

Some inequalities in the arithmetic semigroups

Gintautas BAREIKIS (VU)
e-mail: gintautas.bareikis@maf.vu.lt

1. Introduction

By \mathcal{N} , \mathcal{R} , \mathcal{C} we denote the sets of natural, real, and complex numbers, respectively and $\mathcal{N}_0 := \mathcal{N} \cup \{0\}$.

Let \mathcal{S} be an additive arithmetical semigroup, with degree mapping $\partial : \mathcal{S} \rightarrow \mathcal{N}_0$ (more about it see in [1], [3]), satisfying the following assumption about a total number $S(n)$ of elements of degree n in \mathcal{S} .

Axiom: *There exist the constants $A > 0$, $\gamma > 1$ and $q > 1$ (all depending on \mathcal{S}), such that*

$$S(n) = Aq^n + O\left(\frac{q^n}{n^\gamma}\right) \text{ as } n \rightarrow \infty.$$

We shall construct a probability space, which enable to consider the arithmetical functions as sums of independent random variables.

Define the set \mathcal{S}_n , $n \in \mathcal{N}$ by

$$\mathcal{S}_n = \{a \in \mathcal{S}; \forall p|a \Rightarrow \partial(p) \leq n\}.$$

where p throughout the paper means a prime element of \mathcal{S} .

By \mathcal{A} we denote the class of all subsets of \mathcal{S} . Then for each $A \in \mathcal{A}$ we write

$$\nu_n(A) = \gamma_n \sum_{a \in A \cap \mathcal{S}_n} \frac{1}{q^{\partial(a)}}, \quad \gamma_n := \prod_{\partial(p) \leq n} \left(1 - \frac{1}{q^{\partial(p)}}\right). \quad (1)$$

It is clear, that

$$\nu_n(\mathcal{S}_n) = 1.$$

Thus, the tripple $\{\mathcal{S}, \mathcal{A}, \nu_n\}$ is a probability space.

By \bar{U} we denote the supplement of the set U . It follows from our Axiom, that $q = 1 + s$, $s > 0$. Let with some fixed u

$$\mathcal{Q}_1 := \{p^k \in \mathcal{S}; \partial(p) \leq n, k \geq 1, \exists u \in U, (u, p) = 1, p^k u \in U\}.$$

Lemma 1. Suppose that $U \subset S$ is such, that

$$a := \nu_n(\bar{U}) < \frac{1}{48} \left(\frac{s}{s+1} \right)^2.$$

Then we have

$$\sum_{p^k \in Q_2} \frac{1}{q^{k\partial(p)}} \leq \frac{3(s+1)}{s} a,$$

where

$$Q_2 = \{p^k \in S; \partial(p) \leq n, k \geq 1, p^k \notin Q_1\}.$$

Proof of Lemma 1. Let $p_1^{k_1} \in Q_2$. We set

$$V_{p_1} = \{p_1^{k_1} u; u \in U, (p_1, u) = 1\}.$$

It is clear, that $V_{p_1} \cap U = \emptyset$. Then $V_{p_1} \subset \bar{U}$. It follows that

$$\nu_n(V_{p_1}) \leq a.$$

Further, for $p_1^{k_1} \in Q_2$ we have

$$\begin{aligned} \nu_n(V_{p_1}) &= \gamma_n \sum_{u \in U \cap S_n} \frac{1}{q^{k_1\partial(p_1) + \partial(u)}} - \gamma_n \sum_{\substack{u \in U \cap S_n, \\ u \equiv 0 \pmod{p_1}}} \frac{1}{q^{\partial(u) + \partial(p_1)k_1}} \\ &\geq \frac{\nu_n(U)}{q^{k_1\partial(p_1)}} - \frac{1}{q^{(k_1+1)\partial(p_1)}}. \end{aligned}$$

From the last relation and equality

$$\nu_n(U) = 1 - \nu_n(\bar{U}) = 1 - a,$$

we obtain

$$\nu_n(V_{p_1}) \geq \frac{1}{q^{k_1\partial(p_1)}} \left\{ 1 - a - \frac{1}{q} \right\}. \quad (2)$$

Thus, it follows

$$a \geq \frac{s}{q^{k_1\partial(p_1)} 2(1+s)}.$$

The assumptions of the Lemma and the last inequality yield

$$\frac{1}{q^{k_1 \theta(p_1)}} \leq \frac{2(1+s)a}{s} \leq \frac{s}{24(1+s)}. \quad (3)$$

Let p_1, p_2 be two distinct prime elements. If $p_1 = p_2$ and $k_1 \neq k_2$, here $p_1^{k_1}, p_2^{k_2} \in Q_2$, then $V_{p_1} \cap V_{p_2} = \emptyset$. In the case $p_1 \neq p_2$ we have,

$$V_{p_1} \cap V_{p_2} \subset \{m \in \mathcal{S}; m \equiv 0 \pmod{r^{k_1}}\},$$

here $r = p_1 p_2$. Therefore in both cases we have

$$\begin{aligned} \nu_n(V_{p_1} \cap V_{p_2}) &\leq \nu_n(m \in \mathcal{S}; m \equiv 0 \pmod{r^{k_1}}) \leq \gamma_n \sum_{l \in \mathcal{S}_n} \frac{1}{q^{\theta(l)} q^{k_1 \theta(r)}} \\ &= \frac{1}{q^{k_1 \theta(r)}}. \end{aligned} \quad (4)$$

Let Q be an arbitrary subset of Q_2 . Define

$$V = \bigcup_{p^k \in Q} V_p, \quad b := \sum_{p^k \in Q} \frac{1}{q^{k \theta(p)}}.$$

It follows from the estimations (2) and (4) that

$$\nu_n(V) \geq \sum_{p_1^{k_1} \in Q} \nu_n(V_{p_1}) - \sum_{p_1^{k_1}, p_2^{k_2} \in Q} \nu_n(V_{p_1} \cap V_{p_2}) \geq b \left(\frac{s}{2(1+s)} - b \right). \quad (5)$$

The equality $V \cap U = \emptyset$ yields that $\nu_n(V) \leq a$.

Suppose, that

$$b \leq \frac{s}{8(1+s)}.$$

Using this assumption in (5) we arrive at the relation

$$a \geq b \left(\frac{s}{2(1+s)} - \frac{s}{8(1+s)} \right) \geq \frac{sb}{3(1+s)}.$$

Choosing $Q = Q_2$, from the last estimation we conclude

$$\sum_{p^k \in Q_2} \frac{1}{q^{k \theta(p)}} \leq \frac{3(1+s)a}{s}.$$

We see, that the last relation yields validity of the Lemma 1.

Further, assume that

$$b > \frac{s}{8(1+s)}.$$

Let the set $Q^* \subset Q_2$ satisfy the following inequality:

$$b = b(Q^*) = \sum_{p^k \in Q^*} \frac{1}{q^{k\theta(p)}} \leq \frac{s}{8(1+s)} \quad (6)$$

and for all

$$p^k \in Q_2 \setminus Q^*, \quad b + \frac{1}{q^{k\theta(p)}} > \frac{s}{8(1+s)}.$$

Applying the estimation (3) we have

$$b > \frac{s}{8(1+s)} - \frac{1}{q^{k\theta(p)}} \geq \frac{s}{12(1+s)}.$$

The last estimation together with (5) and (6) yield

$$a \geq \frac{1}{32} \left(\frac{s}{1+s} \right)^2.$$

Thus, we arrive at the contradiction, because

$$a \geq \left(\frac{s}{1+s} \right)^2 \frac{1}{32} > \left(\frac{s}{1+s} \right)^2 \frac{1}{48}.$$

Lemma 1 is proved.

For the set $U \subset \mathcal{S}$ we define the set $V = V(U)$ by

$$V = \left\{ a \in \mathcal{S}; \begin{aligned} & au_3 = u_1 u_2, \quad u_2 = u_3 k, \quad (u_3, k) = 1, \quad (u_1, k) = 1, \\ & u_i \in U, \quad i = 1, 2, 3 \end{aligned} \right\}. \quad (7)$$

Lemma 2. Suppose that $U \subset \mathcal{S}$ and the set V is defined in (7). Then the equality

$$\mu_n(\overline{V}) = O_s(\nu_n(\overline{U}))$$

holds. Here $s = q - 1$, $\mu_n(A) = (1/S(n)) \sum_{\substack{m \in A \cap \mathcal{S}, \\ \theta(m) = n}} 1$.

Proof of Lemma 2. We begin proving the lemma under an auxiliary condition

$$\nu_n(\bar{U}) = a < \frac{1}{48} \left(\frac{s}{s+1} \right)^2.$$

Assume that $m = p^k l$, when $(p, l) = 1$, $p^k \in Q_1$, $l \in U$. Then for each $u \in U$ with conditions: $(p, u) = 1$ and $p^k u \in U$ we have

$$mu = lup^k.$$

Therefore $m \in V$. It is clear, that for each $m = p^k l$, where $(p, l) = 1$, $p^k \in Q_2$ or $l \in \bar{U}$ we have $m \in \bar{V}$. Further

$$\begin{aligned} \sum_{\substack{\partial(m) \leq n \\ m \in \bar{V}}} \partial(m) &\leq \sum_{\substack{\partial(l) \leq n \\ l \in \bar{U}}} \sum_{\substack{\partial(p^k) \leq n - \partial(l)}} \partial(p^k) + \sum_{\substack{p^k \in Q_2 \\ \partial(p^k) \leq n}} \partial(p^k) \sum \partial(l) \leq n - \partial(p^k) 1 \\ &= S_1 + S_2. \end{aligned} \tag{8}$$

Applying Lemma 1 we obtain

$$\sum_{k \partial(p) \leq n - \partial(l)} \partial(p^k) \leq q^{n - \partial(l)}, \quad \sum_{\substack{m \in \bar{U} \\ \partial(m) \leq n}} \frac{1}{q^{\partial(m)}} \leq \gamma_n^{-1} \left(\gamma_n \sum_{m \in S_n \cap \bar{U}} \frac{1}{q^{\partial(m)}} \right) = O(na).$$

It follows from these two inequalities, that

$$S_1 = O(q^n an). \tag{9}$$

Let us consider the sum S_2 . Making use of Lemma 8 [3] and Lemma 1 we assert

$$S_2 \leq \sum_{\substack{p^k \in Q_2 \\ \partial(p^k) \leq n}} \partial(p^k) q^{n - \partial(p^k)} = O_s(nq^n a).$$

This estimation together with (9) and (8) yield

$$\sum_{\substack{m \in \bar{V} \\ \partial(m) \leq n}} \partial(m) = O_s(q^n an).$$

Further, making use of the last estimation we obtain

$$\mu_n(\bar{V}) \leq \frac{1}{nS(n)} \sum_{\substack{m \in \bar{V} \cap S \\ \partial(m) \leq n}} \partial(m) = O(a).$$

It follows

$$\mu_n(\bar{V}) = O_s(\nu_n(\bar{U})).$$

The case

$$\nu_n(\bar{U}) > \frac{1}{48} \left(\frac{s}{s+1} \right)^2$$

is obvious. Lemma 2 is proved.

We define

$$\xi_p = \xi_p(m) = g_n(p^k); \text{ if } p^k || m, k \geq 0, m \in \mathcal{S}, n \in \mathcal{N}. \quad (10)$$

Here $g_n(m)$ is some sequence of real-valued arithmetic functions. The reader can easily prove the following

Theorem 1. *The functions ξ_p ; $\partial(p) \leq n$, defined in (10), are independent random variables in the space $\{\mathcal{S}, \mathcal{A}, \nu_n\}$, distributed by*

$$\nu_n(\xi_p = g_n(p^k)) = \frac{1}{q^{k\partial(p)}} (1 - q^{-\partial(p)}), \quad (11)$$

$$k \geq 0, \partial(p) \leq n, n \in \mathcal{N}.$$

We set

$$G_n(m, u) = \prod_{\partial(p) \leq z(u), p^\alpha || m} g_n(p^\alpha), \quad m \in \mathcal{S};$$

$$Z_n(u) = \prod_{\partial(p) \leq z(u)} |\xi_p|^{\frac{1}{\partial(p)}} \operatorname{sgn} \xi_p e^{-\alpha(u)},$$

where the function $z(u)$, $u \in [0, 1]$ is some real-valued monotonic function with condition $z(1) = n$, $\beta(n) \rightarrow \infty$ as $n \rightarrow \infty$, and $\xi(p)$, $\partial(p) \leq n$ are random variables defined by (11).

Theorem 2. *For each $0.5 < \epsilon < 1$ we have*

$$\mu_n \left(\sup_{0 \leq u \leq 1} |G_n(m, u) - 1| \geq \epsilon \right) = O_s \left(\nu_n \left(\sup_{0 \leq u \leq 1} |Z_n(u) - 1| \geq \frac{\epsilon}{8} \right) \right).$$

Proof of Theorem 2. Set

$$U = \left\{ m \in \mathcal{S}; \sup_{0 \leq u \leq 1} |G_n(m, u) - 1| < \frac{\epsilon}{8} \right\},$$

here $u \in [0, 1]$ is a fixed number. Let the set V be defined as in (7). It means, that for each $m \in V$ there exist the numbers $n_1, n_2, n_3 \in U$ such, that $mn_3 = n_1n_2$, where $n_3 | n_2, n_2 = n_3l, (n_1, l) = 1$. Since each number of \mathcal{S} has unique decomposition into the power of prime numbers, then multiplicity of $G_n(o, u)$ implies relation

$$G_n(m, u) = G_n(n_1, u)G_n(l, u) = G_1G_2G_3^{-1},$$

where $G_i = G_n(n_i, u); n_i \in U, i = 1, 2, 3$.

Use of the identity

$$1 - \prod_{i=1}^k b_i = \sum_{j=1}^k (1 - b_j) \prod_{l=j+1}^k b_l$$

leads to the inequalities

$$\begin{aligned} |G_n(m, u) - 1| &\leq |1 - G_1||G_2G_3^{-1}| + |1 - G_2||G_1G_3^{-1}| + |1 - G_3||G_2G_1G_3^{-1}| \\ &\leq (1 + \epsilon)^2(|1 - G_1| + |1 - G_2| + 2|1 - G_3|) \leq \epsilon. \end{aligned}$$

Therefore $|G_n(m, u) - 1| \leq \epsilon, \forall m \in U$. It yields

$$V \subset \{m \in \mathcal{S}; \sup_{0 \leq u \leq 1} |G_n(m, u) - 1| \leq \epsilon\}.$$

The last relation together with Lemma 2 complete the proof of the Theorem 2.

References

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Keletas nelygybių aritmetinėse pusgrupėse

G. Bareikis

Šiame darbe pateiktas Ružos nelygybės analogo aritmetinėse pusgrupėse įrodymas. Be to, pateiktas pavyzdys, kaip galima pritaikyti šią nelygybę aritmetinių bei atsitiktinių multiplikacinių procesų tikimybių palyginimui.