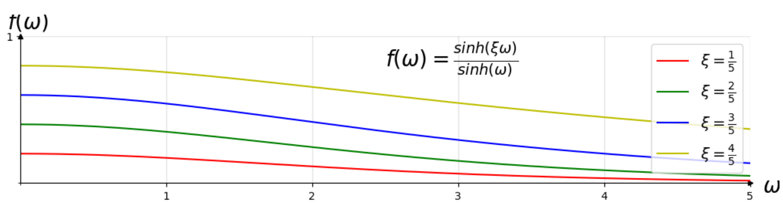

Figure 7: The function $f_\xi(\omega) = \frac{\sinh(\xi\omega)}{\sinh(\omega)}$ for several values of ξ. The function is monotonically decreasing and $f_{\xi_1}(\omega) < f_{\xi_2}(\omega)$ whenever $\xi_1 < \xi_2$.

Next, we will investigate the curve $g(\omega; \xi_1, \xi_2)$. We have investigated the function $\frac{\sinh(\omega)}{\sinh(\xi\omega)}$ in 1.4.5 and the curve g is composed of the reciprocals of that function, as depicted in 7. Thus, we can immediately see that

$$\lim_{\omega \to 0} g(\omega; \xi_1, \xi_2) = (\xi_1, \xi_2),$$

$$\lim_{\omega \to \infty} g(\omega; \xi_1, \xi_2) = (0, 0).$$

Since both functions g_1, g_2 are monotonic, the curve g will monotonically interpolate between the points (ξ_1, ξ_2) and $(0, 0)$. Because $0 < \xi_1 \leq \xi_2 < 1$, the end-point (ξ_1, ξ_2) can only lie within the triangle restricted by the lines $\{x = 0, y = 1, y = x\}$.

From Figure 7 we can see that

$$\frac{\sinh(\xi_1\omega)}{\sinh(\omega)} \leq \frac{\sinh(\xi_2\omega)}{\sinh(\omega)},$$

whenever $\xi_1 \leq \xi_2$ and the equality is only achieved when $\xi_1 = \xi_2$. Therefore the x-coordinate of the parametric curve g will always be less than or equal to the y-coordinate. That is, if $\xi_1 \leq \xi_2$, then

$g_1(\omega) \leq g_2(\omega)$.

We can also compute the slope of the curve $g(\omega)$ by calculating

$$\frac{dg_2}{dg_1} = \frac{\frac{dg_2}{d\omega}}{\frac{dg_1}{d\omega}} = \frac{\xi_2 \sinh(\omega)\cosh(\xi_2\omega) - \sinh(\xi_2\omega)\cosh(\omega)}{\xi_1 \sinh(\omega)\cosh(\xi_1\omega) - \sinh(\xi_1\omega)\cosh(\omega)}. \tag{2.36}$$

If $\xi_1 = \xi_2$, then $\frac{dg_2}{dg_1}(\omega) = 1$ for all $\omega \in (0, \infty)$. It is possible to evaluate the limits

$$\lim_{\omega\to 0} \frac{dg_2}{dg_1}(\omega) = \frac{\xi_2(1-\xi_2^2)}{\xi_1(1-\xi_1^2)},$$
$$\lim_{\omega\to\infty} \frac{dg_2}{dg_1}(\omega) = \infty, \quad \text{for } 0 < \xi_1 < \xi_2 < 1.$$

The function $\frac{dg_2}{dg_1}(\omega)$ is positive for positive ξ_1, ξ_2 and is monotonically increasing as can be observed from Figure 8. Therefore, given that $g(\omega)$ monotonically interpolates between (ξ_1, ξ_2) and $(0, 0)$, where $\xi_1, \xi_2 > 0$, the restriction that $g_1(\omega) \leq g_2(\omega)$, and that $\lim_{\omega\to 0} \frac{dg_2}{dg_1}(\omega) > 0$, $\lim_{\omega\to\infty} \frac{dg_2}{dg_1}(\omega) = \infty$ for $0 < \xi_1 < \xi_2 < 1$, we can say that the curve $g(\omega; \xi_1, \xi_2)$ lies in the triangle restricted by the lines $\{x = 0, y = \xi_2, y = \frac{\xi_2}{\xi_1}x\}$. This suggests that the curve g is concave down whenever $0 < \xi_1 < \xi_2 < 1$.

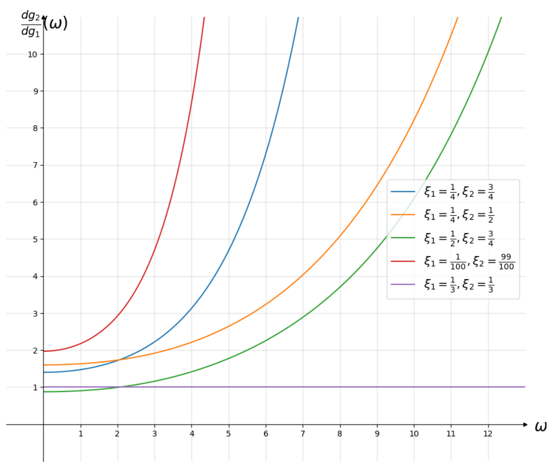

Figure 8: Equation (2.36) for various values of $0 < \xi_1 \leq \xi_2 < 1$. We can see that the function equals 1 whenever $\xi_1 = \xi_2$ and is a monotonically increasing positive function when $0 < \xi_1 < \xi_2 < 1$. We can also notice that the function increases faster for greater values of the difference $\xi_2 - \xi_1$.

We can compute the second derivative to see the concavity of the the curve g:

$$\begin{aligned}
\frac{d^2 g_2}{dg_1^2} =& \frac{\frac{d}{d\omega}\left(\frac{dg_2}{dg_1}\right)}{\frac{dg_1}{d\omega}} \\
=& \Big(-\xi_1^2\xi_2 \sinh(\omega\xi_1)\cosh(\omega\xi_2) + \frac{\xi_1^2 \sinh(\omega\xi_1)\sinh(\omega\xi_2)}{\tanh(\omega)} + \xi_1\xi_2^2 \sinh(\omega\xi_2)\cosh(\omega\xi_1) \\
&-\xi_1 \sinh(\omega\xi_2)\cosh(s\omega\xi_1) - \frac{\xi_2^2 \sinh(\omega\xi_1)\sinh(\omega\xi_2)}{\tanh(\omega)} + \xi_2 \sinh(\omega\xi_1)\cosh(\omega\xi_2)\Big) \\
&\cdot(\xi_1 \sinh(\omega)\cosh(\omega\xi_1) - \sinh(\omega\xi_1)\cosh(\omega))^{-1}
\end{aligned}$$

Numerical exploration shows that $\frac{d^2 g_2}{dg_1^2}(\omega)$ has one root at $\omega = 0$ and is negative otherwise whenever $0 < \xi_1 < \xi_2 < 1$. Therefore, the curve g has negative curvature as it can be seen from Figure 9. In other words, the curve g is concave down.

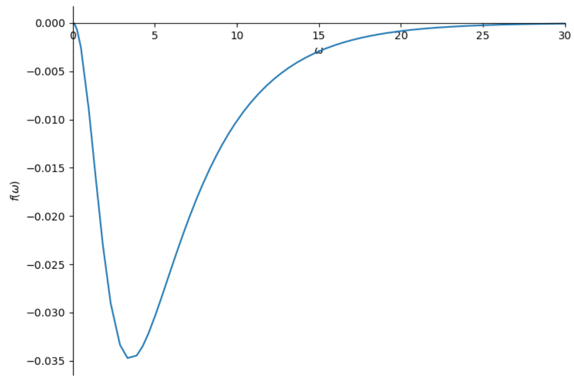

Figure 9: The curvature of the curve g as a function of $\omega > 0$. The curvature is negative for all tested values of $0 < \xi_1 < \xi_2 < 1$ and asymptotically approaches 0 from below. In this figure $\xi_1 = \frac{1}{4}$, $\xi_2 = \frac{3}{4}$.

A few examples of curves g are depicted in Figure 10.

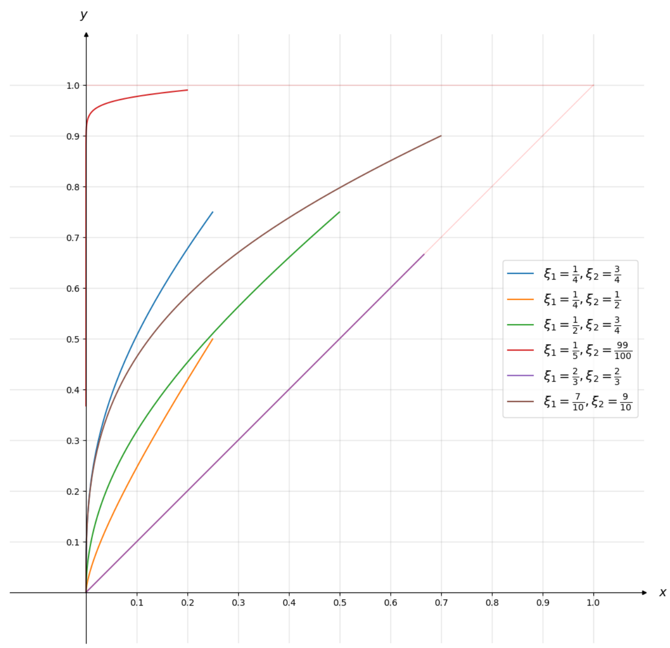

Figure 10: Curves (2.31) for various values of $0 < \xi_1 \leq \xi_2 < 1$. The curves start at their respective point (ξ_1, ξ_2) and approach the point $(0, 0)$ as $\omega \to \infty$. The curves are confined to the triangle defined by the lines $\{x = 0, y = 1, y = x\}$, which are depicted in red. The purple curve depicts the case when $\xi_1 = \xi_2$: the curve is a straight line with slope 1. Otherwise, the curves are concave.

Having investigated the line l (2.32) and the curve g (2.31) separately, let us combine two as their intersection will determine the roots of (2.29).

First we will consider the case when $\xi_1 = \xi_2 = \xi$. Then the curve g becomes a straight line segment $y = x$ between the points (ξ, ξ) and $(0, 0)$. The line l cannot cross the point $(0, 0)$, therefore the line l and curve g cannot coincide; they can only cross at one point at most.

If $\xi_1 = \xi_2 = \xi$, then we will have an intersection if and only if the system of equations

$$x - y = 0,$$
$$\frac{\gamma_1}{d} x + \gamma_2 y = 1$$

has a solution $x \in (0, \xi)$ and $y \in (0, \xi)$. This corresponds to the condition that

$$y = x = \frac{d}{\gamma_1 + d\gamma_2} \in (0, \xi).$$

This is satisfied whenever

$$0 < \frac{d}{\xi(\gamma_1 + d\gamma_2)} < 1.$$

This holds even if $\gamma_1 = 0$ or $\gamma_2 = 0$. Figure 11 shows examples of this situation.

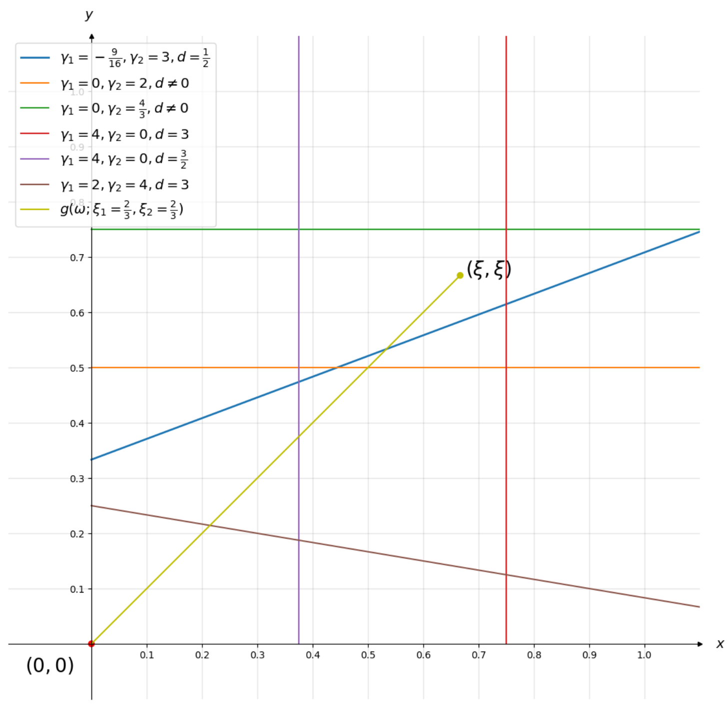

Figure 11: Lines (2.32) for various values of γ_1, γ_2, d and a curve (2.31) in yellow, when $\xi_1 = \xi_2 = \xi$. Plotted are the curve's starting point (ξ, ξ) in yellow and the end-point $(0, 0)$ in red. The lines cannot cross the red point. We will have a root $\omega > 0$ when a line and the curve cross. When $\gamma_1 = 0$, then we have a horizontal line. When $\gamma_2 = 0$, we have a vertical line. Blue, orange, red, purple and brown lines cross the yellow curve. Red and green lines do not.

Now we will turn to the case when $0 < \xi_1 < \xi_2 < 1$. We will have a solutions if

$$x = \frac{\sinh(\xi_1 \omega)}{\sinh(\omega)},$$
$$y = \frac{\sinh(\xi_2 \omega)}{\sinh(\omega)}$$

satisfies

$$\frac{\gamma_1}{d}x + \gamma_2 y = 1$$

for some $\omega > 0$.

From the concavity of the curve g we can see that we can have two, one or no intersections between the line l and curve g. Figures 13 and 12 show some examples.

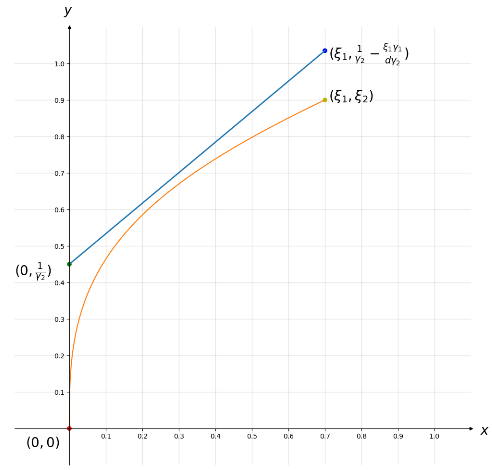

Figure 12: The line (2.32) in blue and curve (2.31) in yellow for $\xi_1 < \xi_2$. We can see that for some values of $\xi_1,\ \xi_2,\ \gamma_1,\ \gamma_2,\ d,\ t_d$ the line and the curve do not cross. In this figure $\xi_1 = 0.7,\ \xi_2 = 0.9,\ d = 0.5,\ t_d = 0.8,\ \gamma_1 = -\frac{13}{14},\ \gamma_2 = \frac{20}{9}$.

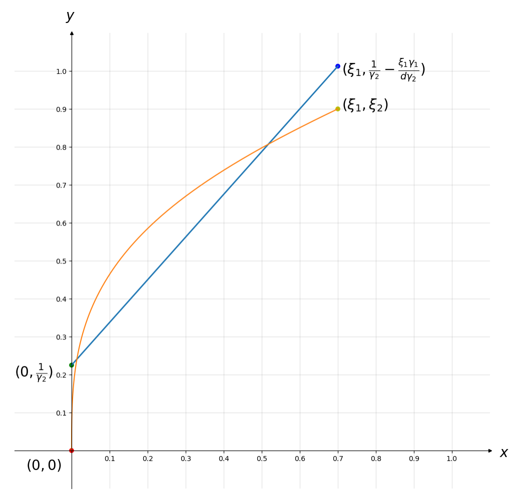

Figure 13: The line (2.32) in blue and curve (2.31) in yellow for $\xi_1 < \xi_2$ can cross at two points. The yellow point (ξ_1, ξ_2) represents the start of the curve and the red point $(0, 0)$ represents the end of the curve. Also depicted is y-intercept point of the line in blue and the blue point is the point that the line (2.32) taken when $x = \xi_1$. In this figure $\xi_1 = 0.7,\ \xi_2 = 0.9,\ d = 0.5,\ t_d = 0.8,\ \gamma_1 = -2.5,\ \gamma_2 = \frac{40}{9}$.

Let us check what needs to be satisfied for there to be two intersections. As can be seen from Figure 13, a necessary condition for the line to intersect the curve at two points is that the y-intercept of the line is lower than the maximum height of the curve, that is, $\frac{1}{\gamma_2} < \xi_2$ or, equivalently, that $\frac{1}{\xi_2} < \gamma_2$. Another necessary condition is that the slope of the line is positive, that is, $-\frac{\gamma_1}{d\gamma_2} > 0$. Since we have to have $\gamma_2 > \frac{1}{\xi_2} > 0$, we see that d and γ_1 have to have opposite signs. Also, as discussed previously, we have to have $\xi_1 < \xi_2$. Finally, by Figure 13, the y-value of the line at the point $x = \xi_1$ has to be higher than ξ_2. Substituting $x = \xi_1$ into (2.32), we see that

$$\frac{1}{\gamma_2} - \frac{\gamma_1}{d\gamma_2}\xi_1 < \xi_2,$$

which, after rearranging, becomes

$$\frac{\gamma_1}{\frac{d}{\xi_1}} + \frac{\gamma_2}{\frac{1}{\xi_2}} < 1.$$

This implies that the crossing point is beneath the line (2.21) corresponding to the zero eigenvalue in the γ_1, γ_2-space.

In summary, the necessary conditions to have two negative eigenvalues are:

- $\frac{\gamma_1}{\frac{d}{\xi_1}} + \frac{\gamma_2}{\frac{1}{\xi_2}} < 1$,
- $\frac{1}{\gamma_2} < \xi_2$,
- $\xi_1 < \xi_2$,
- d and γ_1 have opposite signs.

All of these results can also be verified by looking at Figures 4 and 5.

However, these are not sufficient conditions, as can be seen from the example in Figure 12. While all of the necessary conditions are satisfied, the line l does not cross the curve g. Provided that the necessary conditions already presented are kept fulfilled, we can see that there will not be any crossing when the line is above the curve and there will be two crossing points when the line is below the curve. Therefore, we will investigate when the line is tangent to the curve. The line (2.32) will be tangent to the curve g if

$$x = \frac{\sinh(\xi_1\omega)}{\sinh(\omega)}, \tag{2.37}$$

$$y = \frac{\sinh(\xi_2\omega)}{\sinh(\omega)}, \tag{2.38}$$

$$m = \frac{\frac{d}{ds}\left(\frac{\sinh(\xi_2\omega)}{\sinh(\omega)}\right)}{\frac{d}{ds}\left(\frac{\sinh(\xi_1\omega)}{\sinh(\omega)}\right)}. \tag{2.39}$$

The first two equations (2.37)–(2.38) evaluate to

$$\frac{\sinh(\xi_2\omega)}{\sinh(\omega)} = m\frac{\sinh(\xi_1\omega)}{\sinh(\omega)} + b, \tag{2.40}$$

whereas the equation (2.39) evaluates to

$$m = \frac{\xi_2\cosh(\xi_2\omega)\sinh\omega - \sinh(\xi_2\omega)\cosh\omega}{\xi_1\cosh(\xi_1\omega)\sinh\omega - \sinh(\xi_1\omega)\cosh\omega}. \tag{2.41}$$

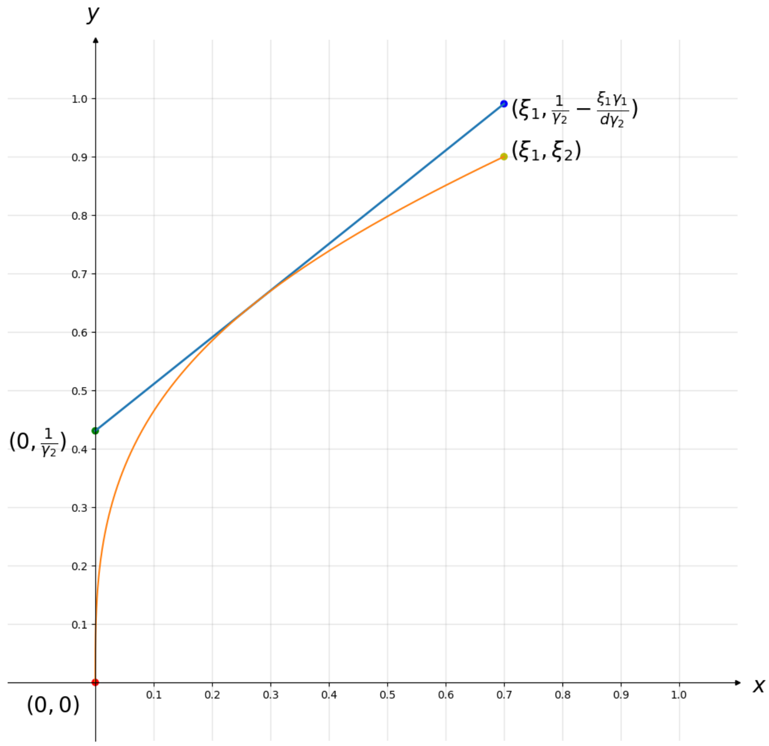

Figure 14: For some values of $\gamma_1, \gamma_2, d, \xi_1, \xi_2$, the line (2.32) in blue and curve (2.31) in yellow for $\xi_1 < \xi_2$ are tangent and therefore we have a double root. In this figure $\xi_1 = 0.7,\ \xi_2 = 0.9,\ d = 0.5,\ t_d = 0.8,\ \gamma_1 = -\frac{13}{14},\ \gamma_2 = \frac{209}{90}$.

Here m and b are as in (2.34)–(2.35).

The graphs of (2.41) are depicted in Figure 8. Whenever the equations (2.37)–(2.39) are satisfied, we will have a double root of (2.29). This situation is depicted in Figure 14.

In short, given a pair ω_0, m_0 that satisfies (2.41), we can find a value $b_0 = \frac{1}{\gamma_2}$ that satisfies (2.40). We will have two different roots if we lower the line below the tangent point. For that we need to lower the y-intercept of the line, which is $\frac{1}{\gamma_2}$. We can pick γ_2 such that

$$0 < \frac{1}{\gamma_2} < b_0$$

and we will have two different roots. Having admissable γ_2, we can use (2.34) to find the required values of d, γ_1. From (2.34) we can deduce that

$$\frac{\gamma_1}{d} < -\frac{m_0}{b_0}.$$

Alternatively, we can select the two roots. Call the two values of ω where we have intersections as ω_1 and ω_2. Then define the points

$$(x_1, y_1) = \Big(\frac{\sinh(\xi_1\omega_1)}{\sinh(\omega_1)}, \frac{\sinh(\xi_2\omega_1)}{\sinh(\omega_1)}\Big),$$
$$(x_2, y_2) = \Big(\frac{\sinh(\xi_1\omega_2)}{\sinh(\omega_2)}, \frac{\sinh(\xi_2\omega_2)}{\sinh(\omega_2)}\Big).$$

From

$$y_1 = mx_1 + b,$$
$$y_2 = mx_2 + b$$

we get that

$$m = \frac{y_2 - y_1}{x_2 - x_1},$$
$$b = y_1 - mx_1$$

where m and b are as in (2.34)–(2.35). Then we have the values for the line l (2.33).

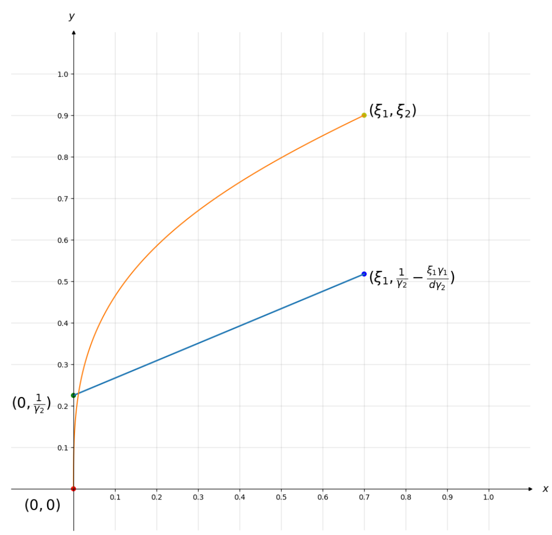

Figure 15: The line (2.32) in blue and curve (2.31) in yellow for $\xi_1 < \xi_2$ can cross at one point when the y-value $< \xi_2$ at the point $x = \xi_1$. In this figure $\xi_1 = 0.7,\ \xi_2 = 0.9,\ d = 0.5,\ t_d = 0.8,\ \gamma_1 = -\frac{13}{14},\ \gamma_2 = \frac{40}{9}$.

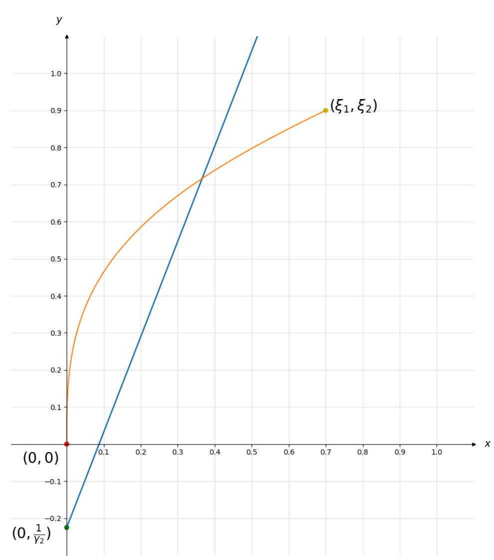

Figure 16: The line (2.32) in blue and curve (2.31) in yellow for $\xi_1 < \xi_2$ can cross at one point when the y-value < 0 at the point $x = 0$. In this figure $\xi_1 = 0.7,\ \xi_2 = 0.9,\ d = 0.5,\ t_d = 0.8,\ \gamma_1 = \frac{40}{7},\ \gamma_2 = -\frac{40}{9}$.

Now let us investigate when we only have one root $\omega > 0$. Figures 15 and 16 depict the two scenarios where we have a single root. Let us investigate the first case in Figure 15. Define the points

$$(x_0, y_0) = (0, b),$$
$$(x_1, y_1) = (\xi_1, m\xi_1 + b).$$

From Figure 15 we see that we must have $0 < y_0$ and $y_1 < \xi_2$. From this we can deduce that

$$0 < \gamma_2,$$
$$1 < \frac{\gamma_1}{d}\xi_1 + \gamma_2\xi_2.$$

For the case depicted in Figure 16, define the points

$$(x_0, y_0) = (0, b),$$
$$(x_1, y_1) = (\xi_1, m\xi_1 + b).$$

Then we will have a single crossing if $y_0 < 0$ and $y_1 > \xi_2$. From this we conclude that

$$0 > \gamma_2,$$
$$1 > \frac{\gamma_1}{d}\xi_1 + \gamma_2\xi_2$$

However, we can combine these two cases into a single condition. Observe that we will have a single root if and only if the line l crosses the line segment connecting the points $(0, 0)$ and (ξ_1, ξ_2) as it is depicted in Figure 17

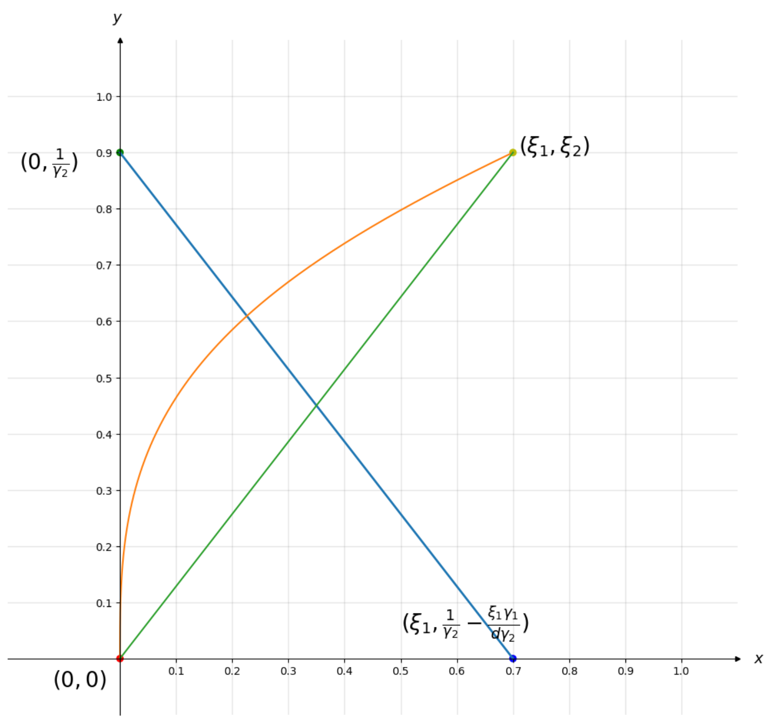

Figure 17: The line (2.32) in blue and curve (2.31) in yellow for $\xi_1 < \xi_2$ can cross at one point if and only if the line (2.32) crosses the green line segment $y = \frac{\xi_2}{\xi_1}x$ for $x \in (0, \xi_1)$. In this figure $\xi_1 = 0.7,\ \xi_2 = 0.9,\ d = 0.5,\ t_d = 0.8,\ \gamma_1 = \frac{5}{7},\ \gamma_2 = \frac{10}{9}$.

The equation for the line segment is $y = \frac{\xi_2}{\xi_1}x$. The line (2.32) crosses $y = \frac{\xi_2}{\xi_1}x$ when the system of equations

$$\begin{aligned} 1 &= \frac{\gamma_1}{d}x + \gamma_2 y, \\ y &= \frac{\xi_2}{\xi_1}x \end{aligned}$$

has a solution $x \in (0, \xi_1),\ y \in (0, \xi_2)$.

In other words, the solution

$$\begin{aligned} x &= \frac{d\xi_1}{\xi_1\gamma_1 + \xi_2 d\gamma_2} \in (0, \xi_1), \\ y &= \frac{d\xi_2}{\xi_1\gamma_1 + \xi_2 d\gamma_2} \in (0, \xi_2), \end{aligned}$$

if and only if

$$0 < \frac{d}{\xi_1\gamma_1 + \xi_2 d\gamma_2} < 1. \tag{2.42}$$

This applies even when $\gamma_2 = 0$ or $\gamma_1 = 0$. Indeed, if $\gamma_2 = 0$ or $\gamma_1 = 0$, then we cannot have two roots as horizontal or vertical lines cross the curve g at most once and are never tangent to the curve. Therefore, only (2.42) condition is relevant for these cases.

Thus, we can say that we will have a single root whenever (2.42) is satisfied.

3 Sturm–Liouville Problems with More General Transmission and Non-Local Boundary Conditions

3.1 Problem 1

Consider a generalizatoin of the problem (2.1)–(2.5):

$$-u'' = \lambda u, \text{ for } t \in I_1 \cup I_2, \tag{3.1}$$

$$u(0) = 0, \tag{3.2}$$

$$u(t_b) = du(t_a), \tag{3.3}$$

$$u'(t_b) = \delta u'(t_a), \tag{3.4}$$

$$< n_1, u > =< l_1, u > + < l_2, u >, \tag{3.5}$$

where $I_1 = (0, t_a)$ and $I_2 = (t_b, 1)$, $d, \delta \in \mathbb{R}$ not both equal to zero, $0 < t_a \leq t_b < 1$, $\gamma_1, \gamma_2 \in \mathbb{R}$ some constants, n_1 is a local boundary condition at the point $t = 1$:

$$< n_1, u >:= \alpha u(1) + \beta u'(1), \;\; \alpha, \beta \in \mathbb{R}. \tag{3.6}$$

l_1 is a non-local boundary condition in the interval I_1 and l_2 is a non-local boundary condition in the interval I_2.

If we repeat the solution procedure outlined in subsection 2.1, then we can find that the eigenfunctions

$$u_1(t) = \frac{\sin(st)}{s}, \qquad \text{for } t \in I_1,$$

$$u_2(t) = M(s)\cos(st) + N(s)\frac{\sin(st)}{s}, \text{ for } t \in I_2,$$

where

$$M(s) = \cos(st_b)\Big(d\frac{\sin(st_a)}{s} - \delta\frac{\sin(st_b)}{s}\Big), \tag{3.7}$$

$$N(s) = d\frac{\sin(st_a)}{\sin(st_b)} - \cos^2(st_b)\Big(d\frac{\sin(st_a)}{s} - \delta\frac{\sin(st_b)}{s}\Big), \tag{3.8}$$

exist if and only if s satisfies the condition

$$M(s)\Big\langle n_1 - l_1, \cos(st)\Big\rangle + N(s)\Big\langle n_1 - l_2, \frac{\sin(st)}{s}\Big\rangle = \Big\langle l_1, \frac{\sin(st)}{s}\Big\rangle. \tag{3.9}$$

As before $s^2 = \lambda$, where $s \in \mathbb{C}_s$. $M(s)$ and $N(s)$ are related by

$$N(s) = d\frac{\sin(st_a)}{\sin(st_b)} - \cos(st_b)M(s).$$

Note that if $< n_1, u >= u(1)$, $< l_1, u >= \gamma_1 u(\xi_1)$ for $\xi_1 \in I_1$, $< l_2, u >= \gamma_2 u(\xi_2)$ for $\xi_2 \in I_2$, $d = \delta$ and $t_a = t_b$, then $M(s) = 0$, $N(s) = d$ and (3.9) becomes (2.19).

3.1.1 The case when $\lambda = 0$

We will consider when the operator of the problem is singular, that is, when $\lambda = 0$. Evaluating (3.7)–(3.8) at $s = 0$ we get

$$\begin{aligned} M(0) &= dt_a - \delta t_b, \\ N(0) &= d\frac{t_a}{t_b} - M(0), \end{aligned}$$

where we have evaluated the limits

$$\begin{aligned} \lim_{s\to 0} \frac{\sin(st)}{s} &= t, \\ \lim_{s\to 0} \frac{\sin(st_a)}{\sin(st_b)} &= \frac{t_a}{t_b}. \end{aligned}$$

At $s = 0$ (3.9) becomes

$$M(0)\langle n_1 - l_1, 1\rangle + N(0)\langle n_1 - l_2, t\rangle = \langle l_1, t\rangle. \tag{3.10}$$

First, we will consider what happens when $M(0) = 0$ and/or $N(0) = 0$.

$$\begin{aligned} &M(0) = 0 \text{ if and only if } dt_a = \delta t_b, \\ &M(0) = 0 \text{ implies that } N(0) = \delta, \\ &N(0) = 0 \text{ if and only if } dt_a = -\delta\frac{t_b^2}{1-t_b}, \\ &N(0) = 0 \text{ implies that } M(0) = -\delta\frac{t_b}{1-t_b}. \end{aligned}$$

$M(0) = 0$ and $N(0) = 0$ together imply that both $d, \delta = 0$. This is an extreme case that is similar to the one investigated in (2.7)–(2.11) and is ill-posed. Then the eigenfunctions are

$$\begin{aligned} u_1(t) &= t, \text{ for } t \in I_1, \\ u_2(t) &= 0, \text{ for } t \in I_2, \end{aligned}$$

whenever

$$\langle l_1, t\rangle = 0.$$

If $dt_a = \delta t_b$, then $M(0) = 0$ and $N(0) = \delta$, which we will assume is not equal to zero. Since $0 < t_a \le t_b < 1$, we also have that $|d| \ge |\delta|$ and the signs of d, δ must be the same. The eigenfunctions are

$$u_1(t) = t, \quad \text{for } t \in I_1,$$
$$u_2(t) = \delta t, \ \text{for } t \in I_2,$$

and we have a further condition that

$$\langle n_1 - \frac{1}{\delta} l_1 - l_2, t \rangle = 0. \tag{3.11}$$

If n_1 takes the most general form of (3.6), then $< n_1, t >= \alpha + \beta$. If $< n_1, t >= 0$, that is, $\alpha = -\beta$, then the functionals l_1, l_2 are linearly dependent and $< l_2, t >= \frac{1}{\delta} < l_1, t >$. We will assume that $< n_1, t > \neq 0$. Since $\delta \neq 0$ and $< n_1, t > \neq 0$, we can rewrite (3.11) as

$$\frac{1}{\delta} \frac{< l_1, t >}{< n_1, t >} + \frac{< l_2, t >}{< n_1, t >} = 1,$$

which is an equation of a line in the variables $< l_1, t >, \ < l_2, t >$ for fixed $\delta, \ < n_1, t >$. If we further assume that $< n_1, u >= 1, \ < l_1, u >= \gamma_1 u(\xi_1), \ < l_2, u >= \gamma_2 u(\xi_2)$, then we can retrieve (2.21) with d replaced by δ.

If $dt_a = -\delta \frac{t_b^2}{1-t_b}$, then $N(0) = 0$ and $M(0) = -\delta \frac{t_b}{1-t_b}$ or, equivalently, $M(0) = d\frac{t_a}{t_b}$. Since $0 < t_a \leq t_b < 1$, we can conclude that the signs of d and δ must be opposite and that $-|d| \leq |d| \frac{t_a}{t_b} \leq |d|$. This, in turn, implies that $-|d| \leq |\delta| \frac{t_b}{1-t_b} \leq |d|$. Solving for t_b, we get the inequality $t_a \leq t_b \leq \min\{\frac{|d|}{|\delta|+|d|}, 1\}$.

The corresponding eigenfunctions in this case are

$$u_1(t) = t, \qquad \text{for } t \in I_1,$$
$$u_2(t) = M(0), \ \text{for } t \in I_2.$$

A further condition is that

$$M(0) < l_1, 1 > + < l_1, t >= M(0) < n_1, 1 > . \tag{3.12}$$

Note that this condition does not involve the functional l_2. If $< n_1, 1 >= 0$, that is, if $\alpha = 0$, then $< l_1, 1 >= \frac{1}{M(0)} < l_1, t >$. So let us assume that $< n_1, 1 >= \alpha \neq 0$. If we fix $M(0)$ and $< n_1, 1 >$, then we can rewrite (3.12) as

$$\frac{< l_1, 1 >}{< n_1, 1 >} + \frac{1}{M(0)} \frac{< l_1, t >}{< n_1, 1 >} = 1,$$

which is an equation of a line in $< l_1, 1 >, \ < l_1, t >$ for fixed $M(0), \ < n_1, 1 >$. If we take $< n_1, u >= 1, \ < l_1, u >= \gamma_1 u(\xi_1)$, then we have $\gamma_1 = \frac{M(0)}{\xi_1 + M(0)}$.

If $dt_a \neq \delta t_b$ and $dt_a \neq -\delta \frac{t_b^2}{1-t_b}$, then $M(0) \neq 0$ and $N(0) \neq 0$, so we have to consider equation (3.10) in its full generality.

If we assume that $< n_1, u >= 1, \ < l_1, u >= \gamma_1 u(\xi_1), \ < l_2, u >= \gamma_2 u(\xi_2)$, then our condition is

$$\Big(\xi_1 + dt_a - \delta t_b\Big)\gamma_1 + \xi_2 \Big(d\frac{t_a}{t_b} - dt_a + \delta t_b\Big)\gamma_2 = 1.$$

3.2 Problem 2

Consider another generalization when we have several discontinuities at the points $t = t_{d_1}, t_{d_2}, ..., t_{d_n}$:

$$\begin{aligned}
-u'' &= \lambda u,\\
u(0) &= 0,\\
u(t_{d_1}+0) &= d_1 u(t_{d_1}-0),\\
u'(t_{d_1}+0) &= d_1 u'(t_{d_1}-0),\\
u(t_{d_2}+0) &= d_2 u(t_{d_2}-0),\\
u'(t_{d_2}+0) &= d_2 u'(t_{d_2}-0),\\
&...\\
u(t_{d_n}+0) &= d_n u(t_{d_n}-0),\\
u'(t_{d_n}+0) &= d_n u'(t_{d_n}-0),\\
u(1) &= \gamma_0 u(\xi_0) + \gamma_1 u(\xi_1) + ... + \gamma_n u(\xi_n)
\end{aligned}$$

for $t \in I_0 \cup I_1 \cup I_2 \cup ... \cup I_n$, where

$$I_0 = (0, t_{d_1}),\ I_1 = (t_{d_1},\ t_{d_2}), ..., I_{n-1} = (t_{d_{n-1}},\ t_{d_n}),\ I_n = (t_{d_n}, 1).$$

$\lambda \in \mathbb{C}$ is the eigenvalue, $d_i \in \mathring{\mathbb{R}}$, where the remark about $d = 0$ case apply as well.

$$0 < \xi_0 \leq t_{d_1} \leq \xi_1 \leq t_{d_2} \leq ... \leq t_{d_n} \leq \xi_n < 1.$$

The solution set $u(t)$ will be

$$\begin{aligned}
u_0(t) &= \frac{\sin(st)}{s},\ \text{for } t \in I_0,\\
u_1(t) &= d_1\frac{\sin(st)}{s},\ \text{for } t \in I_1,\\
u_2(t) &= d_1 d_2\frac{\sin(st)}{s},\ \text{for } t \in I_2,\\
&...\\
u_n(t) &= d_1 d_2 ... d_n\frac{\sin(st)}{s},\ \text{for } t \in I_n.
\end{aligned}$$

For $\lambda = s^2,\ s \in \mathbb{C}_s$ to be an eigenvalue, s must satisfy

$$d_1 d_2 ... d_n \frac{\sin s}{s} = \gamma_0 \frac{\sin(\xi_0 s)}{s} + \gamma_1 d_1 \frac{\sin(\xi_1 s)}{s} + \gamma_2 d_1 d_2 \frac{\sin(\xi_2 s)}{s} + ... + \gamma_n d_1 d_2 ... d_n \frac{\sin(\xi_n s)}{s}.$$

The main conclusion is that this situation is analogous to the one investigated in the main part of the text, except that this time we must work in a higher-dimensional space.

4 Conclusions

In this work we have looked at a Sturm–Liouville problem for a simple second-order differential operator with transmission and non-local boundary conditions. An analytic solution method was presented as well as a geometric way to investigate the eigenvalues. The conditions that need to be satisfied to have eigenvalues have been determined. The singular case and the case of negative eigenvalues was explored in more depth. It was found that under certain conditions we may have up to two negative eigenvalues, including repeated ones. The analysis of positive and complex eigenvalues proved to be much more difficult and it is postponed to future research. Some generalizations have been considered and preliminary results were presented, although much more research is required for the more general cases.

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