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A One-Parameter Class of Separable Solutions for An Age-Sex-Structured Population Model with an Infinite Range of Reproductive Ages, A Discrete Set of Offspring, and Maternal Care

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Abstract
A mathematical model based on a discrete newborn set is proposed to describe the evolution of a sex-age-structured population, taking into account the temporary pair of sexes, infinite ranges of reproductive age of sexes, and maternal care of offspring. Pair formation is modeled by a weighted harmonic mean type function. The model is based on the concept of density of families composed of mothers with their newborns. All individuals are divided into the pre-reproductive and reproductive age groups. Individuals of the pre-reproductive class are divided into the newborn and teenager groups. Newborns are under maternal care while the teenagers can live without maternal care but cannot mate. Females of the reproductive age group are divided into singles and those who care for their offspring. The model is composed of a coupled system of integro-partial differential equations. Sufficient conditions for the existence of a one-parameter class of separable solutions of this model are found in the case of stationary vital rates.

Keywords: Age-sex-structured population models, Population models with parental care, Two sex population models
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1. Introduction

The purpose of this work is to analyze a mathematical model for a spatially homogeneous population structured by age and sex taking into account temporary (only for the mating period) pairs of sexes, infinite reproductive age ranges of sexes, a discrete set of offspring, and maternal care of them.

In mathematical biology, the Sharpe-Lotka-McKendrick [1], Fredrickson [2], Hoppensteadt-Staroverov [3], [4], and Hadeler [5] models are well known. The first of them is usually used for the evolution description of the age-structured asexual populations. The model [2] describes two-sex age-structured populations with temporary pairs of sexes. The Hoppensteadt–Staroverov model and its Hadeler modification including a maturation period describe the evolution of age-structured two-sex populations with permanent pairs of sexes. The existence of the separable solutions to model [3], [4], [5] is studied in [6] and [7].

But, all these models do not address the child care phenomenon which is native to many species of mammals and birds. Birds and some species of mammals care for their offspring in pairs. In populations of some species of mammals and fishes only the mother cares for her offspring. Several models (see [8]–[12] and literature there) were proposed for description of child care in two-sex populations with temporary and permanent pairs of sexes. In the first case ([9]), only the mother cares for her offspring. In the second case ([8], [10]-[12]), both parents take care of offspring. Models [8], [10], and[11] are based on the idea of the newborn density which is described by a corresponding PDE. However, a problem arises when describing spatially distributed populations using models of this type, because the equations describing the movement of newborns do not guarantee that they follow the mother or both parents. To overcome this problem, some models have been proposed based on a discrete set of newborns and density of the family (mother-newborns [9] or both parents-newborns [12]). In addition to child care, work [9] takes into account the pregnancy of females. It is also assumed that the reproductive age intervals of males and females in model [9] are finite. To the best of our knowledge, there has been no work in the last decade that has examined the dynamics of the caregiver population.

In the present paper, we revise model [9] by dropping the Environmental pressure and female's pregnancy and contrary to model [9] assume that the age reproductive intervals of both parents are infinite. This is the novelty of the model under consideration. As in [9], all individuals are divided into pre-reproductive and reproductive age groups. Individuals of pre-reproductive class are divided into the newborn and teenager groups. Newborns are under maternal care while the teenagers can live without maternal care but cannot mate. Individuals of the reproductive age class are divided into singles and those who care for their offspring. The goal of this paper is to find sufficient conditions for the existence of separable solutions of the proposed model in the case of stationary vital rates.

The plan of this work is the following: In Section 2, the basic notions are given. In Section 3, we describe the model. Separable solutions are studied in section 4. Some concluding remarks in section 5 conclude the paper.

2. Notations

The following notions are used in this paper:

T, τ_{i*}: child care and maturation period, respectively ($i=1$ for males, $i=2$ for females);

$u_i(t,\tau_i)$: density at time t of individuals of age τ_i ($\tau_i \in (T,\tau_{i*})$ for juveniles, $\tau_i \in (\tau_{i*},\infty)$ for adult individuals, $i=1$ for males, $i=2$ for females);

$u_{2k_1k_2}(t,\tau_1,\tau_2,\tau_3)$: density at time t of females aged τ_2 who take care of k_1 sons and k_2 daughters of age τ_3, born from fathers of age τ_1;

$\nu_i(t,\tau_i)$: mortality at time t of individuals aged τ_i ($i=1$ for males, $i=2$ for females);

$\nu_{2k_1k_2}(t,\tau_1,\tau_2,\tau_3)$: mortality at moment t of mothers aged τ_2 caring for k_1 sons and k_2 daughters of age τ_3, born from fathers of age τ_1;

$\nu_{2k_1k_2;s_1s_2}(t,\tau_1,\tau_2,\tau_3)$: mortality at time t of k_1-s_1 sons and k_2-s_2 daughters of age τ_3, born from fathers of age τ_1 and who are under care of mothers aged τ_2;

$p_i(t,\tau_i)u_i(t,\tau_i)$; density of individuals of age τ_i who wish to mate at time t ($i=1$ for males, $i=2$ for females);

$u_i^0(\tau_i)$: initial density of individuals aged τ_i ($i=1$ for males, $i=2$ for females);

$u^0_{2k_1k_2}(\tau_1,\tau_2,\tau_3)$: initial density of females aged τ_2 who take care of k_1 sons and k_2 daughters aged τ_3;

$|k|=k_1+k_2$, $|s|=s_1+s_2$ with integer valued k_1,k_2,s_1,s_2 where $|k|,|s|=0,1,\dots,n$; $\sum\limits_{|k|=1}^{n} a_{k_1k_2} = \sum\limits_{k_1=0}^{n-1}\sum\limits_{k_2=1}^{n-k_1} a_{k_1k_2}$;

$p(t,\tau_1,\tau_2)\alpha_{2k_1k_2}(t,\tau_1,\tau_2)dt$: probability to produce k_1 sons and k_2 daughters in the time interval $[t,t+dt]$ by a temporal pair formed of a male aged $\tau_1$and female of age τ_2;

$p_{k_1k_2}=p_1\,p_2\,p\,\alpha_{2k_1k_2}$.

$[u_2(t,\tau)]$: jump discontinuity of function u_2 at line $\tau_2=\tau$.

3. The Model

In this section, we present a deterministic model to describe the evolution of a population structured by sex and age. We take into account temporary pairs of sexes, a discrete set of offspring, and maternal care for them. By temporary pairs, we mean pairs that exist during the mating period, duration of which is not taken into account. We use a weighted harmonic mean pair formation function and assume that when a mother dies all offspring under her care die. Using the balance law, we derive the following equations for the dynamic description of a population with a discrete set of offspring:

$$\begin{cases} \partial_t u_1 + \partial_{\tau_1} u_1 + \nu_1 u_1 = 0 \quad \text{in } (0,\infty)\times(T,\infty), \\ u_1|_{\tau_1=T} = \int\limits_{\tau_{1*}}^{\infty} d\tau_1 \int\limits_{\tau_{2*}+T}^{\infty} \sum\limits_{|k|=1}^{n} k_1 u_{2k_1k_2}|_{\tau_3=T}\, d\tau_2 \quad \text{in } [0,\infty), \\ u_1|_{t=0} = u_1^0 \quad \text{in } [T,\infty), \end{cases} \tag{3.1}$$

$$\begin{cases} \partial_t u_2 + \partial_{\tau_2} u_2 + \nu_2 u_2 = S_2^u, \\ u_2|_{\tau_2=T} = \int\limits_{\tau_{1*}}^{\infty} d\tau_1 \int\limits_{\tau_{2*}+T}^{\infty} \sum\limits_{|k|=1}^{n} k_2 u_{2k_1k_2}|_{\tau_3=T}\, d\tau_2 \quad \text{in } [0,\infty), \\ [u_2(t,\tau)] = 0 \quad \text{in } [0,\infty),\ \tau = \tau_{2*}, \tau_{2*}+T, \\ u_2|_{t=0} = u_2^0 \quad \text{in } [T,\infty) \end{cases} \tag{3.2}$$

where

$$S_2^u = \begin{cases} 0 \quad \text{in } (0,\infty)\times(T,\tau_{2*}), \\ \sum\limits_{|k|=0}^{n} \int\limits_{\tau_{1*}}^{\infty} d\tau_1 \left(\int\limits_0^{\tau_2-\tau_{2*}} \nu_{2k_1k_2;00} u_{2k_1k_2}\, d\tau_3 - u_{2k_1k_2}|_{\tau_3=0} \right) \quad \text{in } (0,\infty)\times(\tau_{2*},\tau_{2*}+T), \\ \sum\limits_{|k|=0}^{n} \int\limits_{\tau_{1*}}^{\infty} d\tau_1 \left(\int\limits_0^{T} \nu_{2k_1k_2;00} u_{2k_1k_2}\, d\tau_3 + u_{2k_1k_2}|_{\tau_3=T} - u_{2k_1k_2}|_{\tau_3=0} \right) \quad \text{in } (0,\infty)\times(\tau_{2*}+T,\infty), \end{cases}$$

$$\begin{cases} \partial_t u_{2k_1k_2} + \sum\limits_{j=1}^{2} \partial_{\tau_j} u_{2k_1k_2} + \Big(\nu_{2k_1k_2} + \sum\limits_{|s|=0}^{|k|-1} \nu_{2k_1k_2;s_1s_2}\Big) u_{2k_1k_2} = S_{2k_1k_2}^u \quad \text{in } (0,\infty)\times[\tau_{1*},\infty)\times(\tau_{2*}+\tau_3,\infty)\times(0,T), \\ u_{2k_1k_2}|_{\tau_3=0} = \dfrac{p_{k_1k_2} u_1 u_2}{\sum\limits_{j=1}^{2} \int\limits_{\tau_{j*}}^{\infty} p_j u_j\, d\tau_j} \quad \text{in } (0,\infty)\times[\tau_{1*},\infty)\times(\tau_{2*},\infty), \\ u_{2k_1k_2}|_{t=0} = u_{2k_1k_2}^0 \quad \text{in } [\tau_{1*},\infty)\times[\tau_{2*}+\tau_3,\infty)\times[0,T] \end{cases} \tag{3.3}$$

where

$$S_{2k_1k_2}^u = \begin{cases} 0, \ |k| = n, \\ \sum\limits_{|s|=|k|+1}^{n} \nu_{2s_1s_2;k_1k_2} u_{2s_1s_2}, \ |k| = n-1, n-2, \ldots, 1. \end{cases}$$

We add to this system the following compatibility conditions:

$$u_i^0|_{\tau_i=T} = \int\limits_{\tau_{1*}}^{\infty} d\tau_1 \int\limits_{\tau_{2*}+T}^{\infty} \sum_{|k|=1}^{n} k_i u_{2k_1k_2}^0|_{\tau_3=T}\, d\tau_2,\ i = 1,2,$$

$$u_{2k_1k_2}^0|_{\tau_3=0} = \frac{p_{2k_1k_2}|_{t=0} u_1^0 u_2^0}{\sum\limits_{j=1}^{2} p_j|_{t=0} u_j^0\, d\tau_j} \ \text{in } [\tau_{1*},\infty)\times[\tau_{2*},\infty).$$

4. Separable Solutions

In this section, we study system (3.1)–(3.3) with the vital rates p, p_1, p_2, ν_1, ν_2, $\nu_{2k_1k_2}$, $\nu_{2k_1k_2;s_1s_2}$, $\alpha_{2k_1k_2}$ independent of time t and look for solutions of the form

$$\begin{cases} u_i(t,\tau_i) = \exp\{\lambda t\} w_i(\tau_i), \\ w_i(\tau_i) = a_i v_i(\tau_i),\ v_i(T) = 1, \quad i = 1,2, \\ u_{2k_1k_2} = \exp\{\lambda t\} w_{2k_1k_2}, \\ w_{2k_1k_2} = a_1 a_2 e^{-\lambda\tau_3} v_1(\tau_1) v_2(\tau_2-\tau_3) v_{2k_1k_2}(\tau_1,\tau_2,\tau_3)/\alpha, \quad |k| = 1,2,\ldots,n, \\ \quad \alpha = a_1 \int_{\tau_{1*}}^{\infty} p_1 v_1\, d\tau_1 + a_2 \int_{\tau_{2*}}^{\infty} p_2 v_2\, d\tau_2 \end{cases} \tag{4.1}$$

where constants λ, $a_1 = w_1(T)$, $a_2 = w_2(T)$, and functions v_i, $v_{2k_1k_2}$ are to be determined. Set:

$$y_i = a_i/\alpha,\ ||v_i|| = \int_{\tau_{i*}}^{\infty} v_i\,d\tau_i,\ i=1,2,\ \gamma = \sum_{|k|=1}^{n} p_{2k_1k_2},\ \ P = \int_{\tau_{1*}}^{\infty} \gamma v_1\,d\tau_1,$$

$$l_2 = \nu_2 + \lambda + y_1 P,\ l_{2k_1k_2} = \nu_{2k_1k_2} + \sum_{|s|=0}^{|k|-1} \nu_{2k_1k_2;s_1s_2},\ |k| = 1,\dots,n,$$

$$r = \sum_{|k|=1}^{n} v_{2k_1k_2}|_{\tau_3=T},\ R = \int_{\tau_{1*}}^{\infty} r v_1\,d\tau_1,\ q = \sum_{|k|=1}^{n} \nu_{2k_1k_2;00}\, v_{2k_1k_2},\ Q = \int_{\tau_{1*}}^{\infty} q v_1\,d\tau_1,$$

$$\beta_i(x) = \int_{\tau_{1*}}^{\infty} v_1(\tau_1) \sum_{|k|=1}^{n} k_i v_{2k_1k_2}(\tau_1, x+T, T)\,d\tau_1,\ i=1,2.$$

Substituting functions (4.1) into system (3.1)–(3.3) and performing calculations, we get the following equations:

$$\begin{cases} v_1' + (\nu_1+\lambda)v_1 = 0 \text{ in } (T,\infty),\ v_1(T)=1, \\ 1 = y_2 e^{-\lambda T} \int_{\tau_{2*}}^{\infty} v_2(x)\beta_1(x)\,dx, \end{cases} \tag{4.2}$$

$$\begin{cases} v_2' + (\nu_2+\lambda)v_2 = S_2^v,\ v_2(T)=1,\ [u_2(\tau)] = 0,\ \tau = \tau_{2*}, \tau_{2*}+T, \\ 1 = y_1 e^{-\lambda T} \int_{\tau_{2*}}^{\infty} v_2(x)\beta_2(x)\,dx, \end{cases} \tag{4.3}$$

where

$$S_2^v = \begin{cases} 0 \quad \text{in } (T,\tau_{2*}), \\ y_1\left(\int_0^{\tau_2-\tau_{2*}} v_2(\tau_2-\tau_3)Q(\tau_2,\tau_3)e^{-\lambda\tau_3}\,d\tau_3 - v_2(\tau_2)P(\tau_2)\right) \quad \text{in } (\tau_{2*}, \tau_{2*}+T), \\ y_1\left(\int_0^{T} v_2(\tau_2-\tau_3)Q(\tau_2,\tau_3)e^{-\lambda\tau_3}\,d\tau_3 + e^{-\lambda T}v_2(\tau_2-T)R(\tau_2) - v_2(\tau_2)P(\tau_2)\right) \quad \text{in } (\tau_{2*}+T,\infty), \end{cases} \tag{4.4}$$

$$\begin{cases} \sum_{j=2}^{3} \partial_{\tau_j} v_{2k_1k_2} + \left(\nu_{2k_1k_2} + \sum_{|s|=0}^{|k|-1} \nu_{2k_1k_2;s_1s_2}\right) v_{2k_1k_2} = S_{2k_1k_2}^v \quad \text{in } [\tau_{1*},\infty)\times(\tau_{2*}+\tau_3,\infty)\times(0,T) \\ v_{2k_1k_2}|_{\tau_3=0} = p_{2k_1k_2} \quad \text{in } [\tau_{1*},\infty)\times[\tau_{2*},\infty), \end{cases} \tag{4.5}$$

where

$$S_{2k_1k_2}^v = \begin{cases} 0,\ |k| = n, \\ \sum_{|s|=|k|+1}^{n} \nu_{2s_1s_2;k_1k_2} v_{2s_1s_2},\ |k| = n-1, n-2, \dots, 1. \end{cases}$$

We also have the equation for λ,

$$y_1||p_1v_1|| + y_2||p_2v_2|| = 1. \tag{4.6}$$

Integrating Eqs. $(4.2)_1$ and $(4.3)_1$, we get $w_i(\tau_i) = w_i(T)v_i(\tau_i)$ where

$$v_1(\tau_1) = \exp\left\{ -\int_T^{\tau_1} (\nu_1+\lambda)\,ds \right\} \text{ in } [T,\infty),$$

$$v_2(\tau_2) = \exp\left\{ -\int_T^{\tau_2} (\nu_2+\lambda)\,ds \right\} \text{ in } [T,\tau_{2*}].$$

Now we transform Eq. (4.3) with a given positive y_1 into a set of Volterra's type integral equations. To do this, we change variables on the right hand side of Eq. (4.4), then integrate Eq. $(4.3)_1$, and after then change the order of integration. As a result, we have

$$v_2(\tau_2) = f(\tau_2) + \int_{\tau_{2*}+jT}^{\tau_2} G(\tau_2,y)v_2(y)\,dy \text{ in } [\tau_{2*}+jT, \tau_{2*}+(j+1)T] \tag{4.7}$$

with $j = 0,1,2,\ldots$, where

$$G(\tau_2,y) = y_1 \int_y^{\tau_2} \exp\Big\{-\int_z^{\tau_2} l_2(s)\,ds - \lambda(z-y)\Big\} Q(z,z-y)\,dz,$$

$$f(\tau_2) = \begin{cases} v_2(\tau_{2*})\exp\Big\{-\int_{\tau_{2*}}^{\tau_2} l_2\,ds\Big\} \text{ in } [\tau_{2*},\tau_{2*}+T], \\ v_2(\tau_{2*}+jT)\exp\Big\{-\int_{\tau_{2*}+jT}^{\tau_2} l_2\,ds\Big\} + y_1\int_{\tau_{2*}+jT}^{\tau_2} \exp\Big\{-\int_z^{\tau_2} l_2\,ds\Big\}dz\Big(\int_{z-T}^{\tau_{2*}+jT} v_2(y)Q(z,z-y)e^{-\lambda(z-y)}\,dy \\ \quad +v_2(z-T)R(z)e^{-\lambda T}\Big) \quad \text{in } [\tau_{2*}+jT, \tau_{2*}+(j+1)T], \end{cases}$$

with $j = 1,2,\ldots$ and

$$v_2(\tau_{2*}) = \exp\Big\{-\int_T^{\tau_{2*}} (\nu_2+\lambda)\,ds\Big\}.$$

Define:

$\nu_{i*} = \inf_{[\tau_{i*},\infty)} \nu_i,\ \nu_i^* = \sup_{[\tau_{i*},\infty)} \nu_i,\ p_{i*} = \inf_{[\tau_{i*},\infty)} p_i, p_i^* = \sup_{[\tau_{i*},\infty)} p_i,$
$\nu_{2k_1k_2*} = \inf_{[\tau_{1*},\infty)\times[\tau_{2*}+\tau_3,\infty)\times[0,T]} \nu_{2k_1k_2},\ p_* = \inf_{[\tau_1*,\infty)\times[\tau_{2*},\infty)} p,$
$\nu_{2k_1k_2}^* = \sup_{[\tau_{1*},\infty)\times[\tau_{2*}+\tau_3,\infty)\times[0,T]} \nu_{2k_1k_2},\ p_* = \sup_{[\tau_1*,\infty)\times[\tau_{2*},\infty)} p,$
$\nu_{2k_1k_2;s_1s_2*} = \inf_{[\tau_{1*},\infty)\times[\tau_{2*}+\tau_3,\infty)\times[0,T]} \nu_{2k_1k_2;s_1s_2},$
$\nu_{2k_1k_2;s_1s_2*} = \sup_{[\tau_{1*},\infty)\times[\tau_{2*}+\tau_3,\infty)\times[0,T]} \nu_{2k_1k_2;s_1s_2},$
$\alpha_{2k_1k_2*} = \inf_{[\tau_{1*},\infty)\times[\tau_2*,\infty)} \alpha_{2k_1k_2},\ \alpha_{2k_1k_2}^* = \sup_{[\tau_{1*},\infty)\times[\tau_2*,\infty)} \alpha_{2k_1k_2},$
$l_{2k_1k_2*} = \nu_{2k_1k_2*} + \sum_{s=0}^{|k|-1} \nu_{2k_1k_2;s_1s_2*},\ l_{2k_1k_2}^* = \nu_{2k_1k_2}^* + \sum_{s=0}^{|k|-1} \nu_{2k_1k_2;s_1s_2}^*,$
$\gamma^* = \sum_{|k|=1}^n p_{2k_1k_2}^*,\ \gamma_* = \sum_{|k|=1}^n p_{2k_1k_2*},\ \ \nu_* = \min(\nu_{1*},\nu_{2*}),$
$l_{2*} = \nu_{2*}+\lambda+\gamma_*||v_1||,\ l_2^* = \nu_2^*+\lambda+\gamma^*||v_1||.$

Consider two functions:

$$\overline{v}_{2k_1k_2}(\tau_3) = \begin{cases} p_{2k_1k_2}^*\exp\{-l_{2k_1k_2*}\tau_3\}, \quad |k| = n, \\ p_{2k_1k_2}^*\exp\{-l_{2k_1k_2*}\tau_3\} + \int_0^{\tau_3}\exp\{-(\tau_3-z)l_{2k_1k_2*}\}\sum_{|s|=|k|+1}\nu_{2s_1s_2;k_1k_2}^*\overline{v}_{2k_1k_2}(z)\,dz, \quad |k| = n-1,n-2,\ldots,1, \end{cases} \tag{4.8}$$

and

$$\underline{v}_{2k_1k_2}(\tau_3) = \begin{cases} p_{2k_1k_2*}\exp\{-l_{2k_1k_2}^*\tau_3\}, \quad |k| = n, \\ p_{2k_1k_2*}\exp\{-l_{2k_1k_2}^*\tau_3\} + \int_0^{\tau_3}\exp\{-(\tau_3-z)l_{2k_1k_2}^*\}\sum_{|s|=|k|+1}\nu_{2s_1s_2;k_1k_2*}\underline{v}_{2k_1k_2}(z)\,dz, \quad |k| = n-1,n-2,\ldots,1, \end{cases}$$

(4.9)

in $[0,T]$. Functions $\bar{v}_{2k_1k_2}$ and $\underline{v}_{2k_1k_2}$ for $|k|=n-1,n-2,\dots,1$ can be found recurrently starting from $|k|=n-1$ since $\bar{v}_{2k_1k_2}$ and $\underline{v}_{2k_1k_2}$ for $|k|=n$ are known.

Lemma 4.1. *Let $\nu_*+\lambda>0$ be a given positive constant. Assume that functions $\nu_{2k_1k_2}$ and $\nu_{2k_1k_2;s_1s_2}$ lie in $C^{0,1,1}([\tau_{1*},\infty)\times(\tau_{2*}+\tau_3,\infty)\times(0,T))\cap C([\tau_{1*},\infty)\times[\tau_{2*}+\tau_3,\infty)\times[0,T])$, $\alpha_{2k_1k_2}\in C^{0,1}([\tau_{1*},\infty)\times(\tau_{2*},\infty))\cap C([\tau_{1*},\infty)\times]\tau_{2*},\infty))$, $p_{2k_1k_2}\in C^{0,1}([\tau_{1*},\infty)\times(\tau_{2*},\infty))\cap C([\tau_{1*},\infty)\times[\tau_{2*},\infty))$ and let they be nonnegative bounded functions in domains of their definition. Then problem (4.5) has a unique nonnegative solution $v_{2k_1k_2}\in C^{0,1,1}([\tau_{1*},\infty)\times(\tau_{2*}+\tau_3,\infty)\times(0,T))\cap C([\tau_{1*},\infty)\times[\tau_{2*}+\tau_3,\infty)\times[0,T])$ such that $\underline{v}_{2k_1k_2}\le v_{2k_1k_2}\le\bar{v}_{2k_1k_2}$ in $[\tau_{1*},\infty)\times[\tau_{2*}+\tau_3,\infty)\times[0,T]$ where $\bar{v}_{2k_1k_2}$ and $\underline{v}_{2k_1k_2}\in C^1([0,T])$ are determined by formulas (4.8) and (4.9), respectively.*

Proof. Conditions of this lemma let us to solve linear equation (4.5) to have

$$v_{2k_1k_2}(\tau_1,\tau_2,\tau_3)=\begin{cases} p_{2k_1k_2}(\tau_1,\tau_{23})\exp\left\{-\int\limits_0^{\tau_3} l_{2k_1k_2}(\tau_1,s+\tau_{23},s)\,ds\right\}, \quad |k|=n,\\ p_{2k_1k_2}(\tau_1,\tau_{23})\exp\left\{-\int\limits_0^{\tau_3} l_{2k_1k_2}(\tau_1,s+\tau_{23},s)\,ds\right\}\\ \quad+\int\limits_0^{\tau_3}\exp\left\{-\int\limits_z^{\tau_3} l_{2k_1k_2}(\tau_1,s+\tau_{23},s)\,ds\right\}\sum\limits_{|s|=|k|+1}\nu_{2s_1s_2;k_1k_2}(\tau_1,z+\tau_{22},z)v_{2k_1k_2}(\tau_1,z+\tau_{23},z)\,dz,\\ \quad |k|=n-1,n-2,\dots,1,\end{cases}\tag{4.10}$$

in $[\tau_{1*},\infty)\times[\tau_{2*}+\tau_3,\infty)\times[0,T]$, where $\tau_{23}=\tau_2-\tau_3$.

Function (4.10) for $|k|=n-1,n-2,\dots,1$ can be found recurrently starting from $|k|=n-1$ since $v_{2k_1k_2}$ for $|k|=n$ is known. Note that function $v_{2k_1k_2}$ is independent of parameters y_1 and λ. Direct comparison of Eq. (4.8) with (4.10) and Eq. (4.9) with (4.10) proves the inequality $\underline{v}_{2k_1k_2}\le v_{2k_1k_2}\le\bar{v}_{2k_1k_2}$ in $[\tau_{1*},\infty)\times[\tau_{2*}+\tau_3,\infty)\times[0,T]$. Differentiability of $\bar{v}_{2k_1k_2}$ and $\underline{v}_{2k_1k_2}$ in $[0,T]$ follows from Eqs.(4.8) with (4.9). □

Let

$q_*=\sum\limits_{k|=1}^{n}\nu_{2k_1k_2,00*}\min\limits_{[0,T]}\underline{v}_{2k_1k_2},\ q^*=\sum\limits_{k|=1}^{n}\nu^*_{2k_1k_2,00}\max\limits_{[0,T]}\bar{v}_{2k_1k_2},$

$r^*=\sum\limits_{|k|=1}^{n}\bar{v}_{2k_1k_2}(T),\ r_*=\sum\limits_{|k|=1}^{n}\underline{v}_{2k_1k_2}(T).$

Then $\gamma_*||v_1||\le P\le\gamma^*||v_1||,\ q_*||v_1||\le Q\le q^*||v_1||,\ r_*||v_1||\le R\le r^*||v_1||$.

Consider two following systems:

$$\bar{v}_2'+l_{2*}\bar{v}_2=\begin{cases} y_1||v_1||\int\limits_0^{\tau_2-\tau_{2*}}\bar{v}_2(\tau_2-\tau_3)e^{-\lambda\tau_3}\,d\tau_3\,q^* \quad \text{in } (\tau_{2*},\tau_{2*}+T), \quad \bar{v}_2(\tau_{2*})=\exp\{-(\tau_{2*}-T)(\nu_{2*}+\lambda)\},\\ y_1||v_1||\left(\int\limits_0^T\bar{v}_2(\tau_2-\tau_3)e^{-\lambda\tau_3}\,d\tau_3\,q^*+\bar{v}_2(\tau_2-T)e^{-\lambda T}r^*\right) \quad \text{in } (\tau_{2*}+T,\infty), \quad [\bar{v}_2(\tau_{2*}+T)]=0\end{cases}\tag{4.11}$$

and

$$\underline{v}_2'+l_{2*}\underline{v}_2=\begin{cases} y_1||v_1||\int\limits_0^{\tau_2-\tau_{2*}}\underline{v}_2(\tau_2-\tau_3)e^{-\lambda\tau_3}\,d\tau_3\,q_* \quad \text{in } (\tau_{2*},\tau_{2*}+T), \quad \underline{v}_2(\tau_{2*})=\exp\{-(\tau_{2*}-T)(\nu_2^*+\lambda)\},\\ y_1||v_1||\left(\int\limits_0^T\underline{v}_2(\tau_2-\tau_3)e^{-\lambda\tau_3}\,d\tau_3\,q_*+\underline{v}_2(\tau_2-T)e^{-\lambda T}r_*\right) \quad \text{in } (\tau_{2*}+T,\infty), \quad [\underline{v}_2(\tau_{2*}+T)]=0.\end{cases}\tag{4.12}$$

Applying the argument used to construct Eq. (4.7), Eqs. (4.11) and (4.12) on each interval $[\tau_{2*}+jT,\tau_{2*}+(j+1)T]$, $j=0,1,\dots$, can be transformed to Volterra integral equations having unique positive solutions.

Lemma 4.2. *Assume that function $\nu_2\in C([\tau_{2*},\infty))$ and parameter y_1 are positive and let conditions of Lemma 4.1 be fulfilled. Then Eq. (4.7) has a unique positive solution $v_2\in C^1((\tau_{2*},\infty))\cap C([\tau_{1*},\infty))$. Moreover, $\underline{v}_2\le v_2\le\bar{v}_2$ in $[\tau_{2*},\infty)$ where $\bar{v}_2$ and $\underline{v}_2$ are unique positive solutions of Eqs. (4.11) and (4.12), respectively.*

Proof. The proof of the existence and uniqueness of the solution is based on the existence and uniqueness theorem of the Volterra linear integral equation. It remains to prove the inequality $\underline{v}_2 \le v_2 \le \overline{v}_2$. Set $Z = \overline{v}_2 - v_2$. Subtracting Eq. (4.3)$_1$ from Eq. (4.11) we get the equation

$$\begin{cases} Z' + l_{2*}Z = y_1 \int\limits_0^{\tau_2-\tau_{2*}} Z(\tau_2-\tau_3)\,d\tau_3 Q(\tau_2,\tau_3)e^{-\lambda\tau_3}\,d\tau_1 + f(\tau_2) \quad \text{in } (\tau_{2*},\tau_{2*}+T), \\ Z(\tau_{2*}) = \exp\Big\{-\int\limits_T^{\tau_{2*}}(\nu_{2*}+\lambda)\,ds\Big\} - \exp\Big\{-\int\limits_T^{\tau_{2*}}(\nu_2+\lambda)\,ds\Big\} \end{cases}$$

with a known nonnegative term

$$f(\tau_2) = (l_2 - l_{2*})v_2 + y_1 \int\limits_0^{\tau_2-\tau_{2*}} v_2(\tau_2-\tau_3)\Big(q^*||v_1|| - Q(\tau_2,\tau_3)\Big)e^{-\lambda\tau_3}\,d\tau_3.$$

This equation can be easily transformed into the Volterra integral equation with a nonnegative kernel and nonnegative known term. Hence it has a unique nonnegative solution $\overline{v}_2 - v_2$ in $[\tau_{2*},\tau_{2*}+T]$ and therefore $v_2 \le \overline{v}_2$. Similarly, we prove that $\underline{v}_2 \le v_2$ in $[\tau_{2*},\tau_{2*}+T]$. Subtracting (4.12) from Eq. (4.3)$_1$ and arguing similarly as above, we prove the inequality $\underline{v}_2 \le v_2 \le \overline{v}_2$ in $[\tau_{2*}+T,\infty)$. □

It is well known that a solution to the linear Volterra integral equation with a parameter that has a continuous kernel and a continuous known term with respect to the (argument, parameter) variable is also continuous with respect to the same variable. Hence, functions v_2, $\overline{v}_2$, and $\underline{v}_2$ are continuous with respect to (τ_2, y_1, λ).

Now we prove that $||v_2||$ is continuous with respect to parameters y_1 and λ. We integrate Eq. (4.3)$_1$ to have

$$\overline{v}_2(\tau_2) = \overline{v}_2(\tau_{2*})\exp\Big\{-\int\limits_{\tau_{2*}}^{\tau_2} l_{2*}\,ds\Big\} + \begin{cases} I_1 \text{ in } (\tau_{2*},\tau_{2*}+T), \\ I_2+I_3+I_4 \text{ in } (\tau_{2*}+T,\infty) \end{cases} \tag{4.13}$$

where

$$I_1 = y_1 \int\limits_{\tau_{2*}}^{\tau_2} \exp\Big\{-\int\limits_z^{\tau_2} l_{2*}\,ds\Big\}dz \int\limits_0^{z-\tau_{2*}} v_2(z-\tau_3)e^{-\lambda\tau_3}\,d\tau_3\,q^*||v_1|| = y_1 \int\limits_0^{\tau_2-\tau_{2*}} e^{-\lambda\tau_3}\,d\tau_3 \int\limits_{\tau_3+\tau_{2*}}^{\tau_2} v_2(z-\tau_3)\exp\Big\{-\int\limits_z^{\tau_2} l_{2*}\,ds\Big\}dz\,q^*||v_1||,$$

$$I_2 = y_1 \int\limits_{\tau_{2*}}^{\tau_{2*}+T} \exp\Big\{-\int\limits_z^{\tau_2} l_{2*}\,ds\Big\}dz \int\limits_0^{z-\tau_{2*}} v_2(z-\tau_3)e^{-\lambda\tau_3}\,d\tau_3\,q^*||v_1|| = y_1 \int\limits_0^{T} e^{-\lambda\tau_3}\,d\tau_3 \int\limits_{\tau_3+\tau_{2*}}^{\tau_{2*}+T} v_2(z-\tau_3)\exp\Big\{-\int\limits_z^{\tau_2} l_{2*}\,ds\Big\}dz\,q^*||v_1||,$$

$$I_3 = y_1 \int\limits_{\tau_{2*}+T}^{\tau_2} \exp\Big\{-\int\limits_z^{\tau_2} l_{2*}\,ds\Big\}dz \int\limits_0^{T} v_2(z-\tau_3)e^{-\lambda\tau_3}\,d\tau_3\,q^*||v_1|| = y_1 \int\limits_0^{T} e^{-\lambda\tau_3}\,d\tau_3 \int\limits_{T+\tau_{2*}}^{\tau_2} v_2(z-\tau_3)\exp\Big\{-\int\limits_z^{\tau_2} l_{2*}\,ds\Big\}dz\,q^*||v_1||,$$

$$I_4 = y_1 \int\limits_{\tau_{2*}+T}^{\tau_2} \exp\Big\{-\int\limits_z^{\tau_2} l_{2*}\,ds\Big\}e^{-\lambda T}v_2(z-T)\,dz\,r^*||v_1||.$$

Observe that

$$I_2 + I_3 = y_1 \int\limits_0^{T} e^{-\lambda\tau_3}d\tau_3 \int\limits_{\tau_3+\tau_{2*}}^{\tau_2} v_2(z-\tau_3)\exp\Big\{-\int\limits_z^{\tau_2} l_{2*}\,ds\Big\}dz\,q^*||v_1||.$$

Integrating Eq. (4.13) we find

$$||\overline{v}_2|| = \int\limits_{\tau_{2*}}^{\infty} \overline{v}_2(\tau_{2*})\exp\Big\{-\int\limits_{\tau_{2*}}^{\tau_2} l_{2*}\,ds\Big\}d\tau_2||v_1|| + J_1 + J_2 + J_3,$$

where

$$J_1 = y_1 \int_{\tau_{28}}^{\tau_{2*}+T} d\tau_2 \int_0^{\tau_2-\tau_{2*}} e^{-\lambda\tau_3}\,d\tau_3 \int_{\tau_3+\tau_{2*}}^{\tau_2} \overline{v}_2(z-\tau_3)\exp\Big\{-\int_z^{\tau_2} l_{2*}\,ds\Big\}\,dz\,q^*||v_1||,$$

$$J_2 = y_1 \int_{\tau_{2*}+T}^{\infty} d\tau_2 \int_0^{T} e^{-\lambda\tau_3}d\tau_3 \int_{\tau_3+\tau_{2*}}^{\tau_2} \overline{v}_2(z-\tau_3)\exp\Big\{-\int_z^{\tau_2} l_{2*}\,ds\Big\}e^{-\lambda\tau_3}\,dz\,q^*||v_1||,$$

$$J_3 = y_1 \int_{\tau_{2*}+T}^{\infty} d\tau_2 \int_{\tau_{2*}+T}^{\tau_2} \exp\Big\{-\int_z^{\tau_2} l_{2*}\,ds\Big\}e^{-\lambda T}\overline{v}_2(z-T)\,dz\,r^*||v_1||.$$

Changing the order of integration we have

$$J_1 = y_1 \int_0^{T} e^{-\lambda\tau_3}\,d\tau_3 \int_{\tau_{2*}+\tau_3}^{\tau_{2*}+T} d\tau_2 \int_{\tau_3+\tau_{2*}}^{\tau_2} \overline{v}_2(z-\tau_3)\exp\Big\{-\int_z^{\tau_2} l_{2*}\,ds\Big\}\,dz\,q^*||v_1||,$$

$$J_2 = y_1 \int_0^{T} e^{-\lambda\tau_3}\,d\tau_3 \int_{\tau_{2*}+T}^{\infty} d\tau_2 \int_{\tau_3+\tau_{2*}}^{\tau_2} \overline{v}_2(z-\tau_3)\exp\Big\{-\int_z^{\tau_2} l_{2*}\,ds\Big\}e^{-\lambda\tau_3}\,dz\,q^*||v_1||,$$

$$J_3 = y_1 e^{-\lambda T} \int_{\tau_{2*}}^{\infty} \overline{v}_2(x)\,dx \int_{x+T}^{\infty} \exp\Big\{-\int_{x+T}^{\tau_2} l_{2*}\,ds\Big\}\,d\tau_2\, r^*||v_1||$$

and then

$$J_1+J_2 = y_1 \int_0^{T} e^{-\lambda\tau_3}\,d\tau_3 \int_{\tau_{2*}+\tau_3}^{\infty} d\tau_2 \int_{\tau_3+\tau_{2*}}^{\tau_2} \overline{v}_2(z-\tau_3)\exp\Big\{-\int_z^{\tau_2} l_2^{\lambda}\,ds\Big\}\,dz\,q^*||v_1||.$$

Thus

$$\begin{cases} ||\overline{v}_2|| = \overline{v}_2(\tau_{2*}) \int_{\tau_{2*}}^{\infty} \exp\Big\{-\int_{\tau_{2*}}^{\tau_2} l_{2*}\,ds\Big\}d\tau_2 + y_1\int_0^{T} e^{-\lambda\tau_3}\,d\tau_3 \int_{\tau_{2*}+\tau_3}^{\infty} d\tau_2 \int_{\tau_3+\tau_{2*}}^{\tau_2} \overline{v}_2(z-\tau_3)\exp\Big\{-\int_z^{\tau_2} l_{2*}\,ds\Big\}\,dz\,q^*||v_1|| \\ \quad +y_1 e^{-\lambda T}\int_{\tau_{2*}}^{\infty}\overline{v}_2(x)\,dx\int_{x+T}^{\infty}\exp\Big\{-\int_{x+T}^{\tau_2} l_{2*}\,ds\Big\}\,d\tau_2\, r^*||v_1||. \end{cases} \tag{4.14}$$

and after changing the order of integration

$$\begin{cases} ||\overline{v}_2|| = \overline{v}_2(\tau_{2*}) \int_{\tau_{2*}}^{\infty} \exp\Big\{-\int_{\tau_{2*}}^{\tau_2} l_{2*}\,ds\Big\}d\tau_2 + y_1\int_0^{T} e^{-\lambda\tau_3}\,d\tau_3 \int_{\tau_{2*}+\tau_3}^{\infty} \overline{v}_2(z-\tau_3)\,dz\int_z^{\infty}\exp\Big\{-\int_z^{\tau_2} l_{2*}\,ds\Big\}\,d\tau_2\,q^*||v_1|| \\ \quad +y_1 e^{-\lambda T}\int_{\tau_{2*}}^{\infty}\overline{v}_2(x)\,dx\int_{x+T}^{\infty}\exp\Big\{-\int_{x+T}^{\tau_2} l_{2*}\,ds\Big\}\,d\tau_2\, r^*||v_1|| \end{cases} \tag{4.15}$$

If $\nu_* + \lambda > 0$, then

$$||\overline{v}_2||l_{2*} = \exp\Big\{-(\tau_{2*}-T)(\nu_{2*}+\lambda)\Big\} + \Big(\int_0^{T} e^{-\lambda\tau_3}\,d\tau_3 q^* + e^{-\lambda T}r^*\Big)||\overline{v}_2||||v_1||y_1.$$

Hence

$$||\overline{v}_2|| := \omega_r(y_1,\lambda) = \frac{\exp\{-(\tau_{2*}-T)(\nu_{2*}+\lambda)\}}{A(y_1,\lambda)} \tag{4.16}$$

provided that

$$A(y_1,\lambda) := \nu_{1*} + \lambda + y_1||v_1||N(\lambda) > 0 \tag{4.17}$$

where

$$N(\lambda) := \gamma_* - \int_0^T e^{-\lambda \tau_3} d\tau_3\, q^* - e^{-\lambda T}\, r^*.$$

Observe that condition (4.17) is fulfilled if $N(\lambda) \geq 0$ and $\nu_* + \lambda \geq \varepsilon$, $\varepsilon > 0$. Eq. (4.16) under condition (4.17) shows that $||\bar{v}_2||$ is continuous in (y_1, λ). This, the positivity and continuity of $\bar{v}_2$ with respect to (τ_2, y_1, λ) show that $||\bar{v}_2||$ converges uniformly with respect to $\lambda \in [-\nu_* + \varepsilon, \lambda']$ and $y_1 \in [0, y_1']$ where $\lambda' < \infty$, $y_1' < \infty$. Then Lemma 4.2 shows that $||v_2||$ converges uniformly too and the continuity of v_2 with respect to (τ_2, y_1, λ) proves the continuity of $||v_2||$ with respect to $\lambda \in [-\nu_* + \varepsilon, \lambda']$ and $y_1 \in [0, y_1']$.

Define

$$\tilde{q}_1(y_1, \lambda) = e^{-\lambda T} \int\limits_{\tau_{2*}}^{\infty} v_2(x) \beta_2(x)\, dx,$$

$$\tilde{q}_2(y_1, \lambda) = e^{-\lambda T} \int\limits_{\tau_{2*}}^{\infty} v_2(x) \beta_1(x)\, dx.$$

Using Lemma 4.1, we can prove that functions $\tilde{q}_1(y_1, \lambda)$ and $\tilde{q}_2(y_1, \lambda)$ are continuous in $\lambda \geq -\nu_* + \varepsilon$ and $y_1 > 0$. Eqs. $(4.2)_2$ and $(4.3)_2$ can be rewritten as follows:

$$\begin{cases} y_1 - \dfrac{1}{\tilde{q}_1(y_1, \lambda)} = 0, \\ y_2 - \dfrac{1}{\tilde{q}_2(y_1, \lambda)} = 0. \end{cases} \tag{4.18}$$

Function

$$z(y_1, \lambda) = y_1 - \frac{1}{\tilde{q}_1(y_1, \lambda)}$$

is continuous with respect to (y_1, λ). Obviously, $z|_{y_1=0} < 0$. Eq. $(4.3)_1$ shows that

$$v_2(\tau_2) \geq \tilde{v}_2(\tau_2) := \exp\Big\{ -\int\limits_T^{\tau_2} (\nu_2(\tau_2) + \lambda)\, d\tau_2 - (\tau_2 - T) p_2^* p^* \sum_{|k|=1}^{n} \alpha^*_{2k_1k_2}\, d\tau_2 \Big\}$$

for all $y_1 \geq 0$. Define:

$$\widehat{q}_1(\lambda) = e^{-\lambda T} \int\limits_{\tau_{2*}}^{\infty} \tilde{v}_2(x) \beta_2(x)\, dx,\ \widehat{q}_2(\lambda) = e^{-\lambda T} \int\limits_{\tau_{2*}}^{\infty} \tilde{v}_2(x) \beta_1(x)\, dx.$$

Then by definition $\tilde{q}_1(y_1, \lambda) > \widehat{q}_1(\lambda)$ for all $y_1 \geq 0$. Hence, $z|_{y_1 = 1/\widehat{q}_1(\lambda)} > 0$. The continuity of z shows that function $z(y_1, \lambda)$ has at least one positive root $\overline{y}_1(\lambda) \in (0, 1/\widehat{q}_1(\lambda))$, which is continuous in λ. Then Eq. $(4.18)_2$ shows that $y_2(\overline{y}_1(\lambda), \lambda)$, is also continuous with respect to $\lambda \geq -\nu_* + \varepsilon$ with small $\varepsilon > 0$.

Now we find constant λ. Set:

$$\overline{h}_i = \sum_{||k||=1}^{n} k_i \overline{v}_{2k_1k_2}(T), \quad \underline{h}_i = \sum_{|k|=1}^{n} k_i \underline{v}_{2k_1k_2}(T), \quad i = 1, 2, \quad B(\lambda) = \overline{y}_1(\lambda)||p_1 v_1|| + y_2(\overline{y}_1(\lambda), \lambda)||p_2 v_2||.$$

It is evident that

$$\widehat{q}_i(\lambda) \geq \underline{h}_i\, e^{-\lambda T} ||\tilde{v}_2|| ||v_1||,\ i = 1, 2.$$

Then using Eqs. (4.6), (4.16), and (4.18), we get

$$\overline{y}_1(\lambda)||p_1 v_1|| \leq \frac{p_1^*}{\widehat{q}_2}||v_1|| \leq \frac{p_1^* e^{\lambda T}}{\underline{h}_2 ||\tilde{v}_2||},\ \overline{y}_2(\lambda)||p_2 v_2|| \geq \frac{p_2^* e^{\lambda T}}{\overline{h}_2 ||v_1||},$$

Hence

$$H_l(\lambda) := \frac{p_{2*} e^{\lambda T}}{\overline{h}_2 ||v_1||} \leq B(\lambda) \leq H_r(\lambda) := \frac{p_1^* e^{\lambda T}}{\underline{h}_2 ||\tilde{v}_2||} + \frac{p_2^* e^{\lambda T}}{\underline{h}_1 ||v_1||} \tag{4.19}$$

provided that condition (4.17) with $y_1 = 1/\widehat{q}_1(\lambda)$ (i.e., $A(1/\widehat{q}_1(\lambda), \lambda) > 0$) is satisfied. Analysis of inequalities (4.19) allows us to formulate the following assertion:

Lemma 4.3. *Let conditions of Lemmas 4.1 and 4.2 be satisfied. Assume that* $\lambda_0 \geq -\nu_* + \varepsilon$ *with a small* $\varepsilon > 0$ *and* $\lambda_1 > \lambda_0$ *are such that* $H_r(\lambda_0) < 1$*,* $H_l(\lambda_1) > 1$*, and* $N(\lambda_0) \geq 0$*. Then function* $B(\lambda) - 1$ *has at least one real root* λ_2*.*

The proof of lemma is obvious, since H_l, H_r, and N are monotonous functions of λ.

Based on Lemmas 4.1–4.3, we formulate the following proposition:

Theorem 4.1. *Let conditions of Lemmas 4.1–4.3 be satisfied. Then system* (3.1)–(3.3) *has a one-parametric class of separable solutions.*

5. Conclusion

We proposed a deterministic model for two-sex population with a discrete set of offspring and maternal care assuming that pairs of sexes exist only during the period of mating, which is disregarded. The Environmental pressure is also neglected in our model. The reproductive age intervals in model [9] are finite. Contrary to model [9], we let the reproductive age intervals be infinite. The existence of the separable solutions is proved under some conditions on the model data.

To close the paper, we discuss conditions that led to the existence of the solutions to characteristic equation (4.6) of our model and equation (4.9) for exponent λ of the model [9] in the case of the absence of the Environmental pressure. Equation (4.9) of model [9] has at least one real solution without any additional restriction on the model data. As shown in Theorem 1 of our model, the proof of the solvability of characteristic equation (4.6) is based on the proof of the continuity of the norm $||v_2||$ with respect to parameter λ. Knowing this, some robust restrictions on the model data were formulated, that are sufficient for the existence of the solution to characteristic equation (4.6).

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References

[1] G. F. Webb, *Theory of Nonlinear Age-Dependent Population Dynamics*, New York: M. Dekker, 1985. ISBN 9780824772901, 0824772903.

[2] A. G. Fredrickson, *A mathematical theory of age structure in sexual population: random mating and monogamous marriage models*, Math. Biosci., **10** (1971), 117-143.

[3] F. C. Hoppensteadt, *Mathematical theories of populations, genetics, and epidemics*, In CBMS Appl. Math. SIAM, 20, Philadelphia, 1979, 1-71.

[4] O. V. Staroverov, *Reproduction of the structure of the population and marriages*, Ekonomika i matematicheskije metody, (1977), 77-82 (in Russian).

[5] K. P. Hadeler, *Pair formation with maturation period*, J. Math. Biol., **32** (1993), 1-15.

[6] J. Pruss, W. Schappacher, *Persitent age-distributions for a pair-formation model*, Math. Biol., **33** (1994), 17–33.

[7] R. Zacher, *Persistent solutions for age-dependent pair-formation models*, J. Math. Biol., **42** (2001), 507-531.

[8] V. Skakauskas, *A population dynamics model with parental care*, Lith. Math. J., **42**(1) (2002), 71-80.

[9] V. Skakauskas, *An age-structured population dynamics model with females pregnancy and child care*, Lith. Math. J., **44**(3) (2004), 251-271.

[10] V. Skakauskas, *A two sex population dynamics model with strong parental care*, Nonlinear Analysis: Real World Applications, **6** (2005), 609-636.

[11] V. Skakauskas, *A two sex population dynamics model with strong maternal care*, Lith. Math. J., **46**(2) (2006), 172-201.

[12] V. Skakauskas, *A pair formation model with a discrete set of offspring and child care*, Lith. Math. J., **47**(1) (2007), 78-111.