

# On differential geometry of tangent and cotangent bundles of Riemannian manifold

Angelė BAŠKIENĖ (ŠU)  
*e-mail: baskiene@fm.su.lt*

A large number of mathematicians investigate differential geometry of tangent and cotangent bundles of Riemannian manifold  $V_n$ . In works of japanesse investigators Yano, Sasaki and others [1, 2, 3] we may find excellent results on prolongations or lifts of some tensor fields and connections of  $V_n$  in these bundles.

In this paper we shall examine the tangent bundle  $T(V_n)$  and cotangent bundle  $T^c(V_n)$  of  $n$ -dimensional Riemannian manifold  $V_n$  with metric  $g$ , which allows us to define the diffeomorphism  $f : T(V_n) \rightarrow T^c(V_n)$ . The diffeomorphism  $f$  is used to transfer tensor fields of  $T(V_n)$  to ones in  $T^c(V_n)$  and converse.

On the other hand, there exist lifts or continuations of functions, tensor fields of  $V_n$  to  $T(V_n)$  and  $T^c(V_n)$ . We shall find some relations between these lifts in  $T(V_n)$  and  $T^c(V_n)$ .

1. Let us have the  $n$ -dimensional Riemannian manifold  $V_n$  with local system of coordinates  $x^i, i, j, \dots = 1, 2, \dots, n$ , and metric  $g_{ij}$ . Denote by  $\nu_x(\nu^i) \in T_x V_n$  and  $\tilde{\nu}_x(\tilde{\nu}_i) \in T_x^c V_n$  a vector and a covector, respectively, at a point  $x(x^i) \in V_n$ . Then a point  $X$  of the tangent bundle

$$T(V_n) = \bigcup_{x \in V_n} T_x V_n$$

has the coordinates  $(X^A) = (x^i, \nu^i)$ , and a point  $Y$  of the cotangent bundle

$$T^c(V_n) = \bigcup_{x \in V_n} T_x^c V_n$$

has the coordinates  $(Y^A) = (x^i, \tilde{\nu}_i)$ , where  $A, B, \dots = 1, 2, \dots, n, n+1, \dots, 2n$ .

Now we can define the diffeomorphism  $f : T(V_n) \rightarrow T^c(V_n)$  by equations

$$\begin{aligned} Y^i &= X^i (= x^i), \\ Y^{n+i} &= g_{ij} X^{n+j} (\tilde{\nu}_i = g_{ij} \nu^j). \end{aligned} \tag{1}$$

Let us have in  $T(V_n)$  the curve  $k : X^A = X^A(t)$ . Then in  $T^c(V_n)$  we have a corresponding curve  $f \circ k$ :

$$\begin{aligned} Y^i &= X^i(t), \\ Y^{n+i} &= g_{ij}(t) X^{n+j}(t). \end{aligned}$$

The tangent vector  $V^A = \frac{dX^A}{dt}$  to the curve  $k$  and the tangent vector  $\tilde{V}^A = \frac{dY^A}{dt}$  to the corresponding curve  $f \circ k$  in view of (1) are connected by the formulas

$$\begin{aligned}\tilde{V}^i &= V^i, \\ \tilde{V}^{n+i} &= \partial_j g_{ik} V^j X^{n+k} + g_{ik} V^{n+k}.\end{aligned}\quad (2)$$

A mapping  $f^{-1} : T^c(V_n) \rightarrow T(V_n)$ , defined by formulas

$$\begin{aligned}X^i &= Y^i (= x^i), \\ X^{n+i} &= \sum_j g^{ij} Y^{n+j} (v^i = g^{ij} \tilde{v}_j),\end{aligned}$$

transfers every vector  $\tilde{V}^A$  to the vector  $V^A$ , where

$$\begin{aligned}V^i &= \tilde{V}^i, \\ V^{n+i} &= \sum_l g^{il} \tilde{V}^{n+l} + \partial_j g^{il} g_{ik} \tilde{V}^j X^{n+k}.\end{aligned}\quad (3)$$

Let us have the vector  $u^i(x^j)$  in a basic manifold  $V_n$ . Then in  $T(V_n)$  there are some vector fields:

- 1) a complete lift  $U_1(u^i, X^{n+j} ds \frac{\partial u^i}{\partial x^j})$ ;
- 2) a vertical lift  $U_2(0, u^i)$ ;
- 3) a horizontal lift  $U_3(u^i, -\Gamma_{jk}^i u^j X^{n+k})$ ,

where  $\Gamma_{jk}^i$  are Christoffel's symbols defined by the metric  $g_{ij}$ .

We shall find vectors  $\tilde{U}_1, \tilde{U}_2, \tilde{U}_3$  in  $T^c(V_n)$  corresponding to the vectors  $U_1, U_2, U_3$  with respect to diffeomorphism  $f$ .

From (2) we can easily find the vector  $\tilde{U}_2(0, g_{ij} u^j)$ . Analogously,  $\tilde{U}_1^i = u^i$ ,

$$\begin{aligned}\tilde{U}_1^{n+i} &= \partial_j g_{ik} u^j X^{n+k} + g_{ik} X^{n+j} \frac{\partial u^k}{\partial x^j} = \partial_j g_{ik} u^i \sum_l g^{kl} Y^{n+l} \\ &\quad + g_{ik} \frac{\partial u^k}{\partial x^j} \sum_l g^{jl} Y^{n+l} = \sum_l g_{ik} Y^{n+l} (-\partial_j g^{kl} u^j + g^{lj} \frac{\partial u^k}{\partial x^j}), \\ \tilde{U}_3^i &= u^i, \quad \tilde{U}_3^{n+i} = \partial_j g_{ik} u^j X^{n+k} - g_{ik} \Gamma_{jl}^k X^{n+j} u^l \\ &= \sum_l g^{lk} Y^{n+l} u^j (\partial_j g_{ik} - g_{im} \Gamma_{jk}^m) = \sum_k \Gamma_{ji}^k Y^{n+k} u^i.\end{aligned}$$

**PROPOSITION 1.** The vectors  $\tilde{U}_1, \tilde{U}_2, \tilde{U}_3$  in  $T^c(V_n)$ , corresponding with respect to diffeomorphism  $f$  to the complete, vertical, horizontal lifts  $U_1, U_2, U_3$  in  $T(V_n)$  of the vector field  $u^i(x^i)$

in the basic manifold  $V_n$ , are the following

$$\begin{aligned} \tilde{U}_1(u^i, \sum_l g_{ik} Y^{n+l} (-\partial_j g^{kl} u^j + g^{lj} \frac{\partial u^k}{\partial x^j})), \quad \tilde{U}_2(0, g_{ij} u^j), \\ \tilde{U}_3(u^i, \sum_k \Gamma_{ji}^k Y^{n+k} u^j), \end{aligned}$$

respectively.

2. Let us examine a complete lift  $G_{AB}$  of metric  $g_{ij}$  from manifold  $V_n$  to  $T(V_n)$  [4]:

$$(G_{AB}) = \begin{pmatrix} X^{n+k} \frac{\partial g_{ij}}{\partial x^k} & g_{ij} \\ g_{ij} & 0 \end{pmatrix}. \tag{4}$$

Wel search such a metric  $\tilde{G}_{AB}$  in  $T^c(V_n)$  for which the equality  $\tilde{G}_{AB} \tilde{U}^A \tilde{V}^B = G_{AB} U^A V^B, \forall U, V \in T(V_n)$  holds. From (2-4)

$$\begin{aligned} G_{AB} U^A V^B &= G_{ij} U^i V^j + G_{i,n+j} U^i V^{n+j} + G_{n+i,j} U^{n+i} V^j \\ &+ G_{n+i,n+j} U^{n+i} V^{n+j} = X^{n+k} \partial_k g_{ij} U^i V^j + g_{ij} (U^i V^{n+j} + U^i V^{n+j}) \\ &= X^{n+k} \partial_k g_{ij} \tilde{U}^i \tilde{V}^j + g_{ij} \tilde{U}^i \left( \partial_k g^{lj} \tilde{V}^k g_{lm} X^{n+m} + \sum_l g^{lj} \tilde{V}^{n+l} \right) \\ &+ g_{ij} \tilde{V}^j \left( \partial_k g^{li} \tilde{U}^k g_{lm} X^{n+m} + \sum_l g^{lj} \tilde{U}^{n+l} \right) \\ &= \tilde{U}^i \tilde{V}^j (X^{n+k} \partial_k g_{ij} + g_{ib} \partial_j g^{bl} g_{lm} X^{n+m} + g_{ja} \partial_i g^{al} g_{lm} X^{n+m}) \\ &+ \tilde{U}^i \tilde{V}^{n+i} + \tilde{U}^{n+j} \tilde{V}^j = \tilde{U}^i \tilde{V}^j X^{n+m} (\partial_m g_{ij} - \partial_j g_{im} - \partial_i g_{jm}) + \delta_{ij} \tilde{U}^i \tilde{V}^{n+j} \\ &+ \delta_{ij} \tilde{U}^{n+i} \tilde{V}^j = -2 \sum_l Y^{n+l} \Gamma_{ji}^l \tilde{U}^i \tilde{V}^j + \delta_{ij} \tilde{U}^i \tilde{V}^{n+j} + \delta_{ij} \tilde{U}^{n+i} \tilde{V}^j. \end{aligned}$$

Thus, matrix of the metric  $\tilde{G}_{AB}$  is

$$(\tilde{G}_{AB}) = \begin{pmatrix} -2 \sum_k Y^{n+k} \Gamma_{ij}^k & \delta_{ij} \\ \delta_{ij} & 0 \end{pmatrix}. \tag{5}$$

PROPOSITION 2. The complete lift  $G_{AB}$  of the metric  $g_{ij}$  to the manifold  $T(V_n)$  the diffeomorphism (1) transfers into the continuation of the metric  $g_{ij}$  to  $T^c(V_n)$  (5) in the Yano-Patterson sence [3].

3. Let us consider the Sasaki metric [1] in  $T(V_n)$  with a quadratic form  $d\sigma^2 = ds^2 + g_{ij} D\nu^i D\nu^j$ , where  $ds^2 = g_{ij} dx^i dx^j, D\nu^j = d\nu^j + \Gamma_{ik}^j \nu^i dx^k$  is the covariant differential of

the vektor  $\nu^j$ . We find the components of the Sasaki metric  ${}^sG_{AB}$ . We have

$$\begin{aligned} d\sigma^2 &= {}^sG_{AB}dX^A dX^B = g_{ij}dX^i dX^j + g_{ij}(dX^{n+i} + \Gamma_{km}^i dX^k \nu^m) \\ &\quad \times (dX^{n+j} + \Gamma_{lr}^j dX^l \nu^r) = g_{ij}dX^i dX^j + g_{ij}dX^{n+i} dX^{n+j} \\ &\quad + g_{ij}\Gamma_{km}^j \Gamma_{lr}^i \nu^m \nu^r dX^k dX^l + g_{ij}\Gamma_{km}^i \nu^m dX^k dX^{n+j} + g_{ij}dX^{n+i} dX^l \Gamma_{lr}^j \nu^r \\ &= (g_{ij} + g_{kl}\Gamma_{im}^k \Gamma_{jr}^l \nu^m \nu^r) dX^i dX^j + g_{kj}\Gamma_{rm}^k \nu^m dX^i dX^{n+j} \\ &\quad + g_{il}\Gamma_{jr}^l \nu^r dX^{n+i} dX^j + g_{ij}dX^{n+i} dX^{n+j}. \end{aligned}$$

Thus,

$$({}^sG_{AB}) = \begin{pmatrix} g_{ij} + g_{kl}\Gamma_{im}^k \Gamma_{jr}^l \nu^m \nu^r & \Gamma_{il}^k g_{kj} \nu^l \\ \Gamma_{jl}^i g_{ik} \nu^l & g_{ij} \end{pmatrix}. \quad (6)$$

We search for a metric  ${}^s\tilde{G}_{AB}$  in  $T^c(V_n)$  such that  ${}^s\tilde{G}_{AB}dY^A dY^B = {}^sG_{AB}dX^A dX^B$ . From (6), (1) we have

$$\begin{aligned} {}^sG_{AB}dX^A dX^B &= g_{ij}dX^i dX^j + g_{ij}dX^{n+i} dX^{n+j} \\ &= g_{ij}dX^i dX^j + g_{ij}(dX^{n+i} + \Gamma_{km}^i dx^k X^{n+m})(dX^{n+j} + \Gamma_{lr}^j dx^l X^{n+r}). \end{aligned}$$

Moreover, in virtue of (1)

$$\begin{aligned} D\nu^i &= dX^{n+i} = dX^{n+i} + \Gamma_{km}^i dx^k X^{n+m} = d(g^{ia}Y^{n+a}) + \Gamma_{km}^i dx^k g^{mb}Y^{n+b} \\ &= \partial_k g^{ia} dX^k Y^{n+a} + g^{ia} dY^{n+a} + \Gamma_{km}^i dx^k g^{mb} Y^{n+b} \\ &= g^{ia}(dY^{n+a} - \Gamma_{ka}^b dx^k Y^{n+b}) + dX^k Y^{n+k}(\partial_k g^{ia} + \Gamma_{km}^i g^{ma} + \Gamma_{kb}^a g^{ib}) \\ &= g^{ia} dY^{n+a} + dX^k Y^{n+a} \nabla_k g^{ia} = g^{ia} dY^{n+a}. \end{aligned}$$

Similarly  $dX^{n+j} + \Gamma_{lr}^j dx^l X^{n+r} = g^{jb} dY^{n+b}$ . Hence

$$\begin{aligned} {}^sG_{AB}dX^A dX^B &= g_{ij}dX^i dX^j + g_{ij}g^{ia} dY^{n+a} g^{jb} dY^{n+b} \\ &= g_{ij}dX^i dX^j + g^{ab} dY^{n+a} dY^{n+b} \\ &= g_{ij}dY^i dY^j + g^{ab}(dY^{n+a} - \Gamma_{ka}^m dX^k Y^{n+m})(dY^{n+b} - \Gamma_{bl}^r dX^l Y^{n+r}) \\ &= g_{ij}dY^i dY^j + g^{ab}\Gamma_{ia}^m \Gamma_{bj}^r Y^{n+m} Y^{n+r} dY^i dY^j - g^{aj}\Gamma_{ia}^m Y^{n+m} dY^i dY^{n+j} \\ &\quad - g^{ib}\Gamma_{bj}^r Y^{n+r} dY^{n+i} dY^j + g^{ab} dY^{n+a} dY^{n+b}. \end{aligned}$$

This shows that a matrix of the metric  ${}^s\tilde{G}_{AB}$  is

$$({}^s\tilde{G}_{AB}) = \begin{pmatrix} g_{ij} + g^{ab}\Gamma_{ia}^m \Gamma_{bj}^r Y^{n+i} Y^{n+j} & -g^{aj}\Gamma_{ia}^m Y^{n+m} \\ -g^{bj}\Gamma_{bj}^m Y^{n+m} & g^{ij} \end{pmatrix}. \quad (7)$$

PROPOSITION 3. The diffeomorphism (1) maps the Sasaki metric (6) in  $T(V_n)$  into the fundamental metric  ${}^s\tilde{G}_{AB}$  (7) in  $T^c(V_n)$ .

4. Finally we consider an 1-form  $p = Y^{n+j}dY^j$  in  $T^c(V_n)$ , which is called the fundamental form [2]. The mapping  $f^{-1} : T^c(V_n) \rightarrow T(V_n)$  transfers  $p$  into the 1-form  $q = g_{ij}X^{n+i}dX^j$ . Exterior differential  $dp = dY^{n+j} \wedge dY^j = \frac{1}{2}E_{AB}dY^A \wedge dY^B$  defines the skew-symmetric tensor  $E_{AB}$  with a matrix  $(E_{AB}) = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ , where  $(I)$  is the unit  $n$ -dimensional matrix.

We shall find the corresponding exterior 2-form  $E'$  and the skew-symmetric tensor  $E'_{AB}$  in  $T(V_n)$ . In view of (1)

$$\begin{aligned} dY^{n+j} \wedge dY^j &= d(g_{jm}X^{n+m}) \wedge dX^j = (\partial_k g_{jm} dX^k X^{n+m} + g_{jm} dX^{n+m}) \wedge dX^j \\ &= \frac{1}{2} \partial_{[k} g_{j]m} X^{n+m} dX^k \wedge dX^j + \frac{1}{2} g_{jm} (dX^{n+m} \wedge dX^j - dX^j \wedge dX^{n+m}) = E'. \end{aligned}$$

From this

$$(E'_{AB}) = \begin{pmatrix} \partial_{[k} g_{i]m} X^{n+m} & -g_{ik} \\ g_{ik} & 0 \end{pmatrix}. \quad (8)$$

PROPOSITION 4. The skew-symmetric tensors  $E'_{AB}$  (8) in  $T(V_n)$  and  $E_{AB}$  (7) in  $T^c(V_n)$  defined by the exterior differential of the fundamental form  $p$  are the corresponding tensors with respect to the diffeomorphism  $f$ .

## References

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## Rymano daugdaros liečiamojo ir koliečiamojo pluoštų diferencialinės geometrijos klausimu

A. Baškienė

Nagrinėjama Rymano daugdara  $V_n$ , jos liečiamasis pluoštas  $T(V_n)$  ir koliečiamasis pluoštas  $T^c(V_n)$ . Difeomorfizmas  $f : T(V_n) \rightarrow T^c(V_n)$  apibrėžiamas kanoniniu  $V_n$  izomorfizmu. Darbe nagrinėjami tam tikrų daugdaros  $V_n$  tenzorinių liftai daugdarose  $T(V_n)$  ir  $T^c(V_n)$  bei ieškomas ryšys tarp tų liftų.

Irodyta teorema: difeomorfizmas  $f$   $V_n$  metrikos pilną liftą daugdaroje  $T(V_n)$  atvaizduoja į tos metrikos pratęsimą (Jano-Patersono prasme) daugdaroje  $T^c(V_n)$ .