DOI 10.1007/s10986-024-09631-5
Lithuanian Mathematical Journal, Vol. 64, No. 2, April, 2024, pp. 125–137

Notes on universality in short intervals and exponential shifts

Johan Andersson [a], **Ramūnas Garunkštis** [b,1], **Roma Kačinskaitė** [b,c], **Keita Nakai** [d], **Łukasz Pańkowski** [e,2], **Athanasios Sourmelidis** [f,3,4], **Rasa Steuding** [g], **Jörn Steuding** [g], **and Saeree Wananiyakul** [h]

[a] Division of Mathematics, Institute of Science and Technology, Örebro University
[b] Institute of Mathematics, Faculty of Mathematics and Informatics, Vilnius University, Naugarduko str. 24, 03225 Vilnius, Lithuania
[c] Department of Mathematics and Statistics, Faculty of Informatics, Vytautas Magnus University, Universiteto str. 10, Akademija, LT-53361 Kaunas District, Lithuania
[d] Graduate school of Mathematics, Nagoya University, Chikusa-Ku, Nagoya 464-8602, Japan
[e] Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Uniwersytetu Poznańskiego 4, 61-614 Poznań, Poland
[f] Institute of Analysis and Number Theory, Graz University of Technology, Steyrergasse 30, 8010 Graz, Austria
[g] Department of Mathematics, Würzburg University, Emil Fischer-Str. 40, 97 074 Würzburg, Germany
[h] Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University, 10330 Bangkok, Thailand
(e-mail: johan.andersson@oru.se; ramunas.garunkstis@mif.vu.lt; roma.kacinskaite@mif.vu.lt; m21029d@math.nagoya-u.ac.jp; lpan@amu.edu.pl; sourmelidis@math.tugraz.at; joern.steuding@uni-wuerzburg.de; s.wananiyakul@hotmail.com)

Dedicated to Antanas Laurinčikas on the occasion of his 75th birthday

Received December 8, 2023; revised February 27, 2024

Abstract. We improve a recent universality theorem for the Riemann zeta-function in short intervals due to Antanas Laurinčikas with respect to the length of these intervals. Moreover, we prove that the shifts can even have exponential growth. This research was initiated by two questions proposed by Laurinčikas in a problem session of a recent workshop on universality.

MSC: 11M06

Keywords: universality, zeta-functions, exponent pairs, exponential shifts

1 Introduction

In 2019, Antanas Laurinčikas [8] proved that Voronin's celebrated universality theorem holds for shifts restricted to short intervals. More precisely, the real shifts τ for which the Riemann zeta-function $\zeta(s + i\tau)$

[1] The author is funded by the Research Council of Lithuania (LMTLT), agreement No. S-MIP-22-81.
[2] The author is partially supported by the grant No. 2021/41/B/ST1/00241 from the National Science Centre.
[3] The author is supported by the Austrian Science Fund (FWF) project No. M 3246-N.
[4] Corresponding author.

approximates an admissible target function (from a large class of functions) can be found in any short interval $[T,\, T + T^{1/3+\epsilon}]$ (for sufficiently large T), and the set of these shifts has positive lower density as $T \to \infty$; see Theorem A. Note that the case of *weighted* universality is related but different; the concept of weighted universality was also introduced by Laurinčikas [7] in 1995.

Before we recall the precise statement, we introduce some language. We say that a function $\mathcal{F}$ is *universal in an interval* $[T,\, T+H]$ if for every $\varepsilon > 0$ and every admissible target function g defined on some admissible set $\mathcal{K}$, the inequality

$$\liminf_{T\to\infty} \frac{1}{H} \operatorname{meas}\Big\{\tau \in [T,\, T+H]\colon\ \max_{s\in\mathcal{K}} \big|\mathcal{F}(s+\mathrm{i}\tau) - g(s)\big| < \varepsilon\Big\} > 0$$

holds; if here H can be significantly smaller than T, then $\mathcal{F}$ is said to be *universal in short intervals*. We may restrict ourselves to the case of universality for the Riemann zeta-function $\mathcal{F} = \zeta$; Dirichlet L-functions can be treated analogously, and for further universal zeta- or L-functions, the setting can be adjusted. The original universality theorem for ζ was proved by Voronin [22] for intervals $[0, T]$ or, equivalently, for $[T, 2T]$ in place of $[T,\, T + H]$; generalized and extended versions of Voronin's theorem were established by Steven Gonek, Bhaskar Bagchi, Axel Reich, Laurinčikas, and others (see, for example, [12]); however, the case of universality in short intervals, that is, $[T,\, T+H]$ with $H = o(T)$ is new and was introduced by Laurinčikas [8]. This concept was also studied for discrete universality with respect to arithmetic progressions and some other sequences [9]; our reasoning applies to this setting too. Moreover, Andersson [1] has recently shown, among other findings, that continuous universality in short intervals is equivalent to its discrete analogue in arithmetic progressions.

For the case of $\zeta(s)$, note that the set $\mathcal{K}$ is admissible if it is a compact subset of the strip $1/2 < \sigma < 1$ with connected complement, where, here and in the sequel, we write $s = \sigma + \mathrm{i}t$. Moreover, in this case a function $g : \mathcal{K} \to \mathbb{C}$ is admissible if g is continuous, analytic, and nonvanishing in the interior of $\mathcal{K}$. For the logarithm of ζ, however, the target function does not need to be without zero; the nonvanishing restriction is related to the Riemann hypothesis (see [12, Sect. 9]).

In this setting, Laurinčikas [8] obtained the following result.

Theorem A. *The Riemann zeta-function is universal in short intervals* $[T,\, T + H]$ *for every* H *satisfying*

$$T^{1/3}(\log T)^{26/15} \leqslant H \leqslant T.$$

Very recently,[5] Laurinčikas suggested to look for improvements of this result. In particular, he asked for a proof with $H = T^{\epsilon}$, where $\epsilon > 0$ is a fixed constant that can be arbitrarily small. Moreover, he asked whether

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\Big\{\tau \in [T, 2T]\colon\ \max_{s\in\mathcal{K}} \big|\mathcal{F}\big(s + \mathrm{i}e^{\tau}\big) - f(s)\big| < \varepsilon\Big\} > 0, \tag{1.1}$$

where $\mathcal{K}$ and f are admissible, and $\mathcal{F}$ is a universal function. This question probably originates from [14], where a joint universality result for Dirichlet L-functions is proved, and the vertical shifts are of the form $\tau^{\alpha}(\log \tau)^{\beta}$ for a wide variety of values for α and β. This result has been generalized by Laurinčikas et al. [10] to vertical shifts generated by a function $\varphi(\tau)$ that have a polynomial-like behavior.

In the subsequent sections of this note, we will (i) improve the exponent $1/3$ in Theorem A, (ii) give an affirmative answer to the question whether $H = T^{\epsilon}$ is possible subject to a certain restriction of the range of universality, resp. the yet unsolved Lindelöf hypothesis, (iii) go beyond the latter result under the assumption of the open Riemann hypothesis, and (iv) give an affirmative answer to the second proposed question by introducing an alternative approach and addressing a more general problem.

We use the Landau notation $f(x) = O(g(x))$ and the Vinogradov notation $f(x) \ll g(x)$ meaning that there exists some constant $C > 0$ such that $|f(x)| \leqslant Cg(x)$ for all admissible values of x (where the meaning of "admissible" will be clear from the context).

[5] in a video address during a workshop on *Zeta-Functions, Universality, and Chaotic Operators* at CIRM, Luminy, 7–11 August 2023.

2 Unconditional results I – Bourgain & Watt

As can be seen from [8], the only crucial point in proving the universality for short intervals $[T, T+H]$ is a bounded mean-square for $\zeta(s)$, that is,

$$\frac{1}{H}\int_T^{T+H} \left|\zeta(\sigma+\mathrm{i}t)\right|^2 \mathrm{d}t \ll_\sigma 1, \quad \sigma > \frac{1}{2}. \tag{2.1}$$

It is worth mentioning that if we can prove (2.1) for some $\sigma_0 > 1/2$, then the implicit constant appearing therein is uniform on $\sigma \geqslant \sigma_0$ for any sufficiently large $T = T(\sigma_0) \geqslant 0$. We describe this rigorously in Lemma 2. Therefore our task to prove the universality of $\zeta(s)$ in short intervals $[T, T+H]$ reduces to proving (2.1) for any fixed $\sigma > 1/2$. To that end, the method of exponent pairs is a rather useful tool (which originates from an old work by Johannes van der Corput, starting with [21]), and we refer to the next section for a more detailed discussion. In his paper on the universality in short intervals [8], Laurinčikas used the following result of Aleksandar Ivić [6, Thm. 7.1].

Proposition 1. *Let (κ, λ) be an exponent pair satisfying $1+\lambda-\kappa \geqslant 2\sigma$ for $\sigma \in (1/2, 1)$. Then* (2.1) *holds uniformly for*

$$T^{(\kappa+\lambda+1-2\sigma)/(2\kappa+2)}(\log T)^{(\kappa+2)/(\kappa+1)} \leqslant H \leqslant T.$$

Using the exponent pair $(9/26, 7/13)$ due to Heath-Brown [5], it follows that estimate (2.1) holds with $H = T^{23/70}(\log T)^{61/35}$. Incorporating this in [8] yields a slight improvement of Theorem A; note that $23/70 = 0.32857\,142\ldots < 1/3$.

However, we can do better by making use of a recent result of Bourgain and Watt [4], namely

$$\frac{1}{2U}\int_{T-U}^{T+U} \left|\zeta\left(\frac{1}{2}+\mathrm{i}t\right)\right|^2 \mathrm{d}t \ll \log T \quad \text{for } U = T^{1273/4053+\epsilon}. \tag{2.2}$$

We will show that this mean-square estimate on the critical line implies the desired mean-square bound (2.1) for the same range to the right of the critical line (up to a negligible factor T^ϵ). This leads to the following:

Theorem 1. *The Riemann zeta-function is universal in short intervals $[T, T+H]$ for every H satisfying*

$$T^{1273/4053} \leqslant H \leqslant T.$$

Note that $1273/4053 = 0.31408\,832\ldots$.

The proof relies on the following lemmata.

Lemma 1. *Let $T \geqslant 1$, $1 \leqslant H \leqslant T$, and $1/2 \leqslant \sigma_0 < \sigma \leqslant 1$. Then*

$$\int_T^{T+H} \left|\zeta(\sigma+\mathrm{i}t)\right|^2 \mathrm{d}t \ll H + \frac{H^{2(\sigma_0-\sigma)}}{(\sigma-\sigma_0)^2} \int_{T-\log T}^{T+H+\log T} \left|\zeta(\sigma_0+\mathrm{i}t)\right|^2 \mathrm{d}t.$$

Lemma 1 resembles the result of Ivić [6, Lemma 7.1], and the proofs of these results follow the same lines. An immediate consequence of Lemma 1 is that the mean-square of $\zeta(s)$ in short intervals on a vertical line implies a better mean-square estimate result to the right of this line.

Lemma 2. *Suppose that there exist $\theta \in (0,1)$ and $\sigma_0 \geqslant 1/2$ such that for any $\epsilon > 0$, $T \geqslant 1$, and $T^{\theta+\epsilon} \leqslant H \leqslant T$, we have*

$$\int_T^{T+H} \left|\zeta(\sigma_0 + \mathrm{i}t)\right|^2 \mathrm{d}t \ll_\epsilon HT^\epsilon.$$

Then, for any $\eta \in (0,1/2)$ and any function $\omega(T) \to 0^+$ as $T \to \infty$, we have

$$\int_T^{T+H} \left|\zeta(\sigma + \mathrm{i}t)\right|^2 \mathrm{d}t \ll_{\eta,\omega} H$$

for all $T > 0$, $T^{\theta-\omega(T)} \leqslant H \leqslant T$, and $\sigma_0 + \eta \leqslant \sigma \leqslant 1$.

Proof. It is sufficient to prove this for $H = T^{\theta-\omega(T)}$. To that end, let $\epsilon = \min(1-\theta,\, \eta\theta)$. By Lemma 1 we have

$$\begin{aligned}\int_T^{T+H} \left|\zeta(\sigma + \mathrm{i}t)\right|^2 \mathrm{d}t &\ll_\eta H + H^{2(\sigma_0-\sigma)} \int_{T-\log T}^{T+H+\log T} \left|\zeta(\sigma_0 + \mathrm{i}t)\right|^2 \mathrm{d}t \\ &\ll_\eta H + H^{-2\eta} \int_{T-\log T}^{T-\log T+T^{\theta+\epsilon}} \left|\zeta(\sigma_0 + \mathrm{i}t)\right|^2 \mathrm{d}t.\end{aligned}$$

In view of our assumptions, it follows that the right-hand side of the above relation is bounded by

$$\ll_\eta H + H^{-2\eta}T^{\theta+\epsilon} \ll_\eta H + T^{-2\eta(\theta-\omega(T))+\theta+\eta\theta} \ll_\eta H\big(1 + T^{-\eta\theta+(1+2\eta)\omega(T)}\big).$$

Since $\omega(T) \to 0^+$ as $T \to \infty$, we deduce that the right-hand side of this relation is $O_{\eta,\omega}(H)$, and the lemma follows. □

Proof of Lemma 1. Let

$$\zeta_H(s) = \sum_{n=1}^{\infty} n^{-s}\mathrm{e}^{-n/H}$$

be a smooth truncation of the Dirichlet series of the Riemann zeta-function. By a variant of Perron's formula (see, for example, [6, (4.60)]) we have that

$$\zeta_H(s) = \frac{1}{2\pi\mathrm{i}} \int_{2-\infty\mathrm{i}}^{2+\infty\mathrm{i}} \Gamma(z)\zeta(s+z)H^z \,\mathrm{d}z.$$

By moving the integration path to the line $-1/2 < \mathrm{Re}(z) = \sigma_0 - \sigma < 0$ we pick up the residues from the gamma function at $z = 0$ and from the zeta-function at $z = 1 - s$ and obtain

$$\zeta_H(s) = \frac{1}{2\pi\mathrm{i}} \int_{\sigma_0-\sigma-\infty\mathrm{i}}^{\sigma_0-\sigma+\infty\mathrm{i}} \Gamma(z)\zeta(s+z)H^z \,\mathrm{d}z + \zeta(s) + \Gamma(1-s)H^{1-s}. \tag{2.3}$$

We now use the change of variables $\tau = -\mathrm{i}(z + \sigma - \sigma_0)$ and divide the integral in (2.3) into parts with $|\tau| \leqslant \log T$ and $|\tau| \geqslant \log T$. Rearranging the terms and multiplying out a factor $H^{\sigma_0-\sigma}$ from the integrals, we obtain

$$\zeta(s) = \zeta_H(s) - H^{\sigma_0-\sigma} R_H(s) - H^{\sigma_0-\sigma} E_H(s), \tag{2.4}$$

where

$$R_H(s) = \frac{1}{2\pi} \int\limits_{-\log T}^{\log T} \Gamma(\sigma_0 - \sigma + \mathrm{i}\tau)\zeta(\sigma_0 + \mathrm{i}t + \mathrm{i}\tau)H^{\mathrm{i}\tau}\,\mathrm{d}\tau,$$

and

$$E_H(s) = \Gamma(1 - \sigma - \mathrm{i}t)H^{1-\sigma_0+\mathrm{i}t} + \frac{1}{2\pi} \int\limits_{|\tau|\geqslant\log T} \Gamma(\sigma_0 - \sigma + \mathrm{i}\tau)\zeta(\sigma_0 + \mathrm{i}t + \mathrm{i}\tau)H^{\mathrm{i}\tau}\,\mathrm{d}\tau.$$

In view of Stirling's formula [6, (A.34)]

$$\bigl|\Gamma(x + \mathrm{i}t)\bigr| \ll \mathrm{e}^{-|t|}\bigl(x + |t|\bigr)^{-1}, \quad 0 < x \leqslant 1,\ t \in \mathbb{R}, \tag{2.5}$$

and the bound

$$\zeta(\sigma_0 + \mathrm{i}t) \ll |t|^{1/4}, \quad |t| \geqslant 1,$$

it follows that

$$\begin{aligned} E_H(s) &\ll \mathrm{e}^{-|t|}H^{1-\sigma_0} + \int\limits_{\log T}^{\infty} \mathrm{e}^{-\tau}\bigl(|t| + \tau\bigr)^{1/4}\,\mathrm{d}\tau \\ &\ll \mathrm{e}^{-|t|}H^{1-\sigma_0} + |t|^{1/4}T^{-1} + T^{-3/4}, \quad |t| \geqslant 1. \end{aligned}$$

Therefore, in view of (2.4), we obtain that

$$\int\limits_{T}^{T+H} \bigl|\zeta(\sigma + \mathrm{i}t)\bigr|^2\,\mathrm{d}t \ll I_1 + H^{2(\sigma_0-\sigma)} I_2 + H^{2(\sigma_0-\sigma)+1}T^{-3/2}, \tag{2.6}$$

where

$$I_1 = \int\limits_{T}^{T+H} \bigl|\zeta_H(\sigma + \mathrm{i}t)\bigr|^2\,\mathrm{d}t \quad \text{and} \quad I_2 = \int\limits_{T}^{T+H} \bigl|R_H(\sigma + \mathrm{i}t)\bigr|^2\,\mathrm{d}t.$$

By the Montgomery–Vaughan inequality [13, Cor. 3] we get that

$$I_1 \ll H \sum_{n=1}^{\infty} n^{-2\sigma}\mathrm{e}^{-2n/H} + \sum_{n=1}^{\infty} n^{1-2\sigma}\mathrm{e}^{-2n/H} \ll H, \quad H \geqslant 1. \tag{2.7}$$

We now consider the integral I_2. By $|R_H(s)|^2 = R_H(s)\overline{R_H(s)}$ we can write I_2 as

$$I_2 = \frac{1}{(2\pi)^2} \int_T^{T+H} \int_{-\log T}^{\log T} \int_{-\log T}^{\log T} H^{\mathrm{i}(\tau_1-\tau_2)}\Gamma(\sigma_0-\sigma+\mathrm{i}\tau_1)\Gamma(\sigma_0-\sigma-\mathrm{i}\tau_2) \times \zeta\big(\sigma_0+\mathrm{i}(t+\tau_1)\big)\zeta\big(\sigma_0-\mathrm{i}(t+\tau_2)\big)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\mathrm{d}t. \tag{2.8}$$

We change the integration order so that t is the innermost variable of integration. The integral in t may now be estimated by the Cauchy–Schwarz inequality:

$$\left|\int_T^{T+H} \zeta\big(\sigma_0+\mathrm{i}(t+\tau_1)\big)\zeta\big(\sigma_0-\mathrm{i}(t+\tau_2)\big)\,\mathrm{d}t\right|^2 \leqslant \int_{T+\tau_1}^{T+H+\tau_1} \big|\zeta(\sigma_0+\mathrm{i}t)\big|^2\,\mathrm{d}t \int_{T+\tau_2}^{T+H+\tau_2} \big|\zeta(\sigma_0+\mathrm{i}t)\big|^2\,\mathrm{d}t \leqslant \left(\int_{T-\log T}^{T+H+\log T} \big|\zeta(\sigma_0+\mathrm{i}t)\big|^2\,\mathrm{d}t\right)^2,$$

where the last inequality follows from $-\log T \leqslant \tau_1, \tau_2 \leqslant \log T$. By estimating the remaining factors in (2.8) by their absolute values and using that the innermost integral is now independent of τ_1, τ_2, we get that

$$I_2 \ll \left(\frac{1}{2\pi}\int_{-\log T}^{\log T} \big|\Gamma(\sigma_0-\sigma+\mathrm{i}\tau)\big|\,\mathrm{d}\tau\right)^2 \int_{T-\log T}^{T+H+\log T} \big|\zeta(\sigma_0+\mathrm{i}t)\big|^2\,\mathrm{d}t, \tag{2.9}$$

and the first factor in this relation is bounded by $(\sigma-\sigma_0)^{-2}$ in view of Stirling's formula (2.5). Now the conclusion of Lemma 1 follows from (2.6), (2.7), and (2.9). □

Applying Lemmas 1 and 2 in combination with (2.2) implies the statement of Theorem 1. The method of proof also shows that any mean-square bound on the critical line of the form

$$\frac{1}{H}\int_T^{T+H} \left|\zeta\left(\frac{1}{2}+\mathrm{i}t\right)\right|^2 \mathrm{d}t \ll T^{\epsilon} \quad \text{for } H = T^{\eta}$$

with some $\eta > 0$ implies the universality for ζ in short intervals $[T, T+H]$ for every H satisfying $T^{\eta} \leqslant H \leqslant T$. We will return to this observation in Section 4.

3 Unconditional results II – restricted universality

There is a variation in using Ivić's Proposition 1 that improves the exponents for a certain prize. For this purpose, we introduce the concept of restricted universality as follows: we say that a function $\mathcal{F}$ is σ_0-*restricted universal in an interval* $[T, T+H]$ if it is universal for the same interval and for every admissible set $\mathcal{K}$ located in the restricted strip $\sigma_0 < \sigma < 1$, where $\sigma_0 \in (1/2, 1)$. Examples of L-functions that are σ_0-restricted universal in $[T, 2T]$ are elements of the Selberg class with large degree (see, for example, [12, Thm. 4]). Again we shall consider only the case of the Riemann zeta-function since our methods can be adjusted to more general L-functions as well.

As we have discussed in the previous section, in order to show that $\zeta(s)$ is σ_0-restricted universal in $[T, T+H]$, it suffices to obtain (2.1) for σ_0. If σ_0 is "far" from $1/2$ (the prize we have to pay) then we can prove (2.1) for even shorter intervals.

For a better understanding we first recall some theory of exponent pairs (see [6, Sect. 2.3]). A pair of nonnegative real numbers (κ, λ) is said to be an *exponent pair* if $0 \leqslant \kappa \leqslant 1/2 \leqslant \lambda \leqslant 1$ and

$$\sum_{B<n\leqslant B+h} \exp\bigl(2\pi \mathrm{i} f(n)\bigr) \ll A^{\kappa} B^{\lambda},$$

where $A > 1/2$, $B \geqslant 1$, $1 < h \leqslant B$, and f is a differentiable function satisfying

$$f'(x) \asymp A \quad \text{for } B \leqslant x \leqslant 2B.$$

For some *advanced* exponent pairs, we may have to assume similar bounds for higher derivatives, but this is not relevant for our application. More essential in our context is the question of how to find exponent pairs.

It is not difficult to verify that $(0, 1)$ and $(1/2, 1/2)$ are exponent pairs. In addition, if (κ, λ) is an exponent pair, then the so-called A- and B-*processes* (or else Weyl's differencing method, resp., van der Corput's method) produce further exponent pairs,

$$\left(\frac{\kappa}{2\kappa+2}, \frac{1}{2} + \frac{\lambda}{2\kappa+2}\right) \quad \text{and} \quad \left(\lambda - \frac{1}{2}, \kappa + \frac{1}{2}\right),$$

respectively. Iteration of these processes leads to an infinitude of new exponent pairs. For the exponent in the restricted universality due to Laurinčikas [8], the exponent pair $(\kappa, \lambda) = (4/11, 6/11)$ has been used in Ivić's Proposition 1 with $\sigma = 1/2 + \epsilon$. Note that $(1/2, 1/2)$ is not applicable since it does not meet the condition $1 + \lambda - \kappa \geqslant 2\sigma$. Actually, $(4/11, 6/11)$ does not result from the A- or B-process but from convexity. If (κ_1, λ_1) and (κ_2, λ_2) are exponent pairs, then also

$$\bigl(t\kappa_1 + (1-t)\kappa_2,\, t\lambda_1 + (1-t)\lambda_2\bigr) \tag{3.1}$$

is an exponent pair for any $t \in [0, 1]$. We observe that then $(4/11, 6/11)$ arises from applying A and B to $(1/2, 1/2)$ to get $(2/7, 4/7)$ in combination with $(1/2, 1/2)$ and the parameter $t = 12/33$. Note that all three exponent pairs here give the exponent $1/3$ at T but the latter exponent pair $(4/11, 6/11)$, chosen by Laurinčikas [8], gives a smaller exponent for the $\log$-term than $(2/7, 4/7)$.

There are further exponent pairs known that do not arise by one of the processes above; for example, the pair $(13/84 + \epsilon,\, 55/84 + \epsilon)$ found by Bourgain [3] led him to the so far best bound for the Riemann zeta-function on the critical line. In our case, however, this exponent pair does not produce a T-exponent below $1/3$. This is another example showing that one exponent pair may be good for one application but less good for another.

Recently, Trudgian and Yang [20] came up with an update of Rankin's approach for finding the *best possible* exponent pair to a given problem [15]. We found the exponent pair $(9/26, 7/13)$ by checking the boundary of the convex set of all known exponent pairs considered in their paper (see Fig. 1). The gray area is the set $\mathcal{E}$ of all known exponent pairs; the straight line passing through consists of the set of exponent pairs that yield the T-exponent $1/3$, including Laurinčikas' exponent pair $(\kappa, \lambda) = (4/11, 6/11)$ represented as a thick dot. All points in the gray-colored set below the line yield a T-exponent $< 1/3$. The best choice, however, follows from the exponent pair $(\kappa, \lambda) = (9/26, 7/13)$ represented by another thick dot a little below this line. To see that, we observe, by calculating the directional derivatives of $(\kappa, \lambda) \mapsto (\kappa+\lambda)/(2\kappa+2)$, that the minimum for the T-exponent is taken on the lower boundary of $\mathcal{E}$; since this part of the boundary consists of line segments, verifying that given $\mathcal{E}$ our choice is optimal is only a matter of a straightforward computation.

We consider the condition $1+\lambda-\kappa \geqslant 2\sigma$ in Ivić's Proposition 1, which, for the exponent pair $(9/26, 7/13)$, implies as necessary the inequality

$$\sigma \leqslant \frac{1}{2}\left(1 + \frac{7}{13} - \frac{9}{26}\right) = \frac{31}{52}.$$

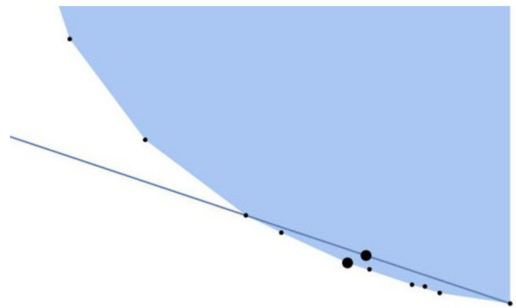

Figure 1. The set $\mathcal{E}$ of all known exponent pairs.

In combination with Lemma 1, this gives the existence of the mean-square (2.1) in the half-plane $\sigma > 31/52$ for short intervals $[T, T+H]$ with $H \geqslant T^{9/35}(\log T)^{61/35}$. We arrive, therefore, at the following:

Theorem 2. *The Riemann zeta-function is* $31/52$*-restricted universal in short intervals* $[T, T+H]$ *for every* H *satisfying*

$$T^{9/35}(\log T)^{61/35} \leqslant H \leqslant T.$$

Moreover, for every fixed $\epsilon \in (0, 1/2)$*, the Riemann zeta-function is* $1-\epsilon$*-restricted universal in short intervals* $[T, T+H]$ *for every* H *satisfying*

$$T^{\epsilon} \leqslant H \leqslant T.$$

Note that $9/35 = 0.25714\,285\ldots$ and $31/52 = 0.59615\,384\ldots$. The second statement follows by applying (3.1) to the trivial exponent pairs $(0, 1)$ and $(1/2, 1/2)$ to derive an exponent pair (κ, λ) satisfying $0 < \kappa < \epsilon$ and $1 - \epsilon < \lambda < 1$.

4 Conditional results – Lindelöf & Riemann hypotheses

It is a folklore conjecture that every $(\epsilon, 1/2+\epsilon)$ is an exponent pair. If so, then the reasoning from the previous section would imply the unrestricted universality for the Riemann zeta-function in short intervals $[T, T+H]$ with $H = T^{1/4+\epsilon}$ and the 3/4-restricted universality with $H = T^{\epsilon}$. This conjecture also implies the Lindelöf hypothesis (which states that $\zeta(\sigma+\mathrm{i}t) \ll t^{\epsilon}$ as $t \to +\infty$ for every fixed $\sigma \geqslant 1/2$). Assuming the latter open conjecture, however, we can show a stronger result:

Theorem 3. *If the Lindelöf hypothesis is true, then the Riemann zeta-function is universal in short intervals* $[T, T+H]$ *for every* H *satisfying*

$$T^{\epsilon} \leqslant H \leqslant T.$$

Proof. Once more, it suffices to prove (2.1) for every fixed $\sigma > 1/2$ and H as in the theorem. Observe that in view of Lemma 1,

$$\int_T^{T+H} \left|\zeta(\sigma+\mathrm{i}t)\right|^2 \mathrm{d}t \ll_{\sigma} H + H^{1-2\sigma} \int_0^{3T} \left|\zeta\left(\frac{1}{2}+\mathrm{i}t\right)\right|^2 \mathrm{d}t.$$

However, the Lindelöf hypothesis implies that [19, Thm. 13.2]

$$\int_0^{3T} \left|\zeta\left(\frac{1}{2}+\mathrm{i}t\right)\right|^2 \mathrm{d}t \ll_{\epsilon} T^{\epsilon}, \quad T \geqslant 1.$$

Thus, for $0 < \epsilon < 2\sigma - 1$, we obtain

$$\int_T^{T+H} \left|\zeta(\sigma + \mathrm{i}t)\right|^2 \mathrm{d}t \ll_\sigma H,$$

and the theorem follows. □

It is remarkable, however, that even shorter intervals are possible, at least if we want to assume another unproven conjecture. Assuming the Riemann hypothesis (that is, the nonvanishing of $\zeta(\sigma + \mathrm{i}t)$ for $\sigma > 1/2$), Sankaranarayanan and Srinivas [17] proved for

$$\frac{1}{2} + \frac{2A_1}{\log\log T} \leqslant \sigma \leqslant 1 - \delta$$

and $\exp((\log T)^{2-2\sigma}) \leqslant H \leqslant T$ with sufficiently large T that

$$\frac{1}{H}\int_T^{T+H} \left|\zeta(\sigma + \mathrm{i}t)\right|^2 \mathrm{d}t = \zeta(2\sigma) + O\left(\exp\left(-A_2\frac{(\log T)^{2-2\sigma}}{\log\log T}\right)\right),$$

where the implicit constant depends on $\delta > 0$, and $A_2 > 0$ depends on $A_1 > 0$. Since the lower bound decreases to $\lim_{T\to\infty} 1/2 + 2A_1/\log\log T = 1/2$, we obtain the following:

Theorem 4. *If the Riemann hypothesis is true, then the Riemann zeta-function is universal in short intervals* $[T,\, T + H]$ *for every* H *satisfying*

$$\exp\bigl((\log T)^{1-\epsilon}\bigr) \leqslant H \leqslant T.$$

Note that $\exp((\log T)^{2-2\sigma_0}) = o(T^\epsilon)$ for every $\sigma_0 > 1/2$.

It is worth mentioning that Lee [11] replaced the role of the mean-square in his proof of universality for Hecke L-functions by a density theorem or a density hypothesis. More precisely, he proved (see [11, Thm. 3]) that the logarithm of a given Hecke L-function can be well approximated in the mean on the vertical segment $\sigma + \mathrm{i}t$ with $t \in [T,\, T + H]$ by a suitable Dirichlet polynomial, provided that $T^a \leqslant H \leqslant T$, $1/2 < a \leqslant 1$, and

$$N_L(\sigma, T,\, T + H) \ll H\left(\frac{H}{\sqrt{T}}\right)^{c(1/2-\sigma)} \log T \quad \text{uniformly for } \sigma \geqslant \frac{1}{2}; \tag{4.1}$$

here $N_L(\sigma, T, T + H)$ counts the number of zeros of $L(s)$ in the region $\mathrm{Re}(s) > \sigma, T < \mathrm{Im}(s) = t < T + H$. Then, using a straightforward modification of Voronin's proof of universality, he succeeded in showing that (4.1) with $H = T$ implies the universality for $L(s)$ in the entire strip $1/2 < \mathrm{Re}(s) < 1$. By the same argument we can show that (4.1) implies also the universality for $L(s)$ in short intervals $[T,\, T + H]$. In particular, we can prove unconditionally that the Riemann zeta-function is universal in short intervals $[T,\, T + H]$, since (4.1) with $L(s) = \zeta(s)$ is known due to Selberg [18]. Thus improving Selberg's zero density estimate is an alternative strategy to prove the universality in short intervals. The best result in this direction seems to be due to Balasubramanian [2, Thm. 6], who proved that for every H satisfying $T^{27/82} \leqslant H \leqslant T$,

$$N_\zeta(\sigma,\, T,\, T + H) \ll H \cdot H^{2(1/2-\sigma)/(3-2\sigma)} (\log T)^{100} \quad \text{uniformly for } \sigma \geqslant \frac{1}{2}. \tag{4.2}$$

Note that the only reason why Theorem 3 in [11] holds only for $H \geqslant T^a$ with $1/2 < a < 1$ is the presence of the factor $H/\sqrt{T}$ in the density hypothesis (4.1). Hence we can adopt Lee's argument, essentially replacing

(4.1) by (4.2), to improve Theorem 3 from [11] and, in consequence, show the universality of the Riemann zeta-function in short intervals $[T, T+H]$ with $H = T^{27/82+\varepsilon}$ for every $\varepsilon > 0$. Moreover, if we extend (4.2) to $H = T^{\delta}$ with $0 < \delta < 27/82$, then we will immediately be able to prove the universality in short intervals for the same H.

5 Exponential shifts

We conclude this note by answering positively the second question of Laurinčikas regarding (1.1). To that end, we define the class $\Phi \subseteq C^1(0, +\infty)$ of functions $\phi(\tau) > 0$ with $\psi(\tau) := \phi'(\tau)/\phi(\tau)$ satisfying for any sufficiently large $T > 0$ and any $\tau \geqslant T$ the following properties:

(i) $\phi'(\tau)$ is an increasing function with $\phi'(\tau + 1/\psi(\tau)) \ll \phi'(\tau)$,
(ii) $\psi(\tau)$ is an increasing function with $A^{-1} \leqslant \psi(\tau) \leqslant \psi(\tau + 1/\psi(\tau)) \leqslant A + \psi(\tau)$ for some absolute constant $A > 0$ or a decreasing function with $\psi(\tau) \geqslant B/\tau$ for some absolute constant $B > 0$.

This definition is tailor-made for

(1) polynomials $\phi(\tau)$ of degree $\geqslant 1$,
(2) functions $\phi(\tau) \asymp \alpha^{p(\tau)}$ for some polynomial $p(\tau)$ and $\alpha > 1$,
(3) functions $\phi(\tau) \asymp \alpha^{\beta^{p(\tau)}}$ for some polynomial $p(\tau)$ and $\alpha, \beta > 1$,

and so on. More examples can be generated from these cases by multiplying the terms of polynomials with powers of logarithms.

Having introduced the class Φ, we will establish some basic properties of its elements. The mean-value theorem and axiom (i) imply that for any $C > 0$,

$$\frac{\psi(\tau)}{C}\left[\phi\left(\tau + \frac{C}{\psi(\tau)}\right) - \phi(\tau)\right] \geqslant \phi'(\tau) = \psi(\tau)\phi(\tau)$$

and hence

$$\phi\left(\tau + \frac{C}{\psi(\tau)}\right) \geqslant (C+1)\phi(\tau), \quad \tau \in [T, 2T]. \tag{5.1}$$

In the case where $\psi(\tau)$ is an increasing function, we set, for any integer $k \geqslant 0$,

$$T_0 = T \quad \text{and} \quad T_k = T_{k-1} + \frac{1}{\psi(T_{k-1})}.$$

It then follows by induction from axiom (ii) that $A^{-1} \leqslant \psi(T_k) \leqslant \psi(T) + kA$, $k \geqslant 0$. Therefore the series $\sum_k 1/\psi(T_{k-1})$ diverges to $+\infty$, and each of its terms contributes at most A. Hence there is $K = K(T, A) \in \mathbb{N}$ such that

$$\sum_{k \leqslant K} \frac{1}{\psi(T_{k-1})} = T + O(1) \quad \text{and} \quad T_K = T_0 + \sum_{0 \leqslant m \leqslant K-1} \frac{1}{\psi(T_m)} = 2T + O(1). \tag{5.2}$$

In conclusion, if $\psi(\tau)$ is an increasing function, then we can find points $T_0, T_1, \dots, T_K$ that form, up to an $O(1)$ error, a partition of the interval $[T, 2T]$.

Theorem 5. *Let $\mathcal{K}$ and f be admissible, and let $\varepsilon > 0$. If $\phi \in \Phi$, then*

$$\liminf_{T \to \infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [T, 2T] \colon \max_{s \in \mathcal{K}} \left|\zeta\big(s + \mathrm{i}\phi(\tau)\big) - f(s)\right| < \varepsilon\right\} > 0.$$

Proof. We will mainly need Voronin's universality theorem in the following form:

$$\operatorname{meas}\Bigl\{\tau \in [T, DT]\colon \max_{s\in\mathcal{K}} \bigl|\zeta(s+\mathrm{i}\tau) - f(s)\bigr| < \varepsilon\Bigr\} \gg T \tag{5.3}$$

for a fixed $D > 1$ and any sufficiently large $T > 0$. We will also employ a classic result from measure theory [16, Thm. 7.26 or p. 156]:

Proposition 2. *Let $\phi : [a, b] \to [\alpha, \beta]$ be absolutely continuous and monotonic with $\phi(a) = \alpha$ and $\phi(b) = \beta$, and let $f \geqslant 0$ be a Lebesgue-measurable function. Then*

$$\int_{\alpha}^{\beta} f(t)\,\mathrm{d}t = \int_{a}^{b} f\bigl(\phi(x)\bigr)\phi'(x)\,\mathrm{d}x.$$

We set

$$\mathcal{E} := \Bigl\{\tau \geqslant 0\colon \max_{s\in\mathcal{K}} \bigl|\zeta(s+\mathrm{i}\tau) - f(s)\bigr| < \varepsilon\Bigr\},$$
$$S_T := \Bigl\{\tau \in [T, 2T]\colon \max_{s\in\mathcal{K}} \bigl|\zeta\bigl(s+\mathrm{i}\phi(\tau)\bigr) - f(s)\bigr| < \varepsilon\Bigr\},$$

and let $\mathbf{1}_{\mathcal{E}}$ be the characteristic function of $\mathcal{E}$.

If $\psi(\tau)$ is a decreasing function, then axiom (i) and Proposition 2 imply

$$\operatorname{meas}(S_T) \geqslant \frac{1}{\phi'(2T)} \int_{T}^{2T} \mathbf{1}_{\mathcal{E}}\bigl(\phi(\tau)\bigr)\phi'(\tau)\,\mathrm{d}\tau = \frac{1}{\phi'(2T)} \int_{\phi(T)}^{\phi(2T)} \mathbf{1}_{\mathcal{E}}(\tau)\,\mathrm{d}\tau.$$

Observe that $\phi(2T) \geqslant \phi(T + B/\psi(T)) \geqslant (1+B)\phi(T)$ by (5.1). Therefore by (5.3) with $D = 1 + B$ we obtain that

$$\operatorname{meas}(S_T) \gg \frac{1}{\phi'(2T)} \int_{\phi(2T)/(1+B)}^{\phi(2T)} \mathbf{1}_{\mathcal{E}}(\tau)\,\mathrm{d}\tau \gg \frac{\phi(2T)}{\phi'(2T)} \gg \frac{1}{\psi(2T)} \gg T.$$

If $\psi(\tau)$ is an increasing function, then we construct the partition $T_0, \dots, T_K$ of $[T, 2T]$ as described above Theorem 5, and we see that

$$\operatorname{meas}(S_T) = \sum_{k\leqslant K} \int_{T_{k-1}}^{T_k} \mathbf{1}_{\mathcal{E}}\bigl(\phi(\tau)\bigr)\,\mathrm{d}\tau + O(1).$$

Once more, from axiom (i) and Proposition 2 it follows that

$$\phi'(T_k) \int_{T_{k-1}}^{T_k} \mathbf{1}_{\mathcal{E}}\bigl(\phi(\tau)\bigr)\,\mathrm{d}\tau \geqslant \int_{\phi(T_{k-1})}^{\phi(T_k)} \mathbf{1}_{\mathcal{E}}(\tau)\,\mathrm{d}\tau.$$

Observe that $\phi(T_k) \geqslant 2\phi(T_{k-1})$ by (5.1). Therefore by (5.3) with $D = 2$, axiom (i), and relation (5.2) we obtain that

$$\operatorname{meas}(S_T) \gg \sum_{k\leqslant K} \frac{1}{\phi'(T_k)} \int_{\phi(T_{k-1})}^{2\phi(T_{k-1})} \mathbf{1}_{\mathcal{E}}(\tau)\,\mathrm{d}\tau \gg \sum_{k\leqslant K} \frac{\phi(T_{k-1})}{\phi'(T_{k-1})} \gg \sum_{k\leqslant K} \frac{1}{\psi(T_{k-1})} \gg T,$$

which concludes the proof of the theorem. □

Acknowledgment. We are grateful to the referees for helping us improve our work.

Funding. Open access funding provided by Graz University of Technology.

References

1. J. Andersson, Discrete universality, continuous universality and hybrid universality are equivalent, 2023, https://doi.org/10.48550/arxiv.2310.03619.
2. R. Balasubramanian, An improvement on a theorem of Titchmarsh on the mean square of $|\zeta(\frac{1}{2}+it)|$, *Proc. Lond. Math. Soc. (3)*, **36**(3):540–576, 1978, https://doi.org/10.1112/plms/s3-36.3.540.
3. J. Bourgain, Decoupling, exponential sums and the Riemann zeta function, *J. Am. Math. Soc.*, **30**(1):205–224, 2016, https://doi.org/10.1090/jams/860.
4. J. Bourgain and N. Watt, Decoupling for perturbed cones and the mean square of $|\zeta(\frac{1}{2}+it)|$, *Int. Math. Res. Not.*, **2018**(17):5219–5296, 2017, https://doi.org/10.1093/imrn/rnx009.
5. D.R. Heath-Brown, A new kth derivative estimate for exponential sums via Vinogradov's mean value, *Proc. Steklov Inst. Math.*, **296**(1):88–103, 2017, https://doi.org/10.1134/s0081543817010072.
6. A. Ivić, *The Riemann zeta-function. The theory of the Riemann zeta-function with applications*, John Wiley & Sons, New York, 1985.
7. A. Laurinčikas, On the universality of the Riemann zeta-function, *Lith. Math. J.*, **35**(4):399–402, 1995, https://doi.org/10.1007/bf02348827.
8. A. Laurinčikas, Universality of the Riemann zeta-function in short intervals, *J. Number Theory*, **204**:279–295, 2019, https://doi.org/10.1016/j.jnt.2019.04.006.
9. A. Laurinčikas, Discrete universality of the Riemann zeta-function in short intervals, *Appl. Anal. Discrete Math.*, **14**(2):382–405, 2020, https://doi.org/10.2298/aadm190704019l.
10. A. Laurinčikas, R. Macaitienė, and D. Šiaučiūnas, A generalization of the Voronin theorem, *Lith. Math. J.*, **59**(2): 156–168, 2019, https://doi.org/10.1007/s10986-019-09418-z.
11. Y. Lee, The universality theorem for Hecke L-functions, *Math. Z.*, **271**(3–4):893–909, 2011, https://doi.1007/s00209-011-0895-6.

12. K. Matsumoto, A survey on the theory of universality for zeta and L-functions, in M. Kaneko, S. Kanemitsu, and J. Liu (Eds.), *Number Theory: Plowing and Starring Through High Wave Forms. Proceedings of the 7th China-Japan Seminar, Fukuoka, Japan, October 28–November 1, 2013*, Ser. Number Theory Appl., Vol. 11, World Scientific, Hackensack, NJ, 2015, pp. 95–144, `https://doi.org/10.1142/9789814644938_0004`.

13. H.L. Montgomery and R.C. Vaughan, Hilbert's inequality, *J. Lond. Math. Soc.*, **8**(2):73–82, 1974, `https://doi.org/10.1112/jlms/s2-8.1.73`.

14. Ł. Pańkowski, Joint universality for dependent L-functions, *Ramanujan J.*, **45**(1):181–195, 2017, `https://doi.org/10.1007/s11139-017-9886-5`.

15. R.A. Rankin, Van der Corput's method and the theory of exponent pairs, *Q. J. Math., Oxf. II. Ser.)*, **6**(1):147–153, 1955, `https://doi.org/10.1093/qmath/6.1.147`.

16. W. Rudin, *Real and Complex Analysis,* 3rd ed., McGraw-Hill, New York, 1987.

17. A. Sankaranarayanan and K. Srinivas, Mean-value theorem of the Riemann zeta-function over short intervals, *J. Number Theory*, **45**(3):320–326, 1993, `https://doi.org/10.1006/jnth.1993.1081`.

18. A. Selberg, Contributions to the theory of the Riemann zeta-function, *Arch. Math. Naturvid.*, **48**(5):89–155, 1946.

19. E.C. Titchmarsh, *The Theory of the Riemann Zeta-Function,* 2nd ed., Clarendon Press, Oxford, 1986. Revised by D.R. Heath-Brown.

20. T.S. Trudgian and A. Yang, On optimal exponent pairs, 2023, `https://doi.org/10.48550/arxiv.2306.05599`.

21. J.G. van der Corput, Verschärfung der Abschätzung beim Teilerproblem, *Math. Ann.*, **89**(3–4):161–178, 1923, `https://doi.org/10.1007/bf01455975`.

22. S.M. Voronin, Theorem on the "universality" of the Riemann zeta-function, *Math. USSR, Izv.*, **9**(3):443–453, 1975, `https://doi.org/10.1070/im1975v009n03abeh001485`.