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A multidimensional limit theorem for powers of the Riemann zeta-function

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Let $s = \sigma + it$ be a complex variable. The Riemann zeta-function $\zeta(s)$ is defined, for $\sigma > 1$, by

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}.$$

Denote by $\mathcal{B}(S)$ the class of Borel sets of the space S, and let, for $T > 0$,

$$\nu_T(\ldots) = \frac{1}{T}\text{meas}\{t \in [0,T], \ldots\},$$

where $\text{meas}\{A\}$ stands for the Lebesgue measure of the set A, and in place of dots some condition satisfied by t is to be written.

Let $k_1, \ldots, k_n$ be natural numbers, $k = \max(k_1, \ldots, k_n)$ and $D_k = \{s \in \mathbb{C} : 1 - \frac{1}{k} < \sigma < 1\}$. Denote by $H(D_k)$ the space of analitic on D_k functions equipped with the topology of uniform convergence on compacta. Denote by γ the unit circle on $\mathbb{C}$, i.e. $\gamma = \{s \in \mathbb{C} : |s| = 1\}$, and let

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \gamma$ for each prime number p. Whith the product topology and pointwise multiplication Ω is a compact Abelian topological group. Then there exists the probability Haar measure m_H on $(\Omega, \mathcal{B}(\Omega))$. This yields a probability space $(\Omega, \mathcal{B}(\Omega), m_H)$.

Let $\omega(p)$ stand for the projection of $\omega \in \Omega$ to the coordinate space γ_p. Setting

$$\omega(m) = \prod_{p^\alpha || m} \omega^\alpha(p),$$

where $p^\alpha || m$ means that $p^\alpha | m$, but $p^{\alpha+1} \nmid m$, we obtain an extension of $\omega(p)$ to the set $\mathbb{N}$ as a completely multiplicative unimodular function. Define on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ an $H(D_k)$-valued random element

$$\zeta^k(s, \omega) = \sum_{m=1}^{\infty} \frac{\omega(m) d_k(m)}{m^s}, \quad s \in D_k,\ \omega \in \Omega,$$

where $d_k(m) = \sum_{m=m_1 m_2 \dots m_k} 1$.

Let n be a natural number and $H^n(D_k) = \underbrace{H(D_k)\dots H(D_k)}_{n}$. Define on $(\Omega, \mathcal{B}(\Omega), m_H)$ an $H^n(D_k)$-valued random element

$$\zeta_n(s,\omega) = (\zeta^{k_1}(s,\omega), \zeta^{k_2}(s,\omega), \dots, \zeta^{k_n}(s,\omega)),$$

and let P_{ζ_n} denote the disrtibution of $\zeta_n(s,\omega)$.

We will prove a limit theorem for the probability measure

$$P_T(A) = \frac{1}{T}\text{meas}\Big\{ t \in [0,T],\ (\zeta^{k_1}(s+i\tau), \zeta^{k_2}(s+i\tau), \dots, \zeta^{k_n}(s+i\tau)) \in A, \\ A \in \mathcal{B}(H^n(D_k))\Big\}.$$

Theorem. *The probability measure P_T converges weakly to P_{ζ_n} as $T \to \infty$.*

Lemma 1. *The probabality measure*

$$\nu_T(\zeta^k(s+i\tau) \in A), \quad A \in \mathcal{B}(H(D_k)),$$

converges weakly to the disribution of the random element $\zeta^k(s,\omega)$ as $T \to \infty$.

Proof. The proof of Lemma 1 is similar to that of analogous statement for $k = 1$. Therefore we will give the sketch of proof only. We begin the proof of the Lemma 1 by a limit theorem for Dirichlet polynomials

$$p_{n,k}(s) = \sum_{m=1}^{n} \frac{d_k(s)}{m^s}.$$

Let G denote some open subset of $\mathbb{C}$. Define a probability mesaure on $(H(G), \mathcal{B}(H(G)))$ by

$$P_{T,p_n,k}(A) = \nu_T(p_{n,k}(s+i\tau) \in A), \quad A \in \mathcal{B}(H(G)).$$

Then we prove that there exists a probability measure $P_{p_n,k}$ on $(H(G), \mathcal{B}(H(G)))$ such that the probability measure $P_{T,p_n,k}$ converges weakly to $P_{p_n,k}$ as $T \to \infty$. After this we define

$$p_{n,k}(s,g) = \sum_{m=1}^{n} \frac{d_k(s)g(m)}{m^s}$$

and

$$\widetilde{P}_{T,p_n,k}(A) = \nu_T(p_{n,k}(s+i\tau, g) \in A), \quad A \in \mathcal{B}(H(G)),$$

where $g(m)$ is an unimodular completely multiplicative function, and show that the probability measures $P_{T,p_n,k}$ and $\widetilde{P}_{T,p_n,k}$ converge weakly to the same measure as $T \to \infty$.

Now we prove a similar assertion for absolutely convergent Dirichlet series. Let $\sigma_1 > 1 - \frac{1}{k}$, $k \geqslant 2$. We define the function

$$l_n(s) = \frac{s}{\sigma_1}\Gamma\left(\frac{s}{\sigma_1}\right) n^s, \quad n \in \mathbb{N}$$

in the strip $-\sigma_1 \leqslant \sigma \leqslant \sigma_1$. Here $\Gamma(s)$ stands for the Euler gamma-function. Suppose $\sigma > 1 - \frac{1}{k}$ and

$$\zeta_{2,n,k}(s) = \frac{1}{2\pi i} \int\limits_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \zeta^k(s+z) l_n(z) \frac{\mathrm{d}z}{z}.$$

We approximate by mean the function $\zeta^k(s)$, i.e. if K be a compact subset of the half-plane $\sigma > 1 - \frac{1}{k}$, then

$$\lim_{n\to\infty} \overline{\lim_{T\to\infty}} \int\limits_0^T \sup_{s\in K} \left|\zeta^k(\sigma + i\tau) - \zeta_{2,n,k}(s + i\tau)\right| \mathrm{d}\tau = 0.$$

Let

$$\zeta_{2,n,k}(s,\omega) = \sum_{m=1}^{\infty} \frac{d_k(m)\omega(m)}{m^s} \exp\left\{-\left(\frac{m}{n}\right)^{\sigma_1}\right\}.$$

We define two probability measures on $\big(H(D_k), \mathcal{B}(H(D_k))\big)$

$$P^1_{T,n,k}(A) = \nu_T\big(\zeta_{2,n,k}(s + i\tau) \in A\big),$$
$$Q^1_{T,n,k}(A) = \nu_T\big(\zeta_{2,n,k}(s + i\tau, \omega) \in A\big),$$

and show that both these probability measures converge weakly to the same probability measure $P^1_{n,k}$ as $T \to \infty$.

Let Ω_1 be a subset of Ω suth that for the $\omega \in \Omega_1$ the series

$$\sum_{m=1}^{\infty} \frac{\omega(m) d_k(m)}{m^s}$$

converges uniformly on compact subsets of D_k and for $\sigma > 1 - \frac{1}{k}$ the estimate

$$\int\limits_0^T |\zeta^k(\sigma + it, \omega)|^2 \mathrm{d}t = BT$$

is valid. Let

$$Q_{T,k}(A) = \nu_T\big(\zeta^k(s+i\tau,\omega_1) \in A\big), \quad A \in \mathcal{B}(H(D_k)).$$

Then we prove that there exists a probability mesure P_k^1 on $(H(D_k), \mathcal{B}(H(D_k)))$ such that both the probability measures $P_{T,k}$ ir $Q_{T,k}$ converge weakly to P_k^1 as $T \to \infty$.

Then using this fact and applying elements of ergodic theory we complete proof of the Lemma 1, proving that P_k^1 is the distribution of $\zeta^k(s,\omega)$.

Let S and S_1 be two metric spaces and let $h : S \to S_1$ is measurable function. Then every probability measure P on $(S, \mathcal{B}(S))$ induces on $(S_1, \mathcal{B}(S_1))$ the unique probability measure Ph^{-1} defined by equality $Ph^{-1}(A) = P(h^{-1}A), \quad A \in \mathcal{B}(S_1)$.

Lemma 2. *Let $h : S \to S_1$ be a continuous function. If P_n converges weakly to P, then $P_n h^{-1}$ coverges weakly to Ph^{-1} as $n \to \infty$.*

Proof can be found [1].

Lemma 3. *The family of probability measures $\{P_T, T > 0\}$ is relatively compact.*

Proof. From Lemma 1 the probability mesure

$$P_{T,k_i}(A) = \nu_T\big(\zeta^{k_i}(s+i\tau) \in A\big), \quad A \in \mathcal{B}(H(D_k)),$$

converges weakly to the distribution of the random element $\zeta^{k_i}(s,\omega)$ as $T \to \infty$. From this it follows that the family of the probability measures $\{P_{T,k_i}, T > 0\}$ is relatively compact. Since $H(D_k)$ is a complete separable space, hence we obtain by the second Prochorov theorem that the family $\{P_{T,k_i}\}$ is tight, i.e. for an arbitry $\epsilon > 0$ there exists a compact set $K_k \subset H(D_k)$ such that

$$P_{T,k_i}(H(D_k) \backslash K_{k_i}) < \frac{\epsilon}{n} \tag{1}$$

for all $T > 0$. Define on a probability space $(\widetilde{\Omega}, \mathcal{F}, Q)$ a random element η_T by

$$Q(\eta_T \in A) = \frac{1}{T}\int_0^T I_A \mathrm{d}t, \quad A \in \mathcal{B}(\mathbb{R}),$$

where A is the indicator function of set A. Consider the $H(D_k)$-valued random element

$$\zeta_{T,k_i}(s) = \zeta^{k_i}(s+i\eta_T),$$

and let

$$\zeta_T(s) = (\zeta_{T,k_1}, \zeta_{T,k_2}, \dots, \zeta_{T,k_n}).$$

Then, by (1)

$$Q\big(\zeta_{T,k_i}(S) \in H(D_k)\backslash K_{k_i}\big) < \frac{\epsilon}{n}.$$

Let $K = K_{k_1} \times K_{k_2} \times \dots \times K_{k_n}$, then

$$P_T\big(H^n(D_k)\backslash K\big) = Q\big(\zeta_T(s) \in H^n(D_k)\backslash K\big)$$
$$= Q\left(\bigcup_{k=1}^{n}(\zeta_{T,k_i}(s) \in H^n(D_k)\backslash K)\right) \leqslant \sum_{k=1}^{n} Q\big(\zeta_{T,k_i}(s) \in H^n(D_k)\backslash K\big) < \epsilon$$

for all $T > 0$. Consequently, the family $\{P_T\}$ is tight. From the first Prokorov theorem it is relatively compact. Lemma 3 is proved.

Let $s_1, \dots, s_r \in D_k, \widetilde{D} = \{s \in \mathbb{C}, \sigma > 1 - \frac{1}{k} - \min_{1 \leqslant l \leqslant r} \Re s_l\}, u_{kl} \in \mathbb{C}$, where $1 \leqslant k \leqslant n$, $1 \leqslant l \leqslant r$. Define a function $h : H^n(D_k) \to H(\widetilde{D})$ by the formula

$$h(f_1, \dots, f_n) = \sum_{k=1}^{n} \sum_{l=1}^{r} u_{kl} f_k(s_l + s), \quad s \in D_k,\ f_j \in H(D_k),\ j = 1, \dots, n.$$

Morever, let

$$\zeta_h(s) = h\big(\zeta^{k_1}(s), \zeta^{k_2}(s), \dots, \zeta^{k_n}(s)\big).$$

The functions $\zeta_h(s)$ and $\zeta^k(s)$ have the same analytic properties. Therefore, reasoning similary as in the proof of Lemma 1, we obtain

$$\zeta_h(s + i\eta_T) \underset{T\to\infty}{\overset{\mathcal{D}}{\longrightarrow}} h\big(\zeta_n(s, \omega)\big). \tag{2}$$

Proof of Theorem. By Lemma 3 there exists a squence $T_1 \to \infty$ such that P_{T_1} converges weakly to some probability measure P. Let P is the distribution of $H^n(D_k)$-valued random element

$$\widetilde{\zeta}(s) = \big(\widetilde{\zeta}_1(s), \dots, \widetilde{\zeta}_n(s)\big),$$

i.e.

$$\zeta_{T_1} \underset{T_1\to\infty}{\overset{\mathcal{D}}{\longrightarrow}} \widetilde{\zeta}.$$

Hence and from Lemma 2 we have that

$$h(\zeta_{T_1}) \underset{T_1\to\infty}{\overset{\mathcal{D}}{\to}} h(\widetilde{\zeta}),$$

or

$$\zeta_h(s+i\eta_{T_1}) \underset{T_1\to\infty}{\overset{\mathcal{D}}{\to}} h(\widetilde{\zeta}). \tag{3}$$

Then by (2)

$$\zeta_h(s+i\eta_{T_1}) \underset{T_1\to\infty}{\overset{\mathcal{D}}{=}} h(\zeta_n). \tag{4}$$

Now it follows from (3) and (4) that

$$h(\zeta_n) \overset{\mathcal{D}}{=} h(\widetilde{\zeta}). \tag{5}$$

Let a function $h_1 : H(\widetilde{D}) \to \mathbb{C}$ be given by the formula

$$h_1(f) = f(0),\ f \in H(\widetilde{D}).$$

Then from (5) we have

$$h_1(h(\zeta_n)) \overset{\mathcal{D}}{=} h_1(h(\widetilde{\zeta})),$$

or

$$h(\zeta_n)(0) \overset{\mathcal{D}}{=} h(\widetilde{\zeta})(0).$$

This yields

$$\sum_{k=1}^{n}\sum_{l=1}^{r} u_{kl}\zeta^k(s,\omega) \overset{\mathcal{D}}{=} \sum_{k=1}^{n}\sum_{l=1}^{r} u_{kl}\widetilde{\zeta}_k(s_l) \tag{6}$$

for arbitrary $u_{kl} \in \mathbb{C}$.

Hiperplanes in the space $\mathbb{R}^{2nk}$ form a determining class. Therefore, the hiperplanes also form a determining class in the space $\mathbb{C}^{nk}$. Taking into account (6), we obtain that the random elements $\zeta^k(s,\omega)$ and $\widetilde{\zeta}_k(s_l)$ have the same distribution.

Let K be a compact subset of D_k, $f_1,\dots,f_n \in H(\widetilde{D})$, and let a sequence$\{s_l\}$ be dense in K. For an arbitrary $\epsilon > 0$ we set

$$G = \Big\{(g_1,\dots,g_n) \in H^n(D_k) : \sup_{s\in K} \big|g_j(s) - f_j(s)\big| \leqslant \epsilon\Big\},\quad j = 1,\dots,n,$$

$$G_r = \Big\{(g_1,\dots,g_n) \in H^n(D_k) : \big|g_j(s) - f_j(s)\big| \leqslant \epsilon\Big\}.$$

From the properties of random elements $\zeta^k(s,\omega)$ and $\widetilde{\zeta}_k(s_l)$ it follows that

$$m_H\big(\omega \in \Omega : \zeta_n(s,\omega) \in G_r\big) = P\big(\widetilde{\zeta}(s) \in G_r\big). \tag{7}$$

Since the sequence $\{s_l\}$ is dense in K, we have $G_1 \supset G_2 \supset \ldots$, and $G_l \to G$ as $l \to \infty$. Thus, letting $r \to \infty$ in (7) we find

$$m_H\big(\omega \in \Omega : \zeta_n(s,\omega) \in G\big) = P\big(\widetilde{\zeta}(s) \in G\big).$$

From this we have

$$\zeta_n \overset{\mathcal{D}}{=} \widetilde{\zeta}.$$

Thus

$$\zeta_{T_1} \underset{T_1 \to \infty}{\overset{\mathcal{D}}{\longrightarrow}} \zeta_n.$$

These means that the probability measure P_T converges weakly to the distribution of the random element ζ_n as $T_1 \to \infty$. Since $\{P_T\}$ is relatively compact and random element ζ_n is independent of the choice of the sequence T_1 the assertion of the theorem follows.

References

[1] P. Billingsley, *Convergence of Probability Measures*, John Wiley, New York (1968).

[2] A. Laurinčikas, *Limit Theorems for the Riemann Zeta-function*, Kluwer, Dordrecht, Boston, London (1996).

Daugiamatė ribinė teorema Rymano dzeta funkcijos laipsniams

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Straipsnyje įrodoma daugiamatė ribinė teorema Rymano dzeta funkcijos laipsniams analizinių funkcijų erdvėje.