

Turinys

1. Įvadas.....	2
2. Uždaviniai	4
3. Įrodymas	5
4. Išvados.....	17
5. Summary	18
6. Literatūra.....	19

1. Įvadas

Tarkime yra nagrinėjama skaičių

$$y_1, y_2, \dots, y_N$$

generalinė visuma, kurios vidurkiui $\bar{y} = \frac{1}{N} \sum_{j=1}^N y_j$ įvertinti išrenkama tūrio n paprasta atsitiktinė imtis be grąžinimo:

$$y_{i_1}, y_{i_2}, \dots, y_{i_n},$$

kur $1 \leq i_1 < i_2 < \dots < i_n \leq N$. Jeigu skaičiai y_1, y_2, \dots, y_N yra gaminio charakteristikos, kurias matavimo prietaisai rodo su paklaida ε , tai kiekvienas skaičius y_1, y_2, \dots, y_N virsta atsitiktiniu dydžiu

$$y_1 + \varepsilon_1, y_2 + \varepsilon_2, \dots, y_N + \varepsilon_N.$$

čia $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_N$ yra vienodai pasiskirstę nepriklausomi atsitiktiniai dydžiai. Jų pasiskirstymo funkciją pažymėkime $F(x)$. Jei matavimo prietaisai be sisteminės paklaidos, tai jų matematinė viltis $M\varepsilon_i$ yra lygi nuliui, t.y.

$$M\varepsilon_i = 0,$$

viesiems $i = 1, 2, \dots, N$. ε_i dispersiją $D\varepsilon_i$ žymėsime σ^2 , t.y. $\sigma^2 = D\varepsilon_i$, $i=1, 2, \dots, N$. Atsitiktinio dydžio $y_i + \varepsilon_i$ tikimybinis skirstinys yra:

$$G_i(x) = P\{y_i + \varepsilon_i < x\} = P\{\varepsilon_i < x - y_i\} = F(x - y_i),$$

kur $i = 1, 2, \dots, N$.

Pažymėkime $\xi_i = y_i + \varepsilon_i$, $i = 1, 2, \dots, N$.

Dabar mūsų stebėjimo objektas yra nepriklausomų atsitiktinių dydžių

$$\xi_1, \xi_2, \dots, \xi_N$$

generalinė visuma, kurios tyrimams panaudosime paprastą atsitiktinę imtį be grąžinimo. Imties išrinkimui pasinaudosime Bengt von Bahr [1] pasiūlytu metodu.

Tarkime $\vec{I} = (I_1, I_2, \dots, I_N)$ yra indikatorių vektorius nepriklausomas nuo $\xi_1, \xi_2, \dots, \xi_N$.

Kiekvienam nulių - vienetų rinkiniui $\vec{v} = (v_1, v_2, \dots, v_N)$, čia $v_j = 0$ arba 1 , viesiems $j = 1, 2, \dots, N$, priskiriam tikimybę

$$P(\vec{I} = \vec{v}) = \binom{N}{n}^{-1},$$

kai sekoje v_1, v_2, \dots, v_N yra n vienetų ir $N-n$ nulių.

Paprastą atsitiktinę imtį pažymėkime

$$\xi_{i_1}, \xi_{i_2}, \dots, \xi_{i_n},$$

kur $1 \leq i_1 < i_2 < \dots < i_n \leq N$. Nagrinėsime sumos

$$S_n = I_1 \xi_1 + I_2 \xi_2 + \dots + I_N \xi_N,$$

kur $P\{I_j = 1\} = \frac{n}{N}$ ir $P\{I_j = 0\} = 1 - \frac{n}{N}$, tikimybinį skirstinį

$$F_{Nn}(x) = P\{S_n < x\}.$$

2. Uždaviniai

Bengt von Bahr [1] įrodė

Teorema. Tegul X_1, X_2, \dots, X_N yra nepriklausomi atsitiktiniai dydžiai, iš kurių atsitiktinai išrenkame n elementų, o jų suma žymima S_n .

Jei $EX_k = \mu_k$, $EX_k^2 = \beta_k$, $E|X_k|^3 = \gamma_k$, kur $\sum_{k=1}^N \mu_k = 0$, $\frac{1}{N} \sum_{k=1}^N \beta_k = 1$, $\alpha^2 = \frac{1}{N} \sum_{k=1}^N \mu_k^2$

ir $\gamma = \max_{1 \leq k \leq N} \gamma_k$, tada

$$\left| P\left(\frac{S_n}{\sqrt{n(1-f\alpha^2)}} \leq x\right) - \Phi(x) \right| \leq \frac{60\gamma}{\sqrt{n(1-f\alpha^2)}^{\frac{3}{2}}}$$

kur $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-y^2/2} dy$ yra normuota Gauso pasiskirstymo funkcija ir $f = \frac{n}{N}$.

Dešinėje nelygybės pusėje yra reiškinys

$$\frac{60\gamma}{\sqrt{n(1-f\alpha^2)}^{\frac{3}{2}}},$$

kurio optimalumas yra nežinomas. Šio uždavinio sprendimui yra reikalingas pasiskirstymo funkcijos $F_{Nn}(x)$ asimptotinis skleidinys. Mes įrodėme, kad

$$\begin{aligned} P\{S_n < x\} &= \Phi(x) - \frac{\alpha_3}{3!\sqrt{n}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{u^2}{2}} (u^3 - 3u) du - \frac{(\alpha_4 - \frac{3}{N})}{4!n} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{u^2}{2}} (u^4 - 6u^2 + 3) du + \dots = \\ &= \Phi(x) - \frac{\alpha_3}{3!\sqrt{n}} \frac{1}{\sqrt{2\pi}} \left(x^2 e^{-\frac{x^2}{2}} - e^{-\frac{x^2}{2}} \right) + \frac{(\alpha_4 - \frac{3}{N})}{4!n} \frac{1}{\sqrt{2\pi}} \left(3xe^{-\frac{x^2}{2}} + x^3 e^{-\frac{x^2}{2}} \right) + \dots \end{aligned}$$

Šioje teoremoje yra atsakymas nuo ko priklauso skirtumo

$$P\{S_n < x\} - \Phi(x)$$

įvertis.

Pastebėkime, kad $S_n = I_1(y_1 + \varepsilon_1) + I_2(y_2 + \varepsilon_2) + \dots + I_N(y_N + \varepsilon_N) = \sum_{j=1}^N I_j y_j + \sum_{j=1}^N I_j \varepsilon_j$.

3. Įrodymas

$F_{Nn}(x)$ nagrinėsime charakteringųjų funkcijų metodu. Tarkime ξ_j charakteringoji funkcija yra

$$f_j(t) = Me^{it\xi_j}, \quad j = 1, 2, \dots, N.$$

Yra įrodyta [2], kad sumos S_n charakteringoji funkcija yra

$$f_{S_n}(t) = \binom{N}{n}^{-1} \sum_{1 \leq i_1 < i_2 < \dots < i_n \leq N} f_{i_1}(t) f_{i_2}(t) \dots f_{i_n}(t). \quad (1)$$

Šią formulę Bengt von Bahr [1] panaudoja $F_{Nn}(x)$ aproksimavimui normalaus dėsnio charakteringąja funkcija $e^{ita - \frac{t^2}{2}\sigma^2}$, kur $\sigma > 0$ ir $a \in (-\infty, +\infty)$.

Patogesnė ir paprastesnė formulė yra gauta darbe [2]:

$$f_{S_n}(t) = \frac{1}{2\pi P_N(n)} \int_{-\pi}^{\pi} e^{-i\theta n} \prod_{k=1}^N (q + pe^{i\theta} f_k(t)) d\theta.$$

Ją galime užrašyti ir taip:

$$f_{S_n}(t) = \binom{N}{n}^{-1} \sum_s^* (-1)^{n-s} \prod_{m=1}^n \frac{1}{k_m! m^{k_m}} \left(\sum_{k=1}^N f_k^m(t) \right)^{k_m},$$

kur $P_N(n) = \binom{N}{n} p^n q^{N-n}$, $p = \frac{n}{N}$, $q = 1-p$, o \sum_s^* žymi sumavimą pagal visus neneigiamus sveikuosius lygties

$$k_1 + k_2 + \dots + k_n = s,$$

$$k_1 + 2k_2 + \dots + nk_n = n$$

sprendinius.

Jeigu $g(t) = Me^{it\varepsilon_1}$ yra a.d. ε_1 charakteringoji funkcija, tai

$$f_j(t) = \int_{-\infty}^{\infty} e^{itx} dG_j(x) = \int_{-\infty}^{\infty} e^{itx} dF(x - y_j) = \left[\begin{array}{l} x - y_j = u \\ x = y_j + u \end{array} \right] = \int_{-\infty}^{\infty} e^{it(y_j + u)} dF(u) = e^{ity_j} Me^{it\varepsilon_1}$$

Gavome, kad

$$f_j(t) = e^{ity_j} g(t). \quad (2)$$

Iš (1) ir (2) lygybių išplaukia, kad

$$\begin{aligned}
f_{S_n}(t) &= \binom{N}{n}^{-1} \sum_{1 \leq j_1 < j_2 < \dots < j_n \leq N} \left(e^{ity_{j_1}} g(t) \right) \left(e^{ity_{j_2}} g(t) \right) \dots \left(e^{ity_{j_n}} g(t) \right) = \\
&= g^n(t) \frac{1}{C_N^n} \sum_{1 \leq j_1 < j_2 < \dots < j_n \leq N} e^{it(y_{j_1} + y_{j_2} + \dots + y_{j_n})} = g^n(t) \cdot f_{Z_n}(t),
\end{aligned} \tag{3}$$

$$\text{kur } f_{Z_n}(t) = \frac{1}{C_N^n} \sum_{1 \leq j_1 < j_2 < \dots < j_n \leq N} e^{it(y_{j_1} + y_{j_2} + \dots + y_{j_n})}.$$

R. Erdős ir A. Rényi [3] yra įrodę, kad

$$f_{Z_n}(t) = \frac{1}{P_N(n)} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\theta n} \prod_{j=1}^N (q + p e^{ity_j + i\theta}) d\theta.$$

Iš (3) lygybės seka, kad a.d.

$$\varepsilon = \sum_{j=1}^N \varepsilon_j I_j = \varepsilon_1(\omega) + \varepsilon_2(\omega) + \dots + \varepsilon_n(\omega) \quad \text{ir} \quad Z_n = \sum_{j=1}^N y_j I_j(\omega),$$

kur $I_1 + I_2 + \dots + I_N = n$, yra nepriklausomi a.d., kadangi

$$f_{S_n}(t) = g^n(t) \cdot f_{Z_n}(t) \quad \text{ir} \quad S_n = \varepsilon + Z_n.$$

Gausime S_n / \sqrt{n} charakteringosios funkcijos $\overline{f}_n(t)$ tikslią išraišką, kuri bus naudinga kitų paklaidų įvertinimui.

Tarkim $f_k(t)$ yra X_k , $k = 1, 2, \dots, N$ charakteringoji funkcija. Tada

$$\overline{f}_n(t) = \binom{N}{n}^{-1} \sum_{1 \leq i_1 < i_2 < \dots < i_n \leq N} \prod_{j=1}^n f_{i_j} \left(\frac{t}{\sqrt{n}} \right),$$

$$\text{kur } f_{i_j} \left(\frac{t}{\sqrt{n}} \right) = e^{i \frac{t}{\sqrt{n}} y_{i_j}} M e^{i \frac{t}{\sqrt{n}} \varepsilon}.$$

Dauginame abi puses iš tos pačios funkcijos $e^{t^2/2n}$ ir gauname

$$e^{t^2/2n} \overline{f}_n(t) = \binom{N}{n}^{-1} \sum_{1 \leq i_1 < i_2 < \dots < i_n \leq N} \prod_{j=1}^n \left[e^{t^2/2n} f_{i_j} \left(\frac{t}{\sqrt{n}} \right) \right].$$

Tarkime $z \in \mathbb{C}$ ir $g(z)$ yra funkcija pavidalo

$$g(z) = \binom{N}{n}^{-1} \prod_{k=1}^N \left(1 + z e^{t^2/2n} f_k \left(\frac{t}{\sqrt{n}} \right) \right)$$

$$\text{tuomet } g(z) = g(0) + \sum_{\nu=1}^N \frac{z^\nu}{\nu!} g^{(\nu)}(0) = \binom{N}{n}^{-1} + \sum_{\nu=1}^N \frac{z^\nu}{\nu!} g^{(\nu)}(0)$$

$$\text{čia } g^{(\nu)}(0) = \binom{N}{n}^{-1} \sum_{1 \leq i_1 < i_2 < \dots < i_\nu \leq N} \prod_{j=1}^{\nu} e^{t^2/2n} f_{i_j} \left(\frac{t}{\sqrt{n}} \right), \quad \frac{d^\nu}{dx^\nu} g(x) \Big|_{x=0} = g^{(\nu)}(x) \Big|_{x=0} = g^{(\nu)}(0).$$

Funkciją $g(z)$ užsirašysime mums patogesniu pavidalu:

$$\begin{aligned} g(z) &= \binom{N}{n}^{-1} \prod_{k=1}^N \left(1 + z \left[\left(e^{t^2/2n} f_k \left(\frac{t}{\sqrt{n}} \right) - 1 \right) + 1 \right] \right) = \binom{N}{n}^{-1} \prod_{k=1}^N \left((1+z) + z \left(e^{t^2/2n} f_k \left(\frac{t}{\sqrt{n}} \right) - 1 \right) \right) = \\ &= \binom{N}{n}^{-1} \prod_{k=1}^N (1+z) \prod_{k=1}^N \left(1 + \frac{z}{1+z} \left(e^{t^2/2n} f_k \left(\frac{t}{\sqrt{n}} \right) - 1 \right) \right) = \binom{N}{n}^{-1} (1+z)^N \prod_{k=1}^N \left(1 + \frac{z}{1+z} \left(e^{t^2/2n} f_k \left(\frac{t}{\sqrt{n}} \right) - 1 \right) \right) = \\ &= \binom{N}{n}^{-1} (1+z)^N \exp \left\{ \ln \prod_{k=1}^N \left(1 + \frac{z}{1+z} \left(e^{t^2/2n} f_k \left(\frac{t}{\sqrt{n}} \right) - 1 \right) \right) \right\} = \\ &= \binom{N}{n}^{-1} (1+z)^N \exp \left\{ \sum_{k=1}^N \ln \left(1 + \frac{z}{1+z} \left(e^{t^2/2n} f_k \left(\frac{t}{\sqrt{n}} \right) - 1 \right) \right) \right\} \end{aligned}$$

Kadangi $\ln(1+x) = \sum_{j=1}^{\infty} (-1)^{j+1} \frac{x^j}{j}$, kai $-1 < x \leq 1$, gauname

$$\begin{aligned} g(z) &= \binom{N}{n}^{-1} (1+z)^N \exp \left\{ \sum_{k=1}^N \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} \left(\frac{z}{1+z} \right)^j \left(e^{t^2/2n} f_k \left(\frac{t}{\sqrt{n}} \right) - 1 \right)^j \right\} = \\ &= \binom{N}{n}^{-1} (1+z)^N \exp \left\{ \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} \left(\frac{z}{1+z} \right)^j \sum_{k=1}^N \left(e^{t^2/2n} f_k \left(\frac{t}{\sqrt{n}} \right) - 1 \right)^j \right\}. \end{aligned}$$

Toliau pasinaudosime sekančia lema.

LEMA. Tegul laipsninė eilutė $\sum_{j=1}^{\infty} a_j x^j$ konverguoja į funkciją $f(x)$ taško $x = 0$ aplinkoje

(arba yra jos asimptotinis skleidinys, kai $x \rightarrow 0$). Tada

$$e^{f(x)} = 1 + \sum_{j=1}^{\infty} b_j x^j,$$

kur

$$b_j = \sum \frac{{}^* a_1^{v_1} \dots a_j^{v_j}}{v_1! \dots v_j!}, \quad (4)$$

Suma \sum^* yra imama pagal visus v_1, \dots, v_j ($v_j = 0, 1, 2, \dots$), kurie tenkina lygybę

$$v_1 + 2v_2 + \dots + jv_j = j. \quad (5)$$

$$\text{Mūsų atveju } x = \frac{z}{1+z}, \quad a_j(t) = \frac{(-1)^{j+1}}{j} \sum_{k=1}^N \left(e^{t^2/2n} f_k \left(\frac{t}{\sqrt{n}} \right) - 1 \right)^j.$$

Dabar gauname, kad

$$\begin{aligned} g(z) &\doteq \binom{N}{n}^{-1} (1+z)^N \exp \left\{ \sum_{j=1}^{\infty} \left(\frac{z}{1+z} \right)^j \frac{(-1)^{j+1}}{j} \sum_{k=1}^N \left(e^{t^2/2n} f_k \left(\frac{t}{\sqrt{n}} \right) - 1 \right)^j \right\} = \\ &= \binom{N}{n}^{-1} (1+z)^N \left(1 + \sum_{j=1}^{\infty} \left(\frac{z}{1+z} \right)^j \sum \frac{{}^* a_1^{v_1}(t) \dots a_j^{v_j}(t)}{v_1! \dots v_j!} \right). \end{aligned}$$

Pažymėkime

$$b_k(t) = e^{t^2/2n} f_k \left(\frac{t}{\sqrt{n}} \right) - 1, \quad k = 1, 2, \dots, N,$$

ir

$$B_j(t) = \frac{(-1)^{j+1}}{j} \sum_{k=1}^N b_k^j(t), \quad j = 1, 2, \dots$$

Pasinaudodami šiais pažymėjimais gauname, kad

$$\begin{aligned} g(z) &= \binom{N}{n}^{-1} (1+z)^N \left\{ 1 + \sum_{j=1}^{\infty} \left(\frac{z}{1+z} \right)^j \sum \frac{*a_1^{v_1}(t) \dots a_j^{v_j}(t)}{v_1! \dots v_j!} \right\} = \\ &= \binom{N}{n}^{-1} (1+z)^N + \sum_{j=1}^{\infty} z^j (1+z)^{N-j} \sum \frac{*a_1^{v_1}(t) \dots a_j^{v_j}(t)}{v_1! \dots v_j!} \binom{N}{n}^{-1} \end{aligned} \quad (6)$$

Pagal (4), (5) ir (6) gauname

$$\begin{aligned} g(z) &= \binom{N}{n}^{-1} \left\{ (1+z)^N \left(1 + \sum_{j=1}^{\infty} \sum * \left(\frac{z}{1+z} \right)^{i_1} \frac{B_1^{i_1}}{i_1!} \left(\frac{z}{1+z} \right)^{2i_2} \frac{B_2^{i_2}}{i_2!} \dots \left(\frac{z}{1+z} \right)^{j i_j} \frac{B_j^{i_j}}{i_j!} \right) \right\} = \\ &= \binom{N}{n}^{-1} \left\{ (1+z)^N + \sum_{j=1}^{\infty} \sum * \frac{B_1^{i_1}}{i_1!} \frac{B_2^{i_2}}{i_2!} \dots \frac{B_j^{i_j}}{i_j!} z^{i_1+2i_2+\dots+j i_j} (1+z)^{N-i_1-2i_2-\dots-j i_j} \right\}. \end{aligned}$$

Tarkime, kad $z = e^{i\theta}$ ir $B_j = B_j(t)$. Tada

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\theta n} g(e^{i\theta}) d\theta &= \binom{N}{n}^{-1} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\theta n} \prod_{k=1}^N \left(1 + e^{i\theta} e^{\frac{t^2}{2n}} f_k \left(\frac{t}{\sqrt{n}} \right) \right) d\theta = \\ &= \binom{N}{n}^{-1} \sum_{1 \leq i_1 < i_2 < \dots < i_n \leq N} e^{t^2/2n} f_{i_1} \left(\frac{t}{\sqrt{n}} \right) e^{t^2/2n} f_{i_2} \left(\frac{t}{\sqrt{n}} \right) \dots e^{t^2/2n} f_{i_n} \left(\frac{t}{\sqrt{n}} \right) = \binom{N}{n}^{-1} \sum_{1 \leq i_1 < i_2 < \dots < i_n \leq N} e^{t^2/2} \prod_{j=1}^n f_{i_j} \left(\frac{t}{\sqrt{n}} \right), \end{aligned}$$

$$\text{nes } \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\theta(n-l)} d\theta = \begin{cases} 1, n=l \\ 0, n \neq l \end{cases}.$$

Galime rašyti ir taip:

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\theta n} g(e^{i\theta}) d\theta &= \binom{N}{n}^{-1} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\theta n} (1+e^{i\theta})^N d\theta + \binom{N}{n}^{-1} \sum_{j=1}^N \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\theta n+i\theta j} (1+e^{i\theta})^{N-j} d\theta \sum * \frac{B_1^{i_1}}{i_1!} \frac{B_2^{i_2}}{i_2!} \dots \frac{B_j^{i_j}}{i_j!} = \\ &= \binom{N}{n}^{-1} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\theta n} \sum_{l=0}^N \binom{N}{l} e^{i\theta l} 1^{N-l} d\theta + \binom{N}{n}^{-1} \sum_{j=1}^N \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\theta(n-j)} \sum_{m=0}^{N-j} \binom{N-j}{m} e^{i\theta m} 1^{N-j-m} d\theta \sum * \frac{B_1^{i_1}}{i_1!} \frac{B_2^{i_2}}{i_2!} \dots \frac{B_j^{i_j}}{i_j!} = \\ &= \sum_{l=0}^N \binom{N}{l} \binom{N}{n}^{-1} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\theta(n-l)} d\theta + \binom{N}{n}^{-1} \sum_{j=1}^N \sum_{m=0}^{N-j} \binom{N-j}{m} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\theta(n-j-m)} d\theta \sum * \frac{B_1^{i_1}}{i_1!} \frac{B_2^{i_2}}{i_2!} \dots \frac{B_j^{i_j}}{i_j!} = \frac{\binom{N}{n}}{\binom{N}{n}} + \end{aligned}$$

$$\begin{aligned}
& + \sum_{j=1}^N \sum_{m=0}^{N-j} \binom{N-j}{m} \binom{N}{n}^{-1} \sum * \frac{B_1^{i_1}}{i_1!} \frac{B_2^{i_2}}{i_2!} \dots \frac{B_j^{i_j}}{i_j!} = 1 + \sum_{j=1}^n \binom{N-j}{n-j} \binom{N}{n}^{-1} \sum * \frac{B_1^{i_1}}{i_1!} \frac{B_2^{i_2}}{i_2!} \dots \frac{B_j^{i_j}}{i_j!} = \\
& = 1 + \sum_{j=1}^n \frac{n!}{N!} (N-j)(N-j-1)\dots(n-j+1) \sum * \frac{B_1^{i_1}}{i_1!} \frac{B_2^{i_2}}{i_2!} \dots \frac{B_j^{i_j}}{i_j!} \tag{7}
\end{aligned}$$

$$\text{Kai } j = 1, \text{ turime } \binom{N-j}{n-j} \binom{N}{n}^{-1} = \frac{n!}{N!} (N-1)(N-2)\dots n = \frac{n}{N},$$

$$\text{kai } j = 2, \quad \binom{N-j}{n-j} \binom{N}{n}^{-1} = \frac{n!}{N!} (N-2)(N-3)\dots n(n-1) = \frac{n(n-1)}{N(N-1)},$$

$$\text{kai } j = 3, \quad \binom{N-j}{n-j} \binom{N}{n}^{-1} = \frac{n!}{N!} (N-3)(N-4)\dots n(n-1)(n-2) = \frac{n(n-1)(n-2)}{N(N-1)(N-2)}.$$

Įsistatę į (7), gauname

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\theta n} g(e^{i\theta}) d\theta = 1 + \frac{n}{N} \frac{B_1}{1!} + \frac{n(n-1)}{N(N-1)} \left(\frac{B_1^2}{2!} + \frac{B_2}{1!} \right) + \frac{n(n-1)(n-2)}{N(N-1)(N-2)} \left(\frac{B_1^3}{3!} + \frac{B_1}{1!} \frac{B_2}{1!} + \frac{B_3}{1!} \right) + \dots$$

$$\binom{N}{n}^{-1} \sum_{1 \leq i_1 < i_2 < \dots < i_n \leq N} \prod_{j=1}^n f_{i_j} \left(\frac{t}{\sqrt{n}} \right) = e^{-\frac{t^2}{2}} + \sum_{j=1}^n \frac{n!}{N!} (N-j)(N-j-1)\dots(n-j+1) *$$

$$* \sum \frac{B_1^{i_1}}{i_1!} \frac{B_2^{i_2}}{i_2!} \dots \frac{B_j^{i_j}}{i_j!} e^{-\frac{t^2}{2}}$$

Charakteringoji funkcija $f_{i_j}\left(\frac{t}{\sqrt{n}}\right)$ yra lygi:

$$f_{i_j}\left(\frac{t}{\sqrt{n}}\right) = \int_{-\infty}^{\infty} e^{itx} p_{i_j}(x) dx.$$

Sudauginame abi puses pagal j . Tada:

$$\prod_{j=1}^n f_{i_j}\left(\frac{t}{\sqrt{n}}\right) = \int_{-\infty}^{\infty} e^{itx} p_{i_1, i_2, \dots, i_n}(x) dx$$

ir

$$p_{i_1, i_2, \dots, i_n}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \prod_{j=1}^n f_{i_j}\left(\frac{t}{\sqrt{n}}\right) dt.$$

$$P\left\{\frac{\xi_{i_1} + \xi_{i_2} + \dots + \xi_{i_n}}{\sqrt{n}} < x\right\} = \int_{-\infty}^x p_{i_1, i_2, \dots, i_n}(u) du.$$

$$\binom{N}{n}^{-1} \sum_{1 \leq i_1 < i_2 < \dots < i_n \leq N} \prod_{j=1}^n f_{i_j}\left(\frac{t}{\sqrt{n}}\right) = \int_{-\infty}^x \binom{N}{n}^{-1} \sum_{1 \leq i_1 < i_2 < \dots < i_n \leq N} p_{i_1, i_2, \dots, i_n}(u) e^{iu} du.$$

$$\begin{aligned} \binom{N}{n}^{-1} \sum_{1 \leq i_1 < i_2 < \dots < i_n \leq N} p_{i_1, i_2, \dots, i_n}(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \binom{N}{n}^{-1} \sum_{1 \leq i_1 < i_2 < \dots < i_n \leq N} \prod_{j=1}^n f_{i_j}\left(\frac{t}{\sqrt{n}}\right) dt \approx \\ &\approx \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} e^{-\frac{t^2}{2}} dt + \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} e^{-\frac{t^2}{2}} \frac{n}{N} \frac{B_1}{1!} dt + \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} e^{-\frac{t^2}{2}} \frac{n(n-1)}{N(N-1)} \left(\frac{B_1^2}{2!} + \frac{B_2}{1!}\right) dt + \dots \end{aligned} \quad (8)$$

$$B_1(t) = \frac{(-1)^2}{1} \sum_{k=1}^N b_k(t) = \sum_{k=1}^N \left(e^{\frac{t^2}{2n}} f_k\left(\frac{t}{\sqrt{n}}\right) - 1 \right) = \sum_{k=1}^N \frac{f_k\left(\frac{t}{\sqrt{n}}\right) - e^{-\frac{t^2}{2n}}}{e^{\frac{t^2}{2n}}}$$

$$f_k\left(\frac{t}{\sqrt{n}}\right) - e^{-\frac{t^2}{2n}} = \int_{-\infty}^{\infty} e^{\frac{it}{\sqrt{n}}x} dF_k(x) - \int_{-\infty}^{\infty} e^{\frac{it}{\sqrt{n}}x} d\Phi(x) = \int_{-\infty}^{\infty} e^{\frac{it}{\sqrt{n}}x} d[F_k(x) - \Phi(x)],$$

kur $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{u^2}{2}} du$ – standartinio normalaus skirstinio pasiskirstymo funkcija.

$$\begin{aligned} \frac{f_k\left(\frac{t}{\sqrt{n}}\right) - e^{-\frac{t^2}{2n}}}{e^{-\frac{t^2}{2n}}} &= \int_{-\infty}^{\infty} e^{\left(\frac{it}{\sqrt{n}}\right)x - \frac{1}{2}\left(\frac{it}{\sqrt{n}}\right)^2} d[F_k(x) - \Phi(x)] = \int_{-\infty}^{\infty} \sum_{l=0}^{\infty} \frac{1}{l!} P_l(u) \left(\frac{it}{\sqrt{n}}\right)^l d[F_k(u) - \Phi(u)] = \\ &= \sum_{l=1}^{\infty} \frac{1}{l!} \left(\frac{it}{\sqrt{n}}\right)^l \int_{-\infty}^{\infty} P_l(u) d[F_k(u) - \Phi(u)], \end{aligned}$$

kur $P_1(u) = \text{Meixner daugianariai}$, $P_0(u) = 1$ ir

$$\int_{-\infty}^{\infty} d[F_k(u) - \Phi(u)] = 1 - 1 = 0.$$

Tada

$$B_1(t) = \sum_{l=1}^{\infty} \frac{1}{l!} \left(\frac{it}{\sqrt{n}}\right)^l \int_{-\infty}^{\infty} P_l(u) d\left(\sum_{k=1}^N [F_k(u) - \Phi(u)]\right)$$

ir

$$\begin{aligned} &\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2} - itx} \frac{n}{N} \sum_{l=1}^{\infty} \left(\frac{it}{\sqrt{n}}\right)^l \frac{1}{l!} \int_{-\infty}^{\infty} P_l(u) d\left(\sum_{k=1}^N [F_k(u) - \Phi(u)]\right) dt = \\ &= \frac{n}{N} \sum_{l=1}^{\infty} \left(\frac{1}{\sqrt{n}}\right)^l \frac{1}{l!} \int_{-\infty}^{\infty} P_l(u) d\left(\sum_{k=1}^N [F_k(u) - \Phi(u)]\right) * \frac{1}{2\pi} \int_{-\infty}^{\infty} (it)^l e^{-\frac{t^2}{2} - itx} dt = \\ &= \sum_{l=1}^{\infty} n \left(\frac{(-1)^l}{\sqrt{n}}\right)^l \frac{d^l}{dx^l} \varphi(x) \frac{1}{l!} \int_{-\infty}^{\infty} P_l(u) d\left(\frac{1}{N} \sum_{k=1}^N [F_k(u) - \Phi(u)]\right) \end{aligned} \quad (9)$$

$$\text{kur } \varphi(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2} - itx} dt \text{ ir } \frac{1}{2\pi} \int_{-\infty}^{\infty} (it)^l e^{-\frac{t^2}{2} - itx} dt = (-1)^l \frac{d^l}{dx^l} \varphi(x).$$

$$e^{\left(\frac{it}{\sqrt{n}}\right)x - \frac{1}{2}\left(\frac{it}{\sqrt{n}}\right)^2} = \sum_{m=0}^{\infty} \frac{P_m(x)}{m!} \left(\frac{it}{\sqrt{n}}\right)^m$$

Mums reikia rasti Meixner daugianarius $P_m(x)$.

Pakeiskime $\frac{it}{\sqrt{n}}$ į ε :

$$e^{\varepsilon x - \frac{\varepsilon^2}{2}} = \sum_{m=0}^{\infty} \frac{P_m(x)}{m!} \varepsilon^m$$

Imame abiejų pusių išvestines $v = 1, 2, \dots$ pagal ε ir gauname

$$e^{\varepsilon x - \frac{\varepsilon^2}{2}} (x - \varepsilon) = \sum_{m=1}^{\infty} \frac{P_m(x)}{m!} m \varepsilon^{m-1} = \frac{P_1(x)}{1!} * 1 + \sum_{m=2}^{\infty} \frac{P_m(x)}{m!} m \varepsilon^{m-1}$$

Kai $\varepsilon = 0$, gauname

$$P_1(x) = x.$$

Rasime išvestinę, kai $v = 2$,

$$\left(e^{\varepsilon x - \frac{\varepsilon^2}{2}} (x - \varepsilon) \right)' = e^{\varepsilon x - \frac{\varepsilon^2}{2}} (x - \varepsilon)(x - \varepsilon) + e^{\varepsilon x - \frac{\varepsilon^2}{2}} (-1) = \frac{P_2(x)}{2!} * 2 * 1 + \sum_{m=3}^{\infty} \frac{P_m(x)}{m!} m(m-1) \varepsilon^{m-2}$$

Kai $\varepsilon = 0$, gauname

$$P_2(x) = x^2 - 1.$$

Ieškome išvestinės, kai $v = 3$,

$$\begin{aligned} \left(e^{\varepsilon x - \frac{\varepsilon^2}{2}} (x - \varepsilon) \right)'' &= \left(e^{\varepsilon x - \frac{\varepsilon^2}{2}} (x - \varepsilon)(x - \varepsilon) + e^{\varepsilon x - \frac{\varepsilon^2}{2}} (-1) \right)' = \left(e^{\varepsilon x - \frac{\varepsilon^2}{2}} (x - \varepsilon)(x - \varepsilon) + e^{\varepsilon x - \frac{\varepsilon^2}{2}} (-1) \right) (x - \varepsilon) + \\ &+ \left(e^{\varepsilon x - \frac{\varepsilon^2}{2}} (x - \varepsilon) \right) (-1) - e^{\varepsilon x - \frac{\varepsilon^2}{2}} (x - \varepsilon) = e^{\varepsilon x - \frac{\varepsilon^2}{2}} (x - \varepsilon)(x - \varepsilon)(x - \varepsilon) - e^{\varepsilon x - \frac{\varepsilon^2}{2}} (x - \varepsilon) - 2e^{\varepsilon x - \frac{\varepsilon^2}{2}} (x - \varepsilon) = \\ &= \frac{P_3(x)}{3!} * 3! + \sum_{m=4}^{\infty} \frac{P_m(x)}{m!} m(m-1)(m-2) \varepsilon^{m-3} \end{aligned}$$

Kai $\varepsilon = 0$, gauname

$$P_3(x) = x^3 - 3x.$$

Rasime išvestinę, kai $v = 4$,

$$\begin{aligned} \left(e^{\varepsilon x - \frac{\varepsilon^2}{2}} (x - \varepsilon) \right)'''' &= \left(\left(e^{\varepsilon x - \frac{\varepsilon^2}{2}} (x - \varepsilon)(x - \varepsilon) + e^{\varepsilon x - \frac{\varepsilon^2}{2}} (-1) \right) (x - \varepsilon) + \left(e^{\varepsilon x - \frac{\varepsilon^2}{2}} (x - \varepsilon) \right) (-1) - e^{\varepsilon x - \frac{\varepsilon^2}{2}} (x - \varepsilon) \right)' = \\ &= \left(\left(e^{\varepsilon x - \frac{\varepsilon^2}{2}} (x - \varepsilon)(x - \varepsilon) + e^{\varepsilon x - \frac{\varepsilon^2}{2}} (-1) \right) (x - \varepsilon) + e^{\varepsilon x - \frac{\varepsilon^2}{2}} (x - \varepsilon) (-1) - e^{\varepsilon x - \frac{\varepsilon^2}{2}} (x - \varepsilon) \right) (x - \varepsilon) - \\ &- \left(e^{\varepsilon x - \frac{\varepsilon^2}{2}} (x - \varepsilon)(x - \varepsilon) + e^{\varepsilon x - \frac{\varepsilon^2}{2}} (-1) \right) - 2 \left(e^{\varepsilon x - \frac{\varepsilon^2}{2}} (x - \varepsilon)(x - \varepsilon) + e^{\varepsilon x - \frac{\varepsilon^2}{2}} (-1) \right) = \frac{P_4(x)}{4!} * 4! + \\ &+ \sum_{m=5}^{\infty} \frac{P_m(x)}{m!} m(m-1)(m-2)(m-3) \varepsilon^{m-4} \end{aligned}$$

Kai $\varepsilon = 0$, gauname

$$P_4(x) = x^4 - 6x^2 + 3.$$

Pažymėsime:

$$\int_{-\infty}^{\infty} u dF_k(u) = \mu_k \quad \text{ir} \quad \int_{-\infty}^{\infty} u^2 dF_k(u) = \beta_k.$$

Dabar padarysime tokias prielaidas:

$$\sum_{k=1}^N \mu_k = 0,$$

$$\frac{1}{N} \sum_{k=1}^N \beta_k = 1.$$

Tada,

kai $l = 1$:

$$\frac{1}{N} \int_{-\infty}^{\infty} u d \left(\sum_{k=1}^N [F_k(u) - \Phi(u)] \right) = \frac{1}{N} \sum_{k=1}^N \int_{-\infty}^{\infty} u dF_k(u) - \frac{1}{N} \sum_{k=1}^N \int_{-\infty}^{\infty} u d\Phi(u) = 0,$$

kai $l = 2$,

$$\frac{1}{N} \int_{-\infty}^{\infty} (u^2 - 1) d \left(\sum_{k=1}^N [F_k(u) - \Phi(u)] \right) = 0,$$

kai $l = 3$,

$$\frac{1}{N} \int_{-\infty}^{\infty} (u^3 - 3u) d \left(\sum_{k=1}^N [F_k(u) - \Phi(u)] \right) = \frac{1}{N} \sum_{k=1}^N \int_{-\infty}^{\infty} u^3 dF_k(u) = \alpha_3,$$

kai $l = 4$,

$$\frac{1}{N} \int_{-\infty}^{\infty} (u^4 - 6u^2 + 3) d \left(\sum_{k=1}^N [F_k(u) - \Phi(u)] \right) = \frac{1}{N} \sum_{k=1}^N \int_{-\infty}^{\infty} u^4 dF_k(u) - \frac{1}{N} \sum_{k=1}^N \int_{-\infty}^{\infty} u^4 d\Phi(u) = \alpha_4 - \frac{3}{N},$$

čia

$$\begin{aligned} \int_{-\infty}^{\infty} u^4 d\Phi(u) &= \int_{-\infty}^{\infty} u^4 \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u^3 e^{-\frac{u^2}{2}} d \left(-\frac{u}{2} \right)^2 = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u^3 de^{-\frac{u^2}{2}} = -\frac{1}{\sqrt{2\pi}} u^3 e^{-\frac{u^2}{2}} \Big|_{-\infty}^{\infty} + \\ &+ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}} 3u^2 du = \lim_{n \rightarrow \infty} \left(-\frac{1}{\sqrt{2\pi}} u^3 e^{-\frac{u^2}{2}} \right) - \lim_{n \rightarrow \infty} \left(-\frac{1}{\sqrt{2\pi}} u^3 e^{-\frac{u^2}{2}} \right) + 3 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u^2 e^{-\frac{u^2}{2}} du = 3. \end{aligned}$$

Dabar pratęsime (9) lygybę:

$$\begin{aligned}
& n \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx - \frac{t^2}{2}} \sum_{l=3}^{\infty} \frac{1}{l!} \left(\frac{it}{\sqrt{n}} \right)^l \int_{-\infty}^{\infty} P_l(u) d \left(\frac{1}{N} \sum_{k=1}^N (F_k(u) - \Phi(u)) \right) dt = n \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx - \frac{t^2}{2}} \frac{1}{3!} \left(\frac{it}{\sqrt{n}} \right)^3 \alpha_3 dt + \\
& + n \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx - \frac{t^2}{2}} \frac{1}{4!} \left(\frac{it}{\sqrt{n}} \right)^4 \left(\alpha_4 - \frac{3}{N} \right) dt + \dots = \frac{\alpha_3}{3! \sqrt{n}} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx - \frac{t^2}{2}} (it)^3 dt + \frac{(\alpha_4 - \frac{3}{N})}{4! n} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx - \frac{t^2}{2}} (it)^4 dt = \\
& = -\frac{\alpha_3}{3! \sqrt{n}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} (x^3 - 3x) - \frac{(\alpha_4 - \frac{3}{N})}{4! n} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} (x^4 - 6x^2 + 3), \\
& \text{kur } (-1)^3 \frac{d^3}{dx^3} \varphi(x) = -\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} x(x^2 - 3) = -\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} (x^3 - 3x) \quad \text{ir}
\end{aligned}$$

$$(-1)^4 \frac{d^4}{dx^4} \varphi(x) = -\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} (x^4 - 6x^2 + 3).$$

$$B_1^2(t) = \left(\frac{1}{N} \sum_{k=1}^N \frac{f_k \left(\frac{t}{\sqrt{n}} \right) - e^{-\frac{t^2}{2n}}}{e^{-\frac{t^2}{2n}}} \right)^2 N^2 = N^2 \sum_{l_1=1}^{\infty} \frac{1}{l_1!} \left(\frac{it}{\sqrt{n}} \right)^{l_1} \int_{-\infty}^{\infty} P_{l_1}(u) d \left(\frac{1}{N} \sum_{k=1}^N [F_k(u) - \Phi(u)] \right)^*$$

$$* \sum_{l_2=1}^{\infty} \frac{1}{l_2!} \left(\frac{it}{\sqrt{n}} \right)^{l_2} \int_{-\infty}^{\infty} P_{l_2}(u) d \left(\frac{1}{N} \sum_{k=1}^N [F_k(u) - \Phi(u)] \right) = N^2 \sum_{l_1=1}^{\infty} \sum_{l_2=1}^{\infty} \frac{1}{l_1! l_2!} \left(\frac{it}{\sqrt{n}} \right)^{l_1+l_2} *$$

$$\begin{aligned}
& * \int_{-\infty}^{\infty} P_{l_1}(u) d \left(\frac{1}{N} \sum_{k=1}^N [F_k(u) - \Phi(u)] \right) \int_{-\infty}^{\infty} P_{l_2}(u) d \left(\frac{1}{N} \sum_{k=1}^N [F_k(u) - \Phi(u)] \right) = \\
& = N^2 \sum_{l_1=3}^{\infty} \sum_{l_2=3}^{\infty} \frac{1}{l_1! l_2!} \left(\frac{it}{\sqrt{n}} \right)^{l_1+l_2} \int_{-\infty}^{\infty} P_{l_1}(u) d \left(\frac{1}{N} \sum_{k=1}^N [F_k(u) - \Phi(u)] \right) * \int_{-\infty}^{\infty} P_{l_2}(u) d \left(\frac{1}{N} \sum_{k=1}^N [F_k(u) - \Phi(u)] \right) = \\
& = \frac{1}{36} \left(\frac{it}{\sqrt{n}} \right)^6 N^2 \alpha_3^2 + \dots
\end{aligned}$$

$$\begin{aligned}
B_2(t) &= -\frac{1}{2} \sum_{k=1}^N b_k^2(t) = -\frac{1}{2} \sum_{k=1}^N \left(\sum_{l=3}^{\infty} \frac{1}{l!} \left(\frac{it}{\sqrt{n}} \right)^l \int_{-\infty}^{\infty} P_l(u) d([F_k(u) - \Phi(u)]) \right)^2 = \\
&= -\frac{1}{2} \sum_{k=1}^N \left(\frac{1}{3!} \right)^2 \left(\frac{it}{\sqrt{n}} \right)^6 \left(\int_{-\infty}^{\infty} P_3(u) d([F_k(u) - \Phi(u)]) \right)^2 + \dots = \\
&= -\frac{1}{2} * \frac{1}{36} N \left(\frac{it}{\sqrt{n}} \right)^6 \frac{1}{N} \sum_{k=1}^N \left(\int_{-\infty}^{\infty} P_3(u) d([F_k(u) - \Phi(u)]) \right)^2 + \dots = -\frac{1}{2} * \frac{1}{36} N \left(\frac{it}{\sqrt{n}} \right)^6 \alpha_{l_3} + \dots,
\end{aligned}$$

$$\text{čia } \alpha_{l_3} = \frac{1}{N} \sum_{k=1}^N \left(\int_{-\infty}^{\infty} P_3(u) d([F_k(u) - \Phi(u)]) \right)^2.$$

Užrašysime trečią (8) lygybės dėmenį:

$$\begin{aligned}
&\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} e^{-\frac{t^2}{2}} \frac{n(n-1)}{N(N-1)} \left(\frac{B_1^2}{2!} + \frac{B_2}{1!} \right) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} e^{-\frac{t^2}{2}} \frac{n(n-1)}{N(N-1)} * \\
&* \left(\frac{1}{72} \left(\frac{it}{\sqrt{n}} \right)^6 N^2 \alpha_3^2 + \dots - \frac{1}{2} * \frac{1}{36} N \left(\frac{it}{\sqrt{n}} \right)^6 \alpha_{l_3} + \dots \right) dt = \frac{1}{72} \frac{1 - \frac{1}{n}}{1 - \frac{1}{N}} \frac{1}{n} \alpha_3^2 \frac{d^6}{dx^6} \varphi(x) + \dots \\
&+ \dots - \frac{1}{72} \frac{1 - \frac{1}{n}}{(N-1)n} \alpha_{l_3} \frac{d^6}{dx^6} \varphi(x) + \dots = \frac{1}{72} \frac{1}{n} \left(1 - \frac{1}{n} \right) \frac{d^6}{dx^6} \varphi(x) \left(\frac{1}{1 - \frac{1}{N}} \alpha_3^2 - \frac{1}{N-1} \alpha_{l_3} \right) + \dots
\end{aligned}$$

4. Išvados

Dabar galime suformuluoti tokias teoremas:

1 TEOREMA. Teisinga lygybė

$$\begin{aligned} P\{S_n < x\} &= \Phi(x) - \frac{\alpha_3}{3!\sqrt{n}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{u^2}{2}} (u^3 - 3u) du - \frac{(\alpha_4 - \frac{3}{N})}{4!n} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{u^2}{2}} (u^4 - 6u^2 + 3) du + \dots = \\ &= \Phi(x) - \frac{\alpha_3}{3!\sqrt{n}} \frac{1}{\sqrt{2\pi}} \left(x^2 e^{-\frac{x^2}{2}} - e^{-\frac{x^2}{2}} \right) + \frac{(\alpha_4 - \frac{3}{N})}{4!n} \frac{1}{\sqrt{2\pi}} \left(3xe^{-\frac{x^2}{2}} - x^3 e^{-\frac{x^2}{2}} \right) + \dots \end{aligned}$$

kur $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{u^2}{2}} du$ – standartinio normalaus skirstinio pasiskirstymo funkcija,

$$\frac{1}{N} \sum_{k=1}^N \int_{-\infty}^{\infty} u^3 dF_k(u) = \alpha_3 \text{ ir } \frac{1}{N} \sum_{k=1}^N \int_{-\infty}^{\infty} u^4 dF_k(u) = \alpha_4.$$

2 TEOREMA. Teisinga lygybė

$$\begin{aligned} \binom{N}{n}^{-1} \sum_{1 \leq i_1 < i_2 < \dots < i_n \leq N} p_{i_1, i_2, \dots, i_n}(x) &= \varphi(x) - \frac{\alpha_3}{3!\sqrt{n}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} (x^3 - 3x) - \frac{(\alpha_4 - \frac{3}{N})}{4!n} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} (x^4 - 6x^2 + 3) + \\ &+ \frac{1}{72} \frac{1}{n} \left(1 - \frac{1}{n} \right) \frac{d^6}{dx^6} \varphi(x) \left(\frac{1}{1 - \frac{1}{N}} \alpha_3^2 - \frac{1}{N-1} \alpha_{13} \right) + \dots, \end{aligned}$$

$$\text{kur } p_{i_1, i_2, \dots, i_n}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \prod_{j=1}^n f_{i_j} \left(\frac{t}{\sqrt{n}} \right) dt.$$

Iš šios teoremos išplaukia 1 teoremos tvirtinimas.

Summary

Bengt von Bahr in [1] proved the following theorem:

THEOREM. Let X_1, X_2, \dots, X_N be independent random variables and let S_n be the sum of n of them chosen at random. If $EX_k = \mu_k$, $EX_k^2 = \beta_k$, $E|X_k|^3 = \gamma_k$, where $\sum_{k=1}^N \mu_k = 0$,

$\frac{1}{N} \sum_{k=1}^N \beta_k = 1$, $\alpha^2 = \frac{1}{N} \sum_{k=1}^N \mu_k^2$ and $\gamma = \max_{1 \leq k \leq N} \gamma_k$, then

$$\left| P\left(\frac{S_n}{\sqrt{n(1-f\alpha^2)}} \leq x\right) - \Phi(x) \right| \leq \frac{60\gamma}{\sqrt{n(1-f\alpha^2)}^{\frac{3}{2}}}$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{y^2}{2}} dy$ is the normalized Gaussian distribution funktion and $f = \frac{n}{N}$.

We proved that

$$\begin{aligned} P\{S_n < x\} &= \Phi(x) - \frac{\alpha_3}{3!\sqrt{n}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{u^2}{2}} (u^3 - 3u) du - \frac{(\alpha_4 - \frac{3}{N})}{4!n} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{u^2}{2}} (u^4 - 6u^2 + 3) du + \dots = \\ &= \Phi(x) - \frac{\alpha_3}{3!\sqrt{n}} \frac{1}{\sqrt{2\pi}} \left(x^2 e^{-\frac{x^2}{2}} - e^{-\frac{x^2}{2}} \right) + \frac{(\alpha_4 - \frac{3}{N})}{4!n} \frac{1}{\sqrt{2\pi}} \left(3xe^{-\frac{x^2}{2}} + x^3 e^{-\frac{x^2}{2}} \right) + \dots \end{aligned}$$

This is the answer what the difference's $P\{S_n < x\} - \Phi(x)$ estimator depends on.

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