VILNIUS UNIVERSITY

Julija Karaliūnaitė

VALUE DISTRIBUTION THEOREMS FOR THE PERIODIC ZETA FUNCTION

Doctoral dissertation
Physical sciences, mathematics (01P)

Vilnius, 2010

The scientific work was carried out in 2006–2010 at Vilnius University.
Scientific supervisor:
Prof. dr. habil. Antanas Laurinčikas (Vilnius University, Physical sciences, Mathematics - 01P)

VILNIAUS UNIVERSITETAS

Julija Karaliūnaitė

REIKŠMIŲ PASISKIRSTYMO TEOREMOS PERIODINEI DZETA FUNKCIJAI

Daktaro disertacija Fiziniai mokslai, Matematika (01P) Vilnius, 2010

Disertacija rengta 2006–2010 metais Vilniaus Universitete.

Mokslinis vadovas:
Prof. habil. dr. Antanas Laurinčikas (Vilniaus Universitetas, fiziniai mokslai, matematika - 01P)

Turinys

Introduction

In the thesis, the Atkinson formula for the periodic zeta-function $\zeta_\lambda(s)$ on the critical line and in the critical strip, and limit theorems in the sense of weak convergence of probability measures in various spaces are considered.

Actuality

The functions of complex variable in some half-plane defined by Dirichlet series with coefficients having a certain arithmetical sense are called the zeta-functions or L-functions. They play an important role in the analytic number theory. For example, the properties of the classical Riemann zeta-function $\zeta(s)$ and Dirichlet L-functions are directly connected to the distribution of prime numbers. Therefore, it is clear why many famous mathematicians studied and study zeta-functions. The works of F. V. Atkinson, H. Bohr, E. Bombieri, B. Conrey, S. M. Gonek, R. Garunkštis, G. H. Hardy, D. R. Heath-Brown, M. V. Huxley, A. E. Ingham, A. Ivič, H. Iwaniec, M. Jutila, A. Laurinčikas, A. F. Lavrik, N. N. Levinson, J. E. Littlewood, K. Matsumoto, H. L. Montgomery, Y. Motohashi, A. Perelli, P. Sarnak, A. Selberg, J. Steuding, E. C. Titchmarsh, S. M. Voronin and others influenced the investigations in this field.

The moment problem plays an important role in the theory of zeta-functions. It consists (in the case of $\zeta(s)$) of finding the asymptotics or estimates for

$$I_k(\sigma, T) \stackrel{\text{def}}{=} \int_0^T |\zeta(\sigma + it)|^{2k} \mathrm{d}t, \quad \sigma \geq \frac{1}{2}, \quad k \geq 0,$$

as $T \to \infty$. The problem is very complicated. For, example, the asymptotics for $I_k(\frac{1}{2}, T)$ are known only for $k = 1$ [8], $k = 2$ [10] and $k = c(\log\log T)^{-\frac{1}{2}}$, $c > 0$, [19].

On the other hand, in many problems, the mean values $I_k(\sigma, T)$ replace individual values of $\zeta(s)$ which are not known. For example, the famous Lindelöf hypothesis asserting that, for every $\varepsilon > 0$,

$$\zeta(\frac{1}{2}+it)=\mathrm{O}(t^{\varepsilon}), \quad t \geq t_0 > 0,$$

is equivalent to the estimate [31]

$$I_k(\frac{1}{2},T)=\mathrm{O}(T^{1+\varepsilon}), \quad k \in \mathbb{N}.$$

The Atkinson formula gives the explicit formula for the error term in the asymptotic formula for $I_1(\frac{1}{2},T)$. This result is not only interesting itself but also has a series of applications, for example, in the investigation of higher moments $I_k(\frac{1}{2},T)$.

Probabilistic limit theorems are used for the characterization of asymptotic behavior of zeta-functions. Recently, it was observed that theorems of such a kind are the principal component in the proof of universality for zeta-functions.

Finally, the periodic zeta-function $\zeta_\lambda(s)$ is not classical, however, it occurs in various problems of analytic number theory. For example, it appears [6], [20] in the asymptotic formula for the mean square with respect to the parameter of Hurwitz and Lerch zeta-functions.

On the other hand, the majority works of the mentioned above authors are devoted to the classical zeta-functions, and the results for the function $\zeta_\lambda(s)$ are not numerous. All those remarks show the actuality of the subject of the thesis.

Aims and problems

The aim of the thesis is to obtain the Atkinson formula and prove limit theorems in the sense of weak convergence of probability measures for the periodic zeta-function $\zeta_\lambda(s)$, more precisely, to solve the following problems:

1. To obtain the Atkinson formula on the critical line for the periodic zeta-function $\zeta_\lambda(s)$.

2. To obtain the Atkinson formula in the critical strip for the periodic zeta-function $\zeta_\lambda(s)$.

3. To prove limit theorems on the complex plane in the sense of weak convergence of probability measure for the periodic zeta-function $\zeta_\lambda(s)$.

4. To prove limit theorems in the space of analytic functions for the periodic zeta-function $\zeta_\lambda(s)$.

Methods

Analytical and probabilistic methods are applied. For the proof of Atkinson formula, we use properties of the function $\Delta(x)$, the error term in the Dirichlet divisor problem, and classical Voronoi formula for $\Delta(x)$. For the proof of limit theorems , the theory of weak convergence of probability measures, in particular, the Prokhorov theory is applied.

Novelty

All results obtained in the thesis are new. The Atkinson formula for the periodic zeta-function $\zeta_\lambda(s)$ was not known. The same is true for limit theorems for periodic zeta-function.

History of the problem and main results

Let $\zeta(s)$, $s = \sigma + it$, as usual, denote the Riemann zeta-function. In [1], F.V. Atkinson discovered a famous formula for the mean square of the function $\zeta(s)$. Let

$$E(T) = \int_0^T |\zeta(\frac{1}{2} + it)|^2 \mathrm{d}t - T \log \frac{T}{2\pi} - (2\gamma_0 - 1)T,$$

where γ_0 is the Euler constant. The Atkinson formula gives an explicit expression for $E(T)$ involving some elementary functions. Let $0 < c_1 < c_2$ be any two fixed constants such that $c_1 T < N < c_2 T$, and let, for brevity,

$$N_1 = N_1(T) = \frac{T}{2\pi} + \frac{N}{2} - \sqrt{\frac{N^2}{4} + \frac{NT}{2\pi}}.$$

Moreover, define

$$f(T, m) = 2T \operatorname{arsinh}(\sqrt{\frac{\pi m}{2T}}) + \sqrt{2\pi m T + \pi^2 m^2} - \frac{\pi}{4},$$

where

$$\operatorname{arsinh}(x) = \log(x + \sqrt{1 + x^2}).$$

Then F.V. Atkinson [1] proved that

$$E(T) = \frac{1}{\sqrt{2}} \sum_{m \le N} \frac{(-1)^m d(m)}{\sqrt{m}}$$

$$\times \left(\operatorname{arsinh} \left(\sqrt{\frac{\pi m}{2T}} \right) \right)^{-1} \left(\frac{T}{2\pi m} + \frac{1}{4} \right)^{-\frac{1}{4}} \cos(f(T, m))$$

$$-2 \sum_{m \le N_1} \frac{d(m)}{\sqrt{m}} \left(\log \frac{T}{2\pi m} \right)^{-1} \cos \left(T \log \frac{T}{2\pi m} - T + \frac{\pi}{4} \right) + \mathrm{O}(\log^2 T).$$

Here, as usual, $d(m)$ is the divisor function. The proof of the Atkinson formula is also given in [11]. M. Jutila proposed [13] a new version of the Atkinson formula with a weaker restriction for N, $c_3 T^\delta \le N \le c_4 T^2$, $\delta > 0$, and with the error term dependent on N. T. Meurman gave [27] an averaged version of the Atkinson formula with the error term $O(T^{-\frac{1}{4}} \log T)$. Y. Motohashi also proposed [28] a new version of the Atkinson formula with a small error term. M. Jutila proposed [14] a new proof of the Atkinson formula based on the Laplace transforms. For more comments, see, a very informative paper [23].

T. Meurman obtained [26] an analogue of Atkinson's formula for Dirichlet L-functions $L(s, \chi)$. Meurman's formula gives an explicit expression for

$$E(q, T) = \sum_{\chi (\mathrm{mod} q)} \int_0^T |L(\frac{1}{2} + it, \chi)|^2 \mathrm{d}t$$

$$- \frac{\varphi^2(q)}{q} T \left(\log \frac{qT}{2\pi} + \sum_{p|q} \frac{\log p}{p-1} + 2\gamma_0 + 1 \right),$$

where the sum runs over all Dirichlet characters modulo q, and $\varphi(q)$ denotes the Euler function.

Define the function $\zeta_\lambda(s)$, for $\sigma > 1$, by the series

$$\zeta_\lambda(s) = \sum_{m=1}^{\infty} \frac{e^{2\pi i \lambda m}}{m^s}, \quad \lambda \in \mathbb{R},$$

and by analytic continuation elsewhere. Clearly,

$$\zeta_\lambda(s) = e^{2\pi i\lambda} L(\lambda, 1, s),$$

where $L(\lambda, \alpha, s)$, $0 < \alpha \le 1$, is the Lerch zeta-function defined, for $\sigma > 1$, by

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i\lambda m}}{(m+\alpha)^s},$$

and by analytic continuation elsewhere. Moreover, for $\lambda \in \mathbb{Z}$, the function $\zeta_\lambda(s)$ reduces to the Riemann zeta-function. In view of the periodicity, we can suppose that $0 < \lambda \le 1$.

The aim of the thesis is to obtain the Atkinson formula for the periodic zeta-function $\zeta_\lambda(s)$ in the critical strip $\frac{1}{2} \le \sigma < 1$. First, in Chapter 1, we obtain the Atkinson formula for $\sigma = \frac{1}{2}$.

Define the constant $c(\alpha)$ by

$$\sum_{m=0}^{n} \frac{1}{m+\alpha} = \log n + c(\alpha) + \mathrm{O}(\frac{1}{n}), \quad n \to \infty.$$

Then in [7] it was proved that, for all $0 < \lambda \le 1$,

$$\int_0^T |\zeta_\lambda(\frac{1}{2} + it)|^2 \mathrm{d}t = T \log T + T(\gamma_0 + c(\lambda)$$

$$-1 - \log 2\pi) + O(T^{\frac{1}{2}} \log T).$$

Let $\lambda = \frac{a}{q}$ with given integers $1 \le a \le q$, and let

$$I(q, T) = \sum_{a=1}^{q} \int_0^T |\zeta_{\frac{a}{q}}(\frac{1}{2} + it)|^2 \mathrm{d}t.$$

Define

$$E(q, T) = I(q, T) - qT(\log \frac{qT}{2\pi} + 2\gamma_0 - 1).$$

In terms of these functions, define

$$\Sigma_1(q,T)=2^{-\frac{1}{2}}\sum_{n\le N}(-1)^{qn}d(n)(qn)^{-\frac{1}{2}}$$

$$\times(\operatorname{arsinh}\sqrt{\frac{\pi qn}{2T}})^{-1}\left(\frac{T}{2\pi qn}+\frac{1}{4}\right)^{-\frac{1}{4}}\cos f(T,qn),$$

$$\Sigma_2(q,T)=-2\sum_{n\le N'}d(n)(qn)^{-\frac{1}{2}}\left(\log\frac{qT}{2\pi n}\right)^{-1}$$

$$\cos(T\log\frac{qT}{2\pi n}-T+\frac{2\pi n}{q}-\frac{\pi}{4})$$

and

$$N'=N'(q,T,N)=q\left(\frac{T}{2\pi}+\frac{qN}{2}-\left(\left(\frac{qN}{2}\right)^2+\frac{qNT}{2\pi}\right)^{\frac{1}{2}}\right).$$

Theorem 1.2.1. *If q is a positive integer and $T\ll N\ll T$, then*

$$E(q,T)=q(\Sigma_1(q,T)+\Sigma_2(q,T))+\mathrm{O}(q^{\frac{1}{2}}\log^2 T)+\mathrm{O}(qT^{-1}).$$

Corollary 1.5.1. *For $q\ll T^2\log^4 T$, we have*

$$E(q,T)=q(\Sigma_1(q,T)+\Sigma_2(q,T))+\mathrm{O}(q^{\frac{1}{2}}\log^2 T).$$

Here and in the sequel, $n_1\ll m\ll n_2$ means that there exist positive constants c_1 and c_2 such that $c_1n_1\le m\le c_2n_2$.

The Atkinson formula allows to estimate the error term in the mean square formula. **Corollary 1.5.2.** *For $T^{\varepsilon}\ll H\ll T^{\frac{1}{2}}$ and $0<\varepsilon<\frac{1}{2}$, we have*

$$E(q,T)\ll qH\log T+q^{\frac{3}{4}}T^{\frac{1}{2}+\varepsilon}H^{-\frac{1}{2}}.$$

K. Matsumoto [22] jointly with T. Meurman [25] obtained the analogue of Atkinson formula in the critical strip $\frac{1}{2}<\sigma<1$.Let

$$E_\sigma(T) = \int_0^T |\zeta(\sigma+it)|^2 \mathrm{d}t - \zeta(2\sigma)T - (2\pi)^{2\sigma-1}\frac{\zeta(2-2\sigma)}{2-2\sigma}T^{2-2\sigma}.$$

The Atkinson formula gives an explicit expression for $E_\sigma(T)$ involving some elementary functions. Let N and N_1 be the same as above, and $\frac{1}{2} < \sigma < 1$. Denote

$$\Sigma_{1,\sigma}(T) = 2^{\sigma-1}(\frac{\pi}{T})^{\sigma-\frac{1}{2}} \sum_{m\le N} (-1)^m \sigma_{1-2\sigma}(m) m^{\sigma-1}$$

$$\times \left(\operatorname{arsinh}\left(\sqrt{\frac{\pi m}{2T}}\right)\right)^{-1} \left(\frac{T}{2\pi m}+\frac{1}{4}\right)^{-\frac{1}{4}}$$

$$\times \cos(f(T,m))$$

and

$$\Sigma_{2,\sigma}(T) = -2(\frac{2\pi}{T})^{\sigma-\frac{1}{2}} \sum_{m\le N_1} \sigma_{1-2\sigma}(m) m^{\sigma-1}$$

$$\times \left(\log\frac{T}{2\pi m}\right)^{-1} \cos\left(T\log\frac{T}{2\pi m} - T + \frac{\pi}{4}\right).$$

Then it is proved [22], [25] that

$$E_\sigma(T) = \Sigma_{1,\sigma}(T) + \Sigma_{2,\sigma}(T) + \mathrm{O}(\log T).$$

where

$$\sigma_a(n) = \sum_{d|n} d^a$$

is the generalized divisor function.

In Chapter 2, we obtain the Atkinson formula for the function $\zeta_\lambda(s)$ in the case $\frac{1}{2} < \sigma < 1$. Let $\lambda = \frac{a}{q}$ with given integers $1 \le a \le q$. For $\frac{1}{2} < \sigma < 1$, let

$$E_\sigma(q,T)=\sum_{a=1}^{q}\int_0^T|\zeta_{\frac{a}{q}}(\sigma+it)|^2\mathrm{d}t-q\zeta(2\sigma)T$$

$$+\frac{\zeta(2\sigma-1)\Gamma(2\sigma-1)\sin(\pi\sigma)}{1-\sigma}(qT)^{2-2\sigma}.$$

Define

$$\Sigma_{1,\sigma}(q,T)=q^{\sigma-1}2^{-\frac{1}{2}}\left(\frac{T}{2\pi}\right)^{\frac{1}{2}-\sigma}\sum_{n\le N}(-1)^{qn}$$

$$\times\sigma_{1-2\sigma}(n)n^{\sigma-1}(\operatorname{arsinh}\sqrt{\frac{\pi qn}{2T}})^{-1}$$

$$\times\left(\frac{T}{2\pi qn}+\frac{1}{4}\right)^{-\frac{1}{4}}\cos(f(T,qm))$$

and

$$\Sigma_{2,\sigma}(q,T)=-2q^{\sigma-1}\left(\frac{T}{2\pi}\right)^{\frac{1}{2}-\sigma}\sum_{n\le N'}\sigma_{1-2\sigma}(n)n^{\sigma-1}$$

$$\times\Big(\log\frac{qT}{2\pi n}\Big)^{-1}\cos\Big(T\log\frac{qT}{2\pi n}-T+\frac{\pi}{4}\Big).$$

The main result of Chapter 2 is the following statement. **Corollary 2.0.2.** *Let q be a positive integer, $T \ll N \ll T$, and let σ be a fixed number satisfying $\frac{1}{2}<\sigma<1$. Then*

$$E_\sigma(q,T)=\Sigma_{1,\sigma}(q,T)+\Sigma_{2,\sigma}(q,T)$$

$$+\mathrm{O}(q^{2\sigma-\frac{7}{4}}\log(qT))+\mathrm{O}(q).$$

Using functional equation for the Riemann zeta function $\zeta(s)$ we can obtain the Atkinson formula in the form of [22] and [25].

An idea of application of probabilistic methods in the theory of functions belongs to H. Bohr. In [3], he together with B. Jessen proved that, for $\sigma > 1$, there exists the limit

$$\lim_{T\to\infty} \frac{1}{T}\mu\{t \in [0,T] : \log\zeta(\sigma+it) \in R\},$$

where μ denotes the Jordan measure, and R is a rectangle which edges parallel to the axes. In [4], this result was extended to the region $\sigma > \frac{1}{2}$.

Later, Bohr-Jessen's theory was developed by many authors. Among them A. Wintner, A. Selberg, A. Gosh, P.D.T.A. Elliott, B. Bagchi, A. Laurinčikas, K. Matsumoto, B. Garunkštis, J. Steuding, J. L. Mauclaire and other famous mathematicians, for history and results, see [10], [12]. At the middle of the last centure, a new theory of convergence of the probability measures was developed, and it became convenient to use this theory for the probabilistic results for zeta-functions.

We need some notation. Let $meas\{A\}$ denote the Lebesgue measure of a measurable set $A \in \mathbb{R}$, and let,for $T > 0$,

$$\nu_T(...) = \frac{1}{T}meas\{t \in [0,T] : ...\},$$

where in place of dots a condition satisfied by t is to be written.

Denote by $\mathcal{B}(S)$ the class of Borel sets of a space S. Let P_n, $n \in \mathbb{N}$, and P be probability measures on on $(S, \mathcal{B}(S))$. We recall that P_n converges weakly to P as $n \to \infty$ with every real bounded continuous function f in the space S we have that

$$\lim_{n\to\infty} \int_S f\mathrm{d}P_n = \int_S f\mathrm{d}P.$$

Define

$$\Omega = \prod_p \gamma_p,$$

where γ_p is the unit circle $\gamma = \{s \in \mathbb{C} : |s| = 1\}$ for all primes p. By the Tikhonov theorem, the torus Ω is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$ the probability Haar measure m_H can be defined. This gives a probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(p)$ the projection of $\omega \in \Omega$ to the coordinate space γ_p. We extend the function $\omega(p)$ to the set

$\mathbb{N}$ by the formula

$$\omega(m) = \prod_{p^\alpha \| m} \omega^\alpha(p), \quad m \in \mathbb{N},$$

where $p^\alpha \| m$ means that $p^\alpha | m$ but $p^{\alpha+1} \nmid m$. For $\sigma > \frac{1}{2}$, define

$$\zeta_\lambda(\sigma, \omega) = \sum_{m=1}^{\infty} \frac{e^{2\pi i \lambda m} \omega(m)}{m^\sigma}, \quad \sigma > \frac{1}{2}.$$

Then $\zeta_\lambda(\sigma, \omega)$ is a complex - valued random element defined on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $P_{\zeta_\lambda}^{\mathbb{C}}$ the distribution of $\zeta_\lambda(\sigma, \omega)$, i.e.,

$$P_{\zeta_\lambda}^{\mathbb{C}}(A) = m_H\{\omega \in \Omega : \zeta_\lambda(\sigma, \omega) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}).$$

The main result of Chapter 3 is the following theorem.

Theorem 0.0.1. *Let $\sigma > \frac{1}{2}$. Then the probability measure*

$$\nu_T(\zeta_\lambda(\sigma + it) \in A), A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to $P_{\zeta_\lambda}^{\mathbb{C}}$, as $T \to \infty$.

Now let G be a region on the complex plane. Denote by $H(G)$ the space of analytic on G functions equipped with the topology of uniform convergence on compacta. Let $D = \{s \in \mathbb{C} : \sigma > \frac{1}{2}\}$.

In Chapter 4, a limit theorem in the space $H(D)$ for the function $\zeta_\lambda(s)$ is proved. On the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H(D)$ - valued random element $\zeta_\lambda(s, \omega)$ by the formula

$$\zeta_\lambda(s, \omega) = \sum_{m=1}^{\infty} \frac{e^{2\pi i \lambda m} \omega(m)}{m^s},$$

and denote by $P_{\zeta_\lambda}^{H}$ its distribution, i.e.,

$$P_{\zeta_\lambda}^{H}(A) = m_H\{\omega \in \Omega : \zeta_\lambda(s, \omega) \in A\}, \quad A \in \mathcal{B}(H(D)).$$

Then in Chapter 4 the following statement is obtained.

Theorem 0.0.2. *The probability measure*

$$\nu_T\{\zeta_\lambda(s + i\tau) \in A\}, \quad A \in \mathcal{B}(H(D)),$$

converges weakly to $P_{\zeta_\lambda}^{H}$ as $T \to \infty$.

Approbation

The results of the thesis were presented at the 4th International Conference "Analytic and Probabilistic Methods in Number Theory" (Palanga, 2006), at the 4th International Conference on Analytic Number Theory and Spatial Tesseletions (Kyiv, 2008), at the International Number Theory Conference (Šiauliai, 2008), at the Conferences of Lithuanian Mathematical Society (2007, 2008, 2009), as well as at the seminars of number theory in Vilnius University.

Principal publications

The main results of the thesis are published in the following papers.

1. J. Karaliūnaitė, *A limit theorem for the function* $\zeta_\lambda(s)$, Fizikos ir matematikos fakulteto mokslinio seminaro darbai, Šiaulių University, **8**, (2005), 56–62.

2. J. Karaliūnaitė, *A survey on limit theorems for the Riemann zeta - function on the complex plane*, in: Anal. Prob. Meth. Number Theory, A. Laurinčikas and E. Manstavičius (Eds), TEV, Vilnius **45** (2005), 48–55.

3. J. Karaliūnaitė, *The Atkinson formula for the periodic zeta-function in the critical strip*, in: Voronoi's Impact on Modern Science, Proc. 4th International Conference Analytic Number Theory and Spatial Tessellations, A. Laurinčikas and J. Steuding (Eds), Institute of Math., Kyiv, (2008), 48–58.

4. J. Karaliūnaitė and A. Laurinčikas, *The Atkinson formula for the periodic zeta - function*, Lietuvos matematikos rinkinys, **47**(4) (2007), 504–516.

Aknowledgements

I thank very much Professor Antanas Laurinčikas for his support, comprehension and help during my doctoral studies. I thank the members of the Faculty of Mathematics and Informatics of Vilnius University for support and useful suggestions. Also I want to thank my mother.

Skyrius 1

Atkinson formula for the periodic zeta-function on the critical line

In this chapter, we prove the Atkinson formula for the periodic zeta-function on the critical line.

1.1 Definition of the periodic zeta-function

The periodic zeta-function $\zeta_\lambda(s)$, $s = \sigma + it$, $\lambda \in \mathbb{R}$, is defined, for $\sigma > 1$, by

$$\zeta_\lambda(s) = \sum_{m=1}^{\infty} \frac{e^{2\pi i \lambda m}}{m^s}.$$

If $\lambda \in \mathbb{Z}$, then the function $\zeta_\lambda(s)$ becomes the Riemann zeta-function $\zeta(s)$. Therefore, in this case, it is meromorphically continuable to the whole complex plane, and the point $s = 1$ is a simple pole with residue 1.

If $\lambda \notin \mathbb{Z}$, then, for $\sigma > 1$,

$$\zeta_\lambda(s) = e^{2\pi i \lambda} \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m+1)^s} = e^{2\pi i \lambda} L(\lambda, 1, s), \tag{1.1}$$

where

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m+\alpha)^s}, \quad \sigma > 1,$$

$0 < \alpha \leq 1$, is the Lerch zeta-function. Since the function $L(\lambda, \alpha, s)$, for $\lambda \notin \mathbb{Z}$, is entire [20], the function $\zeta_\lambda(s)$, in virtue of (1.1), is as well. In this case, in virtue of the periodicity of $e^{2\pi i \lambda m}$, we can suppose that $0 < \lambda < 1$.

1.2 Statement of results

Let $\lambda = \frac{a}{q}$ with given integers a and q, $1 \le a \le q$, and let

$$I(q,T) = \sum_{a=1}^{q} \int_0^T |\zeta_{\frac{a}{q}}(\frac{1}{2} + it)|^2 \mathrm{d}t.$$

Define

$$E(q,T) = I(q,T) - qT(\log \frac{qT}{2\pi} + 2\gamma_0 - 1),$$

where γ_0 is the Euler constant, i.e.,

$$\begin{aligned}\gamma_0 &= \lim_{n\to\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\right) \\ &= 0,577215\ldots.\end{aligned}$$

Let, as usual,

$$d(m) = \sum_{d|m} 1, \quad m \in \mathbb{N}.$$

For two fixed constants c_1 and c_2, $0 < c_1 < c_2$, such that $c_1 T < N < c_2 T$, we put

$$\begin{aligned} N' &= N'(q,T,N) \\ &= q\left(\frac{T}{2\pi} + \frac{qN}{2} - \left(\left(\frac{qN}{2})^2 + \frac{qNT}{2\pi}\right)^{\frac{1}{2}}\right)\right).\end{aligned}$$

Moreover, as usual, let

$$\mathrm{arsinh}(x) = \log(x + \sqrt{1+x^2})$$

and

$$f(T,n) = 2T\mathrm{arsinh}\sqrt{\frac{\pi n}{2T}} + \sqrt{2\pi nT + \pi^2 n^2} - \frac{\pi}{4}).$$

In terms of these functions, define

$$\begin{aligned}\Sigma_1(q,T) &= 2^{-\frac{1}{2}} \sum_{n\le N} (-1)^{qn} d(n)(qn)^{-\frac{1}{2}} \\ &\times \left(\mathrm{arsinh}\sqrt{\frac{\pi qn}{2T}}\right)^{-1} \left(\frac{T}{2\pi qn} + \frac{1}{4}\right)^{-\frac{1}{4}} \\ &\times \cos f(T,qn),\end{aligned}$$

$$\begin{aligned}\Sigma_2(q,T) &= -2 \sum_{n\le N'} d(n)(qn)^{-\frac{1}{2}} \left(\log \frac{qT}{2\pi n}\right)^{-1} \\ &\times \cos(T \log \frac{qT}{2\pi n} - T + \frac{2\pi n}{q} - \frac{\pi}{4}).\end{aligned}$$

The Atkinson formula for the function $\zeta_\lambda(s)$ on the critical line is the following statement.

Theorem 1.2.1. *If q is a positive integer, then*

$$E(q,T) = q(\Sigma_1(q,T) + \Sigma_2(q,T))$$
$$+\mathrm{O}(q^{\frac{1}{2}} \log^2(qT)) + \mathrm{O}(qT^{-1}).$$

Note, that if $q = 1$, then $\zeta_{\frac{a}{q}}(s) = \zeta(s)$, and the formula of Theorem 1.2.1 coincides with that for the function $\zeta(s)$.

1.3 Some lemmas

First, we recall a version of the Poisson summation formula.

Lemma 1.3.1. *Let a and b be integers, and let $f(x)$ be a function of the real variable x with bounded first derivative on $[a,b]$. Then*

$$\sum_{a \le n} {}' f(n) = \int_a^b f(x)\mathrm{d}x + 2\sum_{n=1}^{\infty} \int_a^b f(x)\cos(2\pi n x)\mathrm{d}x.$$

Here, as usual, $\sum'$ means that $\frac{1}{2}f(a)$ and $\frac{1}{2}f(b)$ are to be taken instead of $f(a)$ and $f(b)$ respectively.

Proof of the lemma is given, for example, in [11].

Define

$$g_q(u,v) = \frac{2}{q^{u+v-1}} \sum_{l=1}^{\infty} \frac{1}{l^{u+v-1}} \sum_{m=1}^{\infty} \int_0^{\infty} \frac{\cos(2\pi mlqy)}{y^u(1+y)^v}\mathrm{d}y. \tag{1.2}$$

Lemma 1.3.2. *Suppose that $0 < \mathrm{Re}(u) < 1$. Then the formula*

$$\sum_{a=1}^{q} \zeta_{\frac{a}{q}}(u)\zeta_{-\frac{a}{q}}(1-u)$$
$$= q\left(\frac{1}{2}\left(\frac{\Gamma'}{\Gamma}(1-u) + \frac{\Gamma'}{\Gamma}(u)\right) + 2\gamma_0 - \log\frac{2\pi}{q}\right)$$
$$+q(g_q(u,1-u) + g_q(1-u,u))$$

holds.

Proof. For $\mathrm{Re}(u) > 1$ and $\mathrm{Re}(v) > 1$, we have that

$$\sum_{a=1}^{q} \zeta_{\frac{a}{q}}(u)\zeta_{-\frac{a}{q}}(v)$$
$$= \sum_{a=1}^{q}\sum_{m=1}^{\infty} \frac{e^{2\pi i \frac{a}{q} m}}{m^u} \sum_{n=1}^{\infty} \frac{e^{-2\pi i \frac{a}{q} n}}{n^v} \tag{1.3}$$
$$= q\zeta(u+v) + \sum_{a=1}^{q}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{e^{2\pi i \frac{a}{q}(m-n)}}{m^u n^v}.$$

Since

$$\sum_{a=1}^{q} e^{2\pi i \frac{a}{q}(m-n)} = \begin{cases} q & \text{if } m = n(\mathrm{mod} q), \\ 0 & \text{if } m \neq n(\mathrm{mod} q), \end{cases}$$

from (1.3) we have that

$$\sum_{a=1}^{q} \zeta_{\frac{a}{q}}(u)\zeta_{-\frac{a}{q}}(v) = q(\zeta(u+v) + f_q(u,v) + f_q(v,u)), \tag{1.4}$$

where

$$f_q(u,v) = \sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty} \frac{1}{m_1^u(m_1+qm_2)^v},$$

and the series converges absolutely for $\mathrm{Re}(u+v) > 2$ and $\mathrm{Re}(v) > 1$. Moreover, equality (1.4) shows that the sum $f_q(u,v) + f_q(v,u)$ is analytically continuable, except for poles $u = 1$, $v = 0$ and $u + v = 1$.

Now suppose that $\mathrm{Re}(u+v) > 2$ and $\mathrm{Re}(u) < 0$. In this case, we apply Lemma 1.3.1. This gives

$$\sum_{m_1=1}^{\infty} \frac{1}{m_1^u(m_1+qm_2)^v}$$

$$= \int_0^{\infty} \frac{}{x^u(x+qm_2)^v}\mathrm{d}x + 2\sum_{m=1}^{\infty}\int_0^{\infty} \frac{\cos(2\pi mx)}{x^u(x+qm_2)^v}\mathrm{d}x$$

$$= \frac{1}{(qm_2)^{u+v-1}}\int_0^{\infty} \frac{1}{y^u(1+y)^v}\mathrm{d}y$$

$$+ \frac{2}{(qm_2)^{u+v-1}}\sum_{m=1}\int_0^{\infty} \frac{\cos(2\pi m m_2 q y)}{y^u(1+y)^v}\mathrm{d}y,$$

after the change of variable $x = qm_2y$. Using the well-known formula

$$\int_0^{\infty} \frac{1}{y^u(1+y)^v}\mathrm{d}y = \frac{\Gamma(u+v-1)\Gamma(1-u)}{\Gamma(v)},$$

from this we deduce

$$f_q(u,v) = \frac{\zeta(u+v-1)\Gamma(u+v-1)\Gamma(1-u)}{q^{u+v-1}\Gamma(v)} + g_q(u,v). \tag{1.5}$$

Since the function $\Gamma(s)$ has simple poles at $s = -m$, $m \in \mathbb{N}_0$, (1.5) shows that the function

$$f_q(u,v) - g_q(u,v)$$

is meromorphically continuable in u and v to the whole complex plane. Therefore, the sum

$$(f_q(u,v) - g_q(u,v)) + (f_q(v,u) - g_q(v,u))$$

has the same property as well. From this and from (1.5), we find that

$$\begin{aligned} & f_q(u,v)+f_q(v,u) \\ = & \frac{\zeta(u+v-1)\Gamma(u+v-1)}{q^{u+v-1}} \\ & \times\left(\frac{\Gamma(1-u)}{\Gamma(v)}+\frac{\Gamma(1-v)}{\Gamma(u)}\right) \\ & +g_q(u,v)+g_q(v,u). \end{aligned}$$

Now, by (1.4), we have

$$\begin{aligned} & \sum_{a=1}^{q}\zeta_{\frac{a}{q}}(u)\zeta_{-\frac{a}{q}}(1-u) \\ & = q\left(\zeta(u+v)+\frac{\zeta(u+v-1)\Gamma(u+v-1)}{q^{u+v-1}}\right. \\ & \times\left(\frac{\Gamma(1-u)}{\Gamma(v)}+\frac{\Gamma(1-v)}{\Gamma(u)}\right) \\ & +g_q(u,v)+g_q(v,u). \end{aligned} \tag{1.6}$$

Let $u+v=1+\delta$, $|\delta|<\frac{1}{2}$. Then we have that

$$\begin{aligned} & q\left(\zeta(u+v)+\frac{\zeta(u+v-1)\Gamma(u+v-1)}{q^{u+v-1}}\right. \\ & \left(\frac{\Gamma(1-u)}{\Gamma(v)}+\frac{\Gamma(1-v)}{\Gamma(u)}\right) \\ = & q\left(\zeta(1+\delta)+q^{-\delta}\zeta(\delta)\Gamma(\delta)\left(\frac{\Gamma(1-u)}{\Gamma(1-u+\delta)}+\frac{\Gamma(u-\delta)}{\Gamma(u)}\right)\right) \\ = & q\zeta(1+\delta)+q^{1-\delta}\zeta(1-\delta)\pi^{-\frac{1}{2}+\delta}\frac{\Gamma(\delta)\Gamma(\frac{1-\delta}{2})}{\Gamma(\frac{\delta}{2})} \\ & \left(\frac{\Gamma(1-u)}{\Gamma(1-u+\delta)}+\frac{\Gamma(u-\delta)}{\Gamma(u)}\right) \\ = & q\zeta(1+\delta)+q^{1-\delta}\zeta(1-\delta)\frac{2^{\delta-\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}}\pi^{-\frac{1}{2}+\delta}\frac{\pi}{\cos\frac{\pi\delta}{2}} \\ & \left(\frac{\Gamma(1-u)}{\Gamma(1-u+\delta)}+\frac{\Gamma(u-\delta)}{\Gamma(u)}\right) \\ = & q\zeta(1+\delta)+q^{1-\delta}\zeta(1-\delta)(2\pi)^{\delta}\frac{1}{2\cos\frac{\pi\delta}{2}}\left(\frac{\Gamma(1-u)}{\Gamma(1-u+\delta)}\right. \end{aligned}$$

$$+\frac{\Gamma(u-\delta)}{\Gamma(u)} = q(\frac{1}{\delta}+\gamma_0)$$

$$-(\frac{1}{\delta}-\gamma_0)(1+\delta\log\frac{2\pi}{q}))\frac{1}{2}(1-$$

$$\delta\frac{\Gamma'}{\Gamma}(1-u)+1-\delta\frac{\Gamma'}{\Gamma}(u)+\mathrm{O}(|\delta|).$$

Therefore, if $\delta \to 0$, this and (1.7) give the formula of the lemma.

The next lemma can be found in [1].

Lemma 1.3.3. *Let α, β, γ, a, b, k, T be real numbers such that α, β, γ are positive and bounded, $\alpha \neq 1$, $0 < a < \frac{1}{2}$, $a < \frac{T}{8\pi k}$, $b \geq T$, $k \geq 1$ and $T \geq 1$. Then*

$$\int_a^b y^{-\alpha}(1+y)^{-\beta}(\log\frac{1+y}{y})^{-\gamma} \tag{1.7}$$

$$\times\exp(iT\log\frac{1+y}{y}+2\pi kiy)\mathrm{d}y$$

$$=(2k\pi^{\frac{1}{2}})^{-1}T^{\frac{1}{2}}V^{-\gamma}U^{-\frac{1}{2}}(U-\frac{1}{2})^{-\alpha}(U+\frac{1}{2})^{-\beta}$$

$$\times\exp(iTV+2\pi ikU-\pi ik+\frac{\pi i}{4})$$

$$+\mathrm{O}(a^{1-\alpha}T^{-1})+\mathrm{O}(b^{\gamma-\alpha-\beta}k^{-1})+R(T,k)$$

uniformly for $|\alpha-1|>\varepsilon$,

$$U=\left(\frac{T}{2\pi k}+\frac{1}{4}\right)^{\frac{1}{2}},$$

$$V=2\mathrm{arsinh}\left(\frac{\pi k}{2T}\right)^{\frac{1}{2}}.$$

$$R(T,k)\ll\begin{cases} T^{(\gamma-\alpha-\beta)/2-\frac{1}{4}}k^{-(\gamma-\alpha-\beta)/2-\frac{5}{4}}, & \text{for } 1\leq k\leq T,\\ T^{-\frac{1}{2}-\alpha}k^{\alpha-1}, & \text{for } f\geq T.\end{cases}$$

A similar result holds for the corresponding integral with $-k$ in place of k, except that in that case the main term on the right-hand side of (1.7) is to be omitted.

The following lemma is a modification of Atkinson's Lemma 2.3 from [1].

Lemma 1.3.4. *Let*

$$Z = \frac{T}{2\pi} + \frac{a^2}{2} - a\sqrt{\frac{T}{2\pi} + \frac{a^2}{4}}.$$

For $T \gg 1$, $a \gg \sqrt{T}$, $\nu > 0$ *and* $\alpha > 1$, *we have*

$$\int_a^\infty \left(x^\alpha \text{arsinh}\left(x\sqrt{\frac{\pi}{2T}} \right) \left(\sqrt{\frac{T}{2\pi x^2} + \frac{1}{4}} + \frac{1}{2} \right) \right.$$

$$\times \left(\frac{T}{2\pi x^2} + \frac{1}{4} \right)^{\frac{1}{4}}{}^{-1} \exp 2\pi i(\frac{x^2}{2} - x\sqrt{\frac{T}{2\pi} + \frac{x^2}{4}})$$

$$-\frac{T}{\pi}\text{arsinh}\left(x\sqrt{\frac{\pi}{2T}} \right) + 2x\sqrt{\nu})\mathrm{d}x$$

$$= \frac{4\pi}{T}\nu^{(\alpha-1)/2} \left(\log\frac{T}{2\pi\nu} \right)^{-1} \left(\frac{T}{2\pi} - \nu \right)^{\frac{3}{2}-\alpha}$$

$$\times \exp -i(T\log\frac{T}{2\pi\nu} - T + 2\pi\nu + \frac{\pi}{4}) + i\pi/2$$

$$+\mathrm{O}(a^{-\alpha}\min(\frac{a}{\sqrt{T}}, |\sqrt{\nu} - \sqrt{Z}|^{-1}))$$
$$+\mathrm{O}\left(\nu^{(\alpha-1)/2} \left(\frac{T}{2\pi} - \nu \right)^{1-\alpha} T^{-\frac{3}{2}} \right),$$

provided that $\nu < Z$. *If* $\nu \geq Z$ *or if* $\sqrt{\nu}$ *is replaced by* $-\sqrt{\nu}$, *then the main term and the last error term on the right-hand side are to be omitted.*

Lemma 1.3.5. ***(Stirling's formula)*** *Suppose that a is a constant, $|\arg s| \leq \pi - \delta$, $\delta > 0$, and that $s = 0$ and neighborhoods of poles of $\Gamma(s+a)$ are excluded. Then*

$$\log \Gamma(s+a) = (s + a + \frac{1}{2}) \log s - s + \frac{1}{2} \log 2\pi + \mathrm{O}(|s|^{-1})).$$

Proof of the lemma can be found in [11].

Lemma 1.3.6. *Let*

$$\Delta(x) = \sum_{m \leq x} d(m) - x(\log x + 2\gamma_0 - 1).$$

Then, for every $\varepsilon > 0$,

$$\Delta(x) = \mathrm{O}(x^{\frac{1}{3}+\varepsilon}).$$

The estimate of the lemma was obtained by G. F. Voronoi [33], see also [11]. There exist more precise estimates for $\Delta(x)$, however, the estimate of the lemma is sufficient for our aims.

The Voronoi formula for $\Delta(x)$ will be also useful for the proof of Theorem 1.2.1.

Lemma 1.3.7. *Suppose that x is not an integer. Then*

$$\Delta(x) = (\pi\sqrt{2})^{-1} x^{\frac{1}{4}} \sum_{m=1}^{\infty} \frac{d(m)}{m^{\frac{3}{4}}}$$

$$\times \left(\cos(4\pi\sqrt{mx} - \frac{\pi}{4}) - 3(32\pi\sqrt{mx})^{-1} \sin(4\pi\sqrt{mx} - \frac{\pi}{4}) \right)$$

$$+\mathrm{O}(x^{-\frac{3}{4}}).$$

The lemma is a modification of the Voronoi formula [33]. In the formula of the lemma, the Bessel functions are changed by their asymptotic expressions, see [34].

1.4 Proof of the Atkinson formula for the periodic zeta-function on the critical line

It is not difficult to see that

$$I(q,T)=\sum_{a=1}^{q}\int_0^T |\zeta_{\frac{a}{q}}(\frac{1}{2}+it)|^2\mathrm{d}t$$

$$=\frac{1}{2i}\sum_{a=1}^{q}\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}\zeta_{\frac{a}{q}}(u)\zeta_{-\frac{a}{q}}(1-u)\mathrm{d}u.$$

Therefore, by Lemma 1.3.2,

$$\begin{aligned} I(q,T)=&\frac{q}{2i}\left(\log\frac{\Gamma(\frac{1}{2}+iT)}{\Gamma(\frac{1}{2}-iT)}\right.\\ &+2iT(2\gamma-\log\frac{2\pi}{q})\\ &+\frac{q}{i}\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}g_q(u,1-u)\mathrm{d}u. \end{aligned}\tag{1.8}$$

Since

$$\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}g_q(u,1-u)\mathrm{d}u=\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}g_q(1-u,u)\mathrm{d}u,$$

hence we find that

$$I(q,T)=\frac{q}{2i}\left(\frac{1}{2}\left(-\log\Gamma(\left(\frac{1}{2}-iT\right)\right.\right.$$

$$+\log\Gamma(\left(\frac{1}{2}+iT\right)+\Gamma(\left(\frac{1}{2}+iT\right)-\Gamma(\left(\frac{1}{2}-iT\right)$$

$$+2iT\left(2\gamma_0-\log\frac{2\pi}{q}\right)+\frac{q}{i}\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}g_q(u,1-u)\mathrm{d}u=$$

$$= \frac{q}{2i}\left(\log\frac{\Gamma(\frac{1}{2}+iT)}{\Gamma(\frac{1}{2}-iT)} + 2iT\left(2\gamma_0 - \log\frac{2\pi}{q}\right)\right)$$

$$+\frac{q}{i}\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} g_q(u, 1-u)\mathrm{d}u.$$

An application of Lemma 1.3.5 shows that

$$\log\frac{\Gamma(\frac{1}{2}+iT)}{\Gamma(\frac{1}{2}-iT)} = iT\log 2iT\log(\frac{1}{2}+iT) - \frac{1}{2}$$

$$-iT + \frac{1}{2}\log 2\pi + \mathrm{O}(T^{-1})$$

$$iT\log(\frac{1}{2}-iT) + \frac{1}{2} - iT$$

$$-\frac{1}{2}\log 2\pi + \mathrm{O}(T^{-1})$$

$$= iT(\log i + \log T + \log\left(1+\frac{1}{2iT}\right)) - 2iT + iTiT\log i$$

$$+\log T + \log\left(1-\frac{1}{2iT}\right) + \mathrm{O}(T^{-1})$$

$$= iT\left(\frac{\pi}{2} + \log TR + \frac{1}{2iT} + \mathrm{O}(T^{-2})\right) + \mathrm{O}(T^{-1})$$

$$= 2iT \log T - 2iT + \mathrm{O}(T^{-1}).$$

Therefore, in view of the definition of $E(q,T)$ and (1.8),

$$E(q,T) = \frac{q}{i}\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} g_q(u,1-u)\mathrm{d}u + \mathrm{O}(qT^{-1}). \tag{1.9}$$

Now we consider the convergence of the double series in (1.2). For this, we apply the method of [26]. For $\mathrm{Re}u < 1$, $\mathrm{Re}(u+v) > 0$ and $n \in \mathbb{N}$, we have the estimate

$$\begin{aligned}
&2\int_0^\infty y^{-u}(1+y)^{-v}\cos(2\pi ny)\mathrm{d}y \\
&= \int_0^\infty y^{-u}(1+y)^{-v}(e^{2\pi iy}) + e^{-2\pi iy})\mathrm{d}y \\
&= n^{u-1}\int_0^\infty y^{-u}(1+\frac{y}{n})(e^{2\pi iy}) + e^{-2\pi iy})\mathrm{d}y \\
&= n^{u-1}\int_0^{i\infty} y^{-u}(1+\frac{y}{n})^{-v}e^{2\pi iy})\mathrm{d}y \\
&+n^{u-1}\int_0^{-i\infty} y^{-u}(1+\frac{y}{n})^{-v}e^{-2\pi iy}\mathrm{d}y \\
&= \mathrm{O}\left|\frac{n^{u-1}}{u-1}\right|,
\end{aligned}$$

uniform for bounded u and v. Hence, it follows that the double series (1.2) converges absolutely for $\mathrm{Re}u < 0$, $\mathrm{Re}v > 1$ and $\mathrm{Re}(u+v) > 0$ because it is majorized by the series

$$\sum_{l=1}^\infty \left|\frac{1}{l^v}\right| \sum_{m=1}^\infty \left|\frac{1}{m^{1-u}}\right|,$$

and we take $n = ml$. Thus, we can take $u = v - 1$ with $\mathrm{Re}u < 0$. Using the definition of the function $d(m)$, we obtain that

$$g_q(u,1-u) = 2\sum_{m=1}^\infty d(m)\int_0^\infty \frac{\cos(2\pi mqy)}{y^u(1+y)^{1-u}}\mathrm{d}y. \tag{1.10}$$

Due to (1.9), we need an analytic continuation for $g_q(u,1-u)$ when $\mathrm{Re}u = \frac{1}{2}$.

We take $n \in \mathbb{N}$, and, for brevity, denote

$$v_q(u,x) = 2\int_0^\infty \frac{\cos(2\pi mqy)}{y^u(1+y)^{1-u}}\mathrm{d}y. \tag{1.11}$$

Moreover, let

$$D(x) = \sum_{m \le x} d(m).$$

Then using the definition of $\Delta(x)$ in Lemma 1.3.6 we have that

$$\sum_{m>n} d(m)v(u,m) = \int_{n+\frac{1}{2}}^\infty v_q(u,x)\mathrm{d}D(x)$$
$$= \int_{n+\frac{1}{2}}^\infty (\log x + 2\gamma_0)v_q(u,x)\mathrm{d}x + \int_{n+\frac{1}{2}}^\infty v_q(u,x)\mathrm{d}\Delta(x)$$
$$= \int_{n+\frac{1}{2}}^\infty (\log x + 2\gamma_0)v_q(u,x)\mathrm{d}x + v_q(n,x)\Delta(x)\Big|_{n+\frac{1}{2}}^\infty$$
$$-\int_{n+\frac{1}{2}}^\infty \Delta(x)\frac{\delta v_q(u,x)}{\delta x}\mathrm{d}x = \int_{n+\frac{1}{2}}^\infty (\log x + 2\gamma_0)v_q(u,x)\mathrm{d}x$$
$$-v_q(u,n+\frac{1}{2})\Delta(n+\frac{1}{2}).$$

Consequently, formula (1.10) can be written in the form

$$\begin{aligned} g_q(u,u-1) &= \sum_{m\le n} d(m)v_q(u,m) - v_q(u,n+\frac{1}{2})\Delta(n+\frac{1}{2}) \\ &\int_{n+\frac{1}{2}}^\infty (\log x + 2\gamma_0)v_q(u,x)\mathrm{d}x - \int_{n+\frac{1}{2}}^\infty \Delta(x)\frac{\sigma v_q(u,x)}{\sigma x}\mathrm{d}x \\ &\stackrel{\text{def}}{=} g_{q,1}(u) - g_{q,2}(u) + g_{q,3}(u) - g_{q,4}(u). \end{aligned} \tag{1.12}$$

Note that $g_{q,1}(u)$ and $g_{q,2}(u)$ are analytic functions in the half-plane $\mathrm{Re}u < 1$ because the integral in (1.11) is analytic in this region. So, it remains to consider $g_{q,3}(u)$ and $g_{q,4}(u)$. We start with $g_{q,4}(u)$. Suppose that $\mathrm{Re}u \le 1$ and $u$is bounded. Then, using the formula

$$\cos z = \frac{e^{-iz} + e^{iz}}{2},$$

we find that

$$v(u,x) = \int_0^{i\infty} \frac{e^{2\pi ixqy}}{y^u(1+y)^{1-u}}\mathrm{d}y + \int_0^{-i\infty} \frac{e^{-2\pi ixqy}}{y^u(1+y)^{1-u}}\mathrm{d}y, \tag{1.13}$$

and

$$\begin{aligned}\frac{\partial v(u,x)}{\partial u} &= 2\pi q i \int_0^{i\infty} \frac{e^{2\pi i x q y}}{y^{u-1}(1+y)^{1-u}}\mathrm{d}y - 2\pi q i \int_0^{-i\infty} \frac{e^{-2\pi i x q y}}{y^{u-1}(1+y)^{1-u}}\mathrm{d}y \\ &= 2\pi i q x^{u-2}\Bigg(\int_0^{i\infty} \frac{e^{2\pi i x q y}}{y^{u-1}(1+\frac{y}{x})^{1-u}}\mathrm{d}y - \int_0^{-i\infty} \frac{e^{2\pi i x q y}}{y^{u-1}(1+\frac{y}{x})^{1-u}}\mathrm{d}y \\ &= \mathrm{O}(x^{\mathrm{Re}u-2}).\end{aligned}$$

Therefore, the estimate of the Lemma 1.3.6 shows that the function $g_{q,4}(u)$ is analytic for $\mathrm{Re}u < \frac{2}{3}$.

Denote $b = n + \frac{1}{2}$. Then, in view of (1.13), the function $g_{q,3}(u)$ can be written in the from

$$g_{q,3}(u) = \int_b^{\infty} (\log x + 2\gamma_0)\Big(\int_0^{i\infty} \frac{e^{2\pi i x q y}}{y^u(1+y)^{1-u}}\mathrm{d}y + \int_0^{-i\infty} \frac{e^{-2\pi i x q y}}{y^u(1+y)^{1-u}}\Big)\mathrm{d}y. \tag{1.14}$$

Suppose that $\mathrm{Re}u < 0$. Then the integration by parts yields

$$\int_b^{\infty} \left((\log x + 2\gamma_0)\int_0^{i\infty} \frac{e^{2\pi i x q y}}{y^u(1+y)^{1-u}}\mathrm{d}y\right)\mathrm{d}x \tag{1.15}$$

$$\begin{aligned} &= -\frac{\log b + \gamma_0}{2\pi i q}\int_0^{i\infty} \frac{e^{2\pi i x q y}}{y^{u+1}(1+y)^{1-u}}\mathrm{d}y \\ &= \frac{1}{2\pi i q}\int_b^{\infty}\mathrm{d}x \int_0^{i\infty} \frac{e^{2\pi i x q y}}{y^{u+1}(1+y)^{1-u}}\mathrm{d}y \\ &= \frac{\log b + \gamma_0}{2\pi i q}\int_0^{\infty} \frac{e^{2\pi i b q y}}{y^{u+1}(1+y)^{1-u}}\mathrm{d}y \\ &= \frac{1}{2\pi i q u}\int_0^{i\infty} \frac{e^{2\pi i x q y}}{y^{u+1}(b+y)^{-u}}\mathrm{d}y \\ &= \frac{\log b + \gamma_0}{2\pi i q}\int_0^{\infty} \frac{e^{2\pi i b q y}}{y^{u+1}(1+y)^{1-u}}\mathrm{d}y \\ &= \frac{1}{2\pi i q u}\int_0^{\infty} \frac{e^{2\pi i b q y}}{y^{u+1}(1+y)^{-u}}\mathrm{d}y. \end{aligned}$$

Similarly, we find that

$$\int_b^{\infty} \left((\log x + 2\gamma_0)\int_0^{-i\infty} \frac{e^{-2\pi i x q y}}{y^u(1+y)^{1-u}}\mathrm{d}y\right)\mathrm{d}x$$

$$= \frac{\log b + \gamma_0}{2\pi i q} \int_0^\infty \frac{e^{-2\pi i b q y}}{y^{u+1}(1+y)^{1-u}} \mathrm{d}y$$
$$- \frac{1}{2\pi i q u} \int_0^\infty \frac{e^{2\pi i b q y}}{y^{u+1}(1+y)^{-u}} \mathrm{d}y.$$

This together with (1.14), (1.15) and the formula

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i},$$

leads to

$$\begin{aligned} g_{q,3}(u) = &-\frac{\log b + \gamma_0}{\pi q} \int_0^\infty \frac{\sin(2\pi i b q y)}{y^{u+1}(1+y)^{1-u}} \mathrm{d}y \\ &+ \frac{1}{\pi q u} \int_0^\infty \frac{\sin(2\pi b q y)}{y^{u+1}(1+y)^{-u}} \mathrm{d}y. \end{aligned} \tag{1.16}$$

The integrals in the latter equality converge uniformly on compact subsets of the half-plane $\mathrm{Re}(u) < 1$, therefore, the function $g_{q,3}$ has an analytic continuation which is valid for $\mathrm{Re}(u) = \frac{1}{2}$.

From the above remarks and (1.12), it follows that the function, $g_q(u, 1-u)$ has an analytic continuation which is valid for $\mathrm{Re}(u) = \frac{1}{2}$. Thus, we can integrate in (1.9).

Now, using (1.11) and (1.8), we arrive at

$$E(q,T) = I_{q,1} - I_{q,2} + I_{q,3}3 - I_{q,4} + \mathrm{O}(\frac{q}{T}), \tag{1.17}$$

where, for $j = 1, 2, 3, 4,$

$$I_{q,j} = \frac{q}{i} \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} g_{q,j}(u) \mathrm{d}u. \tag{1.18}$$

Therefore, by (1.12) and (1.16),

$$I_{q,1} = 4 \sum_{n \le N} d(n) \int_0^\infty \frac{\sin(T \log \frac{1+y}{y}) \cos(2\pi q n y)}{y^{\frac{1}{2}}(1+y)^{\frac{1}{2}} \log \frac{1+y}{y}} \mathrm{d}y,$$

$$I_{q,2} = 4\Delta(N) \int_0^\infty \frac{\sin(T \log \frac{1+y}{y}) \cos(2\pi q N y)}{y^{\frac{1}{2}}(1+y)^{\frac{1}{2}} \log \frac{1+y}{y}} \mathrm{d}y,$$

$$I_{q,3}=-\frac{2}{\pi q}(\log N+2\gamma)I_{q,31}+\frac{1}{\pi qi}I_{q,32},$$

where

$$I_{q,31}=\int_0^\infty\frac{\sin(T\log\frac{1+y}{y})\sin(2\pi qNy)}{y^{\frac{3}{2}}(1+y)^{\frac{1}{2}}\log\frac{1+y}{y}}\mathrm{d}y,$$

$$I_{q,32}=\int_0^\infty y^{-1}\sin(2\pi qNy)\mathrm{d}y\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}\left(\frac{1+y}{y}\right)^u u^{-1}\mathrm{d}u,$$

and

$$\begin{aligned}I_{q,4}=4\int_X^\infty\frac{\Delta(x)}{x}\mathrm{d}x\int_0^\infty&\frac{\cos(2\pi qxy)}{y^{\frac{1}{2}}(1+y)^{\frac{3}{2}}\log\frac{1+y}{y}}\\ &\times\{T\cos(T\log\frac{1+y}{y})-\sin(T\log\frac{1+y}{y})\\ &\times(\frac{1}{2}+\log^{-1}\frac{1+y}{y})\}\mathrm{d}y,\end{aligned}$$

where $N>1$.

From now we suppose that $T\ll N\ll T$. To evaluate I_1 we apply Lemma 1.3.3 with $\alpha=\beta=\frac{1}{2}$, $\gamma=1$, $k=qm$, $qm\geq T$, and let $a\to 0$, $b\to\infty$. Then, using the definition of $\Sigma_1(q,T)$, we have

$$I_{q,1}=\Sigma_1(q,T)+\mathrm{O}(q^{-1}\log^2 T)+\mathrm{O}((qT)^{-\frac{1}{2}}\log T). \tag{1.19}$$

Similarly, using the estimate of Lemma 1.3.7, we get

$$I_{q,2}\ll q^{-\frac{1}{2}}T^{-\frac{1}{6}}\log T. \tag{1.20}$$

Next we consider $I_{q,31}$. Divide the range of integration at $(2qN)^{-1}$ and obtain that

$$\begin{aligned}I_{q,31}&=\int_0^{(2qN)^{-1}}+\int_{(2qN)^{-1}}^\infty\frac{\sin(T\log\frac{1+y}{y})\sin(2\pi qNy)}{y^{\frac{3}{2}}(1+y)^{\frac{1}{2}}\log\frac{1+y}{y}}\mathrm{d}y\\ &\ll q^{\frac{1}{2}}T^{-\frac{1}{2}},\end{aligned} \tag{1.21}$$

since by the second mean value theorem

$$\int_0^{(2qN)^{-1}}\frac{\sin(T\log\frac{1+y}{y})\sin(2\pi qNy)}{y^{\frac{3}{2}}(1+y)^{\frac{1}{2}}\log\frac{1+y}{y}}\mathrm{d}y$$

$$
\begin{aligned}
&= 2\pi q N \int_0^{\xi_1} \frac{\sin(T\log\frac{1+y}{y})}{y(1+y)} \left(\frac{y^{\frac12}(1+y)^{\frac12}}{\log\frac{1+y}{y}} \right) \mathrm{d}y \\
&= 2\pi q N \xi_1^{\frac12}(1+\xi_1)^{\frac12} \left(\log\frac{1+\xi_1}{\xi_1} \right)^{-1} \int_{\xi_2}^{\xi_1} \frac{\sin(T\log\frac{1+y}{y})}{y(1+y)} \mathrm{d}y \\
&= 2\pi q N \xi_1^{\frac12}(1+\xi_1)^{\frac12} \left(\log\frac{1+\xi_1}{\xi_1} \right)^{-1} \\
&\quad \times \left(T^{-1}\cos\left(T\log\frac{1+y}{y} \right) \right) \Big|_{\xi_2}^{\xi_1} \\
&\ll q^{\frac12} T^{-\frac12},
\end{aligned}
$$

where $0 \le \xi_2 \le \xi_1 \le (2qN)^{-1}$, and the integral

$$
\int_{(2qN)^{-1}}^{\infty} \ldots \mathrm{d}y
$$

is estimate by Lemma 1.3.3, and is

$$
\ll q^{\frac12} T^{-\frac12}.
$$

We write

$$
\begin{aligned}
I_{q,32} &= \left(\int_0^1 + \int_1^{\infty} \right) \frac{\sin(2\pi q N y)}{y} \mathrm{d}y \int_{\frac12 - iT}^{\frac12 + iT} \left(\frac{1+y}{y} \right) \frac{\mathrm{d}u}{u} \\
&\overset{\text{def}}{=} I'_{q,32} + I''_{q,32}.
\end{aligned} \tag{1.22}
$$

The point $u = 0$ is a simple pole. Therefore, for $0 < y \le 1$, by the residue theorem

$$
\begin{aligned}
&\int_{\frac12 - iT}^{\frac12 + iT} \left(\frac{1+y}{y} \right) \frac{\mathrm{d}u}{u} \\
&= 2\pi i - \left(\int_{\frac12 + iT}^{-\infty + iT} + \int_{\infty - iT}^{\frac12 - iT} \right) \left(\frac{1+y}{y} \right) \frac{\mathrm{d}u}{u} \\
&= 2\pi i + \mathrm{O}\left(\frac{1}{Ty^{\frac12}}\right),
\end{aligned}
$$

since

$$
\begin{aligned}
&\int_{\frac12 \pm iT}^{-\infty \pm iT} \left(\frac{1+y}{y} \right) \frac{\mathrm{d}u}{u} \ll \frac{1}{T} \int_{-\infty}^{\frac12} \left(\frac{1+y}{y} \right)^x \mathrm{d}x \\
&= \frac{1}{T} \left(\frac{1+y}{y} \right)^x \left(\log\frac{1+y}{y} \right)^{-1} \ll \frac{1}{Ty^{\frac12}}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
I'_{q,32} &= 2\pi i \int_0^1 \frac{\sin(2\pi qNy)}{y}\mathrm{d}y + \mathrm{O}\left(\frac{1}{T} \in_0^1 \frac{|\sin(2\pi qNy)|}{y^{\frac{3}{2}}}\mathrm{d}y\right) \\
&= 2\pi i\frac{\pi}{2} + \mathrm{O}(\frac{1}{qN}) + \mathrm{O}\left(\frac{qN}{T}\int_0^{(qN)^{-1}} \frac{\mathrm{d}y}{y^{\frac{1}{2}}}\right) + \mathrm{O}\left(\frac{1}{T}\int_{(qN)^{-1}}^{\infty} \frac{\mathrm{d}y}{y^{\frac{3}{2}}}\right) \qquad (1.23) \\
&= \pi^2 i + \mathrm{O}(q^{\frac{1}{2}}T^{-\frac{1}{2}}).
\end{aligned}
$$

The integration by parts shows that

$$
\begin{aligned}
I''_{q,32} &= \int_1^{\infty} \frac{\sin(2\pi qNy)}{y}\mathrm{d}y \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \left(\frac{1+y}{y}\right)^u \frac{\mathrm{d}u}{u} \\
&= \left(\frac{\cos(2\pi qNy)}{2\pi qNy}\right) \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \left(\frac{1+y}{y}\right) \frac{\mathrm{d}u}{u}\Big|_1^{\infty} \\
&- \int_1^{\infty} \frac{\cos(2\pi qNy)}{2\pi qNy^2}\mathrm{d}y \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \left(\frac{1+y}{y}\right)^u \frac{\mathrm{d}u}{u} \qquad (1.24) \\
&- \int_1^{\infty} \frac{\cos(2\pi qNy)}{2\pi qNy}\mathrm{d}y \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \left(\frac{1+y}{y}\right)^{u-1} \frac{\mathrm{d}u}{y^2} \ll \frac{\log T}{qT}.
\end{aligned}
$$

Here, for $y \geq 1$, we have used the estimate

$$
\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \left(\frac{1+y}{y}\right)^u \frac{\mathrm{d}u}{u} \ll \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \left|\frac{\mathrm{d}u}{u}\right| \ll \log T.
$$

Therefore, in view of (1.22) - (1.24),

$$
I_{q,32} = \pi^2 i + \mathrm{O}\left(\frac{\log T}{(qT)^{\frac{1}{2}}}\right).
$$

This and (1.21) gives the estimate

$$
I_{q,2} = \frac{\pi}{q} + \mathrm{O}\left(\frac{\log T}{(qT)^{\frac{1}{2}}}\right). \qquad (1.25)
$$

Now we will evaluate $I_{q,4}$. To estimate the inner integrals, we use Lemma 1.3.3 with $\alpha = \frac{1}{2}$, $\beta = \frac{3}{2}$, $\gamma = 1, 2$, $k = qx$, letting $a \to 0$, $b \to \infty$. For $qx \gg T$, we obtain in the notation of Lemma 1.3.3 that

$$
\begin{aligned}
&\int_0^{\infty} \frac{\cos(T\log\frac{1+y}{y})\cos 2\pi qxy}{y^{\frac{1}{2}}(1+y)^{\frac{3}{2}}\log\frac{1+y}{y}}\mathrm{d}y \\
&= (4qx)^{-1}(\frac{T}{\pi})^{\frac{1}{2}}V^{-1}U^{-\frac{1}{2}}(U-\frac{1}{2})^{-\frac{1}{2}}(U+\frac{1}{2})^{-\frac{3}{2}} \\
&\times \cos(TV + \pi qx(2U-1) + \frac{\pi}{4}) \\
&+ \mathrm{O}(T^{-1}(qx)^{-\frac{1}{2}}),
\end{aligned}
$$

and similarly, for $\gamma = 1, 2$,

$$\int_0^\infty \frac{\sin(T\log\frac{1+y}{y})\cos 2\pi qxy}{y^{\frac{1}{2}}(1+y)^{\frac{3}{2}}\left(\log\frac{1+y}{y}\right)^\gamma}\mathrm{d}y \ll (qx)^{-\frac{1}{2}}.$$

Hence, we have

$$I_{q,4} = \frac{1}{\sqrt{2}}\int_N^\infty \frac{\Delta(x)}{x}$$
$$\times \left(T(qx)^{-\frac{1}{2}}\frac{\cos(TV + \pi qx(2U-1) + \frac{\pi}{4})}{2\text{arsinh}\left(\frac{\pi qx}{2T}\right)\left(\frac{T}{2\pi qx}+\frac{1}{4}\right)^{\frac{1}{2}}\left(\left(\frac{T}{2\pi qx}+\frac{1}{4}\right)^{\frac{1}{2}}+\frac{1}{2}\right)}\right)$$
$$+\mathrm{O}((qx)^{-\frac{1}{2}})\mathrm{d}x.$$

Using Lemma 1.3.6 and taking $(qx)^{\frac{1}{2}}$ in the place of qx and applying Lemma 1.3.7, we obtain

$$I_{q,4} = \frac{T}{\pi}q^{-\frac{1}{4}}\sum_{n=1}^\infty d(n)n^{-\frac{3}{4}}\int_{\sqrt{qN}}^\infty \frac{\cos(2T\text{arsinh}(x\sqrt{\frac{\pi}{2T}} + x(2\pi T + \pi^2x^2)^{\frac{1}{2}} - \pi x^2 + \frac{\pi}{4})}{x^{\frac{3}{2}}\text{arsinh}(x\sqrt{\frac{\pi x^2}{2T}})\left(\frac{T}{2\pi x^2}+\frac{1}{4}\right)^{\frac{1}{2}}\left(\left(\frac{T}{2\pi x^2}+\frac{1}{4}\right)^{\frac{1}{2}}+\frac{1}{2}\right)}$$
$$\times\left(\cos(4\pi x\sqrt{\frac{n}{q}} - \frac{\pi}{4}) - \frac{3}{32\pi x}\sqrt{\frac{q}{n}}\sin(4\pi x\sqrt{\frac{n}{q}}-\frac{\pi}{4})\right)\mathrm{d}x$$
$$+\mathrm{O}(q^{-\frac{1}{2}}T^{-\frac{1}{6}}).$$

Now we apply Lemma 1.3.4 with $a = \sqrt{qN}$, $\nu = \frac{m}{q}$ and $\alpha = \frac{3}{2}$ or $\alpha = \frac{5}{2}$. Then, in notation of Lemma 1.3.4,

$$qZ = N',$$

where N' is defined in the statement of Theorem 1.2.1. The main term comes from the integral with $\alpha = \frac{3}{2}$. Note also that

$$N' < A\frac{qT}{2\pi},$$

for some $A > 1$. Hence, for $m < N'$,

$$\begin{aligned} I_{q,4} =& 2\sum_{n\le N'} d(n)(nq)^{-\frac{1}{2}}(\log\frac{qT}{2\pi n})^{-1} \\ &\times\cos(T\log\frac{qT}{2\pi n} - T + \frac{2\pi n}{q} - \frac{\pi}{4}) \end{aligned} \tag{1.26}$$

$$+\mathrm{O}(q^{-\frac{1}{2}}\sum_{n\le N'} d(n)n^{-\frac{1}{2}}\left(\log\frac{qT}{2\pi n}\right)^{-1}\left(\frac{T}{2\pi}-\frac{n}{q}\right)^{-1})$$

$$+\mathrm{O}(q^{-\frac{1}{2}}T^{-\frac{1}{2}}\sum_{n\le N'} d(n)n^{-\frac{3}{4}}\left(\frac{T}{2\pi}-\frac{n}{q}\right)^{-\frac{1}{2}})$$

$$+\mathrm{O}(q^{-\frac{1}{4}}T^{\frac{1}{4}}\sum_{n\le N'} d(n)n^{-\frac{3}{4}}\min(1,|\sqrt{\frac{n}{q}}-\sqrt{Z}|^{-1}))$$

$$+\mathrm{O}(q^{-\frac{1}{2}}T^{-\frac{1}{4}}) \stackrel{\text{def}}{=} I_{q,41}+I_{q,42}+I_{q,43}$$

$$+I_{q,44}+O(q^{-\frac{1}{2}}T^{-\frac{1}{6}}),$$

while if $m \ge N'$, then the main term and the last two error terms are to be omitted. We have that

$$I_{q,41} = -\Sigma_2(q,T). \tag{1.27}$$

Since $T \ll N \ll T$, we have $N' \ll qT$. Therefore,

$$I_{q,42} \ll (qT)^{-1}\sum_{m\le N'}\frac{d(m)}{m^{\frac{1}{2}}} \ll (qT)^{-\frac{1}{2}}\log(qT) \tag{1.28}$$

and, similarly,

$$I_{q,43} \ll (qT)^{-\frac{1}{2}}\log(qT). \tag{1.29}$$

The quantity $I_{q,44}$ is estimated as

$$\begin{aligned}
&\ll q^{-\frac{1}{4}}T^{\frac{1}{4}}\sum_{m=1}^{\infty}\frac{d(m)}{m^{\frac{3}{4}}}\min(1,\left|\sqrt{\frac{m}{q}}-\sqrt{Z}\right|)^{-1}\\
&= q^{-\frac{1}{4}}T^{\frac{1}{4}}\left(\sum_{m\le\frac{N'}{2}}+\sum_{\frac{N'}{2}<m\le N'-\sqrt{N'}}+\sum_{N'-\sqrt{N'}<m\le N'+\sqrt{N'}}\right.\\
&+\sum_{N'+\sqrt{N'}<m\le 2N'}+\sum_{m>2N'}\\
&\stackrel{\text{def}}{=} q^{-\frac{1}{4}}T^{\frac{1}{4}}(S_{q,1}+S_{q,2}+S_{q,3}+S_{q,4}+S_{q,5}).
\end{aligned} \tag{1.30}$$

Summing by parts and applying the estimate

$$\sum_{m\le x} d(x) \ll x\log x,$$

we find

$$S_{q,1} = \sum_{m \le \frac{N'}{2}} \frac{d(m)}{m^{\frac{3}{4}}} \left(\frac{N'}{q} \frac{m}{q} \right)^{-1} \le (q)^{\frac{1}{2}} (T)^{-\frac{1}{4}} \log(qT), \tag{1.31}$$

$$\begin{aligned} S_{q,2} &= \sum_{\frac{N'}{2} < m \le N' - \sqrt{N'}} \frac{d(m)}{m^{\frac{3}{4}}} \left(\sqrt{\frac{N'}{q}} \sqrt{\frac{m}{q}} \right)^{-1} \\ &\le \left(\frac{N'}{q} \right)^{-\frac{3}{4}} \sum_{\frac{N'}{2} < m \le N' - \sqrt{N'}} d(m) \left(\frac{N'}{q} \frac{m}{q} \right)^{-1} \\ &\le q \sum_{\sqrt{N'} < l \frac{N'}{2}} \frac{d([N'] - l)}{l} \\ &\le (q)^{-\frac{1}{2}} (T)^{-\frac{1}{4}} \log^2(qT), \end{aligned} \tag{1.32}$$

and, similarly,

$$S_{q,3} = \sum_{N' - \sqrt{N'} < m \le N' + \sqrt{N'}} \frac{d(m)}{m^{\frac{3}{4}}} \left(\frac{N'}{q} \frac{m}{q} \right)^{-1} \le q^{\frac{1}{2}} (T)^{-\frac{1}{4}} \log(qT), \tag{1.33}$$

and

$$S_{q,4} = \sum_{N' - \sqrt{N'} < m \le N' + \sqrt{N'}} \frac{d(m)}{m^{\frac{3}{4}}} \left(\frac{N'}{q} \frac{m}{q} \right)^{-1} \le q^{\frac{1}{2}} (T)^{-\frac{1}{4}} \log^2(qT) \tag{1.34}$$

Moreover,

$$\begin{aligned} S_{q,5} &= \sum_{m \le 2N'} \frac{d(m)}{m^{\frac{3}{4}}} \left(\sqrt{\frac{m}{4}} \sqrt{\frac{N'}{q}} \right)^{-1} \\ &\le q^{\frac{1}{2}} \sum_{m \le 2N'} \frac{d(m)}{m^{\frac{5}{4}}} \\ &\le q^{\frac{1}{2}} (T)^{-\frac{1}{4}} \log(qT). \end{aligned} \tag{1.35}$$

From (1.30)- (1.35) we find that

$$I_{q,44} \ll q^{\frac{1}{2}} (T)^{-\frac{1}{4}} \log^2(qT).$$

This and (1.26) - (1.29) show that

$$I_{q,4} = -\Sigma_2(q,T) + \mathrm{O}(q^{\frac{1}{2}} \log^2(qT)).$$

This, (1.17), (1.19) - (1.21) and (1.25) imply the estimate

$$\begin{aligned} E(q,T) &= q(\Sigma_1(q,T) + \Sigma_2(q,T)) \\ &+\mathrm{O}(q^{\frac{1}{2}} \log^2(qT)) + \mathrm{O}(qT^{-1}). \end{aligned}$$

1.5 Corollaries

In this section, we simplify the error term in Theorem 1.2.1, and obtain an upper estimate for $E(q,T)$.

Corollary 1.5.1. *Suppose that* $q \ll T^2 \log^4 T$. *Then, under hypothesis and notation of Theorem 1.2.1,*

$$E(q,T) = q(\Sigma_1(q,T) + \Sigma_2(q,T)) + \mathrm{O}(q^{\frac{1}{2}} \log^2(qT)).$$

Proof. We have that $q^{\frac{1}{2}} \ll T \log^2 T$. Therefore, $q^{\frac{1}{2}} T^{-1} \ll \log^2 T$, $qT^{-1} \ll q^{\frac{1}{2}} \log^2 T$, and both the error terms in Theorem 1.2.1 coincide.

A more complicated problem is the estimation of the quantity $E(q,T)$.

Theorem 1.5.1. *Let* $0 < \varepsilon < \frac{1}{2}$ *be fixed. Then, for* $T^\varepsilon \ll H \ll T^{\frac{1}{2}}$,

$$\begin{aligned} E(q,T) \ll qH \log T + q \sup_{\frac{T}{2} \le \tau \le 2T} \Bigg| \sum_{m \le T^{1+\varepsilon} H^{-2}} (-1)^{qm} d(m)(qm)^{-\frac{1}{2}} \\ \times \left(\operatorname{arsinh}\sqrt{\frac{\pi qm}{2\tau}}\right)^{-1} \left(\frac{\tau}{2\pi qm} + \frac{1}{4}\right)^{\frac{1}{4}} \cos f(\tau, qm). \end{aligned}$$

Proof. We use Theorem 1.2.1 and apply the average technique. Let $0 \le x \le T$. Then, from the definition of $E(q,T)$, we find that

$$\begin{aligned} E(q,T+x) - E(q,T) &= \sum_{a=1}^{q} \int_T^{T+x} \left|\zeta_{\frac{a}{q}}(\frac{1}{2} + it)\right|^2 \mathrm{d}t \\ &- q(T+x)(\log \frac{q(T+x)}{2\pi} + 2\gamma_0 - 1) \\ &+ qT(\log \frac{qT}{2\pi} + 2\gamma_0 - 1) \\ &= \sum_{a=1}^{q} \int_T^{T+x} \left|\zeta_{\frac{a}{q}}(\frac{1}{2} + it)\right|^2 \mathrm{d}t \\ &+ \mathrm{O}(qx \log(qT)). \end{aligned} \tag{1.36}$$

Moreover, from the mean square formula for $\zeta_\lambda(s)$

$$\int_T^{T+x} \left|\zeta_{\frac{a}{q}}(\frac{1}{2}+it)\right|^2 \mathrm{d}t = \mathrm{O}(T\log T).$$

Thus, (1.36) shows that there exists an absolute constant $c>0$ such that, for $T \geq T_0$,

$$E(q,T+x) - E(q,T) \geq -(qx\log(qT)).$$

Hence,

$$\begin{aligned}\int_T^{T+x} E(q,t)\mathrm{d}t &= \int_T^{T+x} ((E(q,t)-E(q,T)) + E(q,T))\mathrm{d}t \\ \int_T^{T+x} &(E(q,t)-E(q,T))\mathrm{d}t + xE(q,T) \\ &\geq xE(q,T) - Cqx^2\log(qT).\end{aligned}$$

Thus,

$$E(q,T) \leq \frac{1}{x}\int_T^{T+x} E(q,t)\mathrm{d}t + Cqx\log(qT).$$

Let $n \in \mathbb{N}$ be large but fixed. Then, reasoning similarly as above, after n steps we arrive at the bound

$$\begin{aligned}E(q,T) \leq \frac{1}{H^n}\int_0^H \cdots \int_0^H E(q,T+u_1+\cdots+u_n)\mathrm{d}u_1\ldots\mathrm{d}u_n \\ + CqH\log(qT).\end{aligned} \tag{1.37}$$

In Theorem 1.2.1, we take $N=T$ and

$$N' = qT\left(\frac{1}{2\pi} + \frac{q}{2} - \left(\left(\frac{q}{2}\right)^2 + \frac{q}{2\pi}\right)^{\frac{1}{2}}\right).$$

Since

$$\begin{aligned}&\frac{\delta(T\log\frac{qT}{2\pi m} - T + \frac{2\pi m}{q} - \frac{\pi}{4})}{\delta T} \\ &\log\frac{qT}{2\pi m} + \frac{2\pi m}{qT}\frac{q}{2\pi m} - 1 \\ &= \log\frac{qT}{2\pi m} \gg 1,\end{aligned}$$

after n integrations by parts, for n sufficiently large, we obtain that

$$\frac{1}{H^n}\int_0^H \cdots \int_0^H \Sigma_2(q, T + u_1 + \cdots + u_n)\mathrm{d}u_1 \ldots \mathrm{d}u_n \ll qH\log(qH). \tag{1.38}$$

Denote by $\Sigma_1^{(1)}(q, T)$ a part of the sum $\Sigma_1(q, T)$ with $T^{1+\varepsilon}H^{-2} < m \geq T$. Then an integration by parts again and trivial estimates show that

$$\frac{1}{H^n}\int_0^H \cdots \int_0^H \Sigma_1^{(1)}(q, T + u_1 + \cdots + u_n)\mathrm{d}u_1 \ldots \mathrm{d}u_n \ll qH\log(qH). \tag{1.39}$$

Let $\Sigma_1^{(2)}(q, T)$ be a part of the sum $\Sigma_1(q, T)$ with $m \leq T^{1+\varepsilon}H^{-2}$. Then, clearly,

$$\left|\frac{1}{H^n}\int_0^H \cdots \int_0^H \Sigma^(2)_1(q, T + u_1 + \cdots + u_n)\mathrm{d}u_1 \ldots \mathrm{d}u_n\right|$$
$$\sup_{\frac{T}{2}\leq\tau\leq 2T}\left|\sum_{m\leq T^{1+\varepsilon}H^{-2}} (-1)^{qm}d(m)(qm)^{-\frac{1}{2}}\left(\operatorname{arsinh}\sqrt{\frac{\pi qm}{2}}\right)^{-1}\right.$$
$$\times\left(\frac{\tau}{2\pi qm} + \frac{1}{4}\right)^{-\frac{1}{4}} \cos f(\tau, qm).$$

This together with (1.37) - (1.39) proves the theorem.

Corollary 1.5.2. *The estimate*

$$E(q, T) \ll qH\log T + q^{\frac{3}{4}}T^{\frac{1}{2}+\varepsilon}H^{-\frac{1}{2}}$$

holds with $T^\varepsilon \ll H \ll T^{\frac{1}{2}}$ *and every fixed* ε, $0 < \varepsilon < \frac{1}{2}$.

Proof. We have that

$$\sup_{\frac{T}{2}\leq\tau\leq 2T}\left|\sum_{m\leq T^{1+\varepsilon}H^{-2}} (-1)^{qm}d(m)(qm)^{-\frac{1}{2}}\left(\operatorname{arsinh}\sqrt{\frac{\pi qm}{2}}\right)^{-1}\right.$$
$$\times\left(\frac{\tau}{2\pi qm} + \frac{1}{4}\right)^{-\frac{1}{4}} \cos f(\tau, qm)$$
$$\ll (qT)^{-\frac{1}{4}} \sum_{m\leq T^{1+\varepsilon}H^{-2}} \frac{d(m)}{m^{\frac{1}{4}}\log(qT)} \ll \frac{q^{-\frac{1}{4}}}{\log qT}(T^{1+\varepsilon}H^{-2})^{\frac{3}{4}}$$
$$\times \sum_{m\leq T^{1+\varepsilon}H^{-2}} \frac{d(m)}{m} \ll q^{-\frac{1}{4}}T^{\frac{1}{2}+\varepsilon}H^{-2}.$$

Corollary 1.5.3. *We take in Corollary 1.5.2* $H = T^{\frac{1}{3}}$. *Then*

$$\begin{aligned} E(q,T) &\ll qT^{\frac{1}{3}} \log(qT) + q^{\frac{3}{4}} T^{\frac{1}{2}+\varepsilon} T^{-\frac{1}{2}} \\ &\ll qT^{\frac{1}{3}} \log(qT). \end{aligned}$$

We expect that the bound of Corollary 1.5.3 can be improved by the use of exponential sum techniques.

Skyrius 2

Atkinson formula for the periodic zeta-function in the critical strip

Let, as in Chapter 1, $\lambda = \frac{a}{q}$ with given integers a and q, $1 \le a \le q$. For $\frac{1}{2} < \sigma < 1$, define

$$E_\sigma(q,T) = \sum_{a=1}^{q} \int_0^T \left|\zeta_{\frac{a}{q}}(\sigma+it)\right|^2 \mathrm{d}t$$
$$- q\zeta(2\sigma)T + \frac{\zeta(2\sigma-1)\Gamma(2\sigma-1)\sin(\pi\sigma)}{1-\sigma}(qT)^{2-2\sigma},$$

where $\Gamma(s)$ is the Euler gamma-function, and $\zeta(s)$ is the Riemann zeta-function. In this chapter, we give the explicit formula for $E_\sigma(q,T)$.

For $\alpha \in \mathbb{C}$, let

$$\sigma_\alpha(m) = \sum_{d|m} d^\alpha, \quad m \in \mathbb{N},$$

be the generalized divisor function. We use the same notation as in Chapter 1. Define

$$\Sigma_{1,\sigma}(q,T) = q^{\sigma-1}2^{-\frac{1}{2}}\left(\frac{T}{2\pi}\right)^{\frac{1}{2}-\sigma} \sum_{m\le N}(-1)^{qm}$$
$$\sigma_{1-2\sigma}(m)(m)^{\sigma-1}\left(\operatorname{arsinh}\sqrt{\frac{\pi qm}{2T}}\right)^{-1}$$
$$\left(\frac{T}{2\pi qm}+\frac{1}{4}\right)^{-\frac{1}{4}}\cos(f(T,qm)),$$

and

$$\Sigma_{2,\sigma}(q,T) = -2q^{\sigma-1}\Big(\frac{T}{2\pi}\Big)^{\frac{1}{2}-\sigma} \sum_{m\le N'} \frac{\sigma_{1-2\sigma}(m)m^{\sigma-1}}{\log\frac{qT}{2\pi m}}$$
$$\times\cos\Big(T\log\frac{qT}{2\pi m} - T + \frac{\pi}{4}\Big).$$

Theorem 2.0.2. *If q is a positive integer, and $T \ll N \ll T$, then*

$$E_\sigma(q,T) = \Sigma_{1,\sigma}(q,T) + \Sigma_{2,\sigma}(q,T)$$
$$+ \mathrm{O}(q^{2\sigma-\frac{7}{4}}\log(qT))\mathrm{O}(q).$$

2.1 The case $\frac{1}{2} < \sigma < \frac{3}{4}$

Let, as in Section 1.3,

$$g_q(u,v) = \frac{2}{q^{u+v-1}}\sum_{l=1}^{\infty}\frac{1}{l^{u+v-1}}\sum_{m=1}^{\infty}\int_0^\infty \frac{\cos(2\pi mlqy)}{y^u(1+y)^v}\mathrm{d}y.$$

We start with an analogue of Lemma 1.3.2.

Lemma 2.1.1. *Suppose that $0 < \mathrm{Re}(u) < 1$ and $0 < \mathrm{Re}(v) < 1$, $u+v \neq 1$. Then the formula*

$$\sum_{a=1}^{q}\zeta_{\frac{a}{q}}(u)\zeta_{\frac{a}{q}}(v) = q\Big(\zeta(u+v)T$$
$$+\zeta(u+v-1)\Gamma(u+v-1)\Big(\frac{\Gamma(1-u)}{\Gamma(v)} - \frac{\Gamma(1-v)}{u}\Big)$$
$$+g_q(u,v) + g_q(v,u)\Big).$$

Proof. The lemma is the first part of the proof of the Lemma 1.3.2.

Lemma 2.1.2. *Suppose that $0 < \mathrm{Re}(u) < 1$. Then the formula*

$$\sum_{a=1}^{q}\int_0^T \Big|\zeta_{\frac{a}{q}}(\sigma+it)\Big|^2\mathrm{d}t = q\Big(\zeta(2\sigma)T$$
$$+\frac{\zeta(2\sigma-1)\Gamma(2\sigma-1)\sin(\pi\sigma)}{1-\sigma}(qT)^{2-2\sigma}$$
$$iq\int_{\sigma-iT}^{\sigma+iT} g_q(u,2\sigma-u)\mathrm{d}u + \mathrm{O}(q\log(qT))$$

holds.

Proof. We take $u = \sigma + it$ and $v = 2\sigma - u = \sigma - it$ in Lemma 2.1.1. This gives

$$\begin{aligned}
&\sum_{a=1}^{q} \int_0^T \left| \zeta_{\frac{a}{q}}(\sigma + it) \right|^2 \mathrm{d}t = \\
&\frac{1}{2i} \sum_{a=1}^{q} \int_{\sigma - iT}^{\sigma + iT} \zeta_{\frac{a}{q}}(\sigma + it) \zeta_{-\frac{a}{q}}(\sigma - it) \mathrm{d}t \\
&= \frac{1}{2i} \sum_{a=1}^{q} \int_{\sigma - iT}^{\sigma + iT} \zeta_{\frac{a}{q}}(u) \zeta_{-\frac{a}{q}}(v) \mathrm{d}t \\
&= \frac{1}{2i} \int_{\sigma - iT}^{\sigma + iT} \Big(q\zeta(2\sigma) + \frac{\zeta(2\sigma - 1)\Gamma(2\sigma - 1)}{q^{2\sigma - 1}} \Big) \\
&\Big(\frac{\Gamma(1-u)}{\Gamma(2\sigma - u)} + \frac{\Gamma(1 - 2\sigma + u)}{\Gamma(u)} \Big) \\
&+ \frac{q}{i} \int_{\sigma - iT}^{\sigma + iT} g_q(u, 2\sigma - u) \mathrm{d}u,
\end{aligned} \tag{2.1}$$

since

$$\begin{aligned}
&\int_{\sigma - iT}^{\sigma + iT} (g_q(u, 2\sigma - u) + g_q(2\sigma - u, u)) \mathrm{d}u \\
&= 2 \int_{\sigma - iT}^{\sigma + iT} g_q(u, 2\sigma - u) \mathrm{d}u.
\end{aligned}$$

For the integrand of the first integral in (2.1), we apply lemma 1.3.5 (Stirling's formula). This gives

$$\begin{aligned}
\frac{\Gamma(1-u)}{\Gamma(2\sigma - u)} &= \exp\Big\{ \Big(\frac{1}{2} - u \Big) \\
&\times \log(1-u) - (2\sigma - \frac{1}{2} - u) \log(2\sigma - u) - (1-u) \\
&+ (2\sigma - u) + \mathrm{O}(|1-u|^{-1}) + \mathrm{O}(|2\sigma - u|^{-1}) \Big\} \\
&= \exp\Big\{ \Big(\frac{1}{2} - u \Big) \log(1-u) - (2\sigma - \frac{1}{2} - u) \\
&\times \log(2\sigma - u) - (1-u) + (2\sigma - u) + \mathrm{O}(|1-u|^{-1}) + \mathrm{O}(|2\sigma - u|^{-1}) \Big\} \\
&= \exp\Big\{ \Big(\frac{1}{2} - u \Big) (\log(1-u) + \frac{2\sigma - u}{1-u} + (2\sigma - u) \\
&+ \mathrm{O}(|1-u|^{-2}) + (2\sigma - u) + \mathrm{O}(|1-u|^{-1}) + \mathrm{O}(|2\sigma - u|^{-1}) \Big\} \\
&= \exp\Big\{ -(2\sigma - u) \log(1-u) + \mathrm{O}(|1-u|^{-1}) + \mathrm{O}(|2\sigma - u|^{-1}) \Big\}.
\end{aligned}$$

Now integration yields

$$\int_{\sigma-iT}^{\sigma+iT}\frac{\Gamma(1-u)}{\Gamma(2\sigma-u)}\mathrm{d}u$$
$$= -\int_{\sigma-iT}^{\sigma+iT}\frac{\mathrm{d}u}{(1-u)^{2\sigma-1}}+\mathrm{O}(\log T). \tag{2.2}$$

It is easy to see that

$$\begin{aligned}(1-\sigma-iT)^{2-2\sigma}-(1-\sigma+iT)^{2-2\sigma}\\ = T^{2-2\sigma}((-i)^{2-2\sigma}-(i)^{2-2\sigma})+\mathrm{O}(T^{1-2\sigma})\\ = T^{2-2\sigma}(\cos\pi(1-\sigma)-i\sin\pi(1-\sigma)-\cos\pi(1-\sigma)\\ -\sin\pi(1-\sigma))+\mathrm{O}(T^{1-2\sigma})\\ = -2i\sin(\pi\sigma)+\mathrm{O}(T^{1-2\sigma}).\end{aligned}$$

This and (2.2) show that

$$\int_{\sigma-iT}^{\sigma+iT}\frac{\Gamma(1-u)}{\Gamma(2\sigma-u)}\mathrm{d}u = -2i\sin(\pi\sigma)+\mathrm{O}(\log T). \tag{2.3}$$

Obviously,

$$\int_{\sigma-iT}^{\sigma+iT}\frac{\Gamma(1-u)}{\Gamma(2\sigma-u)}\mathrm{d}u = \int_{\sigma-iT}^{\sigma+iT}\frac{\Gamma(1-2\sigma+u)}{\Gamma(u)}\mathrm{d}u.$$

Therefore, the lemma is a result of (2.1) and (2.3).

Remark. *The main term of the lemma can be changed by*

$$q\zeta(2\sigma)T+\frac{(2\pi)^{2\sigma-1}\zeta(2-2\sigma)}{2\sigma-2}(Tq)^{2-2\sigma}.$$

Really, by the functional equation for $\zeta(s)$ [31]

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s),$$

we find that

$$\begin{aligned}\zeta(2\sigma-1) &= \pi^{2\sigma-\frac{3}{2}}\frac{\Gamma(1-\sigma)\Gamma(\sigma)}{\Gamma(\sigma-\frac{1}{2})\Gamma(\sigma)\zeta(2-2\sigma)}\\ &= \frac{(2\pi)^{2\sigma-1}}{(2-2\sigma)\Gamma(2\sigma-1)\sin(\pi\sigma)}\end{aligned}$$

after application of the formulae

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$

and

$$\Gamma(s)\Gamma(s+\frac{1}{2}) = 2\sqrt{\pi}2^{-2s}\Gamma(2s).$$

Define $\Delta_{1-2\sigma}(x)$ by the formula

$$\sum_{n\le x} {}'\sigma_{1-2\sigma}(n) = \zeta(2\sigma)x + (2-2\sigma)^{-1}\zeta(2-2\sigma)x^{2-2\sigma} - \frac{1}{2}\zeta(2\sigma-1) + \Delta_{1-2\sigma}(x), \tag{2.4}$$

where $\sum'$ means that the last term in the sum is to be halved if x is an integer. The analytic continuation of $g_q(u, 2\sigma-u)$ can be obtained by the use of the analogue of Voronoi's classical formula for the function $\Delta_{1-2\sigma}(x)$. We recall that a series is called boundedly convergent if it converges everywhere and its partial sums are bounded. One has

$$\begin{aligned}\Delta_{1-2\sigma}(x) &= \mathrm{O}(x^{-\sigma-\frac{1}{4}}) \\ &+ (\sqrt{2}\pi)^{-1}x^{\frac{3}{4}-\sigma}\sum_{n=1}^{\infty}\sigma_{1-2\sigma}n^{\sigma-\frac{5}{4}}\{\cos(4\pi\sqrt{nx}-\frac{\pi}{4}) \\ &- (32\pi\sqrt{nx})^{-1}(16(1-\sigma)^2-1)\sin(4\pi\sqrt{nx}-\frac{\pi}{4})\}.\end{aligned} \tag{2.5}$$

Here the series is boundedly convergent when x lies in any fixed closed subinterval of $(0,\infty)$, provided that $\frac{1}{2} < \sigma < \frac{3}{4}$. From there, the condition for σ to be from the interval $(\frac{1}{2}, \frac{3}{4})$ appears. One can estimate [25]

$$\Delta_{1-2\sigma}(x) \ll x^{1/(4\sigma+1)+\varepsilon}. \tag{2.6}$$

From Lemma 2.1.2 we find that

$$\begin{aligned}E_\sigma(q,T) &= -iq\int_{\sigma-iT}^{\sigma+iT} g_q(u, 2\sigma-u)\mathrm{d}u + \mathrm{O}(q\log T) \\ &= -iq^{2-2\sigma}(G_{q,1}-G_{q,2}+G_{q,3}-G_{q,4}) + O(q\log T),\end{aligned} \tag{2.7}$$

with

$$G_{q,1} = 4i\sum_{n\le N}\sigma_{1-2\sigma}(n)\int_0^\infty \frac{\cos(2\pi qny)\sin(T\log\frac{1+y}{y})}{y^\sigma(1+y)^\sigma\log\frac{1+y}{y}}\mathrm{d}y,$$

$$G_{q,2} = 4i\Delta_{1-2\sigma}(N)\int_0^\infty \frac{\cos(2\pi qNXy)\sin(T\log\frac{1+y}{y})}{y^\sigma(1+y)^\sigma\log\frac{1+y}{y}}\mathrm{d}y,$$

$$
\begin{aligned}
G_{q,3} = & -\frac{2i}{\pi q}(\zeta(2\sigma) + \zeta(2-2\sigma)X^{2-2\sigma}) \\
& \times \int_0^\infty \frac{\sin(2\pi qNy)\sin(T\log\frac{1+y}{y})}{y^{\sigma+1}(1+y)^\sigma \log\frac{1+y}{y}}\mathrm{d}y \\
& + \frac{(1-2\sigma)\zeta(2-2\sigma)N^{2-2\sigma}}{\pi q} \\
& \times \int_0^\infty \frac{\sin(2\pi qNy)}{y(1+y)^{2\sigma-1}}\mathrm{d}y \int_{\sigma-iT}^{\sigma+iT} (u+1-2\sigma)^{-1}\left(\frac{1+y}{y}\right)^u \mathrm{d}u,
\end{aligned}
$$

$$
\begin{aligned}
G_{q,4} = 4i & \int_N^\infty \frac{\Delta_{1-2\sigma}(x)}{x}\mathrm{d}x \int_0^\infty \frac{\cos(2\pi qxy)}{y^\sigma(1+y)^{\sigma+1}\log\frac{1+y}{y}} \\
& \times \{T\cos(T\log\frac{1+y}{y}) - \sin(T\log\frac{1+y}{y}) \\
& \times ((2\sigma-1)(1+y) - \sigma + \log^{-1}\frac{1+y}{y})\}\mathrm{d}y.
\end{aligned}
$$

where $N > 1$.

To evaluate $G_{q,1}$ we apply Lemma 1.3.2 with $\alpha = \beta = \sigma$, $\gamma = 1$, $k = qn$, $qn \geq T$ and make $a \to 0$, $b \to \infty$. Then

$$
\begin{aligned}
G_{q,1} = & q^{\sigma-1}2^{-\frac{1}{2}}\left(\frac{T}{2\pi}\right)^{\frac{1}{2}-\sigma} \sum_{n\leq N}(-1)^{qn}\sigma_{1-2\sigma}(n)n^{\sigma-1}\left(\operatorname{arsinh}\sqrt{\frac{\pi qn}{2T}}\right)^{-1} \\
& \left(\frac{T}{2\pi qn} + \frac{1}{4}\right)^{-\frac{1}{4}} \cos(2T\operatorname{arsinh}(\sqrt{\frac{\pi qn}{2T}}) + \sqrt{2\pi qnT + \pi^2(qn)^2} - \frac{\pi}{4}) \\
& + \mathrm{O}(q^{\sigma-\frac{7}{4}}T^{-\frac{1}{2}}).
\end{aligned}
$$

Similarly, by using (2.6), we get

$$
G_{q,2} \ll q^{\sigma-\frac{3}{4}}T^{\frac{1-4\sigma}{8\sigma+2}+\varepsilon}. \tag{2.8}
$$

Next consider $G_{q,3}$. Divide the range of integration of the first integral at $(2qX)^{-1}$. The integral over $[0, (2qX)^{-1}]$ gives, as in [22], $\ll q^\sigma T^{\sigma-1}$. The integral over $[(2qX)^{-1}, \infty)$ is estimated by Lemma 1.3.2 with $\alpha = \sigma + 1$, $\beta = \sigma$, $\gamma = 1$, $k = qX$ $a = (2qX)^{-1}$ and making $b \to \infty$. This leads to the estimate $\ll q^{-1}T^{-\frac{1}{2}}$. Consider the second integral. By dividing the range of integration with respect to y at $y=1$, and proceeding like in [22] we have that it equals to

$$i\pi(1-2\sigma)(2\pi)^{2\sigma-1}\frac{\zeta(2-2\sigma)}{\Gamma(2\sigma)\sin(\sigma\pi)}q^{2\sigma-2}+\mathrm{O}(T^{-\sigma})+\mathrm{O}(q^{-1}T^{2-2\sigma})$$

and so, altogether,

$$G_{q,3}=i\pi(1-2\sigma)(2\pi)^{2\sigma-1}\frac{\zeta(2-2\sigma)}{\Gamma(2\sigma)\sin(\sigma\pi)}q^{2\sigma-2}+\mathrm{O}(q^{\sigma}T^{\sigma-1}). \tag{2.9}$$

It remains to evaluate $G_{q,4}$. To estimate the inner integrals, we use Lemma 1.3.2 and the fact that

$$\Delta_{1-2\sigma}(x)\ll x^{\frac{1}{4}+\varepsilon}.$$

Then, for $qx\gg T$, we obtain

$$\begin{aligned}G_{q,4}&=i2^{\sigma}(\pi)^{\sigma-\frac{1}{2}}q^{\sigma-1}T^{\frac{3}{2}-\sigma}\int_{N}^{\infty}\frac{\Delta_{1-2\sigma}(x)}{x^{2-\sigma}}\\&\times\left(\frac{\cos(2T\operatorname{arsinh}\sqrt{\frac{qx\pi}{2T}}+(qx2\pi T+\pi^2(qx)^2)^{\frac{1}{2}}-\pi(qx)^2+\frac{\pi}{4}))}{2\operatorname{arsinh}\left(\frac{\pi qx}{2T}\right)\left(\frac{T}{2\pi qx}+\frac{1}{4}\right)^{\frac{1}{2}}\left(\left(\frac{T}{2\pi qx}+\frac{1}{4}\right)^{\frac{1}{2}}+\frac{1}{2}\right)}\right)\mathrm{d}x\\&+\mathrm{O}(T^{\frac{1}{4}}q^{\sigma-1}).\end{aligned}$$

Changing the variable from x to $(qx)^2$, we find

$$\begin{aligned}G_{q,4}&=2iq^{\sigma-1}(\frac{T}{2\pi})^{\frac{1}{2}-\sigma}\sum_{n=1}^{\infty}\sigma_{1-2\sigma}(n)n^{-\frac{3}{4}}\\&\times\int_{\sqrt{qN}}^{\infty}\frac{\cos(2T\operatorname{arsinh}x\sqrt{\frac{\pi}{2T}}+x(2\pi T+\pi^2x^2)^{\frac{1}{2}}-\pi x^2+\frac{\pi}{4})}{x^{\frac{3}{2}}\operatorname{arsinh}\sqrt{\frac{\pi x^2}{2T}}\left(\frac{T}{2\pi x^2}+\frac{1}{4}\right)^{\frac{1}{4}}\left(\left(\frac{T}{2\pi x^2}+\frac{1}{4}\right)^{\frac{1}{2}}+\frac{1}{2}\right)}\\&\times\left(\cos(4\pi x\sqrt{\frac{n}{q}}-\frac{\pi}{4})-\frac{3}{32\pi x}\sqrt{\frac{q}{n}}\sin(4\pi x\sqrt{\frac{n}{q}}-\frac{\pi}{4})\right)\mathrm{d}x\\&+\mathrm{O}(q^{\sigma-1}T^{\frac{1}{4}-\sigma}).\end{aligned}$$

Now we apply Lemma 1.3.4. The main term comes from the integral with $\alpha=\frac{3}{2}$. Note also that

$$N'<A\frac{qT}{2\pi} \tag{2.10}$$

for some $A > 1$. Hence, for $n < N'$,

$$\begin{aligned}
G_{q,4} = 2iq^{\sigma-1}(\frac{T}{2\pi})^{\frac{1}{2}-\sigma} \sum_{n\le N'} \frac{\sigma_{1-2\sigma}(n)n^{\sigma-\frac{5}{4}}}{\log\frac{qT}{2\pi n}} \\
\times \cos(T\log\frac{qT}{2\pi n} - T + \frac{2\pi n}{q} - \frac{\pi}{4}) \\
+ \mathrm{O}(q^{-\frac{1}{2}}T^{-\frac{3}{2}} \sum_{n\le N'} \sigma_{1-2\sigma}(n)n^{\sigma-1}\left(\frac{T}{2\pi} - \frac{n}{q}\right)^{-\frac{1}{2}}) \\
+ \mathrm{O}(q^{\sigma-1}T^{\frac{3}{4}-\sigma} \sum_{n\le N'} \sigma_{1-2\sigma}(n)n^{\sigma-\frac{5}{4}} \min(1, |\sqrt{n}-\sqrt{N'}|^{-1})) \\
+ \mathrm{O}(q^{\sigma-1}T^{\frac{1}{4}-\sigma}) = G_{q,41} + G_{q,42} + G_{q,43} \\
+ O(q^{\sigma-1}T^{\frac{1}{4}-\sigma}),
\end{aligned}$$

but if $n \ge N'$, the main term and the last two error terms are to be omitted. Since $T \ll N \ll T$, we have $N' \ll qT$. Also we have $\frac{T}{2\pi} - \frac{n}{q} \gg T$. Then from this and (2.10) it is easily seen that

$$G_{q,42} \ll q^{-\frac{1}{2}}T^{\sigma-2}.$$

In $I_{q,43}$ the sum is split up at $\frac{1}{2}N'$, $N' - \sqrt{N'}$, $N' + \sqrt{N'}$ and $2N'$. Then we find easily that

$$G_{q,43} \ll q^{2\sigma-\frac{7}{4}}\log(qT).$$

Hence,

$$\begin{aligned}
G_{q,4} =& 2iq^{\sigma-1}(\frac{T}{2\pi})^{\frac{1}{2}-\sigma} \sum_{n\le N'} \frac{\sigma_{1-2\sigma}(n)n^{\sigma-\frac{5}{4}}}{\log\frac{qT}{2\pi n}} \cos(T\log\frac{qT}{2\pi n} - T + \frac{2\pi n}{q} - \frac{\pi}{4}) \\
&+ O(q^{2\sigma-\frac{7}{4}}\log(qT)).
\end{aligned}$$

Substituting this in (2.7) completes the proof of Theorem 2.0.2 in the case $\frac{1}{2} < \sigma < \frac{3}{4}$.

2.2 The case $\frac{3}{4} \le \sigma < 1$

Define

$$\tilde{D}_{1-2\sigma}(\xi) = \int_0^{\xi} \sum_{n\le t} \sigma_{1-2\sigma}(n)\mathrm{d}t.$$

Instead of (2.5) and (2.6), we will need the Voronoi-type formula for $\tilde{D}_{1-2\sigma}(\xi)$. The crucial point is that the Voronoi series for $\tilde{D}_{1-2\sigma}(\xi)$ converges for any σ satisfying $\frac{1}{2} < \sigma < 1$. The basic principle of the proof of the Theorem 2.0.2

in the case $\frac{3}{4} \le \sigma < 1$ is similar to the proofs in the case $\frac{1}{2} \le \sigma < \frac{3}{4}$ and Theorem 1.2.1. Let $\xi \ge 1$, and define $\tilde{\Delta}_{1-2\sigma}(\xi)$ by

$$\begin{aligned} \tilde{D}_{1-2\sigma}(\xi) = \frac{1}{2}\zeta(2\sigma)\xi^2 + \frac{\zeta(2-2\sigma)}{(2-2\sigma)(3-2\sigma)}\xi^{3-2\sigma} - \frac{1}{2}\zeta(2\sigma-1)\xi \\ + \frac{1}{12}\zeta(2\sigma-2) + \tilde{\Delta}_{1-2\sigma}(\xi). \end{aligned} \tag{2.11}$$

Then the following Voronoi-type formula holds [25].

Lemma 2.2.1. *We have*

$$\begin{aligned} \tilde{\Delta}_{1-2\sigma}(\xi) &= -\frac{1}{2\sqrt{2}\pi^2}\xi^{5/4-\sigma}\sum_{n=1}^{\infty}\sigma_{1-2\sigma}(n)n^{\sigma-7/4}\cos(4\pi\sqrt{n\xi}+\frac{\pi}{4}) \\ &+ \frac{(5-4\sigma)(7-4\sigma)}{64\sqrt{2}\pi^3}\xi^{3/4-\sigma}\sum_{n=1}^{\infty}\sigma_{1-2\sigma}(n)n^{\sigma-9/4}\cos(4\pi\sqrt{n\xi}-\frac{\pi}{4}) \\ &+ \mathrm{O}(\xi^{1/4-\sigma}), \end{aligned} \tag{2.12}$$

where the two infinite series on the right-hand side are uniformly convergent on any finite closed subinterval in $(0,\infty)$.

The proofs of the following two lemmas can be found in [25].

Lemma 2.2.2. *We have* $\tilde{\Delta}_{1-2\sigma}(\xi) = \mathrm{O}(\xi^r \log\xi)$, *where*

$$r = (-4\sigma^2 + 7\sigma - 2)/(4\sigma - 1) \le 1/2.$$

Lemma 2.2.3. *We have*

$$\int_1^x \tilde{\Delta}_{1-2\sigma}(\xi)^2 \ll x^{7/2-2\sigma}.$$

Now analogically to the previous case, using Lemma 2.2.2, we treat the integrals $G_{q,1}$, $G_{q,2}$, $G_{q,3}$. It remains to evaluate the integral $G_{q,4}$. Similarly as in [25], we can show that

$$\begin{aligned} G_{q,4} &= -2\tilde{\Delta}_{1-2\sigma}(X)\int_{\sigma-iT}^{\sigma+iT}\frac{\delta}{\delta X}\int_0^{\infty} y^{-u}(1+y)^{u-2\sigma}\cos(2\pi qXy)\mathrm{d}y\mathrm{d}u \\ &- \int_X^{\infty}\tilde{\Delta}_{1-2\sigma}(\xi)\int_{\sigma-iT}^{\sigma+iT}\frac{\delta^2}{\delta\xi^2}\int_0^{\infty} y^{-u}(1+y)^{u-2\sigma}\cos(2\pi q\xi y)\mathrm{d}y\mathrm{d}u\mathrm{d}\xi \\ &= -G_4^* - G_4^{**} \end{aligned} \tag{2.13}$$

First consider G_4^*. After differentiating the inner integral, we get four integrals. We split up these four integrals at $y = T$. Then we estimate in each case $\int_T^\infty$ by the first derivative test and $\int_0^T$, after the further splitting up into integrals over the intervals $(2^{-k}T, 2^{-k+1}T]$ $(k = 1, 2, ...)$, by the second derivative test (see [11]). This gives

$$\int_{\sigma-iT}^{\sigma+iT} \frac{\delta}{\delta X} \int_0^{\infty} y^{-u}(1+y)^{u-2\sigma} \cos(2\pi qXy) \mathrm{d}y \mathrm{d}u \ll T^{-\frac{1}{2}}.$$

Together with Lemma 2.2.2 and the definition of $G^*_{q,4}$ this gives

$$G^*_{q,4} \ll \log T. \tag{2.14}$$

Now consider $G^{**}_{q,4}$. First we evaluate the inner integral of $G^{**}_{q,4}$ by integrating twice by parts and applying Lemma 1.3.2 and Lemma 2.2.3. Next we apply Lemma 2.2.3, a slight modification of Lemma 1.3.4 (see [25]) and evaluate by a technique we used to evaluate $G_{q,4}$ in the Theorem 1.2.1. Thus we have

$$\begin{aligned} G^{**}_4 = -2q^{-\frac{1}{2}} \left(\frac{T}{2\pi}\right)^{\frac{1}{2}-\sigma} \sum_{n\le N'} \frac{\sigma_{1-2\sigma}(n) n^{\sigma-1}}{\log \frac{qT}{2\pi n}} \\ \cos(T \log \frac{qT}{2\pi n} - T + \frac{\pi}{4}) \\ +\mathrm{O}(q^{2\sigma-4} \log(qT)). \end{aligned}$$

Combining this with (2.14) and (2.13) gives

$$\begin{aligned} G_{q,4} = -2q^{\sigma-1} \left(\frac{T}{2\pi}\right)^{\frac{1}{2}-\sigma} \sum_{n\le N'} \frac{\sigma_{1-2\sigma}(n) n^{\sigma-1}}{\log \frac{qT}{2\pi n}} \\ \cos(T \log \frac{qT}{2\pi n} - T + \frac{\pi}{4}) \\ + \mathrm{O}(q^{2\sigma-4} \log(qT)). \end{aligned} \tag{2.15}$$

This completes the proof of Theorem 2.0.2.

Skyrius 3

Limit theorems on the complex plane for the periodic zeta-function

3.1 Probability theory elements

Let us begin with some well known facts from the theory of weak convergence of probability measures.

Let S be a metric space with its class of Borel sets $\mathcal{B}(S)$, and let P_n and P be probability measures on $(S, \mathcal{B}(S))$. We recall that P_n converges weakly to P as $n \to \infty$ if

$$\int_S f dP_n \to \int_S f dP, \ n \to \infty,$$

for every real bounded continuous function f on S. Let $h : S \to S_1$ be a measurable function. Then every probability measure P on $(S, \mathcal{B}(S))$ induces on $(S_1, \mathcal{B}(S_1))$ the unique probability measure Ph^{-1} defined by the equality $Ph^{-1}(A) = P(h^{-1}A)$, $A \in \mathcal{B}(S_1)$.

Lemma 3.1.1. *Let P_n converge weakly to P and let h be a continuous function. Then $P_n h^{-1}$ also converges weakly to Ph^{-1}.*

Proof. This lemma is a particular case of Theorem 5.1 from [2].

The next two lemmas are called Prokhorov's theorems. They relate compactness with tightness of a family of probability measures, and are often used in applications.

Definition 3.1.2. *The family $\{P\}$ of probability measures on $(S, \mathcal{B}(S))$ is tight if for arbitrary $\varepsilon > 0$ there exists a compact set $K \subset S$ such that $P(K) > 1 - \varepsilon$ for all P from $\{P\}$.*

Definition 3.1.3. *The family $\{P\}$ of probability measures on $(S, \mathcal{B}(S))$ is relatively compact if every sequence of elements of $\{P\}$ contains a weakly convergent subsequence.*

Lemma 3.1.4. *If the family of probability measures $\{P\}$ is tight, then it is relatively compact.*

Lemma 3.1.5. *Let S be a separable complete metric space. If the family of probability measures $\{P\}$ on $(S, \mathcal{B}(S))$ is relatively compact, then it is tight.*

Lemma 3.1.6. *Let P_n and P be probability measures on $(S, \mathcal{B}(S))$. Then the following three assertions are equivalent:*

1° *P_n converges weakly to P;*

2° $\lim_{n\to\infty} P_n(A) = P(A)$ *for all continuity sets of P;*

3° $\liminf_{n\to\infty} P_n(G) \geq P(G)$ *for all open sets G of S.*

Proofs of Lemmas 3.1.4, 3.1.5 and 3.1.6 can be found in [2].

Denote by $\xrightarrow{\mathcal{D}}$ the convergence in distribution.

Lemma 3.1.7. *Let (S, ϱ) be a separable metric space with a metric ϱ, and let $Y_n, X_{1,n}, X_{2,n}, \dots$ be the S-valued random elements defined on the probability space $(\Omega_0, \mathcal{B}(\Omega_0), \mathbb{P})$. Suppose that $X_{k,n} \xrightarrow[n\to\infty]{\mathcal{D}} X_k$ for each k, and also $X_k \xrightarrow[k\to\infty]{\mathcal{D}} X$. If, for every $\varepsilon > 0$,*

$$\lim_{k\to\infty} \limsup_{n\to\infty} \mathbb{P}(\varrho(X_{k,n}, Y_n) \geq \varepsilon) = 0,$$

then $Y_n \xrightarrow[n\to\infty]{\mathcal{D}} X$

The lemma is Theorem 4.2 from [2].

Lemma 3.1.8. *Let random variables $X_1, X_2, \dots$ be pairwise orthogonal and*

$$\sum_{m=1}^{\infty} \mathrm{E}|X_m|^2 \log^2 m < \infty.$$

Then the series

$$\sum_{m=1}^{\infty} X_m$$

converges almost surely.

The lemma is called the Rademacher theorem. Its proof can be found in [21].

Lemma 3.1.9. *Let a process* $X(\tau, \omega)$ *be ergodic,* $E|X(\tau, \omega)| < \infty$*, and let sample paths be integrable almost surely in the Riemann sense over every finite interval. Then*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T X(\tau, \omega) d\tau = EX(0, \omega)$$

almost surely.

Proof of the lemma can be found in [6].

3.2 Definition of a random element

Define

$$\Omega = \prod_p \gamma_p,$$

where γ_p is the unit circle $\gamma = \{s \in \mathbb{C} : |s| = 1\}$ for all primes p. By the Tikhonov theorem, [29], with the product topology and pointwise multiplication, the infinite-dimensional torus Ω is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$ the probability Haar measure m_H can be defined, and this leads to a probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(p)$ the projection of $\omega \in \Omega$ to the coordinate space γ_p. Since the Haar measure m_H is the product of the Haar measures on the coordinate space γ_p, $\{\omega(p)\}$ is a sequence of independent complex-valued random variables uniformly distributed on the circle γ. We extend the function $\omega(p)$ to the set $\mathbb{N}$ by the formula

$$\omega(m) = \prod_{p^\alpha \| m} \omega^\alpha(p),$$

where $p^\alpha \parallel m$ means that $p^\alpha | m$ but $p^\alpha \nmid m$. For $\sigma > 1/2$, define

$$\zeta_\lambda(\sigma, \omega) = \sum_{m=1}^{\infty} \frac{\omega(m) e^{2\pi i \lambda m}}{m^\sigma}.$$

Denote by $\mathrm{E}X$ the expectation of a random element X.

Lemma 3.2.1. $\zeta_\lambda(\sigma, \omega)$ *is an complex valued random element defined on the probability space* $(\Omega, \mathcal{B}(\Omega), m_H)$*.*

Proof. Let $\sigma_1 > 1/2$ be fixed, and let

$$x_m = \frac{\omega(m)e^{2\pi i\lambda m}}{(m)^{\sigma_1}}, \quad m \in \mathbb{N}.$$

Then $\{x_m\}$ is a sequence of complex-valued random variables defined on $(\Omega, \mathcal{B}(\Omega), m_H)$. We have

$$\begin{aligned}
\mathrm{E}|x_m|^2 &= \frac{e^{2\pi i\lambda m}}{m^{2\sigma_1}}, \\
\mathrm{E}x_m\bar{x}_n &= \frac{e^{2\pi i\lambda(m-n)}}{m^{\sigma_1}n^{\sigma_1}} \int_\Omega \omega(m)\bar{\omega}(n)dm_H \\
&= \begin{cases} \frac{1}{m^{2\sigma_1}} & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}
\end{aligned}$$

This shows that $\{x_m\}$ is a sequence of pairwise orthogonal random variables, and

$$\sum_{m=1}^{\infty} \mathrm{E}|x_m|^2 \log^2 m < \infty.$$

Consequently, by Lemma 3.1.8, the series

$$\sum_{m=1}^{\infty} \frac{\omega(m)e^{2\pi i\lambda m}}{m^{\sigma_1}}$$

converges almost surely with respect to the measure m_H. Hence, by the well-known property of Dirichlet series, that if the Dirichlet series converges at the point $s_0 = \sigma_0 + it_0$, then it converges in the half-plane $\sigma > \sigma_0$ [19], the series

$$\sum_{m=1}^{\infty} \frac{\omega(m)e^{2\pi i\lambda m}}{m^{\sigma}}$$

converges almost surely for $\sigma > \sigma_1$. From this, taking $\sigma_1 = 1/2 + 1/r$, $r \in \mathbb{N}$, we deduce that the series converges almost surely for $\sigma > \frac{1}{2}$, and thus it defines a complex valued random element. This ends the proof.

Denote by $P^{\mathbb{C}}_{\zeta_\lambda}$ the distribution of $\zeta_\lambda(s, \omega)$, i.e.,

$$P^{\mathbb{C}}_{\zeta_\lambda}(A) = m_H\left(\omega \in \Omega \, : \, \zeta_\lambda(\sigma, \omega) \in A\right), \, A \in \mathcal{B}(\mathbb{C}).$$

Denote by $meas(A)$ the Lebesgue measure of a measurable set $A \subset \mathbb{R}$, and, for $T > 0$, define

$$\nu_T(\dots) = \frac{1}{T} meas\{t \in [0, T], \dots\},$$

where in place of dots a condition satisfied by t is to be written.

Theorem 3.2.2. *Let $\sigma > \frac{1}{2}$. Then the probability measure*

$$P_T(A) = \nu_T\left(\zeta_\lambda(\sigma + i\tau) \in A\right), \; A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to P_{ζ_λ} as $T \to \infty$.

3.3 A limit theorem on Ω

Theorem 3.3.1. *The probability measure*

$$Q_T(A) = \nu_T\left(\left(p^{-it}, \, p \in \mathcal{P}\right) \in A\right), \; A \in \mathcal{B}(\Omega),$$

converges weakly to the Haar measure m_H as $T \to \infty$.

Proof. The dual group of Ω is

$$\bigoplus_p \mathbb{Z}_p,$$

where $\mathbb{Z}_p = \mathbb{Z}$ for all $p \in \mathcal{P}$. An element

$$\mathbf{k} = (k_2, k_3, \ldots) \in \bigoplus_p^\infty \mathbb{Z}_p,$$

where only a finite number of integers k_p are distinct from zero, acts on Ω by

$$\mathbf{x} \to \mathbf{x}^{\mathbf{k}} = \prod_p x_p^{k_p}, \; \mathbf{x} = (x_2, x_3, \ldots) \in \Omega.$$

Therefore, the Fourier transform $g_T(\mathbf{k})$ of the measure Q_T is

$$\begin{aligned} g_T(\mathbf{k}) &= \int_\Omega \prod_p x_p^{k_p} dQ_T = \frac{1}{T}\int_0^T \prod_p p^{-itk_p} \mathrm{d}t \\ &= \frac{1}{T}\int_0^T \exp\left\{-it\sum_p k_p \log p\right\} \mathrm{d}t. \end{aligned} \tag{3.1}$$

It is well known that the set $\{\log p : p \in \mathcal{P}\}$ is linearly independent over the field of rational numbers $\mathbb{Q}$. Thus, by (3.1),

$$g_T(\mathbf{k}) = \begin{cases} 1 & \text{if } \; \mathbf{k} = \mathbf{0}, \\ \frac{1-\exp\left\{-iT\sum_p k_p \log p\right\}}{iT\exp\left\{-iT\sum_p k_p \log p\right\}} & \text{if } \; \mathbf{k} \neq \mathbf{0}, \end{cases}$$

where in the second case only a finite number of k_p are non zero. Thus,

$$\lim_{T\to\infty} g_T(\mathbf{k}) = \begin{cases} 1 & \text{if } \mathbf{k} = \mathbf{0}, \\ 0 & \text{if } \mathbf{k} \neq \mathbf{0}, \end{cases}$$

and therefore, see, for example, [9], the measure Q_T converges weakly to m_H as $T \to \infty$.

3.4 Limit theorems for absolutely convergent series

Let $\sigma_1 > \frac{1}{2}$ be fixed, and, for $m, n \in \mathbb{N}$,

$$v(m,n) = \exp\Big\{-\Big(\frac{m}{n}\Big)^{\sigma_1}\Big\}.$$

In this section, we consider the series

$$\zeta_{\lambda,n}(s) = \sum_{m=1}^{\infty} \frac{e^{2\pi i\lambda m} v(m,n)}{m^s}.$$

Lemma 3.4.1. *The series for $\zeta_{\lambda,n}(s)$ is absolutely convergent for $\sigma > \frac{1}{2}$.*

Proof. Define

$$l_n(s) = \frac{s}{\sigma_1}\Gamma\Big(\frac{s}{\sigma_1}\Big),$$

and consider the integral

$$a_n(m) = \frac{1}{2\pi i}\int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \frac{l_n(z)}{m^z}\frac{\mathrm{d}z}{z}.$$

Clearly, $a_n(m) = \mathrm{O}(m^{-\sigma_1})$. Therefore, the series

$$\sum_{m=1}^{\infty} \frac{e^{2\pi i\lambda m} a_n(m)}{m^s}$$

converges absolutely for $\sigma_1 > \frac{1}{2}$. Using the Mellin formula

$$\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \Gamma(s)a^{-s}\mathrm{d}s = e^{-a}, \quad a > 0, \quad b > 0,$$

we easily find

$$a_n(m) = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \Gamma(s)\left(\Big(\frac{m}{n}\Big)^{\sigma_1}\right)^{-s} \mathrm{d}s$$
$$\sigma(m,n).$$

This proves the lemma.

The function $\zeta_{\lambda,n}(s)$ also has an integral representation. Really, using the absolute convergence of the series for $\zeta_\lambda(s)$ in the region $\sigma_1 > 1$, we find that

$$\begin{aligned}\zeta_{\lambda,n}(s) &= \sum_{m=1}^{\infty} \frac{e^{2\pi i\lambda m} v(m,n)}{m^s} = \sum_{m=1}^{\infty} \frac{e^{2\pi i\lambda m} a_n(m)}{m^s} \\ &= \sum_{m=1}^{\infty} \frac{e^{2\pi i\lambda m}}{m^s} \Big(\frac{1}{2\pi i} \int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \frac{l_n(z)}{m^z} \frac{\mathrm{d}z}{z} \Big) \\ &= \frac{1}{2\pi i} \int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \sum_{m=1}^{\infty} \frac{e^{2\pi i\lambda m} l_n(z)}{m^{s+z}} \frac{\mathrm{d}z}{z} \\ &= \frac{1}{2\pi i} \int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \zeta_\lambda(s+z) l_n(z) \frac{\mathrm{d}z}{z}.\end{aligned}$$

On $(\mathbb{C}, \mathcal{B}((\mathbb{C}))$, define two probability measures

$$P_{T,n}^{\mathbb{C}}(A) = \nu_T(\zeta_{\lambda,n}(\sigma + it) \in A)$$

and

$$\hat{P}_{T,n}^{\mathbb{C}}(A) = \nu_T(\zeta_{\lambda,n}(\sigma + it, \omega_0) \in A),$$

where ω_0 is a fixed element of Ω, and, for $\sigma > \frac{1}{2}$,

$$\zeta_{\lambda,n}(s) = \sum_{m=1}^{\infty} \frac{e^{2\pi i\lambda m} \omega(m) v(m,n)}{m^s}.$$

Theorem 3.4.2. *Suppose that $\sigma > \frac{1}{2}$. Then both the probability measures $P_{T,n}^{\mathbb{C}}$ and $\hat{P}_{T,n}^{\mathbb{C}}$ converge weakly to the same probability measure $P_n^{\mathbb{C}}$ on $(\mathbb{C}, \mathcal{B}((\mathbb{C}))$ as $T \to \infty$.*

Proof. Consider the function $h_n : \Omega \to \mathbb{C}$ given by the formula

$$h_n(\omega) = \sum_{m=1}^{\infty} \frac{e^{2\pi i\lambda m} \omega(m) v(m,n)}{m^\sigma}, \quad \omega \in \Omega, \quad \sigma > \frac{1}{2}.$$

Since the latter series converge absolutely for $\sigma > \frac{1}{2}$, the function h_n is continuous. Moreover,

$$h_n(p^{-it} : p \in \mathcal{P}) = \sum_{m=1}^{\infty} \frac{e^{2\pi i\lambda m} v(m,n)}{m^{\sigma+it}} = \zeta_{\lambda,n}(\sigma + it).$$

Therefore, $P_{T,n}^{\mathbb{C}} = Q_T h_n^{-1}$, where Q_T is the measure from Theorem 3.3.1. This , the continuity of the function h_n, Theorem 3.3.1 and Lemma 3.1.1 shows that the measure $P_{T,n}^{\mathbb{C}}$ converges weakly to $m_H h_n^{-1}$ as $T \to \infty$.

To prove the weak convergence of the measure $\hat{P}^{\mathbb{C}}_{T,n}$, we define a new function $\hat{h}_n : \Omega \to \mathbb{C}$ by the formula

$$\hat{h}_n(\omega) = \sum_{m=1}^{\infty} \frac{e^{2\pi i \lambda m} \omega_0(m)\omega(m)v(m,n)}{m^{\sigma}}, \quad \omega \in \Omega, \quad \sigma > \frac{1}{2}.$$

Then, similarly as above, we have that the function $\hat{h}_n$ is continuous and $\hat{h}_n((p^{-it} : p \in \mathcal{P})) = \zeta_{\lambda,n}(\sigma_0 + it, \omega_0)$. Using Theorem 3.3.1 and Lemma 3.1.1, we obtain that the measure $\hat{P}^{\mathbb{C}}_{T,n}$ converges weakly to $m_H \hat{h}_n^{-1}$ as $T \to \infty$. However, $\hat{h}_n(\omega) = h_n(\omega\omega_0)$. Since the Haar measure m_H is invariant with respect to translation by points from Ω, hence we obtain that $m_H \hat{h}_n^{-1} = m_H h_n^{-1}$, and the theorem is proved.

3.5 Approximation in the mean

In this section, we approximate in the mean the functions $\zeta_\lambda(s)$ and $\zeta_\lambda(s,\omega)$ by $\zeta_{\lambda,n}(s)$ and $\zeta_{\lambda,n}(s,\omega)$, respectively.

Theorem 3.5.1. *Suppose that $\sigma > \frac{1}{2}$. Then*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T |\zeta_\lambda(\sigma + it) - \zeta_{\lambda,n}(\sigma + it)| \mathrm{d}t = 0.$$

Proof. We start with the integral representation of $\zeta_{\lambda,n}(s)$

$$\zeta_{\lambda,n}(s) = \frac{1}{2\pi i} \int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \zeta_\lambda(s+z) l_n(z) \frac{\mathrm{d}z}{z}.$$

We shift the line of integration in the latter integral to the left. Let $\sigma_2 > \sigma_1$ but $\sigma_2 < \sigma$. The integrand has a simple pole at $z = 0$ with residue $\zeta_\lambda(s)$. Therefore, by the residue theorem,

$$\zeta_{\lambda,n}(s) - \zeta_\lambda(s) = \frac{1}{2\pi i} \int_{\sigma_2 - \sigma - i\infty}^{\sigma_2 - \sigma + i\infty} \zeta_\lambda(s+z) l_n(z) \frac{\mathrm{d}z}{z}. \tag{3.2}$$

Hence, we find that

$$\frac{1}{T} \int_0^T |\zeta_\lambda(\sigma + it) - \zeta_{\lambda,n}(\sigma + it)| \mathrm{d}t$$
$$\ll \int_{-\infty}^{\infty} |l_n(\sigma_2 - \sigma + i\tau)| \Big(\frac{1}{T} \int_0^T |\zeta_\lambda(\sigma_2 + it + i\tau)| \mathrm{d}t \Big) \mathrm{d}\tau.$$

In view of (3.1) and the mean square theorem for the Lerch zeta-function, we find that

$$\frac{1}{T} \int_0^T |\zeta_\lambda(\sigma_2 + it + i\tau)| \mathrm{d}t \ll \Big(\frac{1}{T} \int_{|\tau|}^{T+|\tau|} |\zeta_\lambda(\sigma_2 + it + i\tau)|^2 \mathrm{d}t \Big)^{\frac{1}{2}}$$
$$\ll \Big(\frac{T + |\tau|}{T} \Big)^{\frac{1}{2}} \ll 1 + |\tau|.$$

Thus, substituting this estimate in (3.2), we obtain that

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T |\zeta_\lambda(\sigma+it)-\zeta_{\lambda,n}(\sigma+it)|\mathrm{d}t \ll \int_{-\infty}^{\infty} |l_n(\sigma_2-\sigma+i\tau)|(1+|\tau|)\mathrm{d}\tau. \tag{3.3}$$

However, from the definition of the function $l_n(s)$, it follows easily that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} |l_n(\sigma_2-\sigma+i\tau)|(1+|\tau|)\mathrm{d}\tau = 0.$$

This together with (3.3) proves the theorem.

A more complicated is an approximation in the mean of the function $\zeta_\lambda(s,\omega)$ by $\zeta_{\lambda,n}(s,\omega)$We have to use some facts of ergodic theory.

Let $a_t = (p^{-it} : p \in \mathcal{P}_0)$, $t \in \mathbb{R}$. Then $\{a_t : t \in \mathbb{R}\}$ is a one-parameter group. Define a one-parameter group $\{h_t : t \in \mathbb{R}\}$ of measurable measure preserving transformations of Ω by $h_t(\omega) = a_t\omega$, $\omega \in \Omega$. We recall that a set $A \in B(\Omega)$ is called invariant with respect to $\{h_t : t \in \mathbb{R}\}$ if, for each t, the sets A and $A_t = h_t(A)$ differ at most by a set of zero m_H-measure. All invariant sets form a σ-field. A one-parameter group $\{h_t : t \in \mathbb{R}\}$ is called ergodic if its σ-field of invariant sets consists only of sets having m_H-measure equal to 0 or 1.

Lemma 3.5.2. *The one-parameter group $\{h_t : t \in \mathbb{R}\}$ is ergodic.*

Proof of the lemma can be found in [19].

Lemma 3.5.3. *Suppose that $\sigma > \frac{1}{2}$. Then, for almost all $\omega \in \Omega$,*

$$\int_0^T |\zeta_\lambda(\sigma+it,\omega)|^2\mathrm{d}t = \mathrm{O}(T).$$

Proof. We give only a sketch of the proof because it is similar to that for the Riemann zeta-function [19].

We have that

$$\mathbb{E}\left|\frac{e^{2\pi i\lambda m}\omega(m)}{m^\sigma}\right|^2 = \frac{1}{m^{2\sigma}}.$$

Moreover, using the pairwise orthogonality of random variables $\omega(m)$, we find that

$$\mathbb{E}|\zeta_\lambda(s,\omega)|^2 = \sum_{m=1}^{\infty}\mathbb{E}\left|\frac{e^{2\pi i\lambda m}\omega(m)}{m^\sigma}\right|^2 = \sum_{m=1}^{\infty}\frac{1}{m^{2\sigma}} < \infty, \tag{3.4}$$

since $\sigma > \frac{1}{2}$.

From Lemma 3.5.2, it follows that the process $|\zeta_\lambda(\sigma+it,\omega)|^2$ is ergodic. Therefore, by Lemma 3.1.9,

$$\lim_{T\to\infty}\int_0^T |\zeta_\lambda(\sigma+it,\omega)|^2 \mathrm{d}t = \mathbb{E}|\zeta_\lambda(s,\omega)|^2 < \infty$$

in view of (3.4) for almost all $\omega \in \Omega$.

Theorem 3.5.4. *Suppose that $\sigma > \frac{1}{2}$. Then, for almost all $\omega \in \Omega$,*

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T |\zeta_\lambda(\sigma+it,\omega)-\zeta_{\lambda,n}(\sigma+it,\omega)|^2 \mathrm{d}t = 0.$$

Proof. We use lemma 3.5.3 and repeat the arguments of the proof of Theorem 3.5.1.

3.6 Proof of Theorem 3.2.2

Define one more probability measure

$$\hat{P}_T^{\mathbb{C}}(A) = \nu_T(\zeta_\lambda(\sigma+it,\omega)\in A), \quad A \in \mathcal{B}(\mathbb{C}).$$

Theorem 3.6.1. *Suppose that $\sigma > \frac{1}{2}$. Then both the measures $P_T^{\mathbb{C}}$ and $\hat{P}_T^{\mathbb{C}}$ converge weakly to same probability measure $P^{\mathbb{C}}$ as $T\to\infty$.*

Proof. First we prove that the family of probability measures $\{P_n^{\mathbb{C}} : n \in \mathbb{N}\}$, where $P_n^{\mathbb{C}}$ is the limit measure in Theorem 3.4.2, is tight. Let M be arbitrary positive number. Then, in view of Chebyshev type inequality,

$$\begin{aligned} &P_{T,n}^{\mathbb{C}}(\{z\in\mathbb{C} : |z| > M\}) \\ &= \nu_T(\{|\zeta_{\lambda,n}(\sigma+it)| > M\}) \\ &\le \frac{1}{TM}\int_0^T \sup_{s\in K}|\zeta_{\lambda,n}(\sigma+it)|\mathrm{d}t. \end{aligned} \tag{3.5}$$

Clearly,

$$\begin{aligned} &\sup_{n\in\mathbb{N}}\limsup_{T\to\infty}\frac{1}{T}\int_0^T |\zeta_{\lambda,n}(\sigma+it)|\mathrm{d}t \\ &\le \sup_{n\in\mathbb{N}}\limsup_{T\to\infty}\frac{1}{T}\int_0^T |\zeta_\lambda(\sigma+it)-\zeta_{\lambda,n}(\sigma+it)|\mathrm{d}t \\ &+ \limsup_{T\to\infty}\Big(\frac{1}{T}\int_0^T |\zeta_{\lambda,n}(\sigma+it)|^2\mathrm{d}t\Big)^{\frac{1}{2}}. \end{aligned}$$

This, (3.5), Theorem 3.5.1 and the mentioned in Section 3.4 bound

$$\int_0^T |\zeta_{\lambda,n}(\sigma+it)|^2 \mathrm{d}t = \mathrm{O}(T)$$

imply the inequality

$$\limsup_{T\to\infty} P^{\mathbb{C}}_{T,n}\{z\in\mathbb{C} : |z|>M\}) \le \frac{V}{M} \tag{3.6}$$

with $V<\infty$.

Now let $\varepsilon>0$ be an arbitrary positive number. We take $M=M_\varepsilon=\frac{V}{\varepsilon}$. Then (3.6) shows that

$$\limsup_{T\to\infty} P^{\mathbb{C}}_{T,n}(\{z\in\mathbb{C} : |z|>M_\varepsilon\}) \le \varepsilon. \tag{3.7}$$

The continuity of function $h:\mathbb{C}\to\mathbb{R}$ given by formula $z\to|z|$, Theorem 3.1.5 and Lemma 3.1 imply the weak convergence of

$$\nu_T(\{|\zeta_{\lambda,n}(\sigma+it)|\in A\}), \quad A\in\mathcal{B}(\mathbb{R})$$

to $P^{\mathbb{C}}_n h^{-1}$, as $T\to\infty$. Therefore, by (3.8) and Lemma 3.1.6, we obtain that

$$\begin{aligned} & P^{\mathbb{C}}_n(\{z\in\mathbb{C} : |z|>M_\varepsilon\}) \\ & \le \limsup_{T\to\infty} P^{\mathbb{C}}_{T,n}(\{z\in\mathbb{C} : |z|>M_\varepsilon\}) \\ & \le \varepsilon. \end{aligned}$$

Denoting $K_\varepsilon=\{z\in\mathbb{C} : |z|\le M_\varepsilon\}$, hence we deduce that, for all $n\in\mathbb{N}$,

$$P^{\mathbb{C}}_n(K_\varepsilon)\ge 1-\varepsilon.$$

Since the set K_ε is compact, we have that the family of probability measures $\{P^{\mathbb{C}}_n : n\in\mathbb{N}\}$ is tight. Hence, by Lemma 3.1.4, it is relatively compact. Thus, there exists a sequence $\{P^{\mathbb{C}}_{n_k}\}\subset\{P^{\mathbb{C}}_n\}$ such that the measure $P^{\mathbb{C}}_{n_k}$ converges weakly to a certain probability measure $P^{\mathbb{C}}_n$ as $k\to\infty$.

Let θ be a random variable defined on a certain probability space $(\Omega_0,\mathcal{B}(\Omega_0),\mathbb{P})$ and uniformly distributed on [0,1]. Define

$$X^{\mathbb{C}}_{T,n}(\sigma)=\zeta_{\lambda,n}(\sigma+i\theta T).$$

By Theorem 3.1.5, the measure $P^{\mathbb{C}}_{T,n}$ converges weakly to $P^{\mathbb{C}}_n$ as $T\to\infty$. Therefore

$$X^{\mathbb{C}}_{T,n}(\sigma)\xrightarrow[T\to\infty]{\mathcal{D}} X^{\mathbb{C}}_n(\sigma),$$

where $X^{\mathbb{C}}_n(\sigma)$ is a complex-valued random element with the distribution $P^{\mathbb{C}}_n$. Here and in the sequel, $\xrightarrow{\mathcal{D}}$ denotes the convergence in distribution. From the weak convergence of P_{n_k}, we have that

$$X^{\mathbb{C}}_{n_k}(\sigma)\xrightarrow[k\to\infty]{\mathcal{D}} P^{\mathbb{C}}.$$

Define one more complex-valued random element

$$X_T^{\mathbb{C}}(\sigma) = \zeta_\lambda(\sigma + i\theta T).$$

Then, Theorem 3.1.6 implies that, for every $\varepsilon > 0$ and $\sigma > \frac{1}{2}$,

$$\begin{aligned}
&\lim_{n\to\infty} \limsup_{T\to\infty} \mathbb{P}(|X_T^{\mathbb{C}}(\sigma) - X_{T,n}^{\mathbb{C}}(\sigma) \geq \varepsilon|) \\
&= \lim_{n\to\infty} \limsup_{T\to\infty} \nu_T(|\zeta_\lambda(\sigma + it) - \zeta_{\lambda,n}(\sigma + it) \geq \varepsilon|) \\
&\leq \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T\varepsilon} \int_0^T |\zeta_\lambda(\sigma + it) - \zeta_{\lambda,n}(\sigma + it)| \mathrm{d}t = 0.
\end{aligned}$$

Relations (3.9)-(3.11) show that all hypotheses of Lemma 3.1.7 are satisfacted. Therefore,

$$X_T^{\mathbb{C}}(\sigma) \xrightarrow[T\to\infty]{\mathcal{D}} P^{\mathbb{C}},$$

and we have provd tat the measure $P_T^{\mathbb{C}}$ converges weakly to $P^{\mathbb{C}}$. Moreover, (3.12) shows that the meassure $P^{\mathbb{C}}$ is independant of the choice of the sequence P_{n_k}. Thus,

$$X_n^{\mathbb{C}}(\sigma) \xrightarrow[n\to\infty]{\mathcal{D}} P^{\mathbb{C}}. \tag{3.8}$$

It remains to prove the weak convergence of the measure $\hat{P}_T^{\mathbb{C}}$ to $P^{\mathbb{C}}$ as $T \to \infty$. For this, define

$$\hat{X}_{T,n}^{\mathbb{C}}(\sigma) = \zeta_{\lambda,n}(\sigma + i\theta T, \omega),$$

and

$$\hat{X}_T^{\mathbb{C}}(\sigma) = \zeta_\lambda(\sigma + i\theta T\omega).$$

Then, repeating the above arguments for the random elements $\hat{X}_{T,n}^{\mathbb{C}}(\sigma)$ and $\hat{X}_T^{\mathbb{C}}(\sigma)$, using Theorems 3.1.5 and 3.1.8, Lemma 3.1.7 and (3.13), we find that the measure $\hat{P}_T^{\mathbb{C}}$ also converges weakly to $\hat{P}^{\mathbb{C}}$ as $T \to \infty$.

3.7 Proof of Theorem 3.1.2

In view of Theorem 3.1.9, we have to show that the measure coincides with $P_{\zeta_\lambda}^{\mathbb{C}}$.

We take an arbitrary continuity set A of the measure $P^{\mathbb{C}}$. Then, by Theorem 3.1.9 and Lemma 3.6,

$$\lim_{T\to\infty} \nu_T(\zeta_\lambda(\sigma + it, \omega) \in A) = P^{\mathbb{C}}(A). \tag{3.9}$$

On the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define a random variable θ by the formula

$$\theta(\omega) = \begin{cases} 1, & \text{if } \zeta_\lambda(\sigma, \omega) \in A, \\ 0, & \text{if } \zeta_\lambda(\sigma, \omega) \notin A. \end{cases}$$

Then the expectation of θ is

$$\mathrm{E}(\theta) = \int_\Omega \theta \mathrm{d}m_H = m_H\{\omega : \zeta_\lambda(\sigma, \omega) \in A\} = P^{\mathbb{C}}_{\zeta_\lambda}. \tag{3.10}$$

From Lemma 3.5.2 it follows that the process $\theta(h_t(\omega))$ is ergodic. Therefore, by Lemma 3.1.9,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \theta(h_t(\omega))\mathrm{d}t = E(\theta) \tag{3.11}$$

for almost all $\omega \in \Omega$. On the other hand, by the definition of $h_t(\omega)$,

$$\frac{1}{T} \int_0^T \theta(h_t(\omega))\mathrm{d}t = \nu_T(\zeta_\lambda(\sigma + it, \omega) \in A).$$

This, (3.10) and (3.11) show that

$$\lim_{T\to\infty} \nu_T(\zeta_\lambda(\sigma + it, \omega) \in A) = P^{\mathbb{C}}_{\zeta_\lambda}(A).$$

From this and (3.9) we have that

$$P^{\mathbb{C}}(A) = P^{\mathbb{C}}_{\zeta_\lambda}(A)$$

for all continuity sets A. Since continuity sets constitute a determining class, hence it follows that

$$P^{\mathbb{C}}(A) = P^{\mathbb{C}}_{\zeta_\lambda}(A)$$

for all $A \in \mathcal{B}(\mathbb{C})$. The theorem is proved.

Skyrius 4

Limit theorems in the space of analytic functions for the periodic zeta-function

The function $\zeta_\lambda(s)$ with $\lambda \notin \mathbb{Z}$ is entire one. Therefore, its asymptotic behavior is characterized better by limit theorems in the space of analytic functions. Let $D = \{s \in \mathbb{C} : \sigma > \frac{1}{2}\}$. Denote by $H(D)$ the space of analytic functions on D equipped with the topology of uniform convergence on compacta. We recall that we consider the case $0 < \lambda < 1$.

In this chapter, we use the notation

$$\nu_T(\dots) = \frac{1}{T} meas\{\tau \in [0,T], \dots\},$$

where in place of dots a condition satisfied by τ is to be written, and consider the weak convergence of the probability measure

$$P_T^H(A) = \nu_T(\zeta_\lambda(s + i\tau) \in A), \quad A \in \mathcal{B}(H(D)).$$

We will use the scheme of Chapter 3.

4.1 Definition of $H(D)$-valued random element

Let $(\Omega, \mathcal{B}(\Omega), m_H)$ be the same probability space as in Chapter 3, $\omega(p)$ be the projection of element $\omega \in \Omega$ to γ_p, and let

$$\omega(m) = \prod_{p^\alpha \| m} \omega^\alpha(p).$$

For $\sigma > \frac{1}{2}$, define

$$\zeta_\lambda(s, \omega) = \sum_{m=1}^{\infty} \frac{e^{2\pi i \lambda m}\omega(m)}{m^s}.$$

Lemma 4.1.1. *$\zeta_\lambda(s, \omega)$ is an $H(D)$-valued random element defined on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$.*

Proof. Let $\sigma_1 > \frac{1}{2}$ be a fixed number. Then in the proof of Lemma 3.2.1 it was obtained that the series

$$\sum_{m=1}^{\infty} \frac{e^{2\pi i \lambda m}\omega(m)}{m^{\sigma_1}} \tag{4.1}$$

converges almost surely with respect to the measure m_H. It is well known the if a Dirichlet series converges at the point $s_0 = \sigma_0 + it_0$, then it converges uniformly on compact subsets of the half-plane $\sigma > \sigma_0$ [19]. Therefore, from the convergence of the series (4.1), we obtain that the series

$$\sum_{m=1}^{\infty} \frac{e^{2\pi i \lambda m}\omega(m)}{m^s} \tag{4.2}$$

converges almost surely uniformly on compact subsets of the half-plane $\sigma > \sigma_1$. Taking $\sigma_r = \frac{1}{2} + \frac{1}{r}$, $r \in \mathbb{N}$, we deduce in a standard way that the series (4.2) converges almost surely uniformly on compact subsets of the half-plane $\sigma > \frac{1}{2}$. Therefore, it defines an $H(D)$-valued random element on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$.

Denote by $P^H_{\zeta_\lambda}$ the distribution of the random element $\zeta_\lambda(s, \omega_0)$, i.e.,

$$P^H_{\zeta\lambda}(A) = m_h\{\omega \in \Omega : \zeta_\lambda(s, \omega) \in A\}, \quad A \in \mathcal{B}(H(D)).$$

The main result of this chapter is the following statement.

Theorem 4.1.2. *The probability measure P^H_T converges weakly to $P^H_{\zeta_\lambda}$ as $T \to \infty$.*

4.2 Limit theorems in $H(D)$ for absolutely convergent series

Let $\zeta_{\lambda,n}(s)$ and $\zeta_{\lambda,n}(s,\omega)$ be the same as in Chapter 3. On $(H(D), \mathcal{B}(H(D)),$ define two probability measures

$$P^H_{T,n}(A) = \nu_T(\zeta_{\lambda,n}(s+i\tau) \in A)$$

and

$$\hat{P}^H_{T,n}(A) = \nu_T(\zeta_{\lambda,n}(s+i\tau, \omega_0) \in A), \quad \omega_0 \in \Omega.$$

Theorem 4.2.1. *Both the probability measures $P^H_{T,n}$ and $\hat{P}^H_{T,n}$ converge weakly to the same probability measure P^H_n on $(H(D), \mathcal{B}(H(D))$ as $T \to \infty$.*

Proof. Let the function $u_n : \Omega \to H(D)$ be defined by the formula

$$u_n(\omega) = \sum_{m=1}^{\infty} \frac{e^{2\pi i\lambda m}\omega(m)v(n,m)}{m^s}, \quad \omega \in \Omega.$$

Then u_n is continuous, and

$$u_n((p^{-i\tau} : p \in \mathcal{P})) = \sum_{m=1}^{\infty} \frac{e^{2\pi i\lambda m}v(n,m)}{m^{s+i\tau}} = \zeta_{\lambda,n}(s+i\tau).$$

Thus, $P^H_{T,n} = Q_T u_n^{-1}$, where Q_T is the measure from Theorem 3.3.1. Hence, as in the proof of Theorem 3.4.2, we obtain that the measure $P^H_{T,n}$ converges weakly to $m_H u_n^{-1}$ as $T \to \infty$.

Analogically, define the function $\hat{u}_n : \Omega \to H(D)$ by the formula

$$\hat{u}_n(\omega) = \sum_{m=1}^{\infty} \frac{e^{2\pi i\lambda m}\omega_0(m)v(n,m)}{m^s}, \quad \omega \in \Omega.$$

Then we obtain that the measure $\hat{P}^H_{T,n}$ converges weakly to $m_H \hat{u}_n^{-1}$ as $T \to \infty$. The invariance of the Haar measure m_H shows that $m_H \hat{u}_n^{-1} = m_H u_n^{-1}$, and the theorem is proved.

4.3 Approximation in the mean in the space $H(D)$

We start with a metric in $H(D)$ which induces its topology of uniform convergence on compacts.

It is well known, see, for example [5], that there exists a sequence $\{K_l : l \in \mathbb{N}\}$ of compact subsets of the half-plane D such that

$$D = \bigcup_{l=1}^{\infty} K_l,$$

$K_l \subset K_{l+1}$, and if K is a compact subset of D, then $K \subset K_l$ for some l.

For $f, g \in H(D)$, define

$$\varrho(f, g) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} |f(s) - g(s)|}{1 + \sup_{s \in K_l} |f(s) - g(s)|}.$$

Then ϱ is the desired metric in $H(D)$.

Theorem 4.3.1. *We have*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \varrho(\zeta_\lambda(s + i\tau), \zeta_{\lambda,n}(s + i\tau))\mathrm{d}\tau = 0.$$

Proof. Let K be an arbitrary compact subset of the half-plane D. It follows from the definition of the metric ϱ that it suffices to prove the relation

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \sup_{s\in K} |\zeta_\lambda(s + i\tau) - \zeta_{\lambda,n}(s + i\tau)|\mathrm{d}\tau = 0.$$

Let L be a simple closed contour lying in D and enclosing the set K. Then, by the Cauchy integral formula,

$$\sup_{s\in K} |\zeta_\lambda(s + i\tau) - \zeta_{\lambda,n}(s + i\tau)|$$

$$\ll \int_L \frac{|\zeta_\lambda(z + i\tau) - \zeta_{\lambda,n}(z + i\tau)|}{|z - s|} |\mathrm{d}z|$$

$$\ll \delta^{-1} \int_L |\zeta_\lambda(z+i\tau) - \zeta_{\lambda,n}(z+i\tau)||\mathrm{d}z|,$$

where δ denotes the distance of L from the set K. Denoting by $|L|$ the length of L, hence we find that, for sufficiently large T,

$$\frac{1}{T}\int_0^T \sup_{s\in K} |\zeta_\lambda(s+i\tau) - \zeta_{\lambda,n}(s+i\tau)|\mathrm{d}\tau$$

$$\ll \delta^{-1}\int_L |\mathrm{d}z| \int_{\mathrm{Im}z}^{T+\mathrm{Im}z} |\zeta_\lambda(\mathrm{Re}z+i\tau) - \zeta_{\lambda,n}(\mathrm{Re}z+i\tau)|\mathrm{d}\tau$$

$$\ll |L|\delta^{-1}\sup_{s\in K}\int_0^{2T} |\zeta_\lambda(\sigma+it) - \zeta_{\lambda,n}(\sigma+it)|\mathrm{d}t.$$

Using (3.2), we obtain that

$$\frac{1}{T}\int_0^{2T} |\zeta_\lambda(\sigma+it) - \zeta_{\lambda,n}(\sigma+it)|\mathrm{d}t$$

$$\ll \int_{-\infty}^{\infty} |l_n(\sigma_2-\sigma+i\tau)|\frac{1}{T}\int_{-|\tau|}^{2T+|\tau|} \zeta_\lambda(\sigma_2+it)\mathrm{d}t\mathrm{d}\tau,$$

and the proof is finished in the same way as that of Theorem 3.5.1.

Theorem 4.3.2. *For almost all $\omega\in\Omega$,*

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \varrho(\zeta_\lambda(s+i\tau,\omega) - \zeta_{\lambda,n}(s+i\tau,\omega))\mathrm{d}\tau = 0.$$

Proof. We follow the proof of Theorem 4.3.1 and apply Lemma 3.5.2.

4.4 Proof of Theorem 4.1.2

Theorem 4.4.1. *Both the measures P_T^H and $\hat{P}_T^H$ converge weakly to same probability measure P^H on $(H(D),\mathcal{B}(H(D)))$ as $T\to\infty$.*

Proof. By Theorem 4.2.1 the measures $P_{T,n}^H$ and $\hat{P}_{T,n}^H$ both converge weakly to same probability measure P_n^H on $(H(D),\mathcal{B}(H(D)))$ as $T\to\infty$. Define

$$X_{T,n}^H(s) = \zeta_{\lambda,n}(s+i\theta T).$$

Then we have that

$$X_{T,n}^H(s) \xrightarrow[T\to\infty]{\mathcal{D}} X_n^H(s), \tag{4.3}$$

where $X_n^H(s)$ is an $H(D)$-valued random element with the distribution P_n^H.

Now we will show that the family of probability measures $\{P_n^H, n \in \mathbb{N}\}$ is tight.

Let M_l be arbitrary positive number. Then

$$\begin{aligned}
&P_{T,n}^H(\{g \in H(D) : \sup_{s\in K_l} |g(s)| > M_l\}) \\
&= \nu_T(\{\sup_{s\in K_l} |X_{T,n}^H(s)| > M_l\}) \\
&\le \frac{1}{TM}\int_0^T \sup_{s\in K_l} |\zeta_{\lambda,n}(s+i\tau)|\mathrm{d}\tau.
\end{aligned} \tag{4.4}$$

Let L_l be a simple closed contour that surrounds K_l and lies in D, $l \in \mathbb{N}$. By the Cauchy integral theorem, we have

$$\sup_{s\in K_l} |\zeta_{\lambda,n}(s+i\tau)| \le \frac{1}{\delta_l}\int_{L_l} |\zeta_{\lambda,n}(z+i\tau)||\mathrm{d}z|,$$

where δ_l is the distance from L_l to K_l. We can choose the contour L_l in such a way that $0 < c_l \le \delta_l \le 1$ for some constant c_l. Since K_l is bounded, so is the contour L_l. It follows that

$$\begin{aligned}
&\frac{1}{T}\int_0^T \sup_{s\in K_l} |\zeta_{\lambda,n}(s+i\tau)|\mathrm{d}\tau \\
&\ll \frac{1}{\delta_l}\int_{L_l} |\mathrm{d}z| \frac{1}{T}\int_0^T |\zeta_{\lambda,n}(\mathrm{Re}z + \mathrm{Im}z + i\tau)|\mathrm{d}\tau \\
&\ll \frac{L_l}{\delta_l}\sup_{s\in L_l}\frac{1}{T}\int_{-|\mathrm{Im}z|}^{T+|\mathrm{Im}z|} |\zeta_{\lambda,n}(\mathrm{Re}z + i\tau)|\mathrm{d}\tau.
\end{aligned} \tag{4.5}$$

Since $\zeta_{\lambda,n}(s)$, for $\sigma > \frac{1}{2}$, is given by absolutely convergent series,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T |\zeta_{\lambda,n}(\sigma+it)|^2\mathrm{d}t$$

$$= \sum_{m=1}^{\infty}\frac{v^2(m,n)}{m^{2\sigma}} \le \sum_{m=1}^{\infty}\frac{1}{m^{2\sigma}}.$$

Thus, for $\sigma > \frac{1}{2}$ and sufficiently large T,

$$\frac{1}{T}\int_0^T |\zeta_{\lambda,n}(\sigma+it)|^2 \mathrm{d}t \leq 2\sum_{m=1}^{\infty}\frac{1}{m^{2\sigma}}.$$

Hence, it follows from (4.5) that

$$\frac{1}{T}\int_0^T \sup_{s\in K_l}|\zeta_{\lambda,n}(s+i\tau)|\mathrm{d}\tau$$

$$\ll \frac{|L_l|}{\delta_l}\sup_{s\in L_l}\Big(\frac{1}{T}\int_{-|\mathrm{Im}z|}^{T+|\mathrm{Im}z|}|\zeta_{\lambda,n}(\mathrm{Re}z+i\tau)|\mathrm{d}\tau\Big)^{\frac{1}{2}}$$

$$\ll \frac{|L_l|}{\delta_l}\sup_{s\in L_l}\Big(\sum_{m=1}^{\infty}\frac{1}{m^{2\mathrm{Re}z}}\Big)^{\frac{1}{2}}$$

$$\leq C_l\Big(\sum_{m=1}^{\infty}\frac{1}{m^{2\sigma_l}}\Big)^{\frac{1}{2}}$$

for some constant $C_l > 0$ with $\sigma_l = \min\{\sigma : s \in L_l\} > \frac{1}{2}$. Thus, for every $l, n \in \mathbb{N}$ we have

$$\limsup_{T\to\infty}\frac{1}{T}\int_0^T \sup_{s\in K_l}|\zeta_{\lambda,n}(s+i\tau)|\mathrm{d}\tau \leq C_l V_l < \infty, \tag{4.6}$$

where

$$V_l = \Big(\sum_{m=1}^{\infty}\frac{1}{m^{2\sigma_l}}\Big)^{\frac{1}{2}}.$$

Now let ε be an arbitrary positive number, and put $M_l = M_{l,\varepsilon} = C_l V_j 2^l \varepsilon^{-1}$. Then (4.4) and (4.6) show that

$$\limsup_{T\to\infty} P_{T,n}^H(\{g \in H(D) : \sup_{s\in K_l}|g(s)| > M_{l,\varepsilon}\}) \leq \frac{\varepsilon}{2^l}. \tag{4.7}$$

Define a set

$$H_\varepsilon = \{g \in H(D) : \sup_{s \in K_l} |g(s)| \le M_{l,\varepsilon},\ l \in \mathbb{N}\}.$$

Then the set H_ε is uniformly bounded, therefore, it is compact. Moreover, by (4.7),

$$P_n^H(H_\varepsilon) \ge 1 - \varepsilon \sum_{l=1}^{\infty} \frac{1}{2^l} = 1 - \varepsilon$$

for all $n \in \mathbb{N}$. This means that the family $\{P_n^H : n \in \mathbb{N}\}$ is tight. Thus, by Lemma 3.1.4, it is relatively compact. Therefore, there exists a sequence $\{P_{n_k}^H\} \subset \{P_n^H\}$ such that the measure $P_{n_k}^H$ converges weakly to certain probability measure P^H on $(H(D), \mathcal{B}(H(D))$ as $k \to \infty$. Hence,

$$X_{n_k}^H(s) \xrightarrow[k\to\infty]{\mathcal{D}} P^H. \tag{4.8}$$

Define

$$X_T^H(s) = \zeta_\lambda(s + i\theta T).$$

Then, by Theorem 4.3.1, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \mathbb{P}(\varrho(X_T^H(s), X_{T,n}^H(s)) \ge \varepsilon)$$

$$= \lim_{n\to\infty} \limsup_{T\to\infty} \nu_T(\varrho(\zeta_\lambda(s+i\tau), \zeta_{\lambda,n}(s+i\tau)) \ge \varepsilon)$$

$$\le \lim_{k\to\infty} \limsup_{T\to\infty} \frac{1}{T\varepsilon} \int_0^T \varrho(\zeta_\lambda(s+i\tau), \zeta_{\lambda,n}(s+i\tau)\mathrm{d}\tau = 0.$$

This and (4.8), (4.3) show that all hypotheses of Lemma 3.1 are satisfied. Thus,

$$X_T^H(s) \xrightarrow[T\to\infty]{\mathcal{D}} P^H, \tag{4.9}$$

and this means that the measure P_T^H converges weakly to P^H as $T \to \infty$. Moreover, (4.9) shows that the measure $P^H$is independent on the sequence $\{P_{n_k}^H\}$. Therefore,

$$X_n^H(s) \xrightarrow[n\to\infty]{\mathcal{D}} P^H. \tag{4.10}$$

Now define the random elements

$$\hat{X}_{T,n}^H(s) = \zeta_{\lambda,n}(s + i\theta T, \omega)$$

and

$$\hat{X}_T^H(s) = \zeta_\lambda(s + i\theta T, \omega).$$

Then, using Theorems 4.2.1 and 4.3.2, and (4.10), we obtain similarly as above that the measure $\hat{P}_T^H$ also converges weakly to P^H as $T \to \infty$.

Proof of Theorem 4.1.2 Let A be an arbitrary continuity set of the measure P^H. Then Theorem 4.4.1 together with Lemma 3.1.2 implies

$$\lim_{T\to\infty} \nu_T(\varrho(\zeta_\lambda(s + it, \omega) \in A)) = P^H(A). \tag{4.11}$$

On the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define a random variable θ by the formula

$$\theta(\omega) = \begin{cases} 1, & \text{if } \zeta_\lambda(s,\omega) \in A, \\ 0, & \text{if } \zeta_\lambda(s,\omega) \notin A. \end{cases}$$

Then we have that

$$\mathrm{E}(\theta) = \int_\Omega \theta \mathrm{d}m_H = m_H\{\omega : \zeta_\lambda(s,\omega) \in A\} = P_{\zeta_\lambda}^H(A). \tag{4.12}$$

From Lemma 3.5.2 it follows that the process $\theta(h_\tau(\omega))$ is ergodic, thus, in view of Lemma 3.1.9,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \theta(h_\tau(\omega))\mathrm{d}\tau = E(\theta) \tag{4.13}$$

for almost all $\omega \in \Omega$. However, by the definitions of θ and $h_\tau(\omega)$,

$$\frac{1}{T} \int_0^T \theta(h_\tau(\omega))\mathrm{d}t = \nu_T(\zeta_\lambda(s + i\tau, \omega) \in A).$$

This, (4.12), (4.13) show that

$$\lim_{T\to\infty} \nu_T(\zeta_\lambda(s+i\tau,\omega) \in A) = P^H_{\zeta_\lambda}(A).$$

Combining this with (4.11), we have that $P^H(A) = P^H_{\zeta_\lambda}(A)$ for all continuity sets A of the measure P_H. Hence, $P^H(A) = P^H_{\zeta_\lambda}(A)$ for all $A \in \mathcal{B}((H(D))$. The theorem is proved.

Conclusions

In the thesis the following properties for the periodic zeta - function $\zeta_\lambda(s)$ are established.

1. For the error term in the mean square formula of the function $\zeta_\lambda(s)$ with rational parameter λ on the critical line an averaged version of the Atkinson formula is true.

2. For the error term in the mean square formula of the function $\zeta_\lambda(s)$ with rational parameter λ on the critical strip $\frac{1}{2} < \sigma < 1$ an averaged version of the Atkinson formula is true.

3. For the function $\zeta_\lambda(s)$ in the half-plane $\sigma > \frac{1}{2}$, a limit theorem in the sense of weak convergence of probability measures on the complex plane is valid.

4. For the function $\zeta_\lambda(s)$, a limit theorem in the sense of weak convergence of probability measures in the space of analytic functions equipped with the topology of uniform convergence on compacta is valid.

Notation

$\mathbb{N}_0$	set of all non -negative integers
$\mathbb{N}$	set of all positive integers
$\mathbb{Z}$	set of all integers
$\mathbb{R}$	set of all real numbers
$\mathcal{P}$	set of all prime numbers
$\mathbb{C}$	set of all complex numbers
l, m, n, q	positive integers
i	imaginary unity: $i = \sqrt{-1}$
$s = \sigma + it$	complex variable
$measA$	Lebesgue measure of the set A
$\mathcal{B}(S)$	class of Borel sets of the space S
$\xrightarrow{D}$	convergence in distribution
$\zeta(s)$	Riemann zeta-function defined, for $\sigma > 0$, by $\zeta(s) = \sum\limits_{m=1}^{\infty} \frac{1}{m^s}$, and by analytic continuation elsewhere
$L(s, \chi)$	Dirichlet L - function, defined, for $\sigma > 1$, by $L(s, \xi) = \sum_{m=1}^{\infty} \frac{\xi(m)}{m^s}$, and by analytic continuation elsewhere.
$\Gamma(s)$	Euler gamma-function defined, for $\sigma > 0$, by $\Gamma(s) = \int\limits_0^{\infty} e^{-x} x^{s-1} dx$, and by analytic continuation elsewhere
$L(\lambda, \alpha, s)$	Lerch zeta-function, defined, for $\sigma > 1$, by $L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m+\alpha)^s}$, and by analytic continuation elsewhere.

Literatūra

[1] F.V. Atkinson, *The mean-value of the Riemann zeta-function*, Acta Math., **81** (1949), 353-376.

[2] P. Billingsley, Convergence of Probability Measures, John Wiley & Sons, New York, 1968.

[3] H. Bohr, B. Jessen, *Über die Wertverteilung der Riemannschen Zetafunktion*, Erste, Acta Math. **54**, 1-35 (1930).

[4] H. Bohr, B. Jessen, *Über die Wertverteilung der Riemannschen Zetafunktion*, Zweite Mitteilung, Acta Math. **58**, 1-55 (1932).

[5] J.B. Conway, Functions of One complex Variable, Springer, Berlin, 2001.

[6] H. Cramér, M. R. Ledbetter, Stationary and Related Processes, Wiley, New York, 1967.

[7] R. Garunkštis, A. Laurinčikas, J. Steuding, *On the mean square of Lerch zeta-function*, Liet. Matem. rink. **80** (2003), 47-60.

[8] G. H. Hardy, J. E. Littlewood, *Contributions to the theory of the Riemann zeta-function and the distribution of primes*, Acta Math. **41** (1918), 119-196.

[9] H. Heyer, Probability Measures on Localy Compact Groups, Springer, Berlin (2001).

[10] A. E. Ingham, *Mean-value theorems in the theory of the Riemann zetafunction*, Proc. London Math. Soc. **27** (1926), 273-300.

[11] A. Ivič, *The Riemann Zeta-Function*, Wiley, New York, (1985).

[12] A. Ivič, Mean Values of the Riemann Zeta Function, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1991.

[13] M. Jutila, *Transformation formulae for Dirichlet polynomials*, J. Number Theory, **18**, 135-156 (1984).

[14] 8. M. Jutila, *Atkinson's formula revisited*, in: Voronoi's Impact on Modern Science, Book 1, Proc. Inst. Math. National Acad. Sc. Ukraine, Kiev (1998), **14**, 137-154.

[15] J. Karaliūnaitė, A. Laurinčikas, *The Atkinson formula for the periodic zeta-function*, Lith. matem. J. **47** (3), 504-516 (2007).

[16] M. Katsurada, *An application of Mellin-Barnes type integrals to the mean square of Lerch zeta-functions*, Collect. Math. **48** (1997), 137-153.

[17] A. Laurinčikas, *The Atkinson formula near the critical line*, in: Analytic Probab. Methods in Number Theory, E. Manstavičius, F. Schweiger (eds.), New Trends in Probab. and Statist., **2**, VSP/TEV, (1992), 335-354.

[18] A. Laurinčikas, *The Atkinson formula near the critical line II*, Liet. Matem. Rink., **33** (1993), no. 3, 302-313 (in Russian).

[19] A. Laurinčikas, *Limit Theorems for the Riemann Zeta-Function*, Dordrecht, Kluwer (1996).

[20] A. Laurinčikas, R. Garunkštis, *The Lerch Zeta-Function*, Kluwer, Dordrecht (2002).

[21] M. Loève, *Probability Theory*, Van Nostrand, Toronto, 1955.

[22] K. Matsumoto, *The mean square of the Riemann zeta-function in the critical strip*, Japanese J. Math., **15** (1989), 1-13.

[23] K. Matsumoto, *Recent developments in the mean-square theory of the Riemann zeta and other zeta-functions*, in: Number Theory, Trends Math., Birkhäuser, Boston (2000), 241-286.

[24] K. Matsumoto, *Probabilistic value-distribution theory of zeta-functions*, Sugaku Expositions **17** (2004), 51-71.

[25] K. Matsumoto, T. Meurman, *The mean square of the Riemann zeta-function in the critical strip III*, Acta Math., **64** (1993), 357-382.

[26] T. Meurman, *A generalization of Atkinson's formula to L-functions*, Acta Arithm., **47** (1986), 351-370.

[27] T. Meurman, *On the mean square of the Riemann zeta-function*, Quart. J. Math. Oxford (2), **38** (1987), 337-343.

[28] Y. Motohashi, *A note on the mean value of the zeta and L-functions. IV*, Proc. Japan Acad., **62A** (1986), 311-313.

[29] V. Paulauskas, A. Račkauskas, Funkcinė Analizė. I knyga. Erdvės, UAB "Vaistų žinios", Vilnius (2007).

[30] J. Steuding, *Value distribution of L-functions*, Lect. Notes Math. **1877**, Springer, Berlin (2007).

[31] E. C. Titchmarsh, The Theory of the Riemann Zeta-Function, second edition revised by D. R. Heath-Brown, Clarendon Press, Oxford (1986).

[32] G. F. Voronoi, *Sur une function transcendante et ses applications à la sommation de quelques séries*, Ann. Sci. École. Norm. Sup. **21**(3) (2004) 207-268.

[33] G. F. Voronoi, *Sur une function transcendante et ses applications à la sommation de quelques series*, Ann. Sci. École. Norm. Sup. **21**(3) (2004) 459-533.

[34] G. N. Watson, A Treatise on the Bessel Functions, Cambridge University Press, Cambridge (1966).