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An Outreach Note on the Poincaré Conjecture for Non-specialists

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Abstract. The Poincaré Conjecture, a problem formulated by the French mathematician Henri Poincaré more than a century ago, has been one of the main challenge of modern mathematics. It states that any three-dimensional space which is closed on itself and without holes can be deformed into a sphere of dimension 3.
Even if the conjecture was solved at the beginning of this century, it still remains a mysterious, appealing and intriguing problem worth to be further studied in detail. The purpose of this short popularizing note is, on the one hand, to provide a quick overview for non-experts of what we know today about the Poincaré Conjecture and its related problems in dimension 3, and, on the other hand, to explain why it has represented a central problem in mathematics.

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1. Introduction

On the occasion of the new millennium, and a century after the famous International Congress of Mathematics held in Paris in 1900 where David Hilbert drew up his famous list of 23 unsolved mathematical problems at that time, the Clay Mathematics Institute in Cambridge, Massachusetts, chose a new group of seven difficult problems/conjectures that were still unsolved in the years 2000, awarding a prize of one million dollars for the solution of each one of them. The millennium prizes were announced once again in Paris in the spring of 2000, and among these seven great questions of the new century stand out the Poincaré Conjecture (which is easy to state and a century old), and the all-famous Riemann Hypothesis, formulated in 1859, the only conjecture that was already part of Hilbert's 23 problems of 1900.

Of all these problems, only one has been solved in the meantime: the Poincaré Conjecture, settled by the Russian mathematician Grigori Perelman in 2003 [3, 4]. The resolution of this century-old conjecture, along with the fact that he refused the 1-million prize, has drawn the attention of the general public especially to this problem.

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2. Preliminaries

Since we want to address to an audience of non-experts, we will start from scratch, in order to be able to state and comment the Poincaré Conjecture and its generalizations.

2.1. Manifolds

We need to start by defining and talking about the central topics of interest for us: **manifolds** of dimension n, where, for simplicity, we will consider a manifold as an object considered in a space of dimension N (greater than n). A manifold of dimension 1 ($n = 1$) is a line or a curve (in our standard plane $\mathbb{R}^2$ for example), while a manifold of dimension 2 ($n = 2$) is what we commonly call a surface (in our 3-dimensional space $\mathbb{R}^3$). With more fantasy and abstraction, we can imagine a 3-dimensional manifold M as a subspace of a space of dimension $N \geq 4$ (imagine $\mathbb{R}^4$ as Einstein's space-time), such that, locally, it looks like our 3-dimensional space (as well as, for instance, a local piece of a surface resembles a piece of the real plane), and so on for any natural number n.

Just like prime or complex numbers, it turns out that manifolds are also central objects in the architecture of modern mathematics. They are in fact the basic building blocks of the branch of mathematics called **Topology** (literally the study of "places and forms"). And the Poincaré Conjecture is a cornerstone of the classification of them.

Now, just as a single coordinate is sufficient to identify a point on a curve, two numbers (coordinates) are needed to identify a point on a surface. For example, on the earth's surface, (which is a two-dimensional sphere S^2), we need longitude and latitude. Incidentally, the fact that this parametrization possesses anomalies (e.g. all meridians meet at the north and south poles, where longitude therefore ceases to be well defined) is a sign of a basic topological fact: the sphere S^2 is topologically different (technically not "homeomorphic", see here below) to the torus T^2, which is the surface of a tyre.

Obviously, the sphere S^2 differs also from (is not homeomorphic to) the Euclidean space $\mathbb{R}^2$ either, but this is an easier thing to understand, since the intuitive fact that the sphere is a "closed" surface while the plane $\mathbb{R}^2$ is "open" corresponds to a topological difference that is well encoded (a manifold is said to be **closed** if it has no boundary and takes up a finite region of space).

We end this section by giving an idea of what a homeomorphism is. **Homeomorphisms** are equivalences in the category of topological spaces, more precisely bijective and continuous correspondences in both directions. In particular, homeomorphisms are those functions which preserve all the topological properties of a given manifold (topological space). And it turns out that two manifolds are homeomorphic if one can continuously (i.e. without cutting or glueing) deform the first manifold into the second one.

The Poincaré conjecture states that the 3-sphere is the only three-dimensional compact manifold without boundary and without 'holes', up to homeomomorphisms, i.e. it is the only such manifold where any closed path can be contracted to become a point.

2.2. From the sphere $\mathbb{S}^2$ to the ball $\mathbb{B}^3$

The reader will certainly be familiar with at least one 3-dimensional manifold, namely our three-dimensional space, $\mathbb{R}^3$, but we can construct many other ones in any dimension.

Indeed, also real world mechanics or physics force us to move from dimension 2 to dimensions 3, 4 or higher. These higher dimensional manifolds are objects comparable to surfaces, but where 3, 4 or more (local) coordinates are needed to identify a point. For example, the motion of three bodies subjected to the force of gravity can be studied as an 18-dimensional manifold, each body being defined by three spatial coordinates and three velocity coordinates.

If we imagine the two-dimensional sphere as the (exterior) surface (or boundary) of a three-dimensional ball, then we can conceive spheres in three, four or more dimensions... And so we realize that the n-dimensional sphere is the boundary of the ball of dimension $n+1$. Also, we may figure out that, in each dimension n, the n-sphere is somehow the simplest possible closed manifold to study.

To be more precise, the n-sphere S^n is the set of points of the Euclidean $\mathbb{R}^{n+1}$ space which are at distance 1 from the origin, and elementary analytic geometry helps us by providing its explicit equation $S^n = \{x_1, x_2, \cdots, x_{n+1} \in \mathbb{R}^{n+1} \text{ such that } x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}$; while together with its interior one obtains the $n+1$ ball B^{n+1} expressed in Cartesian coordinates as $B^{n+1} = \{x_1, x_2, \cdots, x_{n+1} \in \mathbb{R}^{n+1}$ such that $x_1^2 + x_2^2 + \cdots + x_{n+1}^2 \leq 1\} \subset \mathbb{R}^{n+1}$.

Let us now imagine a sphere S^2, like the surface of the earth, deprived of the north pole, namely $S^2 - N$. With a little more imagination it is not difficult to see that this punctured sphere is, in some sense, "contractible". That is: each point p of $S^2 - N$ (with p different from the south pole) belongs to a single meridian (this is not true for both the poles), and so it can slide continuously along this meridian until it reaches the south pole.

The same thing can be done for a sphere of any dimension n. And even in this case, the "meridians" of $S^n - N$ meet only at the south pole.

But we can also conceive something more complicated, some kind of spheres Σ^n where the continuous flow from $\Sigma^n - N$ goes towards the south pole but in a much more complicated way, crossing and intersecting not only in the south pole but also elsewhere. These "sorts of spheres" are called **homotopy spheres** (they are objects which are similar to spheres but different from them, because they are shaped in a different way). In other words, a homotopy sphere is a closed manifold Σ^n such that $\Sigma^n - P$ can be deformed continuously (always remaining in $\Sigma^n - P$) to a point (like normal spheres).

3. Formulation of the Poincaré Conjecture

The **Poincaré Conjecture** states that the only homotopy sphere Σ^3 of dimension 3 is the sphere S^3 (up to homeomorphisms) [7]. In other words, this means that one can always find a continuous flow from $\Sigma^3 - N$ (the north pole) towards the south pole without intersections, crossovers or overlaps, except at the south pole. This is also equivalent to saying that $\Sigma^3 - N = \mathbb{R}^3$ (this is clear for S^3, by means of the "stereographic projection").

To be more precise, Poincare's own formulation in 1904 [6], although equivalent to the one just mentioned, was a little bit different: Poincaré indeed conjectured that the only closed and "simply connected" 3-dimensional manifold was just (homeomorphic to) S^3. Where a manifold V is said to be **simply connected** (a new topological notion introduced by Poincaré himself) if every closed curve (i.e. a loop) in V can be deformed continuously to a point, remaining within V.

Note that the first statement given above at the beginning of this section has the advantage of being valid in every dimension. In particular, one can formulate the generalized Poincaré Conjecture, which says that for every n, a homotopy sphere Σ^n is (homeomorphic to) S^n. But the step from $n = 3$ to any n, appeared some thirty years after Poincaré, and moreover, no further progress was made in any dimension greater than 2 until the 1950s.

On the other hand, although formulated in a very different way, the case $n = 2$, which is much more easy, was already known since the middle of the 19th century. More precisely, in dimension 2, the corresponding statement is that in any closed surface which is different from a 2-sphere S^2 one can find at least a loop which cannot be continuously contracted to a point. This result actually follows from a far more detailed and deeper theorem: the classification of closed and connected two-dimensional manifolds, which was proved in different forms since the 1860s, and which says that every compact surface is homeomorphic to a sphere with some number of handles or cross-caps attached.

3.1. TOP versus DIFF

Now, we come back to the sphere S^2, seen as the surface of our globe B^3. This object is obviously homeomorphic, i.e. topologically equivalent, to the surface of an ellipsoid (a rugby ball), but also to the surface of a cube (because we can imagine a play dough sphere, which we can model, without breaking it, as either an ellipsoid or a cube). However, the ellipsoid is smooth, like the sphere, while the surface of the cube is not, because there are edges, corners and points.

Hence, we can say that the equality between the sphere and the ellipsoid is realized by functions that possess continuous derivatives (and these functions are called **diffeomorphisms** = differentiable homeomorphisms, a more restrictive notion than that of homeomorphism). The sphere and the ellipsoid are both homeomorphic and diffeomorphic. On the contrary, the topological equivalence between the sphere and the cube surface is only possible through simple continuous functions that do not admit derivatives: these two surfaces are homeomorphic, but not diffeomorphic. The sphere is smooth and differentiable, while the cube is not.

Now, if we only consider manifolds of dimension less than or equal to 3, then this distinction is an unnecessary pedantry, because in those cases the categories of topological and differentiable manifolds coincide (i.e. we know how to round off edges). Conversely, in dimensions greater than 3, it turns out that there are obstructions to rounding objects, and there exist examples of non-smootable manifolds.

For example, in 1956 the famous mathematician J. Milnor proved that on the sphere of dimension 7 coexist several (actually 28) differentiable structures that are not diffeomorfic to each other. Therefore there exist "exotic" spheres, i.e. spheres admitting differential structures that are different from the standard one (see [3]).

Hence, the general idea at the time was that, as the dimension increases, the difficulties could only increase too. But then, around 1960, S. Smale realized the opposite, and proved at once the Generalized Poincaré Conjecture in any dimension at least 5: a manifold which is a homotopy sphere Σ^n is homeomorphic to a sphere S^n, for every $n \geq 5$.

The reasons why the difficulties decrease can be explained very simplistically as follows: in large dimensions, there is a lot of empty space to manoeuvre around the problems and to develop the needed geometric constructions to solve them and to establish the equality to be proved. While in dimensions smaller than 3 (1 or 2), there is not enough space to create problems. Finally, in dimension 3, namely the starting point of Poincaré, there is both the possibility of having problems but also very little space to act...

The limit situation is dimension 4. This is a world of its own, very different from the others (dimension 2, 3 or higher dimensions, see [5]). In this case it actually took a great tour-de-force to prove the Poincaré Conjecture: in 1982 M. Freedmann proved that Σ^4 is topologically equal (homeomorphic) to S^4. Actually, thanks to Freedmann's work, we know nowadays that precisely in dimension 4 (the dimension of our space-time) there is a great mystery between the topological and the differential pictures. And the smooth 4-dimensional Poincaré Conjecture is still an open problem, far from being solved.

Summarizing, we know that the generalized Poincaré Conjecture can be true or false in the different categories (TOP or DIFF), depending on the dimension, thank to the work of several esteemed mathematicians, such as John Milnor, Steve Smale, Michael Freedman, and Grigori Perelman (see [8]), all of whom have been awarded the Fields Medal, which is the most prestigious prize in mathematics, the equivalent of Nobel Prize.

In particular, in the category TOP (i.e. for topological manifolds) the generalized Poincaré Conjecture is true in all dimensions! Whereas in the DIFF category (i.e. for differential manifolds) it is true in dimensions 1, 2, 3, 5 and 6, it is still open in dimension 4, whereas in the other cases it is generally false.

Notice that, even if the original Poincaré Conjecture in dimension 3 has been solved a century after its first formulation, in all that time, various ways to attack the problem have been developed and tried, as it is related to various areas of mathematics, from group theory to differential equations, from physics to general relativity. And even though it has remained unproven for a hundred years, many profound new results have emerged from the techniques developed to solve it (see [1–4, 8–10]). In few words, all the work devoted to the conjecture improved the deep understanding of the world of 3-manifolds.

4. Geometric Structures in dimension 3

The Poincaré Conjecture is a purely topological problem. Nevertheless, all efforts by topologists to prove it have failed, and in fact no topological proof exists to this day. Therefore, for its resolution, a good idea is to leave the topological framework, and to use geometric or analytic methods in order to have more tools to attack the problem.

One of the most suitable strategies is to equip the manifolds with some **geometric structures**. And in fact, it was just this approach that has proved successful in the end.

This path is very much related to another work by Poincaré himself, his famous Uniformization Theorem. This result concerns surfaces (hence we are in dimension 2) and, roughly speaking, tells us that on each surface we can put a geometry (i.e. a way of measuring distances and angles) that, locally, is like one of the three classical geometries with constant curvature: Euclidean geometry (the plane with zero curvature), non-Euclidean geometry called Lobachevsky's hyperbolic geometry (the one where several lines parallel to a given line pass through a point, called the plane with negative curvature -1), and that of the round sphere (called elliptic geometry with positive curvature +1).

The obvious question is now: can we say something similar in dimension 3? In the 1970s the great American mathematician William Thurston [9] conceived a very spectacular program in order to "geometrize" all closed 3-dimensional manifolds, as Poincaré did in dimension two, with the difference that in dimension three the possible geometries should be 8 instead of 3, and among them, the most important being the hyperbolic one.

4.1. Curvature

In order to provide a geometry to a manifold, a natural procedure is to specify, at each point of the manifold, how the distance between two very close points is expressed. In a more mathematical way, one must specify, at each point, the "metric tensor". Introduced by the German mathematician Bernhard Riemann in the 19th century, this tensor (a generalization of the notion of vector) is used to determine lengths, angles, areas, volumes, etc. on the given manifold. For a manifold of dimension n, it is a table of n^2 numbers (a matrix $n \times n$), which is used to calculate the **curvature** of the manifold.

In dimension 2, the curvature of a surface is an intuitive notion, made rigorous by the German mathematician Friedrich Gauss around 1830. He defined the curvature R as a number obtained by an expression $R(p)$ of the curvature of a surface at its point p. And this number defines and measures the "gap" between the geometry of the surface near the point p and the Euclidean classical geometry (the geometry of the standard plane). For instance, the curvature of a two-dimensional sphere is positive, that of a plane (or a cylinder or a cone) is zero, while that of the surface of a saddle has negative curvature.

If the curvature is independent on the point p of the surface, as in these examples, we speak of elliptic geometry, Euclidean geometry, and hyperbolic geometry. Notice that, these three different geometries differ in the shape of their triangles (in particular in the sum of the inner angles): triangles in the surface of the sphere are fat, those in the plane are standard, while in the hyperbolic case they are slim.

Around 1850, Riemann generalized the notion of curvature to manifolds of any dimension n. But when n is greater than or equal to 3, it is no longer just a number, as in the case of Gauss, but a tensor. Let us suppose that in the neighborhood of a point p of a manifold V^n, we have chosen a local system of coordinates $x_1, x_2, \cdots, x_n$. The Riemann curvature at a point p of V^n is expressed as a table of n^4 numbers, each directly dependent on the (Riemannian) metric defined on the manifold itself, and its derivatives.

Note that Riemann's ideas have long seemed far too abstract. Yet, it is precisely on these notions that the general relativity, Einstein's great masterpiece, is based.

4.2. The Geometrization Conjecture

More than a century after Riemann's innovations, the geometer W. Thurston proposed a new vast classification project in dimension 3, started in the 1970s, in order to prove the Poincaré Conjecture and to deeply understand the set of closed 3-manifolds.

Thurston started highlighting, in dimension 3, eight geometries which are particularly symmetric, three of which are those already defined in the case of surfaces. He therefore devised the so-called **Geometrization Conjecture**, according to which any closed manifold of dimension 3 can be broken, in a unique way, into a finite number of pieces, each of which supports one of the 8 geometries. It is thus a generalization, in dimension 3, of Poincaré's uniformization theorem for surfaces mentioned above. (For his work in the field of topology and geometry in dimension 3, Thurston also received the Fields Medal).

Now, the Geometrization Conjecture is a far more general result than Poincaré Conjecture. In fact, Thurston's conjecture states, among other things, that among the 8 special geometries, the only one that a closed and simply connected 3-manifold may carry is that of constant curvature $+1$. And it is known that a closed and simply connected 3-manifold equipped with a metric of constant curvature $+1$ is topologically equivalent to a sphere. Thus, one can prove the Poincaré Conjecture also by solving Thurston's conjecture.

4.3. The Ricci flow

Consider a manifold equipped with a metric. Is it possible to find a process that modifies its geometry to make it as symmetric as possible?

The idea is to continuously deform the metric at each point p of the manifold so that the average curvature at point p decreases. This brings us to the work of R. Hamilton in the 1980s [2]. He introduced an equation (a non-linear partial differential equation) called the **Ricci Flow**, which turns out to be very useful [4].

On a 3-manifold, we can define a time-dependent metric, and, at each instant of time, we can associate to this metric a certain curvature, the so-called Ricci curvature, which corresponds to a sort of average of Riemann curvatures. The metric and the curvature, being two tensors of the same type, can be entered into an equation that dictates that the instantaneous rate of change of the metric corresponds to the opposite of the change of the Ricci curvature.

Imposing the Ricci flow equation means evolving the metric toward a more regular and symmetric geometry over time. In dimension 2, Hamilton proved that the Ricci flow for any metric in a surface evolves, in finite time, toward a metric of constant curvature.

In dimension 3, things are far more difficult, because the flow may "explode", making infinite quantities appear. Hamilton's abstract program, developed and completed by Perelman, consists just in proving that, as a consequence of these explosions, the manifold V^3 breaks into pieces on which the Ricci flow may continue to evolve, and that, after a finite amount of time and a finite number of explosions, one obtains the starting manifold decomposed into pieces, each endowed with one of Thurston's 8 geometries.

Perelman finally managed, with fine methods of non-linear analysis, to control the explosions of the Ricci flow, and to demonstrate that the whole process of the Ricci flow extinguishes in a finite time, thus completing Hamilton's strategy, and proving both the geometrization of Thurston for 3-manifolds, and the Poincaré Conjecture!

5. Conclusion

Mathematicians call "open problems" those on which they struggle unsuccessfully for a long time. But an open problem is not just a simple unsolved problem. In fact various new results are demonstrated by mathematicians every year, and numerous new questions arise also every year, but (almost) none of them receive such a designation. An open problem is a problem regarded as exceptional, noble, elusive, but whose comprehension is fundamental for the development of the research fields that surround it.

The Poincaré conjecture was the prototype of such a problem. It was really a venerable major question both in classical and modern mathematics. Thanks to it, mathematics has evolved in different branches: from the birth of algebraic topology to the deep and vast world of higher dimensional geometry and topology. And, at the end, with the help of fine and sophisticated analytic tools, to the growth of geometric analysis.

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