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Sheaves on quantaloids

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1. Introduction

This note considers sets valued in possibly non-unital quantaloids (categories enriched in the category of complete sup-lattices subject to certain laws). Quantaloids are a natural and useful categorical generalization of quantales (which are the one-object quantaloids, i.e., complete sup-lattices equipped with a multiplication). In the "logical" approach (see [3]) a sheaf on a topological space U corresponds to the "complete" set valued in the quantale $O(U)$ of open sets of U with the intersection as a multiplication. This approach to sheaves was further developed by U. Höhle [4,5] (based on a symmetric or "right symmetric" (but non-idempotent) quantale) and by F. Borceux and G. van den Bossche [1], C.J. Mulvey and M. Nawaz [6] (based on an idempotent quantale). In the present note, we provide a setting for sheaves on quantaloids (which is more general than ones mentioned above) taking our inspiration in G. van den Bossche's work [2] where sets valued in quantaloids are presented using "matrices". Our results are submitted without proofs. We are going to detail it in a subsequent paper.

2. Preliminaries on quantaloids and matrices over their

We begin by reviewing a few perminent definitions.

DEFINITION 2.1. A *quantaloid* is a locally small category Q (not necessarily having units) such that:

(i) for all u,v objects in Q, the hom-set $Q(u, v)$ is a complete lattice,

(ii) composition of morphisms of Q (in this note denoted by &) preserves arbitrary joins in both variables:

$$p\&\bigvee_i q_i = \bigvee_i p\&q_i \text{ and } (\bigvee_i p_i)\&q = \bigvee_i p_i\&q$$

for all morphisms p,q of Q and for all families (p_i), (q_i) of morphisms of Q (forming respective composable pairs).

Note that we use the unconventional left-to-right direction for composition of morphisms. Examples of the one-object quantaloids (which are called quantales) include frames (and thus complete Boolean algebras) and various ideal lattices of rings or C^*-algebras. Many other examples of quantales and quantaloids can be found in [1]–[8].

From now Q will be an arbitrary quantaloid (not necessarily having units) having a small set of objects. Let Q_0 denote this set and Q_1 the set of morphisms of Q. Let $Sets/Q_0$ denotes the category whose objects are families X of sets X_u indexed by $u \in Q_0$. An element $x \in X_u$ will be called an element over u and we shall sometimes write $d(x)$ for u and $x \in X$ for $x \in X_u$. Morphisms in $Sets/Q_0$ are families of maps $f_u : X_u \to Y_u$.

DEFINITION 2.2 (Definition 1.3 [2]). Let Q be a quantaloid and X and Y be two objects of $Sets/Q_0$. A *matrix* M from X to Y assigns to each pair x,y of $X \times Y$ an element of Q_1: $m_{x,y} : d(x) \to d(y)$. Matrices compose by "matrix multiplication": for $M : X \to Y$, and $N : Y \to Z$, the *composite* $M\&N = L : X \to Z$ has its general element given by

$$l_{x,z} = \bigvee_{y \in Y} m_{x,y} \& n_{y,z}.$$

3. Q-sets and bimodules

The notions of a Q-set and of a bimodule which will be given in this section are taken from [2].

DEFINITION 3.1 (Definition 2.1 [2]). Let Q be a quantaloid. A *Q-set* is an object X of $Sets/Q_0$ provided with a matrix $A : X \to X$ satisfying the following:

Idempotency: $A\&A = A$

A Q-set (X, A) will be called *separated* whenever it satisfies

Separation: if $a_{x,x"} = a_{x',x"}$ and $a_{x",x} = a_{x",x'}$ for all $x" \in X$, then $x = x'$.

An element $x \in X$ of a Q-set (X, A) will be said to be *strict* provided that it satisfies

Strictness: $a_{x,x}\&a_{x,x'} = a_{x,x'}$ and $a_{x',x}\&a_{x,x} = a_{x',x}$ for all $x' \in X$ and a Q-set (X, A) itself will be called *strict* whenever every element $x \in X$ is strict.

We shall usually write a_x for $a_{x,x}$. Note that strict or separated Q-sets were not considered by G. van den Bossche in [2]. Conditions corresponding to Separation appear in [3, 4, 5]. Q-sets as defined in [3,6] are strict Q-sets in our sense.

DEFINITION 3.2 (Definition 2.6 [2]). Let (X, A) and (Y, B) be Q-sets. A *bimodule* (or morphism as called in [2]) $\mathcal{F}$ from (X, A) to (Y, B) , written $\mathcal{F} : (X, A) \to (Y, B)$, is a pair of adjoint matrices $F : X \to Y$, $F^\# : Y \to X$, $F \dashv F^\#$, compatible with the structural matrices A and B, i.e., a pair of matrices satisfying the following:

Compatibility: $F = A\&F = F\&B$ and $F^\# = B\&F^\# = F^\#\&A$,

Adjunction: $A \leqslant F\&F^\#$ (unit) and $F^\#\&F \leqslant B$ (counit).

Bimodules compose just by composition of matrices. Structural matrices are their own adjoints and determine the units for bimodule composition. Thus Q-sets and their bimodules constitute a category denoted by $Q-Sets$. Our first result is the following

PROPOSITION 3.3 (cf. Corollary 13 [6]). Given a bimodule $\mathcal{F} = (F, F^{\#}) : (X, A) \to (Y, B)$ between Q-sets of a quantaloid Q,

(i) if $A \geqslant F\&F^{\#}$, then $\mathcal{F}$ is a monic in $Q-Sets$ (i.e., F can be "canceled on the right" (and hence $F^{\#}$ on the left)),

(ii) if $F^{\#}\&F \geqslant B$, then $\mathcal{F}$ is an epic in $Q-Sets$ (i.e., F is left-cancellable),

(iii) $\mathcal{F}$ is invertible in $Q-Sets$ (of which the inverse is $(F^{\#}, F)$) iff $A \geqslant F\&F^{\#}$ and $F^{\#}\&F \geqslant B$.

4. Singletons of Q-sets and complete Q-sets of a quantaloid Q

DEFINITION 4.1. Given a Q-set (X, A), by a *singleton* $\mathcal{S}$ of (X, A) will be meant a pair of matrices, of a "row" $S : 1 \to X$ and of a "column" $S^{\#} : X \to 1$ (with 1 a singleton set) assigning to each $x \in X$ morphisms $s_x : u \to d(x)$ and $s_x^{\#} : d(x) \to u$ of Q, respectively, and having the following properties:

Reproducing Property:

$$s_x = \bigvee_{x' \in X} s_{x'}\&a_{x',x} \quad \text{and} \quad s_x^{\#} = \bigvee_{x' \in X} a_{x,x'}\&s_{x'}^{\#}$$

for all $x \in X$, or in matrix terms, $S = S\&A$ and $S^{\#} = A\&S^{\#}$;

Singleton Condition: $s_x^{\#}\&s_{x'} \leqslant a_{x,x'}$ for all $x, x' \in X$, i.e., $S^{\#}\&S \leqslant A$;

Totality: $l \leqslant \bigvee_{x \in X} s_x\&s_x^{\#}$ for some idempotent $l : u \to u$ of Q such that $s_x = l\&s_x$ and $s_x^{\#} = s_x^{\#}\&l$ for all $x \in X$, i.e., $\{l\} \leqslant S\&S^{\#}$ with $S = \{l\}\&S$, $S^{\#} = S^{\#}\&\{l\}$.

We shall sometimes write $d(\mathcal{S})$ for u and shall write $\tilde{a_{\mathcal{S}}}$ for $\bigvee_{x \in X} s_x\&s_x^{\#}$. It easily verified that, for any strict element $x \in X$ of a Q-set (X, A), the pair $\mathcal{A}_x = (A_x, A_x^{\#})$ consisting of the row $A_x = (a_{x,x'})_{x' \in X}$ and of the column $A_x^{\#} = (a_{x',x})^{x' \in X}$ of the structural matrix A is a singleton of (X, A) (with $l = a_x$).

PROPOSITION 4.2. Let $\tilde{X} = (\tilde{X}_u)_{u \in Q_0}$ be the family of sets of all singletons of a Q-set (X, A). Let $\tilde{A}$ be the matrix defined by: $\tilde{a}_{\mathcal{S},\mathcal{T}} = S\&T^{\#}$ for all $\mathcal{S} = (S, S^{\#})$, $\mathcal{T} = (T, T^{\#}) \in \tilde{X}$. Then the pair $(\tilde{X}, \tilde{A})$ forms a strict Q-set.

Now we turn to an important class of strict and separated Q-sets of an arbitrary quantaloid Q having the property that each singleton of a Q-set (X, A) is determined by a unique element of X. We need the following analog of Definition 4.15 [3].

DEFINITION 4.3. Let (Y, B) be a Q-set.

(i) We say (X, A) is a *sub-Q-set* of (Y, B) and write $(X, A) \subseteq (Y, B)$ to mean that $X_u \subseteq Y_u$ for each $u \in Q_0$ and the matrix A obtained by restricting B to elements of X makes (X, A) into a Q-set such that

$$b_{x,y} = \bigvee_{x' \in X} a_{x,x'}\&b_{x',y} \quad \text{and} \quad b_{y,x} = \bigvee_{x' \in X} b_{y,x'}\&a_{x',x}$$

for all $x \in X$ and $y \in Y$. (By the way, if (Y, B) is strict, then this condition is always satisfied.)

(ii) We say $(X, A) \subseteq (Y, B)$ *generates* (Y, B) whenever

$$b_{y,y'} = \bigvee_{x \in X} b_{y,x} \& b_{x,y'}$$

for all $y, y' \in Y$.

Note that henceforth we shall keep the notation $(\underline{X}, \underline{A})$ for the sub-Q-set of a Q-set (X, A) of all strict elements of X.

DEFINITION 4.4. If $(\underline{X}, \underline{A}) \subseteq (X, A)$ generates (X, A), then the Q-set (X, A) is said to be *strictly generated*.

PROPOSITION 4.5. For a strictly generated Q-set (X, A), the Q-set $(\tilde{X}, \tilde{A})$ of all singletons of (X, A) is separated, i.e., if $\tilde{a}_{\mathcal{S},\mathcal{S}''} = \tilde{a}_{\mathcal{S}',\mathcal{S}''}$ and $\tilde{a}_{\mathcal{S}'',\mathcal{S}} = \tilde{a}_{\mathcal{S}'',\mathcal{S}'}$ for all $\mathcal{S}'' \in \tilde{X}$, then $\mathcal{S} = \mathcal{S}'$.

PROPOSITION 4.6. Let $(X, A) \subseteq (Y, B)$ be a sub-Q-set of (Y, B) generating it. For any singleton $\mathcal{T}$ of (Y, B), the restriction ${}_X\mathcal{T}$ of $\mathcal{T}$ to X is also a singleton of (X, A) satisfying the condition that $\tilde{a}_{{}_X\mathcal{T}} = \tilde{b}_{\mathcal{T}}$.

PROPOSITION 4.7. Let $(X, A) \subseteq (Y, B)$ be Q-sets. Then every singleton $\mathcal{S}$ of(X, A) extends to a singleton $\mathcal{S}'$ of (Y, B) (with $s'_x = s_x$ and $s'^{\#}_x = s^{\#}_x$ for all $x \in X$). Among possible extensions of a singleton $\mathcal{S} \in \tilde{X}$ there is the "bottom" extension, the singleton ${}^Y\mathcal{S}$ of (Y, B) defined by

$${}^Y s_y = \bigvee_{x \in X} s_x \& b_{x,y} \quad \text{and} \quad {}^Y s^{\#}_y = \bigvee_{x \in X} b_{y,x} \& s^{\#}_x$$

for all $y \in Y$. This singleton has the properties that $\tilde{b}_{{}^Y\mathcal{S}} = \tilde{a}_{\mathcal{S}}$ and that ${}^Y\mathcal{S} \leqslant \mathcal{S}'$ for any singleton $\mathcal{S}'$ of (Y, B) which extends $\mathcal{S}$. The same holds for any pair of such extensions: if ${}^Y\mathcal{S}, {}^Y\mathcal{T} \in \tilde{Y}$ is a pair of bottom extensions of $\mathcal{S}, \mathcal{T} \in \tilde{X}$ to singletons of (Y, B),then $\tilde{b}_{{}^Y\mathcal{S},{}^Y\mathcal{T}} = \tilde{a}_{\mathcal{S},\mathcal{T}}$, while, for any other extensions $\mathcal{S}', \mathcal{T}' \in \tilde{Y}$ of $\mathcal{S}, \mathcal{T} \in \tilde{X}$ (if they exist), $\tilde{b}_{\mathcal{S}',\mathcal{T}'} \geqslant \tilde{a}_{\mathcal{S},\mathcal{T}}$, in particular, $\tilde{b}_{\mathcal{S}'} > \tilde{a}_{\mathcal{S}}$ (for $\mathcal{S}' \neq {}^Y\mathcal{S}$).

We now come to the key property of the Q-set $(\tilde{X}, \tilde{A})$ of all singletons of a strictly generated Q-set presented in the next proposition.

PROPOSITION 4.8. Let $\mathcal{K} = (K, K^{\#})$ be a singleton of the Q-set $(\tilde{X}, \tilde{A})$ of all singletons of a strictly generated Q-set (X, A). Then there exists a unique element $\mathcal{T} \in \tilde{X}$ for which $k_{\mathcal{S}} = \tilde{a}_{\mathcal{T},\mathcal{S}}$ and $k^{\#}_{\mathcal{S}} = \tilde{a}_{\mathcal{S},\mathcal{T}}$ for all $\mathcal{S} \in \tilde{X}$.

This is formalized in the following

DEFINITION 4.9 (cf. Definition 17 [6]). A strict and separated Q-set (X, A) will be said to be complete provided that each singleton $\mathcal{S} = (S, S^{\#})$ of (X, A) is of the form: $\mathcal{S} = \mathcal{A}_x$, i.e., with $S = A_x (= (a_{x,x'})_{x' \in X})$ and $S^{\#} = A^{\#} (= (a_{x',x})^{x' \in X})$, for some (unique) element $x \in X$.

Now we are going to define an adjunction from the full subcategory $SGQ-Sets$ of strictly generated Q-sets of the category $Q-Sets$ to the full subcategory $CQ-Sets$ of complete Q-sets of $Q-Sets$ adapting the construction for quantales considered by C.J. Mulvey and M. Nawaz [6].

PROPOSITION 4.10. Let $\sim$ be the mapping which associates with every strictly generated Q-set (X, A) the complete Q-set $(\tilde{X}, \tilde{A})$ of all singletons of (X, A) and the morphism mapping which associates with every bimodule $\mathcal{F} : (X, A) \to (Y, B)$ the bimodule $\tilde{\mathcal{F}} : (\tilde{X}, \tilde{A}) \to (\tilde{Y}, \tilde{B})$ defined by

$$\tilde{f}_{\mathcal{S},\mathcal{T}} = \bigvee_{\underline{x} \in \underline{X}} \bigvee_{\underline{y} \in \underline{Y}} s_{\underline{x}} \& f_{\underline{x},\underline{y}} \& t^{\#}_{\underline{y}} \text{ and } \tilde{f}^{\#}{}_{\mathcal{T},\mathcal{S}} = \bigvee_{\underline{y} \in \underline{Y}} \bigvee_{\underline{x} \in \underline{X}} t_{\underline{y}} \& f^{\#}_{\underline{y},\underline{x}} \& s^{\#}_{\underline{x}}.$$

Then $\sim$ is a (covariant) functor from the category $SGQ-Sets$ of strictly generated Q-sets to the category $CQ-Sets$ of complete Q-sets.

Finally, we arrive at

Theorem 4.11 (cf. Corollary 19 [6]). *The functors*

$$SGQ-Sets \underset{I}{\overset{\sim}{\rightleftarrows}} CQ-Sets$$

establish an equivalence of categories, where I is the embedding of the subcategory $CQ-Sets$ of complete Q-sets in the category $SGQ-Sets$ of strictly generated Q-sets.

5. Presheaves on a quantaloid Q

Henceforth we shall solely work with strict Q-sets.

DEFINITION 5.1. By a presheaf on Q will be meant a strict Q-set (X, A) together with *restriction*: a partial mapping $\lceil \rfloor : Q_1 \times X \times Q_1 \to X$ (more precisely, a "matrix" $\lceil \rfloor = (\lceil \rfloor_{u,v})^{u \in Q_0}_{v \in Q_0}$ of partial mappings$\lceil \rfloor_{u,v} : Q(u,v) \times X_v \times Q(v,u) \to X_u$) from the triplets $(p, x, p^{\#}) \in Q_1 \times X \times Q_1$ with $dom(p) = cod(p^{\#})$ and $cod(p) = dom(p^{\#}) = d(x)$ (keeping $d(p, x, p^{\#}) = dom(p)$) such that $a_x \& p^{\#} \& p \& a_x \leqslant a_x$, $p \& a_x \leqslant p \& a_x \& p^{\#} \& p \& a_x$, and $a_x \& p^{\#} \leqslant a_x \& p^{\#} \& p \& a_x \& p^{\#}$ (the second and the third of which are actually equalities owing to the first condition), which will be reffered to as *restrictable triplets* of $Q_1 \times X \times Q_1$ to the elements$p \lceil x \rfloor p^{\#}$ of X satisfying the compatibility conditions that: $q \lceil (p \lceil x \rfloor p^{\#}) \rfloor q^{\#} =$

$(q\&p)\lceil x\rfloor(p^{\#}\&q^{\#})$, $a_x\lceil x\rfloor a_x = x$, and $a_{p\lceil x\rfloor p^{\#}, p'\lceil x'\rfloor p'^{\#}} = p\&a_{x,x'}\&p'^{\#}$ for all restrictable triplets $(p,x,p^{\#})$,$(p',x,p'^{\#})$, and $(q,p\lceil x\rfloor p^{\#},q^{\#})$ of $Q_1 \times X \times Q_1$. A presheaf $(X,A,\lceil\rfloor)$ on Q is *separated* whenever the underlying Q-set (X,A) is so.

DEFINITION 5.2 (cf. [6]). By the *canonical presheaf* $(X,A,\lceil\rfloor)$ on Q determined by a complete Q-set (X,A) will be meant the presheaf of which the underlying Q-set is (X,A) itself and of which the restriction is uniquely determined by requiring that $a_{p\lceil x\rfloor p^{\#},x"} = p\&a_{x,x"}$ and $a_{x",p\lceil x\rfloor p^{\#}} = a_{x",x}\&p^{\#}$ for any restrictable triplet $(p,x,p^{\#})$ of $Q_1 \times X \times Q_1$ and for all elements $x" \in X$.

DEFINITION 5.3 (cf. Definition 23 [6]). By a *map of presheaves* $f : (X,A,\lceil\rfloor) \to (Y,B,\lceil\rfloor)$ on Q will be meant a map $f : X \to Y$ in $Sets/Q_0$ satisfying the following:

Strictness: $d(x) = d(fx)$, $b_{fx,y} = a_x\&b_{fx,y}$, and $b_{y,fx} = b_{y,fx}\&a_x$ for all $x \in X$ and $y \in Y$;

Isotonicity: $a_{x,x'} \leqslant b_{fx,fx'}$ for all $x,x' \in X$;

Preservation of Restriction: if $(p,x,p^{\#})$ is restrictable , then $(p,fx,p^{\#})$ is also restrictable and $f(p\lceil x\rfloor p^{\#}) = p\lceil fx\rfloor p^{\#}$.

DEFINITION 5.4 (cf. Definition 24 [6]). By the *canonical map of presheaves*

$$f^{\Gamma} : (X,A,\lceil\rfloor) \to (Y,B,\lceil\rfloor)$$

determined by a bimodule $F : (X,A) \to (Y,B)$ between complete Q-sets will be meant the map $f^{\Gamma} : X \to Y$ in $Sets/Q_0$, every value $f^{\Gamma}x$ of which is uniquely determined by requiring that $b_{f^{\Gamma}x,y} = f_{x,y}$ and $b_{y,f^{\Gamma}x} = f^{\#}_{y,x}$ for all $y \in Y$.

For any quantaloid Q, presheaves on Q together with maps of presheaves on Q form a category, which we shall denote by $Q-Psh$. Moreover, the assignment $\mathcal{F} \mapsto f^{\Gamma}$ determines a functor from the category $CQ{-}Sets$ of complete Q-sets to the category $Q-Psh$ of presheaves on Q, which we shall denote by Γ. We are going to establish the existence of the adjoint to Γ. First, we present several results from [2] (more precisely, their "presheaf" versions).

PROPOSITION 5.5 (Proposition 2.7 [2]). Every map $f : (X,A,\lceil\rfloor) \to (Y,B,\lceil\rfloor)$ of presheaves on Q determines a bimodule $\mathcal{F}^U = (F^U, F^{U\#})$ from (X,A) to (Y,B) by the relations $f^U_{x,y} = b_{fx,y}$ and $f^{U\#}_{y,x} = b_{y,fx}$ for all $x \in X$ and $y \in Y$.

PROPOSITION 5.6 (Proposition 2.9 [2]). The assignment $f \mapsto \mathcal{F}^U$ determines a functor from the category $Q-Psh$ of presheaves on Q to the full subcategory $SQ{-}Sets$ of strict Q-sets of the category $Q{-}Sets$ (or of the category $SGQ{-}Sets$), which we shall denote by $U : Q-Psh \to SQ{-}Sets$.

It is clear that the composite ΓU (rightwards) is none other than the inclusion functor $I : CQ{-}Sets \to SQ{-}Sets$, since the assignment $\mathcal{F} \mapsto f^{\Gamma}$ is one-to-one. The following proposition generalize Theorem 25 [6].

Theorem 5.7. *For any quantaloid Q, the functors*

$$Q\!-\!Psh \overset{\Sigma=U\sim}{\underset{\Gamma}{\rightleftarrows}} CQ\!-\!Sets$$

are adjoint, where $\sim$ is the functor from $SQ-Sets$ to $CQ-Sets$ (introduced in Proposition 4.10).

6. Sheaves on a quantaloid

In this section we describe a novel "sheaf condition" on a separated presheaf, corresponding to that of the completeness of a Q-set.

DEFINITION 6.1. We say that the separated presheaf $(X, A, \lceil\rfloor)$ on Q is the sheaf on Q if it satisfies *sheaf condition*:

(i) for every singleton $\mathcal{S}$ of the underlying Q-set (X, A), there exist "enough" restrictable triplets $(s_x, x, s_x^{\#})$ in the sence that

$$s_x = \bigvee\{s_{x'}\&a_{x',x} | x' \in X, (s_{x'}, x', s_{x'}^{\#})\text{restrictable}\}$$

and

$$s_x^{\#} = \bigvee\{a_{x,x'}\&s_{x'}^{\#} | x' \in X, (s_{x'}, x', s_{x'}^{\#})\text{restrictable}\}$$

for all $x \in X$ (noting that $(s_x, x, s_x^{\#})$ is restrictable iff $s_x = s_x\&s_x^{\#}\&s_x$ and $s_x^{\#} = s_x^{\#}\&s_x\&s_x^{\#}$);

(ii) if the subset $J \subseteq X_u$ (for some $u \in Q_0$) is such that the pair $\mathcal{E} = (E, E^{\#})$ (with $E = (e_x)_{x\in J}$, $E^{\#} = (e_x^{\#})^{x\in J}$, and $e_x = e_x^{\#} = a_x$ ("diagonal" element of A) for $x \in J$) constitutes a singleton of the sub-Q-set $(J, {}_JA) \subseteq (X, A)$, then its "bottom" extension ${}^X\mathcal{E} = ({}^XE, {}^XE^{\#})$ to a singleton of (X, A) obtained by (see Proposition 4.7):

$${}^Xe_{x'} = \bigvee_{x\in J} e_x\&a_{x,x'} (= \bigvee_{x\in J} a_{x,x'}) \text{ and } {}^Xe_{x'}^{\#} = \bigvee_{x\in J} a_{x',x}\&e_x^{\#} (= \bigvee_{x\in J} a_{x',x})$$

for all $x' \in X$, is of the form:

$${}^X\mathcal{E} = \mathcal{A}_y$$

for some (unique) element $y \in X$ (where $\mathcal{A}_y = ((a_{y,x})_{x\in X}, (a_{x,y})^{x\in X})$).

PROPOSITION 6.2. The canonical presheaf $(X, A, \lceil\rfloor)$ determined by any complete Q-set is a sheaf.

With these remarks, denoting for a quantaloid Q by $Q - Sh$ the full subcategory of the category $Q - Psh$ of presheaves on Q determined by sheaves on Q, we see that the functor $\Gamma : CQ-Sets \to Q{-}Psh$ (determined in Definition 5.4) that assigns to each complete Q-set its canonical presheaf may in fact be considered as afunctor into the category of sheaves on Q, and that the functor $U : Q - Psh \to SQ{-}Sets$ (introduced in Proposition 5.6 that assigns to each presheaf its underlying Q-set) restricted to $Q - Sh$ may be regarded as a functor into $CQ-Sets$. We have the following analog of Theorem 29 [6], and also of Theorem 4.13 [3].

Theorem 6.3. *For a quantaloid Q, the categories of complete Q-sets and of sheaves on Q are isomorphic by the functors*

$$CQ{-}Sets \underset{U}{\overset{\Gamma}{\rightleftarrows}} Q{-}Sh$$

that assign respectively to each complete Q-set its canonical presheaf, and to each sheaf its underlying Q-set.

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Pluoštai virš kvantaloidų

R.P. Gylys

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