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On the Addition of a Large Scalar Multiplet to the Standard Model

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We consider the addition of a single $SU(2)$ multiplet of complex scalar fields to the Standard Model (SM). We explicitly consider the various possible values of the weak isospin J of that multiplet, up to and including $J = 7/2$. We allow the multiplet to have arbitrary weak hypercharge. The scalar fields of the multiplet are assumed to have no vacuum expectation value; the mass differences among the components of the multiplet originate in its coupling, present in the scalar potential (SP), to the Higgs doublet of the SM. We derive exact bounded-from-below and unitarity conditions on the SP, thereby constraining those mass differences. We compare those constraints to the ones that may be derived from the oblique parameters.

Subject Index B40, B46, B53, B57

1. Introduction

In this paper, we study the model of New Physics (NP), i.e. of physics beyond the Standard Model (SM), wherein one adds to the SM one gauge-$SU(2)$ multiplet χ with weak isospin J and consisting of $n = 2J + 1$ complex scalar fields. The multiplet has unspecified weak hypercharge Y; therefore, the model enjoys an accidental $U(1)$ symmetry wherein one rephases χ through an arbitrary phase. The scalar fields that compose χ are assumed not to have any vacuum expectation value (VEV), even if one of them—depending on Y and J—may happen to be electrically neutral. There is in the scalar potential (SP) a renormalizable coupling

$$\lambda_4 \sum_{a=1}^{3} \left(H^\dagger \frac{\tau_a}{2} H \right) \left[\chi^\dagger T_a^{(J)} \chi \right] \tag{1}$$

of χ to the Higgs doublet H of the SM. In Eq. (1),

- λ_4 is a dimensionless coefficient,
- the τ_a are the Pauli matrices,
- one conceives of χ as a column vector of n scalar fields,
- the $T_a^{(J)}$ are the $n \times n$ matrices that represent $su(2)$ in the J-isospin representation.

The coupling (1) generates, upon the neutral component of H acquiring VEV v, a squared-mass difference $\Delta m^2 \propto v^2$ between any two components of χ whose third component of isospin differs by one unit.

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This NP model was firstly (to our knowledge) considered 30 years ago [1] as a paradigm for potentially large oblique parameters (OPs). It has later been studied as a model for "minimal" dark matter [2] and, more recently [3], as an explanation for the unexpectedly high value of the $W^\pm$ mass measured by the CDF-II Collaboration. Twelve years ago, Logan and her collaborators [4] showed that n cannot exceed eight, lest perturbative unitarity in the scattering of two scalars of χ to two $SU(2)$ gauge bosons be violated; they also derived mixed constraints on J and Y. Logan's work was revived and expanded very recently [5]. In another recent paper [6], the specific case of the addition of an $SU(2)$ scalar quadruplet to the SM has been considered; the hypercharge of that quadruplet has been restricted to the values $1/2$ or $3/2$,[1] because in those two cases additional quartic couplings—beyond the one of Eq. (1)—of the types χHHH and/or $\chi\chi\chi H$ may be present. (The accidental $U(1)$ symmetry then does not exist, because Y has a well-defined value.) The case studied in Ref. [6] is on the one hand more restricted than the one in this paper, because χ has fixed $J = 3/2$, but on the other hand it is more complicated, because additional quartic terms are allowed in the SP.

In this paper we want to constrain the modulus of the coefficient λ_4 of the term (1) of the SP; in so doing, we place an upper bound on Δm^2. We do this by considering both the unitarity (UNI) and the bounded-from-below (BFB) conditions on the quartic part of the SP. Remarkably, the upper bound on Δm^2 results from both the UNI *and* the BFB conditions, and not just from the former ones. We firstly show this fact, in a simplified version of the SP, in Sect. 2; later on, in Sect. 3, we consider the full SP. Section 4 contains the confrontation of our NP model with the OPs that it generates; we investigate whether the phenomenological OPs constrain Δm^2 more or less than the UNI/BFB conditions. Section 5 contains our conclusions. The Appendix explicitly lists the UNI conditions for all the values of n through eight.

2. Potential without terms four-linear on χ

In our model of NP there is the SM scalar doublet H with hypercharge $1/2$ and an $SU(2)$ scalar multiplet χ with weak isospin J, which is a positive number, either integer or half-integer. The multiplet χ has

$$n = 2J + 1 \tag{2}$$

components χ_I ($I = J,\ J-1,\ J-2,\ \ldots,\ 1-J,\ -J$). Its hypercharge Y remains unspecified, i.e. arbitrary. Together with the charge-conjugate multiplets $\tilde{H}$ and $\tilde{\chi}$, we have the four multiplets

$$H = \begin{pmatrix} a \\ b \end{pmatrix}, \quad \tilde{H} = \begin{pmatrix} b^* \\ -a^* \end{pmatrix}, \quad \chi = \begin{pmatrix} \chi_J \\ \chi_{J-1} \\ \chi_{J-2} \\ \vdots \\ \chi_{1-J} \\ \chi_{-J} \end{pmatrix}, \quad \tilde{\chi} = \begin{pmatrix} \chi^*_{-J} \\ -\chi^*_{1-J} \\ \vdots \\ (-1)^{2J}\, \chi^*_{J-2} \\ -(-1)^{2J}\, \chi^*_{J-1} \\ (-1)^{2J}\, \chi^*_J \end{pmatrix}. \tag{3}$$

Here, a, b, and the χ_I are complex Klein–Gordon fields. Their third components of isospin are

$$a : \frac{1}{2}, \quad b : -\frac{1}{2}, \quad \chi_I : I, \quad \chi^*_I : -I. \tag{4}$$

[1] Other recent papers that consider scalar quadruplets with those specific hypercharges are Refs. [7–9]. They also consider models with additional scalar triplets and five-plets, always with specific hypercharges.

When one multiplies χ by $\tilde{\chi}$ one obtains, among other $SU(2)$ representations, the singlet

$$F_2 \equiv (\chi \otimes \tilde{\chi})_{\mathbf{1}} = \sum_{I=-J}^{J} |\chi_I|^2 \tag{5}$$

and the triplet

$$(\chi \otimes \tilde{\chi})_{\mathbf{3}} = \begin{pmatrix} -\displaystyle\sum_{I=1-J}^{J} \sqrt{\frac{J^2 - I^2 + J + I}{2}}\, \chi_I\, \chi_{I-1}^* \\ \displaystyle\sum_{I=-J}^{J} I\, |\chi_I|^2 \\ \displaystyle\sum_{I=1-J}^{J} \sqrt{\frac{J^2 - I^2 + J + I}{2}}\, \chi_I^*\, \chi_{I-1} \end{pmatrix}. \tag{6}$$

Applying the general Eq. (6) to the specific case of H (i.e. using $J = 1/2$, $\chi_{1/2} = a$, and $\chi_{-1/2} = b$), we obtain

$$\left(H \otimes \tilde{H}\right)_{\mathbf{3}} = \begin{pmatrix} -\dfrac{ab^*}{\sqrt{2}} \\ \dfrac{|a|^2 - |b|^2}{2} \\ \dfrac{a^* b}{\sqrt{2}} \end{pmatrix}. \tag{7}$$

The SP V has a quadratic part V_2 and a quartic part V_4:

$$V = V_2 + V_4. \tag{8}$$

Obviously,

$$V_2 = -\mu_1^2 F_1 + \mu_2^2 F_2, \tag{9}$$

where

$$F_1 \equiv \left(H \otimes \tilde{H}\right)_{\mathbf{1}} = |a|^2 + |b|^2 \tag{10}$$

and F_2 is defined in Eq. (5). We assume both coefficients μ_1^2 and μ_2^2 to be positive, so that H has VEV $\langle 0 \,|b|\, 0\rangle = v$ but χ does not have VEV.

The quartic part of the SP contains

- the term $\left[\left(H \otimes \tilde{H}\right)_{\mathbf{1}}\right]^2$, with coefficient $\dfrac{\lambda_1}{2}$;
- the term $\left(H \otimes \tilde{H}\right)_{\mathbf{1}} (\chi \otimes \tilde{\chi})_{\mathbf{1}}$, with coefficient λ_3;
- the term $\left[\left(H \otimes \tilde{H}\right)_{\mathbf{3}} \otimes (\chi \otimes \tilde{\chi})_{\mathbf{3}}\right]_{\mathbf{1}}$, with coefficient λ_4;
- various terms that are four-linear in the components of χ. We keep those terms unspecified in this section.

Thus,

$$V_4 = \frac{\lambda_1}{2} F_1^2 + \lambda_3 F_1 F_2 + \lambda_4 F_4 + \text{terms four-linear in the } \chi_I, \tag{11}$$

where

$$F_4 \equiv \frac{|a|^2 - |b|^2}{2} \sum_{I=-J}^{J} I\, |\chi_I|^2 + \frac{z + z^*}{2}. \tag{12}$$

We have defined

$$z \equiv ab^* \sum_{I=1-J}^{J} \chi_I^* \, \chi_{I-1} \sqrt{J^2 - I^2 + J + I}. \tag{13}$$

From Eqs. (8), (9), (11), and (12) the mass-squared of the scalar χ_I is

$$m_I^2 = \mu_2^2 + \left(\lambda_3 - \frac{\lambda_4}{2} I \right) |v|^2 . \tag{14}$$

This implies that the difference between the masses-squared of χ_I and χ_{I+1} is

$$\Delta m^2 = \frac{\left| \lambda_4 v^2 \right|}{2}, \tag{15}$$

which is I-independent. An upper bound on $|\lambda_4|$ is therefore equivalent to an upper bound on Δm^2.

The VEV of V is

$$\langle 0 \, | V | \, 0 \rangle = -\mu_1^2 v^2 + \frac{\lambda_1}{2} v^4. \tag{16}$$

Therefore, $\mu_1^2 = \lambda_1 v^2$. The mass-squared of the Higgs particle is $m_H^2 = 2\lambda_1 v^2$. Since experimentally $m_H \approx 125$ GeV and $v \approx 174$ GeV, one has

$$\lambda_1 \approx 0.258. \tag{17}$$

From now on we shall assume Eq. (17) to hold. Contrary to λ_1, the couplings λ_3 and λ_4 are free, but they are constrained by both the UNI and BFB conditions. We next derive those constraints.

2.1. *UNI conditions*

In Ref. [4], and more recently again in Ref. [5], the scattering of two scalars belonging to χ to two gauge bosons of either gauge group $SU(2)$ or $U(1)$ has been considered; therefrom upper bounds on both the isospin J and the hypercharge Y of χ have been derived. Here we consider the scattering of a pair of scalars of χ to another pair of scalars, both pairs having, of course, the same I (third component of isospin) and Y. Whereas in Refs. [4,5] the scattering involves two cubic gauge couplings and the interchange of a virtual particle either in the s, t, or u channel, here the scattering involves no interchange of any virtual particle, rather it takes place directly through a *quartic* coupling in the SP.

Firstly suppose that J is half-integer.

- We consider the scattering of the two two-field states with hypercharge $Y + 1/2$ and null third component of isospin, viz. of $\chi_{-1/2} a$ and $\chi_{1/2} b$. Their scattering matrix is

$$\begin{pmatrix} \lambda_3 - \lambda_4/4 & (2J+1)\,\lambda_4/4 \\ (2J+1)\,\lambda_4/4 & \lambda_3 - \lambda_4/4 \end{pmatrix}. \tag{18}$$

  The eigenvalues of this matrix are

$$\lambda_3 + \frac{J\lambda_4}{2}, \quad \lambda_3 - \frac{(J+1)\,\lambda_4}{2}. \tag{19}$$

- We next consider the scattering of the states with hypercharge $Y - 1/2$ and null third component of isospin, viz. of $\chi_{-1/2} b^*$ and $\chi_{1/2} a^*$. Their scattering matrix is

$$\begin{pmatrix} \lambda_3 + \lambda_4/4 & (2J+1)\,\lambda_4/4 \\ (2J+1)\,\lambda_4/4 & \lambda_3 + \lambda_4/4 \end{pmatrix}. \tag{20}$$

The eigenvalues of this matrix are

$$\lambda_3 + \frac{(J+1)\,\lambda_4}{2}, \quad \lambda_3 - \frac{J\lambda_4}{2}. \tag{21}$$

Let us secondly suppose that J is an integer instead.

- We consider the scattering of the two two-particle states with hypercharge $Y + 1/2$ and third component of isospin $1/2$, viz. of $\chi_0 a$ and $\chi_1 b$. Their scattering matrix is

$$\begin{pmatrix} \lambda_3 & \sqrt{J\,(J+1)}\,\lambda_4 \Big/ 2 \\ \sqrt{J\,(J+1)}\,\lambda_4 \Big/ 2 & \lambda_3 - \lambda_4/2 \end{pmatrix}. \tag{22}$$

  The eigenvalues of this matrix are the ones in Eq. (19).
- We next consider the scattering of the states with hypercharge $Y - 1/2$ and third component of isospin $1/2$, viz. of the states $\chi_1 a^*$ and $\chi_0 b^*$. Their scattering matrix is

$$\begin{pmatrix} \lambda_3 + \lambda_4/2 & \sqrt{J\,(J+1)}\,\lambda_4 \Big/ 2 \\ \sqrt{J\,(J+1)}\,\lambda_4 \Big/ 2 & \lambda_3 \end{pmatrix}. \tag{23}$$

  The eigenvalues of this matrix are in Eq. (21).

Thus, the eigenvalues of the scattering matrices are the same, no matter whether J is integer or half-integer.

We now impose the conditions that the moduli of all the eigenvalues in Eqs. (19) and (21) should be smaller than

$$M = 8\pi. \tag{24}$$

We obtain

$$|\lambda_3| + \frac{J}{2}\,|\lambda_4| < M, \tag{25a}$$

$$|\lambda_3| + \frac{J+1}{2}\,|\lambda_4| < M. \tag{25b}$$

Condition (25b) is of course stronger than condition (25a), therefore one may neglect the latter.

The dispersion of the $2 + n$ states that have zero third component of isospin and zero hypercharge, viz. of the states,[2]

$$|a|^2, |b|^2, |\chi_J|^2, |\chi_{-J}|^2, |\chi_{J-1}|^2, |\chi_{1-J}|^2, \ldots, |\chi_1|^2, |\chi_{-1}|^2, |\chi_0|^2, \tag{26}$$

produces the scattering matrix

$$\mathcal{S} = \begin{pmatrix} \mathcal{A} & \mathcal{B}_J & \mathcal{B}_{J-1} & \cdots & \mathcal{B}_1 & \mathcal{C} \\ \mathcal{B}_J & 0_{2\times 2} & 0_{2\times 2} & \cdots & 0_{2\times 2} & 0_{2\times 1} \\ \mathcal{B}_{J-1} & 0_{2\times 2} & 0_{2\times 2} & \cdots & 0_{2\times 2} & 0_{2\times 1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{B}_1 & 0_{2\times 2} & 0_{2\times 2} & \cdots & 0_{2\times 2} & 0_{2\times 1} \\ \mathcal{C}^T & 0_{1\times 2} & 0_{1\times 2} & \cdots & 0_{1\times 2} & 0 \end{pmatrix}, \tag{27}$$

where

$$\mathcal{A} = \begin{pmatrix} 2\lambda_1 & \lambda_1 \\ \lambda_1 & \lambda_1 \end{pmatrix}, \quad \mathcal{B}_I = \begin{pmatrix} \lambda_3 + I\lambda_4/2 & \lambda_3 - I\lambda_4/2 \\ \lambda_3 - I\lambda_4/2 & \lambda_3 + I\lambda_4/2 \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} \lambda_3 \\ \lambda_3 \end{pmatrix}, \tag{28}$$

[2] In this explicit computation we assume J to be an integer. The final result, viz. Eqs. (34), is also valid for half-integer J.

and $0_{m\times m'}$ denotes the $m \times m'$ matrix that has all its matrix elements equal to zero. The matrix $\mathcal{S}$ is equivalent to

$$\begin{pmatrix} X\mathcal{A}X^T & X\mathcal{B}_J X^T & X\mathcal{B}_{J-1}X^T & \cdots & X\mathcal{B}_1 X^T & X\mathcal{C} \\ X\mathcal{B}_J X^T & 0_{2\times 2} & 0_{2\times 2} & \cdots & 0_{2\times 2} & 0_{2\times 1} \\ X\mathcal{B}_{J-1}X^T & 0_{2\times 2} & 0_{2\times 2} & \cdots & 0_{2\times 2} & 0_{2\times 1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ X\mathcal{B}_1 X^T & 0_{2\times 2} & 0_{2\times 2} & \cdots & 0_{2\times 2} & 0_{2\times 1} \\ \mathcal{C}^T X^T & 0_{1\times 2} & 0_{1\times 2} & \cdots & 0_{1\times 2} & 0 \end{pmatrix}, \tag{29}$$

where

$$X = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \tag{30}$$

and consequently

$$X\mathcal{A}X^T = \begin{pmatrix} 3\lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}, \quad X\mathcal{B}_I X^T = \begin{pmatrix} 2\lambda_3 & 0 \\ 0 & I\lambda_4 \end{pmatrix}, \quad X\mathcal{C}X = \begin{pmatrix} \sqrt{2}\lambda_3 \\ 0 \end{pmatrix}. \tag{31}$$

Thus, the matrix S is equivalent to the direct sum of the two matrices $\mathcal{S}_+$ and $\mathcal{S}_-$, where

$$\mathcal{S}_+ = \begin{pmatrix} 3\lambda_1 & 2\lambda_3 & 2\lambda_3 & \cdots & 2\lambda_3 & \sqrt{2}\lambda_3 \\ 2\lambda_3 & 0 & 0 & \cdots & 0 & 0 \\ 2\lambda_3 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2\lambda_3 & 0 & 0 & \cdots & 0 & 0 \\ \sqrt{2}\lambda_3 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{32a}$$

$$\mathcal{S}_- = \begin{pmatrix} \lambda_1 & J\lambda_4 & (J-1)\lambda_4 & \cdots & \lambda_4 \\ J\lambda_4 & 0 & 0 & \cdots & 0 \\ (J-1)\lambda_4 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_4 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{32b}$$

Computing the eigenvalues of these two matrices and setting their moduli to be smaller than M, we find

$$3\left|\lambda_1\right| + \sqrt{9\lambda_1^2 + 4\left[\sum_{I=1}^{J}\left(4\lambda_3^2\right) + 2\lambda_3^2\right]} < 2M, \tag{33a}$$

$$\left|\lambda_1\right| + \sqrt{\lambda_1^2 + 4\lambda_4^2 \sum_{I=1}^{J} I^2} < 2M. \tag{33b}$$

Performing the sums over I, we obtain

$$3\left|\lambda_1\right| + \sqrt{9\lambda_1^2 + 8\left(2J+1\right)\lambda_3^2} < 2M, \tag{34a}$$

$$\left|\lambda_1\right| + \sqrt{\lambda_1^2 + \frac{2}{3}J\left(J+1\right)\left(2J+1\right)\lambda_4^2} < 2M. \tag{34b}$$

The potential V_4 produces many other scatterings of two-particle states, but they all lead to UNI conditions that either repeat Eq. (25b), or repeat Eq. (34b), or repeat Eq. (34a), or are weaker than one of them.

2.2. *BFB conditions*

In order to evaluate the BFB conditions on V_4, it is handy to use the gauge wherein $b = 0$. We use

$$\left| \sum_{I=-J}^{J} I \left| \chi_I \right|^2 \right| = \left| J \left| \chi_J \right|^2 + (J-1) \left| \chi_{J-1} \right|^2 + \cdots - J \left| \chi_{-J} \right|^2 \right| \tag{35a}$$

$$\leq J F_2 \tag{35b}$$

to write, in that gauge,

$$V_4 \geq \frac{\lambda_1}{2} \left| a \right|^4 + \left(\lambda_3 - J \left| \lambda_4 \right| \big/ 2 \right) \left| a \right|^2 F_2 + \text{terms four-linear in the } \chi_I. \tag{36}$$

Since both $|a|^2 \geq 0$ and $F_2 \geq 0$, the conditions for $(\lambda_1/2) \left| a \right|^4 + \left(\lambda_3 - J \left| \lambda_4 \right| \big/ 2 \right) \left| a \right|^2 F_2$ to be nonnegative, whatever the (nonnegative) values of $|a|^2$ and F_2, are [10]:

$$\lambda_1 \geq 0, \tag{37a}$$

$$\lambda_3 \geq 0, \tag{37b}$$

$$\left| \lambda_4 \right| \leq \frac{2\lambda_3}{J}. \tag{37c}$$

Conditions (37) are *necessary and sufficient* for V_4 to be nonnegative for arbitrary values of the fields a and χ_I. Since a gauge with $b = 0$ can always be obtained, those conditions also hold for V_4 with arbitrary values of a, b, and χ_I.

2.3. *Results*

Transforming all the six relevant UNI and BFB conditions into strict equalities, we have

$$\lambda_3 + \frac{J+1}{2} \left| \lambda_4 \right| = M, \tag{38a}$$

$$\lambda_1 + \sqrt{\lambda_1^2 + \frac{2J \left(2J^2 + 3J + 1 \right) \lambda_4^2}{3}} = 2M, \tag{38b}$$

$$3\lambda_1 + \sqrt{9\lambda_1^2 + 8 \left(2J + 1 \right) \lambda_3^2} = 2M, \tag{38c}$$

$$\left| \lambda_4 \right| = \frac{2}{J} \lambda_3, \tag{38d}$$

where we have taken into account that both λ_1 and λ_3 are nonnegative, cf. Eqs. (17) and (37b), respectively. Equation (38b) gives solution I:

$$\left| \lambda_4 \right| = \sqrt{\frac{6M \left(M - \lambda_1 \right)}{J \left(2J^2 + 3J + 1 \right)}}. \tag{39}$$

Equations (38a) and (38c) together produce solution II:

$$\left| \lambda_4 \right| = \frac{2M}{J+1} - \frac{\sqrt{2M \left(M - 3\lambda_1 \right)}}{\left(J+1 \right) \sqrt{2J+1}}, \tag{40a}$$

$$\lambda_3 = \sqrt{\frac{M \left(M - 3\lambda_1 \right)}{2 \left(2J+1 \right)}}. \tag{40b}$$

Equations (38a) and (38d) together lead to solution III:

$$\left| \lambda_4 \right| = \frac{2M}{2J+1}, \tag{41a}$$

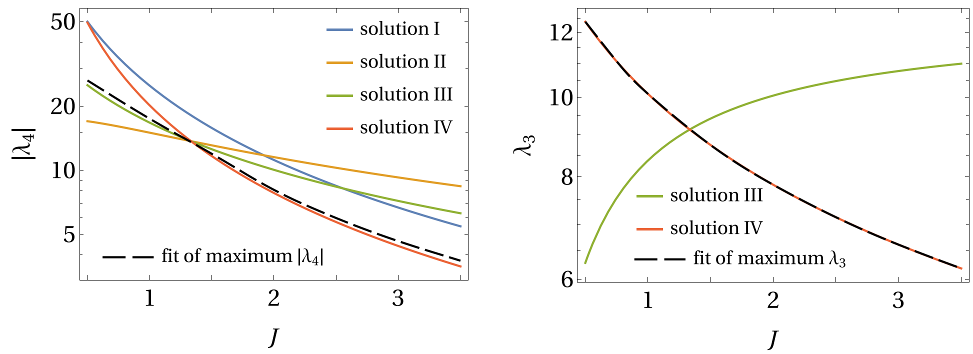

Fig. 1. The solutions (39)–(42) for $|\lambda_4|$ (left) and λ_3 (right) versus J. We have used $M = 8\pi$. The black dashed lines display the maximum allowed values of $|\lambda_4|$ and λ_3 described in Sect. 3.3.

$$\lambda_3 = \frac{J}{2J+1}\, M. \tag{41b}$$

Equations (38c) and (38d) together give solution IV:

$$|\lambda_4| = \frac{\sqrt{2M\,(M - 3\lambda_1)}}{J\sqrt{2J+1}} \tag{42}$$

and Eq. (40b), which is the solution of Eq. (38c).

Solutions I, II, III, and IV for $|\lambda_4|$ and λ_3 are plotted in Fig. 1. Notice that solutions II, III, and IV coincide when

$$2J + 1 = \frac{2MJ^2}{M - 3\lambda_1}, \tag{43}$$

i.e. when $J \approx 1.37$. When J is larger than this value, i.e. when χ is either a quadruplet or a larger multiplet of $SU(2)$, then solution IV—which arises from both the UNI condition (34a) and the BFB condition (37c)—gives the strongest upper bound on both λ_3 and $|\lambda_4|$. Thus, both λ_3 and $|\lambda_4|$ are bounded from above by the interplay of a UNI condition and a BFB condition, for most possible values of J.

We remind the reader that, according to Eq. (37b), the minimum value of λ_3 is 0.

2.4. *Renormalization-group equations*

In a renormalizable field theory, the values of the dimensionless coupling constants evolve with the energy scale μ at which they are measured. That evolution is governed by differential equations named renormalization-group equations (RGEs):[3]

$$16\pi^2 \mu\, \frac{dg}{d\mu} = \beta_g, \tag{44}$$

where g denotes a generic dimensionless coupling and β_g is a function of, in general, all the dimensionless couplings in the theory. Formulas for the functions β_g in a general gauge theory have been presented long ago by Cheng, Eichten, and Li [11]. In the case at hand there is a gauge theory with gauge group $SU(3) \times SU(2) \times U(1)$ and gauge coupling constants g_3 for $SU(3)$, g_2 for $SU(2)$, and g_1 for $U(1)$;[4] in that gauge theory there is an $SU(2)$ doublet with

[3]Here we only consider the one-loop-level RGEs.

[4]We use here the normalization for g_1 usual in the $SU(5)$, $SO(10)$, and E_6 Grand Unified Theories. Still, we keep for the hypercharges of the multiplets the usual normalization given by $Q = I + Y$, where Q is a field's electric charge, I is the third component of isospin, and Y is the hypercharge.

hypercharge $1/2$ and an $SU(2)$ multiplet with isospin J and hypercharge Y. We consider in this subsection the SP

$$V = \sum_{k=1}^{2} \left(\mu_k^2 F_k + \frac{\lambda_k}{2} F_k^2 \right) + \lambda_3 F_1 F_2 + \lambda_4 F_4, \tag{45}$$

i.e. we take into account the presence in V of a quartic term proportional to F_2^2, but we discard all the other terms four-linear in the χ_I (they are studied in some detail in Sect. 3).[5] The dimensionless couplings that we take into account are, thus, g_1, g_2, g_3, λ_1, λ_2, λ_3, and λ_4. Additionally, in the full theory there are fermions with Yukawa couplings to the scalar doublet, and one (and only one) of those Yukawa couplings, viz. y_t—the Yukawa coupling of H to the top quark—is rather large and therefore has a strong influence on the RGEs; so, there is one further dimensionless coupling y_t that we take into account. In order to derive the RGEs for these eight coupling constants we have used a feature of the software SARAH [12]. (We point out that that software only tolerates $SU(2)$ multiplets with isospin up to 3, so we had to edit it and make a modification in order to derive the RGEs for the case $J = 7/2$. We moreover point out that the running time for that software increases *exponentially* with the size of the $SU(2)$ multiplets.) We have obtained

$$\beta_{g_1} = \left(\frac{41}{10} + \frac{4}{5} Y^2 \right) g_1^3, \tag{46a}$$

$$\beta_{g_2} = \left[-\frac{19}{6} + \frac{J(J+1)(2J+1)}{9} \right] g_2^3, \tag{46b}$$

$$\beta_{g_3} = -7 g_3^3, \tag{46c}$$

$$\begin{aligned}\beta_{\lambda_1} = {} & \frac{27}{100} g_1^4 + \frac{9}{10} g_1^2 g_2^2 + \frac{9}{4} g_2^4 + 12 y_t^2 \lambda_1 + 12 \lambda_1^2 + 2(2J+1)\lambda_3^2 + \frac{J(J+1)(2J+1)}{6} \lambda_4^2 \\ & - \left(\frac{9}{5} g_1^2 + 9 g_2^2 \right) \lambda_1 - 12 y_t^4,\end{aligned} \tag{47a}$$

$$\begin{aligned}\beta_{\lambda_2} = {} & \frac{108}{25} (Y g_1)^4 + \frac{72}{5} (Y g_1)^2 (J g_2)^2 + 6 J^2 \left(2J^2 + 1\right) g_2^4 + 2(2J+5)\lambda_2^2 + 4\lambda_3^2 + J^2 \lambda_4^2 \\ & - \left[\frac{36}{5} (Y g_1)^2 + 12 J (J+1) g_2^2 \right] \lambda_2,\end{aligned} \tag{47b}$$

$$\begin{aligned}\beta_{\lambda_3} = {} & \frac{27}{25} \left(Y g_1^2\right)^2 + 3J(J+1) g_2^4 + 6 y_t^2 \lambda_3 + 6 \lambda_1 \lambda_3 + 4(J+1)\lambda_2 \lambda_3 + 4 \lambda_3^2 + J(J+1)\lambda_4^2 \\ & - \left(\frac{9}{10} + \frac{18}{5} Y^2 \right) g_1^2 \lambda_3 - \left[\frac{9}{2} + 6J(J+1) \right] g_2^2 \lambda_3,\end{aligned} \tag{47c}$$

$$\begin{aligned}\beta_{\lambda_4} = {} & \frac{36}{5} Y g_1^2 g_2^2 + 6 y_t^2 \lambda_4 + 2\lambda_1 \lambda_4 + 8 \lambda_3 \lambda_4 \\ & - \left(\frac{9}{10} + \frac{18}{5} Y^2 \right) g_1^2 \lambda_4 - \left[\frac{9}{2} + 6J(J+1) \right] g_2^2 \lambda_4,\end{aligned} \tag{47d}$$

[5] In the numerical part of this section, though, we shall fix $\lambda_2 = 0$ for the sake of simplicity.

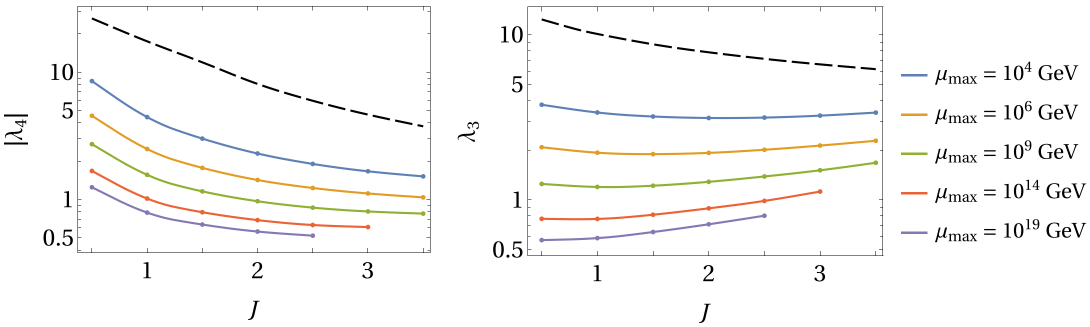

Fig. 2. The maximum allowed values of $|\lambda_4|$ (left) and λ_3 (right) versus J for different cutoff scales $\mu_{\max}$. The black dashed lines are the same as in Fig. 1; they display the maximum values at the electroweak scale.

$$\beta_{y_t} = \left(\frac{9}{2}\, y_t^2 - \frac{17}{20}\, g_1^2 - \frac{9}{4}\, g_2^2 - 8 g_3^2 \right) y_t. \tag{48}$$

We have used a numerical code to solve the differential Eqs. (46)–(48) starting at the scale $\mu = m_t = 173.1$ GeV and letting μ evolve up to the scale $\mu_{\text{Planck}} = 10^{19}$ GeV. We have slightly simplified those equations by setting $\lambda_2 = 0$ at all μ values—even though in general β_{λ_2} is nonzero and therefore a nonzero λ_2 will be generated even if one starts with $\lambda_2 = 0$—and by assuming the hypercharge Y of the additional multiplet to be zero too. At $\mu = m_t$ we have fixed [13]:

$$g_1 = \sqrt{\frac{5}{3}} \times 0.358545, \quad g_2 = 0.64765, \quad g_3 = 1.1618, \tag{49a}$$

$$\lambda_1 = 0.258, \quad y_t = \frac{\sqrt{2} \times 161.98\,\text{GeV}}{246\,\text{GeV}}, \tag{49b}$$

and we have let λ_3 and λ_4 vary freely while obeying the UNI and BFB conditions. Moreover, we have enforced the UNI and BFB conditions at every intermediate scale μ; this indirectly constrains the initial λ_3 and λ_4 because, if they are too large, then at some intermediate $\mu < \mu_{\text{GUT}}$ either the UNI or the BFB conditions will be broken. In Fig. 2 one observes the result of this labor in the form of upper bounds on λ_3 and $|\lambda_4|$ at the scale $\mu = m_t$, depending on the scale μ_{maximum} at which either the UNI or the BFB conditions start being broken. As expected, if one demands that the UNI and BFB conditions are respected for a longer μ range, then one obtains ever stricter upper bounds on the initial values of λ_3 and λ_4. The same upper bounds may be also observed, now in a correlated fashion, for three values of $\mu_{\max}$, in Fig. 3.

The `SARAH` model files and output files, and the expressions of the RGEs for *all* quartic couplings (including the ones mentioned in Sect. 3), both in pdf and `Mathematica` notebook files, are available at https://github.com/jurciukonis/RGEs-for-multiplets.

3. Full potential

The product $\chi \otimes \chi$ of two *identical* multiplets of $SU(2)$ only has a *symmetric* component—the antisymmetric component vanishes because the two multiplets are equal—which consists of:[6]

$$t = \frac{1}{2}\,\text{ceil}\,(n, 2) \tag{50}$$

[6]The ceiling function ceil $(n, 2)$ maps n into the smallest multiple of 2 larger than or equal to n.

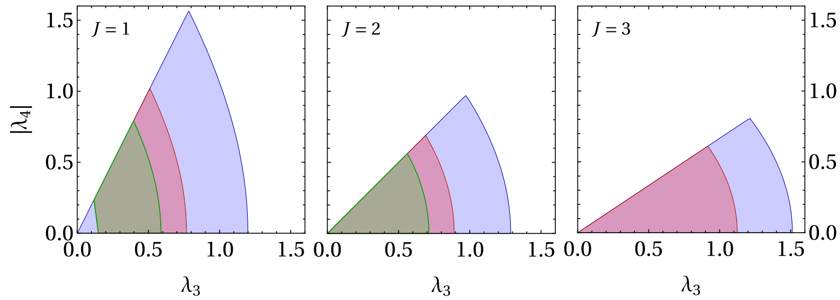

Fig. 3. Regions of stability for $J = 1$ (left), $J = 2$ (middle), and $J = 3$ (right) and for energy ranges from $\mu = m_t$ to 10^9 GeV (blue regions), 10^{14} GeV (red regions), and 10^{19} GeV (green regions).

multiplets of $SU(2)$

$$(\chi \otimes \chi)_{\text{symmetric}} = c \oplus d \oplus e \oplus \cdots \oplus q, \tag{51}$$

where

- c is an $SU(2)$ multiplet with weak isospin $2J$,
- d is an $SU(2)$ multiplet with weak isospin $2J - 2$,
- e is an $SU(2)$ multiplet with weak isospin $2J - 4$,

and so on; lastly, q is either a triplet of $SU(2)$ if J is half-integer, or $SU(2)$-invariant if J is integer. Thus,

$$(\chi \otimes \chi)_{\text{symmetric}} = \begin{pmatrix} c_{2J} \\ c_{2J-1} \\ \vdots \\ c_{-2J} \end{pmatrix} \oplus \begin{pmatrix} d_{2J-2} \\ d_{2J-3} \\ \vdots \\ d_{2-2J} \end{pmatrix} \oplus \begin{pmatrix} e_{2J-4} \\ e_{2J-5} \\ \vdots \\ e_{4-2J} \end{pmatrix} \oplus \cdots , \tag{52}$$

where the sub-indices give the third component of isospin. The two-field states in each multiplet in the right-hand side of Eq. (52) are evaluated by using Clebsch–Gordan coefficients in the standard fashion. Thus,

$$c_I = \sum_{I'=-J}^{J} \sum_{I''=-J}^{J} \delta_{I,I'+I''} \begin{bmatrix} J & J & 2J \\ I' & I'' & I \end{bmatrix} \chi_{I'} \chi_{I''}, \tag{53a}$$

$$d_I = \sum_{I'=-J}^{J} \sum_{I''=-J}^{J} \delta_{I,I'+I''} \begin{bmatrix} J & J & 2J-2 \\ I' & I'' & I \end{bmatrix} \chi_{I'} \chi_{I''}, \tag{53b}$$

$$e_I = \sum_{I'=-J}^{J} \sum_{I''=-J}^{J} \delta_{I,I'+I''} \begin{bmatrix} J & J & 2J-4 \\ I' & I'' & I \end{bmatrix} \chi_{I'} \chi_{I''}, \tag{53c}$$

and so on.

The "terms four-linear in the χ_I" in Eq. (11) are

- a term $\dfrac{\lambda_2}{2} F_2^2$, where F_2 has been defined in Eq. (5);
- a term $\lambda_5 F_5$, where

$$F_5 \equiv \sum_{I=2-2J}^{2J-2} |d_I|^2 ; \tag{54}$$

- a term $\lambda_6 F_6$, where

$$F_6 \equiv \sum_{I=4-2J}^{2J-4} |e_I|^2 \,; \tag{55}$$

- and other analogous terms, up to $\lambda_{t+3} F_{t+3}$, where

$$F_{t+3} \equiv \begin{cases} |q_1|^2 + |q_0|^2 + |q_{-1}|^2 & \Leftarrow \text{half-integer } J, \\ |q_0|^2 & \Leftarrow \text{integer } J. \end{cases} \tag{56}$$

The quartic part of the SP thus is

$$V_4 = \frac{\lambda_1}{2} F_1^2 + \frac{\lambda_2}{2} F_2^2 + \lambda_3 F_1 F_2 + \lambda_4 F_4 + \sum_{i=5}^{t+3} \lambda_i F_i. \tag{57}$$

A term with the invariant

$$F_3 \equiv \sum_{I=-2J}^{2J} |c_I|^2 \tag{58}$$

has not been included in V_4 because F_3 linearly depends on the other invariants. Indeed,

$$F_3 + \sum_{i=5}^{t+3} F_i = F_2^2. \tag{59}$$

3.1. *BFB conditions*

Let us consider again $(\chi \otimes \tilde{\chi})_{\mathbf{3}}$ given in Eq. (6). The $SU(2)$-invariant quantity

$$\left|(\chi \otimes \tilde{\chi})_{\mathbf{3}}\right|^2 \equiv \left(\sum_{I=-J}^{J} I\, |\chi_I|^2 \right)^2 + \left| \sum_{I=1-J}^{J} \chi_I^* \chi_{I-1} \sqrt{J^2 - I^2 + J + I} \right|^2 \tag{60}$$

is four-linear in the χ_I and therefore it must be linearly dependent on F_2^2 and F_i. Indeed, one finds that

$$\left|(\chi \otimes \tilde{\chi})_{\mathbf{3}}\right|^2 = J^2 F_2^2 - \sum_{i=5}^{t+3} \kappa_i F_i, \tag{61}$$

where the numbers κ_i are given by

$$\kappa_i = (i-4)\,(4J + 9 - 2i)\,. \tag{62}$$

Notice that all the κ_i are *positive*. We have explicitly checked, up to $J = 10$, that Eqs. (61) and (62) are correct.

From Eq. (61),

$$\sum_{i=5}^{t+3} \kappa_i F_i - J^2 F_2^2 = - \left(\sum_{I=-J}^{J} I\, |\chi_I|^2 \right)^2 - \left| \frac{z}{ab} \right|^2, \tag{63}$$

where z has been defined in Eq. (13); hence, from the definition of F_4 in Eq. (12),

$$\sum_{i=5}^{t+3} \kappa_i F_i - J^2 F_2^2 = - \left(\frac{2F_4 - z - z^*}{|a|^2 - |b|^2} \right)^2 - \left| \frac{z}{ab} \right|^2. \tag{64}$$

Therefore,

$$\sum_{i=5}^{t+3} \kappa_i F_i - J^2 F_2^2 + \frac{4F_4^2}{F_1^2} = \frac{1}{4\,|ab|^2 \left(|a|^4 - |b|^4\right)^2} \left\{ \left(|a|^4 - |b|^4\right)^2 (z - z^*)^2 - \left[8\,|ab|^2\, F_4 - \left(|a|^2 + |b|^2\right)^2 (z + z^*)\right]^2 \right\}. \tag{65}$$

Thus,

$$\sum_{i=5}^{t+3} \kappa_i F_i - J^2 F_2^2 + \frac{4F_4^2}{F_1^2} \le 0. \tag{66}$$

We now define the dimensionless quantities [14]:[7]

$$r \equiv \frac{F_1}{F_2}, \tag{67a}$$

$$\gamma_i \equiv \frac{F_i}{F_2^2} \quad (i = 5, \dots, t+3), \tag{67b}$$

$$\delta \equiv \frac{2F_4}{J F_1 F_2}. \tag{67c}$$

We then have, from Eq. (57),

$$\frac{V_4}{F_2^2} = \frac{\lambda_1}{2} r^2 + \frac{\lambda_2}{2} + \lambda_3 r + \frac{J}{2} \lambda_4 r \delta + \sum_{i=5}^{t+3} \lambda_i \gamma_i \tag{68a}$$

$$= \frac{1}{2} \left(r,\ 1\right) \begin{pmatrix} \lambda_1 & \lambda_3 + \frac{J}{2} \lambda_4 \delta \\ \lambda_3 + \frac{J}{2} \lambda_4 \delta & \lambda_2 + 2 \sum_{i=5}^{t+3} \lambda_i \gamma_i \end{pmatrix} \begin{pmatrix} r \\ 1 \end{pmatrix}. \tag{68b}$$

It follows from the definitions of F_2 and F_1 in Eqs. (5) and (10), respectively, that $r \ge 0$. Therefore, the conditions for $V_4 \big/ F_2^2$ in Eq. (68b) to be nonnegative are [10]:

$$\lambda_1 \ge 0, \tag{69a}$$

$$\lambda_2 + 2 \sum_{i=5}^{t+3} \lambda_i \gamma_i \ge 0, \tag{69b}$$

$$\lambda_3 + \frac{J}{2} \lambda_4 \delta \ge -\sqrt{\lambda_1 \left(\lambda_2 + 2 \sum_{i=5}^{t+3} \lambda_i \gamma_i\right)}. \tag{69c}$$

Conditions (69b) and (69c) must hold for all possible values of δ and of γ_i. It follows from the definitions of the F_i, cf. Eqs. (54) and (55), that the $\gamma_i \ge 0$. From Eq. (66),

$$\sum_{i=5}^{t+3} \kappa_i \gamma_i - J^2 \left(1 - \delta^2\right) \le 0. \tag{70}$$

Thus, since all the κ_i and the γ_i are positive,

$$0 \le \sum_{i=5}^{t+3} \kappa_i \gamma_i \le J^2 \left(1 - \delta^2\right), \tag{71}$$

[7]The Klein–Gordon fields a, b, and χ_I have mass dimension, hence $[F_1] = [F_2] = M^2$ and $[F_4] = [F_i] = M^4$, for $i = 5, \dots, t+3$.

and therefore $-1 \leq \delta \leq +1$.[8] It is advantageous to define

$$x = \frac{1+\delta}{2} \in [0, 1]\,. \tag{72}$$

Then Eq. (71) reads

$$0 \leq \sum_{i=5}^{t+3} \kappa_i \gamma_i \leq 4J^2 x\,(1-x)\,. \tag{73}$$

This condition determines the domain of the γ_i, which has corners at the points [14]:

$$\gamma_5 = \cdots = \gamma_{t+3} = 0, \tag{74a}$$

$$\gamma_5 = \cdots = \gamma_{t+2} = 0, \quad \gamma_{t+3} = \frac{J^2}{\kappa_{t+3}}, \tag{74b}$$

$$\gamma_5 = \cdots = \gamma_{t+1} = \gamma_{t+3} = 0, \qquad \gamma_{t+2} = \frac{J^2}{\kappa_{t+2}}, \tag{74c}$$

$$\vdots \quad \vdots$$

$$\gamma_6 = \cdots = \gamma_{t+3} = 0, \qquad \gamma_5 = \frac{J^2}{\kappa_5}. \tag{74d}$$

Since $\lambda_2 + 2\sum_{i=5}^{t+3} \lambda_i \gamma_i$ is a linear function of the γ_i, condition (69b) just has to hold at the corners of the domain of the γ_i in order to hold in the whole domain. We thus obtain *necessary* BFB conditions:

$$\lambda_2 \geq 0, \tag{75a}$$

$$\widehat{\lambda}_i \geq 0, \tag{75b}$$

where

$$\widehat{\lambda}_i \equiv \lambda_2 + q_i, \tag{76a}$$

$$q_i \equiv \frac{2J^2}{\kappa_i}\,\lambda_i. \tag{76b}$$

Furthermore, condition (69c) must certainly hold at the point (74a) and for both $\delta = 0$ and $\delta = \pm 1$. Therefore,

$$\lambda_3 \geq -\sqrt{\lambda_1 \lambda_2}, \tag{77a}$$

$$|\lambda_4| \leq \frac{2}{J}\left(\lambda_3 + \sqrt{\lambda_1 \lambda_2}\right). \tag{77b}$$

The *necessary* BFB conditions (77a) and (77b) generalize conditions (37b) and (37c), respectively, when λ_2 is nonzero.

Condition (69c) must also hold at all the other corners (74) of the γ_i domain. Therefore,

$$\lambda_3 - \frac{J}{2}\,\lambda_4 + J\lambda_4 x \geq -\sqrt{\lambda_1\left[\lambda_2 + 4 q_i x\,(1-x)\right]} \tag{78}$$

[8] We shall implicitly assume that the conditions $|\delta| \leq 1$ and (71) completely determine the parameter space, i.e. that no further conditions restrict the parameters δ and γ_i. Equivalently, we assume that, for any parameters $F_2 \geq 0$, $r \geq 0$, $\delta \in [-1, +1]$, and γ_i obeying condition (71), it is possible to find fields a, b, and χ_I satisfying Eqs. (5), (10), (12)–(13), (54)–(56), and (67). We thank Renato Fonseca for calling our attention to this implicit assumption of our work.

must hold for all $i = 5, \ldots, t+3$ and for all $x \equiv [0, 1]$. Thus, the functions

$$\mathrm{f}_i(x) \equiv \lambda_3 - \frac{J}{2}\lambda_4 + J\lambda_4 x + \sqrt{\lambda_1\left[\lambda_2 + 4q_i\left(x - x^2\right)\right]} \quad (i = 5, \ldots, t+3) \tag{79}$$

must be nonnegative $\forall x \in [0, 1]$. Clearly [14],

$$\frac{d\mathrm{f}_i}{dx} = J\lambda_4 + \frac{2\sqrt{\lambda_1}\, q_i(1-2x)}{\sqrt{\lambda_2 + 4q_i\left(x - x^2\right)}}, \tag{80a}$$

$$\frac{d^2\mathrm{f}_i}{dx^2} = \frac{-4q_i\widehat{\lambda}_i\sqrt{\lambda_1}}{\left[\sqrt{\lambda_2 + 4q_i\left(x - x^2\right)}\right]^3}. \tag{80b}$$

Because of Eq. (75b), the second derivative of f_i has the sign opposite to that of q_i, i.e. opposite to that of λ_i. Since we have already ascertained—through condition (77b)—that both $\mathrm{f}_i(0) \geq 0$ and $\mathrm{f}_i(1) \geq 0$, the condition $\mathrm{f}_i(x) \geq 0,\ \forall x \in [0, 1]$ is equivalent to the following:

- either $d^2\mathrm{f}_i / dx^2 < 0$,
- or there is no real number x_0 such that $\mathrm{f}'(x_0) = 0$,
- or such a x_0 exists, but it is outside the interval $[0, 1]$,
- or $\mathrm{f}(x_0) \geq 0$.

This is equivalent to

$$\text{either} \quad \lambda_i > 0, \tag{81a}$$

$$\text{or} \quad \lambda_i \Lambda_i < 0, \tag{81b}$$

$$\text{or} \quad \sqrt{\frac{\widehat{\lambda}_i}{q_i\,\Lambda_i}} > \frac{2}{J\,|\lambda_4|}, \tag{81c}$$

$$\text{or} \quad \lambda_3 \geq -\sqrt{\frac{\widehat{\lambda}_i\,\Lambda_i}{q_i}}, \tag{81d}$$

respectively, where

$$\Lambda_i \equiv \frac{J^2}{4}\lambda_4^2 + q_i\lambda_1. \tag{82}$$

Conditions (69a), (75), (77), and (81) ($i = 5, \ldots, t+3$) are *necessary and sufficient* for the boundedness-from-below of V_4 [14].

3.2. *UNI conditions*

The condition (25b) stays unchanged when there are in V_4 terms four-linear in the χ_I.

The eigenvalues of the scattering matrix of the two-field states with null T_3 and hypercharge $2Y$, viz. the states $\chi_J\chi_{-J},\ \chi_{J-1}\chi_{1-J},\ \chi_{J-2}\chi_{2-J},\ \ldots$, produce the conditions

$$|\lambda_2| < M, \tag{83a}$$

$$|\lambda_2 + 2\lambda_i| < M \quad (i = 5, \ldots, t+3). \tag{83b}$$

The scattering matrix for the two-field states with null hypercharge and null third component of isospin—i.e. for the states $|a|^2$, $|b|^2$, and the n states $|\chi_I|^2$—generalizes the matrix of Eq. (27):

$$\mathcal{S} = \begin{pmatrix} 2\lambda_1 & \lambda_1 & \Sigma_1 \\ \lambda_1 & 2\lambda_1 & \Sigma_2 \\ \Sigma_1^T & \Sigma_2^T & \Lambda \end{pmatrix} \tag{84}$$

where the $1 \times n$ submatrices Σ_1 and Σ_2 are given by

$$(\Sigma_1)_{1k} = \lambda_3 + \frac{\lambda_4}{2} (J + 1 - k) , \tag{85a}$$

$$(\Sigma_2)_{1k} = \lambda_3 - \frac{\lambda_4}{2} (J + 1 - k) , \tag{85b}$$

for $k = 1, \ldots, n$. The $n \times n$ matrix Λ is given by

$$\Lambda_{kl} = \lambda_2 (1 + \delta_{kl}) \tag{86a}$$

$$+4 \sum_{i=5}^{t+3} \lambda_i \sum_{m=1}^{4J+17-4i} \delta_{m,\, k+l+7-2i} \times \left(\begin{bmatrix} J & J & 2J+8-2i \\ J+1-k & J+1-l & 2J+9-2i-m \end{bmatrix} \right)^2 , \tag{86b}$$

for $k, l \in [1, n]$. For all the values of J that we have investigated (i.e. for all integer and half-integer J up to and including 5), the matrix $\mathcal{S}$ of Eq. (84) is equivalent to the direct sum of

- $2J - 1$ 1×1 matrices, i.e. numbers that are linear combinations of λ_2 and the λ_i ($i = 5, \ldots, t + 3$), the coefficient of λ_2 in those linear combinations being 1;
- one 2×2 symmetric matrix with
  - (1, 1) matrix element λ_1,
  - (2, 2) matrix element which is a linear combination of λ_2 and the λ_i, the coefficient of λ_2 in that linear combination being 1,
  - (1, 2) matrix element proportional to λ_4;
- another 2×2 symmetric matrix with
  - (1, 1) matrix element $3\lambda_1$,
  - (2, 2) matrix element which is a linear combination of λ_2 and the λ_i, the coefficient of λ_2 in that linear combination being $2J + 2$,
  - (1, 2) matrix element proportional to λ_3.

The moduli of all the eigenvalues of these $2J + 1$ matrices should be smaller than M. Since the matrices are either 1×1 or 2×2, there are simple analytic expressions for their $2J + 3$ eigenvalues. In particular, from the last two matrices mentioned above one obtains the unitarity conditions

$$|\lambda_1 + A_1| + \sqrt{(\lambda_1 - A_1)^2 + \frac{2}{3} J (J + 1) (2J + 1) \lambda_4^2} < 2M, \tag{87a}$$

$$|3\lambda_1 + A_2| + \sqrt{(3\lambda_1 - A_2)^2 + 8 (2J + 1) \lambda_3^2} < 2M, \tag{87b}$$

where

$$A_1 \equiv \lambda_2 + 4 \sum_{i=5}^{t+3} \frac{J^2 + 16J + 36 + i (2i - 4J - 17)}{J (J + 1)} \frac{4J - 4i + 17}{2J + 1} \lambda_i, \tag{88a}$$

Table 1. The maximum allowed value of $|\lambda_4|$, and the maximum and minimum allowed values of λ_3, for various values of J.

J	1/2	1	3/2	2	5/2	3	7/2
maximum $\|\lambda_4\|$	26.46	17.49	11.96	8.10	5.97	4.65	3.76
maximum λ_3	12.37	10.10	8.75	7.82	7.14	6.61	6.19
minimum λ_3	−1.46	−1.26	−1.13	− 1.03	−0.95	− 0.89	−0.84

$$A_2 \equiv 2\,(J+1)\,\lambda_2 + 4 \sum_{i=5}^{t+3} \frac{4J-4i+17}{2J+1}\,\lambda_i. \tag{88b}$$

We have explicitly checked that Eqs. (87) are correct up to $J = 11/2$.

The unitarity conditions that one obtains from the other $2J$ matrices are explicitly given in the Appendix for all J through 7/2. We point out that our unitarity conditions for the case $J = 3/2$ do not perfectly coincide with the ones given in Ref. [6].

So, the full UNI conditions are:[9]

$$|\lambda_1| < M, \tag{89}$$

condition (25b), conditions (83), conditions (87), and the conditions in the Appendix.

3.3. *Results*

We have generated random sets of values for all the coefficients of V_4 except λ_1, viz. for λ_2, λ_3, λ_4, and the λ_i ($i = 5, \ldots, t+3$). The coefficient λ_1 was kept fixed at the value in Eq. (17). We have then imposed on the generated sets both the BFB and the UNI conditions, thereby discarding most of them. We have made scatter plots of the sets of values that respected both the BFB and the UNI conditions. By carefully scrutinizing those plots, we have arrived at the maximum and minimum allowed values of λ_3, and at the maximum allowed value of $|\lambda_4|$,[10] which are displayed in Table 1. These values were also checked through a fitting procedure, by using both the UNI and BFB conditions.

It turns out that the maximum value of $|\lambda_4|$, when λ_2 and the λ_i are allowed to be nonzero, is slightly larger than Eq. (41a) when J is 1/2 or 1, and slightly larger than Eq. (42) for all larger values of J. This is illustrated in the left panel of Fig. 1.

In Fig. 4 we depict the maximum possible mass of a multiplet of scalars as a function of its minimum mass m. This is simply given by the expression

$$m_{\max} = \sqrt{m^2 + J v^2\,|\lambda_4|_{\text{maximal}}}, \tag{90}$$

where J is the isospin of the multiplet,

$$v^2 = \frac{1}{2\sqrt{2}\,G_F} \approx (174\,\text{GeV})^2\,, \tag{91}$$

and $|\lambda_4|_{\text{maximal}}$ is the maximum allowed value of $|\lambda_4|$ for each J. One sees that heavy scalar multiplets tend to be almost degenerate; for $m \gtrsim 2$ TeV, $m_{\max} - m \sim 100$ GeV. Notice that $m_{\max} - m$ is maximal for $J = 3/2$, i.e. when χ is a quadruplet; if χ is a larger multiplet, then it has more components but, as $|\lambda_4|_{\text{maximal}}$ is smaller, those components are packed into an ever smaller mass range.

[9] Condition (89) is unimportant in practice, because we already know that $\lambda_1 = 0.258$ is quite small.

[10] The coefficient λ_4 can always be zero, i.e. the minimum allowed value of $|\lambda_4|$ is zero.

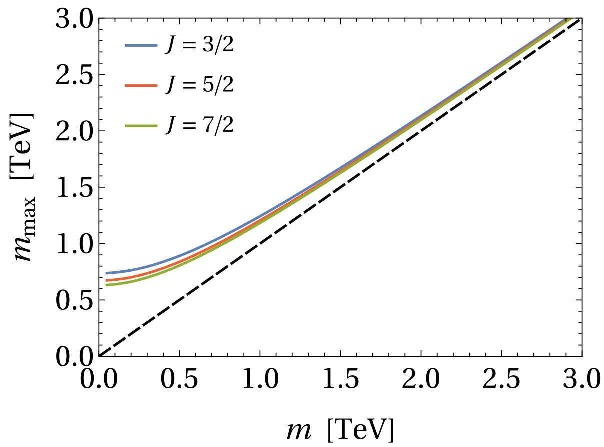

Fig. 4. The maximal mass $m_{\max}$ of a scalar multiplet χ versus its lowest mass m, for three different values of the isospin J of χ.

The maximum allowed value of λ_3 is always attained when λ_2 and all the λ_i vanish, and exactly coincides with Eq. (40b), as is illustrated in the right panel of Fig. 1.[11]

The minimum allowed value of λ_3 is always attained when both λ_4 and all the λ_i are zero, but λ_2 is nonzero. Indeed, the minimum value of λ_3 is determined by the BFB condition (77b)—with λ_4 taken to zero—together with the UNI condition

$$|3\lambda_1 + 2\,(1+J)\,\lambda_2| + \sqrt{\left[3\lambda_1 - 2\,(1+J)\,\lambda_2\right]^2 + 8\,(1+2J)\,\lambda_3^2} \leq 2M, \tag{92}$$

which holds when all the λ_i are taken to zero. Thus, when conditions (77b) and (92) are transformed into equations, they produce the solution

$$\lambda_2 = \frac{1}{2}\,\frac{M\,(M-3\lambda_1)}{(1+J)\,M-(2+J)\,\lambda_1}, \tag{93a}$$

$$\lambda_3 = -\sqrt{\frac{\lambda_1}{2}\,\frac{M\,(M-3\lambda_1)}{(1+J)\,M-(2+J)\,\lambda_1}}. \tag{93b}$$

Equation (93b) gives the minimum value of λ_3 in the fourth row of Table 1.

4. OPs

In our NP model it is possible—depending on the values of J and Y—that the new scalars do not couple to the light fermions at all. If that is so and if, moreover, the new scalars are very heavy, so that they cannot be produced at the LHC—e.g. through the Drell–Yan process—then they will make themselves felt only indirectly through their oblique corrections, i.e. through their contributions to the self-energies of the gauge bosons. Following Maksymyk et al. [15], we parameterize those corrections through six OPs S, T, U, V, W, and X. We use as input of the renormalization process the quantities α (the fine-structure constant), G_F (the Fermi coupling constant), and m_Z (the Z^0 boson mass). Following Ref. [16], we use $\alpha\left(m_Z^2\right) = 1/127.951$, $G_F = 1.1663788 \times 10^{-5}$ GeV^{-2}, and $m_Z = 91.1876$ GeV. We then define the weak mixing angle θ_W through

$$c_W^2 s_W^2 = \frac{\pi\alpha}{\sqrt{2} G_F m_Z^2}, \tag{94}$$

[11] When the coefficient λ_3 attains its maximum allowed value displayed in the third row of Table 1, λ_4 may have various values, including zero.

where $c_W \equiv \cos\theta_W$ and $s_W \equiv \sin\theta_W$. This results in $s_W^2 = 0.23356$. We then use $S, \dots, X$ to parameterize, for each electroweak observable O, the ratio between the prediction of the NP model and the prediction of the SM, by using expressions of the general form

$$\frac{O_{\rm NP}}{O_{\rm SM}} = 1 + c_S^O S + c_T^O T + c_U^O U + c_V^O V + c_W^O W + c_X^O X, \tag{95}$$

where the coefficients $c_S^O, \dots, c_X^O$—given, e.g. in Ref. [17]—are known functions of the input quantities.

4.1. *Formulas for the OPs*

According to Ref. [18], when in the NP model there is *only one* $SU(2)$ multiplet of new scalars with weak isospin J and weak hypercharge Y, the parameter T produced by those scalars is given by

$$T = \frac{G_F}{8\sqrt{2}\pi^2\alpha} \sum_{I=1-J}^{J} \left(J^2 + J - I^2 + I\right) \theta_+ \left(m_I^2, m_{I-1}^2\right), \tag{96}$$

where m_I denotes the mass of the scalar with third component of isospin I. The function $\theta_+(x, y)$ is defined as

$$\theta_+(x, y) \equiv \begin{cases} x + y - \dfrac{2xy}{x-y} \ln\dfrac{x}{y} \Leftarrow x \neq y, \\ 0 \Leftarrow x = y. \end{cases} \tag{97}$$

The parameters V, W, and X are given by

$$V = \frac{G_F m_Z^2}{\sqrt{2}\pi^2\alpha} \sum_{I=-J}^{J} \left(I c_W^2 - Y s_W^2\right)^2 \rho\left(\frac{m_I^2}{m_Z^2}, \frac{m_I^2}{m_Z^2}\right), \tag{98a}$$

$$W = \frac{1}{4\pi s_W^2} \sum_{I=1-J}^{J} \left(J^2 + J - I^2 + I\right) \rho\left(\frac{m_I^2}{m_W^2}, \frac{m_{I-1}^2}{m_W^2}\right), \tag{98b}$$

$$X = -\frac{1}{2\pi} \sum_{I=-J}^{J} (I + Y)\left(I c_W^2 - Y s_W^2\right) \zeta\left(\frac{m_I^2}{m_Z^2}, \frac{m_I^2}{m_Z^2}\right), \tag{98c}$$

respectively. In Eqs. (98),

$$\begin{aligned} \zeta(x, y) = {} & \frac{11}{36} - \frac{5(x+y)}{12} + \frac{xy}{3(x-y)^2} + \frac{(x-y)^2}{6} \\ & + \left[\frac{x^2 - y^2}{4} + \frac{(y-x)^3}{12} + \frac{x^2 + y^2}{4(y-x)} + \frac{x+y}{12(x-y)} + \frac{xy(x+y)}{6(y-x)^3}\right] \ln\frac{x}{y} \\ & + \frac{\Delta(x, y)}{12} \, \mathrm{f}(x, y), \end{aligned} \tag{99a}$$

$$\begin{aligned} \rho(x, y) = {} & \frac{1}{6} - \frac{3(x+y)}{4} + \frac{(x-y)^2}{2} + \left[\frac{(y-x)^3}{4} + \frac{x^2 + y^2}{4(y-x)} + \frac{x^2 - y^2}{2}\right] \ln\frac{x}{y} \\ & + \frac{(x-y)^2 - x - y}{4} \, \mathrm{f}(x, y), \end{aligned} \tag{99b}$$

for $x \neq y$, while

$$\zeta(x, x) = \frac{4}{9} - \frac{4x}{3} + \frac{\Delta(x, x)}{12} \, \mathrm{f}(x, x), \tag{100a}$$

$$\rho\left(x,x\right) = \frac{1}{6} - 2x - \frac{x}{2}\,\mathrm{f}\left(x,x\right). \tag{100b}$$

In Eqs. (99) and (100),

$$\mathrm{f}\left(x,y\right) = \begin{cases} \sqrt{\Delta\left(x,y\right)}\,\ln\dfrac{x+y-1+\sqrt{\Delta\left(x,y\right)}}{x+y-1-\sqrt{\Delta\left(x,y\right)}} \Leftarrow \Delta\left(x,y\right) \geq 0, \\ -2\sqrt{-\Delta\left(x,y\right)}\left[\arctan\dfrac{x-y+1}{\sqrt{-\Delta\left(x,y\right)}} + \left(x \leftrightarrow y\right)\right] \Leftarrow \Delta\left(x,y\right) < 0, \end{cases} \tag{101}$$

where

$$\Delta\left(x,y\right) = 1 - 2\left(x+y\right) + \left(x-y\right)^2. \tag{102}$$

For S and U one has

$$S = S' + S'', \quad U = U' + U''. \tag{103}$$

where

$$S'' = -\frac{2}{\pi}\sum_{I=-J}^{J}\left(Ic_W^2 - Ys_W^2\right)^2 \zeta\left(\frac{m_I^2}{m_Z^2}, \frac{m_I^2}{m_Z^2}\right), \tag{104a}$$

$$U'' = -S'' - \frac{1}{\pi}\sum_{I=1-J}^{J}\left(J^2 + J - I^2 + I\right)\zeta\left(\frac{m_I^2}{m_W^2}, \frac{m_{I-1}^2}{m_W^2}\right), \tag{104b}$$

and

$$S' = -\frac{Y}{3\pi}\sum_{I=-J}^{J} I\,\ln\frac{m_I^2}{\mu^2}, \tag{105a}$$

$$U' = \frac{1}{12\pi}\sum_{I=1-J}^{J}\left(J^2 + J - I^2 + I\right)\mathrm{g}\left(\frac{m_I^2}{m_{I-1}^2}\right) + \frac{1}{6\pi}\sum_{I=-J}^{J}\left(J^2 + J - 3I^2\right)\ln\frac{m_I^2}{\mu^2}. \tag{105b}$$

In Eq. (105b),

$$\mathrm{g}\left(x\right) = \begin{cases} \dfrac{x^3 - 3x^2 - 3x + 1}{\left(x-1\right)^3}\ln x - \dfrac{5x^2 - 22x + 5}{3\left(x-1\right)^2} \Leftarrow x \neq 1, \\ 0 \Leftarrow x = 1 \end{cases} \tag{106}$$

is a function that obeys $\mathrm{g}\left(x\right) = \mathrm{g}\left(1/x\right)$.

We note that the expressions for the OPs are invariant under the transformation $I \to -I,\ Y \to -Y$. This allows one to choose the scalar with $I = -J$ to be the lightest one, provided one keeps Y free, i.e. provided one considers both negative and positive values of Y; that is the procedure that we adopt.

4.2. *Numerical results*

In our numerical work we utilize the set of electroweak observables given in Table 2. For each set of OPs and for each observable O, we have computed $O_{\mathrm{NP}}/O_{\mathrm{SM}}$ by using Eq. (95). We have then computed the residuals, defined as $O_{\mathrm{NP}}/O_{\mathrm{SM}}$ minus the values in the last column of Table 2. The χ^2 function for each set of OPs was defined as $\chi^2 = RC^{-1}R^T$, where R is the row-vector of the residuals and C is the covariance matrix; the latter is evaluated according to the correlations among the observables [16, 19, 20].

For each set of OPs, the pull is evaluated as $r/\delta^{\pm}$, where r is the residual defined above and $\delta^{\pm}$ is the error given in the fourth column of Table 2.

Table 2. First column: the electroweak observables used in our work. Second column: their experimental values, taken from Ref. [16]. Third column: the SM predictions for them. Fourth column: the ratio between the experimental value and the SM prediction.

Observable	Measurement (O_{meas})	SM prediction (O_{SM})	$O_{\mathrm{meas}}/O_{\mathrm{SM}}$
σ^0_{had} [nb]	41.481 ± 0.033	41.482 ± 0.008	0.999976 ± 0.0008186
R_ℓ	20.767 ± 0.025	20.736 ± 0.010	1.00149 ± 0.001299
R_b	0.21629 ± 0.00066	0.21582 ± 0.00002	1.00218 ± 0.003060
R_c	0.1721 ± 0.0030	0.17221 ± 0.00003	0.999361 ± 0.01742
$A_{FB}^{(0,\ell)}$	0.0171 ± 0.001	0.01617 ± 0.00007	1.05751 ± 0.06201
$A_{FB}^{(0,b)}$	0.0996 ± 0.0016	0.1029 ± 0.0002	0.967930 ± 0.01566
$A_{FB}^{(0,c)}$	0.0707 ± 0.0035	0.0735 ± 0.0002	0.961905 ± 0.04769
A_ℓ	0.1513 ± 0.0021	0.1468 ± 0.0003	1.03065 ± 0.01446
A_b	0.923 ± 0.020	0.9347	0.987483 ± 0.02140
A_c	0.670 ± 0.027	0.6677 ± 0.0001	1.00344 ± 0.04044
$\bar{s}^2_\ell$ (LEP-1)	0.2324 ± 0.0012	0.23155 ± 0.00004	1.00367 ± 0.005185
$\bar{s}^2_\ell$ (Tevt.)	0.23148 ± 0.00033	0.23155 ± 0.00004	0.999698 ± 0.001436
$\bar{s}^2_\ell$ (LHC)	0.23129 ± 0.00033	0.23155 ± 0.00004	0.998877 ± 0.001436
m_W [GeV]	80.377 ± 0.012	80.360 ± 0.006	1.00021 ± 0.0001670
Γ_W [GeV]	2.046 ± 0.049	2.089 ± 0.001	0.979416 ± 0.02346
Γ_Z [GeV]	2.4955 ± 0.0023	2.4941 ± 0.0009	1.00056 ± 0.0009903
$g_V^{\nu e}$	-0.040 ± 0.015	-0.0397 ± 0.0001	1.00756 ± 0.37784
$g_A^{\nu e}$	-0.507 ± 0.014	-0.5064	1.00118 ± 0.02765
Q_W(Cs)	-72.82 ± 0.42	-73.24 ± 0.01	0.994265 ± 0.005736
Q_W(Tl)	-116.4 ± 3.6	-116.90 ± 0.02	0.995723 ± 0.03080

Firstly, setting $V = W = X = 0$ and freely adjusting S, T, and U we have accomplished our best fit of the electroweak observables in Table 2. We have obtained $\chi^2 = 14.201$ for $S = -1.2 \times 10^{-2}$, $T = 2.8 \times 10^{-2}$, and $U = 2.0 \times 10^{-3}$.

In our NP model, for each value of the isospin J of the multiplet, there are just three free parameters:

- $|\lambda_4|$, which determines the mass-squared difference $\Delta m^2 = |\lambda_4|\, v^2 / 2$ between any two successive components of the multiplet.
- The mass m of the lightest component of the multiplet; without loss of generality we take that component to be the one with the smallest third projection of isospin. Thus, $m_I^2 = m^2 + (I + J)\,\Delta m^2$ for $I = -J, \ldots,$.
- The hypercharge Y of the multiplet.

For instance, by choosing $J = 2$, $m = 3$ TeV, $\lambda_4 = 3.65$, and $Y = 1.65$ we have obtained $\chi^2 = 14.2015$, which is not very far from our best fit. We thus see that our model is able to fit the electroweak observables just as perfectly as a free fit.

For each value of J up to $7/2$—the upper bound on J found in Ref. [4]—we let Y vary from $-Y_{\max}$ to $Y_{\max}$, where $Y_{\max}$ is the J-dependent upper bound on $|Y|$ determined in Refs. [4,5]. We let m vary from 50 GeV to 3 TeV, and we let $|\lambda_4|$ vary from zero to its maximum allowed value given in Table 1. We keep only the points that either

(1) have χ^2 smaller than 30 *and* all the pulls smaller (in modulus) than three, or
(2) have χ^2 smaller than 17 and *all* the pulls smaller than one, *except, possibly*, the pulls of $A_{FB}^{(0,b)}$, A_ℓ, R_ℓ, and Q_W(Cs).

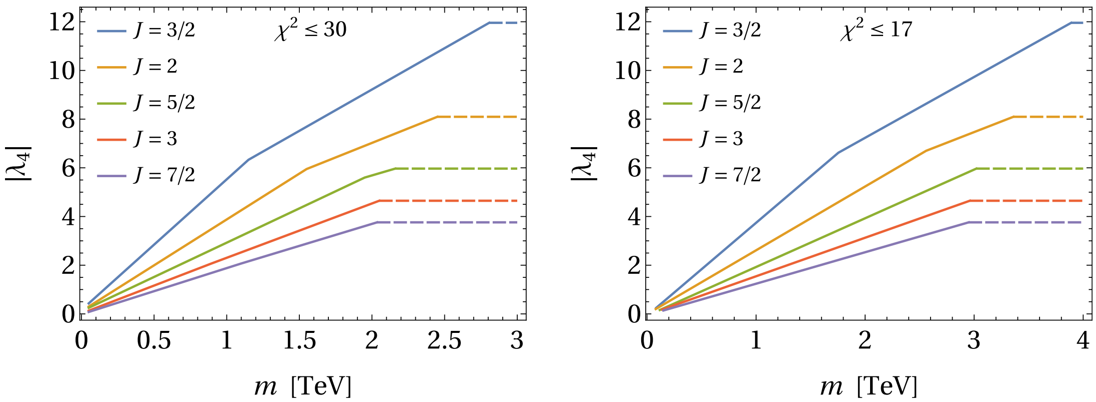

Fig. 5. The maximum allowed value of $|\lambda_4|$versus the lightest mass m, for various values of J and for fits with $\chi^2 \leq 30$ (left) or $\chi^2 \leq 17$ (right). The hypercharge Y was left free. The horizontal dashed lines correspond to the bounds on $|\lambda_4|$ from the UNI and BFB conditions.

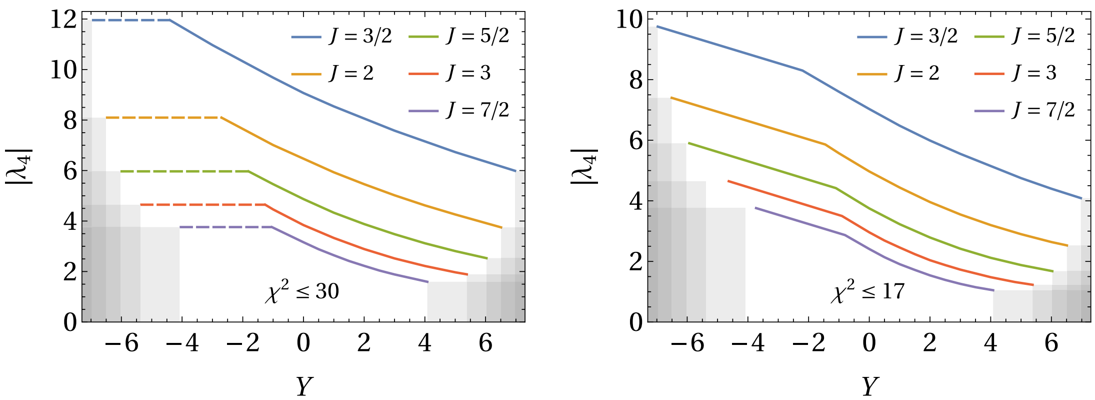

Fig. 6. The upper bound on $|\lambda_4|$versus the hypercharge Y, for various values of J, for $m = 3$ TeV, and for fits with $\chi^2 \leq 30$ (left) or $\chi^2 \leq 17$ (right). The horizontal dashed lines indicate the upper bounds from the UNI and BFB conditions, and the curved lines indicate the upper bounds from the OPs. The gray bands indicate the J-dependent restrictions on Y derived in Refs. [4,5].

In this way we obtain two sets of points, which we use to construct Figs. 5 and 6. Most pulls of the observables are always very small; only a few observables have large pulls. As a consequence, in practice, points with $\chi^2 < 30$ mostly have all the pulls between -3 and $+3$, and points with $\chi^2 < 17$ almost always have all the pulls between -1 and $+1$, except for the observables $A_{FB}^{(0,b)} A_\ell$, R_ℓ, and $Q_W(\mathrm{Cs})$.[12]

One sees in Fig. 5 that, unless m is very large and, therefore, the OPs are very small, the restrictions on $|\lambda_4|$ from the OPs are usually stronger than the UNI and BFB conditions that we have derived in this paper. Indeed, for $\chi^2 \leq 30$ and $m \lesssim 2$ TeV the restrictions from the OPs are stronger, and the same happens for $\chi^2 \leq 17$ and $m \lesssim 3$ TeV.

The relation between the upper bound on $|\lambda_4|$ and the hypercharge Y is quite complex and very much depends on m (because, if m gets larger, then the OPs get smaller and therefore the OPs do not constrain $|\lambda_4|$). In Fig. 6, which was made for $m = 3$ TeV, one observes that, as Y increases, the upper bound on $|\lambda_4|$ slightly decreases. If one requires a smaller χ^2 in the fit of

[12]We make the exception of $Q_W(\mathrm{Cs})$ because, if one forces its pulls to be smaller than one, that noticeably restricts the parameter space, by practically eliminating all the negative values of Y.

the OPs, then the constraint on $|\lambda_4|$ derived therefrom becomes stronger and eventually, as one sees in the right panel of Fig. 6, the UNI+BFB bound becomes completely ineffective.

5. Conclusions and outlook

In this paper we have studied the extension of the SM through a scalar multiplet χ with arbitrary isospin J and hypercharge Y. For every value of J, we have included in the SP just those terms that are present there for any value of Y. We have especially concentrated on the term (1) which fixes the squared-mass difference Δm^2 between the successive components of χ, cf. Eq. (15). We have derived an upper bound on $|\lambda_4|$, hence on Δm^2, from both the UNI and BFB conditions on the SP. We have found that, remarkably, that upper bound depends crucially not just on the UNI conditions, but also on the BFB ones. For instance, the upper bound that we have found is quite a lot more stringent than the one utilized in the recent Ref. [21], which used only UNI conditions.

Remarkably, we have been able to derive *necessary and sufficient* BFB conditions on this model, even when we allowed the presence in the SP of the most general terms four-linear in the components of χ. It so happens that those terms, even if they are quite complicated to account for, end up relaxing only a little bit the upper bound on $|\lambda_4|$, cf. Fig. 1.

Phenomenologically, the model that we have studied is, by itself alone, of little value, because, since we have left Y arbitrary, the multiplet χ does not have Yukawa couplings to any fermions. Moreover, its lightest component is, for arbitrary Y, electrically charged and, moreover, absolutely stable, which is of course incompatible with observation. Therefore, our study can only be understood as a step towards the understanding of more specific models, that will have precise values of J and Y, and probably also extra terms in the SP, viz. higher-dimensional terms.

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Appendix. Explicit UNI conditions

For J through $7/2$, the unitarity conditions that originate in the matrix $\mathcal{S}$ of Eq. (84) are, besides Eqs. (87), the following:

- For $J = 1$,

$$\left| \lambda_2 + \frac{4}{3} \lambda_5 \right| < M. \tag{A.1}$$

- For $J = 3/2$,

$$\left| \lambda_2 + \frac{3}{5} \lambda_5 \right| < M, \tag{A.2a}$$

$$\left|\lambda_2 + \frac{9}{5}\lambda_5\right| < M. \tag{A.2b}$$

- For $J = 2$,

$$\left|\lambda_2 + \frac{16}{7}\lambda_5 - \frac{4}{5}\lambda_6\right| < M, \tag{A.3a}$$

$$\left|\lambda_2 + \frac{8}{7}\lambda_5 + \frac{4}{5}\lambda_6\right| < M, \tag{A.3b}$$

$$\left|\lambda_2 - \frac{6}{7}\lambda_5 + \frac{4}{5}\lambda_6\right| < M. \tag{A.3c}$$

- For $J = 5/2$,

$$\left|\lambda_2 + \frac{79}{45}\lambda_5 - \frac{22}{35}\lambda_6\right| < M, \tag{A.4a}$$

$$\left|\lambda_2 + \frac{19}{9}\lambda_5 - \frac{2}{7}\lambda_6\right| < M, \tag{A.4b}$$

$$\left|\lambda_2 + \frac{5}{9}\lambda_5 + \frac{10}{7}\lambda_6\right| < M, \tag{A.4c}$$

$$\left|\lambda_2 - \frac{29}{15}\lambda_5 + \frac{46}{35}\lambda_6\right| < M. \tag{A.4d}$$

- For $J = 3$,

$$\left|\lambda_2 + \frac{102}{77}\lambda_5 + \frac{25}{21}\lambda_6 - \frac{4}{7}\lambda_7\right| < M, \tag{A.5a}$$

$$\left|\lambda_2 + \frac{18}{77}\lambda_5 + \frac{25}{21}\lambda_6 + \frac{4}{7}\lambda_7\right| < M, \tag{A.5b}$$

$$\left|\lambda_2 + \frac{6}{7}\lambda_5 + \frac{10}{21}\lambda_6 - \frac{4}{7}\lambda_7\right| < M, \tag{A.5c}$$

$$\left|\lambda_2 - \frac{18}{7}\lambda_5 + \frac{19}{21}\lambda_6 + \frac{4}{7}\lambda_7\right| < M, \tag{A.5d}$$

$$\left|\lambda_2 + \frac{194}{77}\lambda_5 - \frac{10}{7}\lambda_6 + \frac{4}{7}\lambda_7\right| < M. \tag{A.5e}$$

- For $J = 7/2$,

$$\left|\lambda_2 - \frac{1}{14}\lambda_5 + \frac{31}{22}\lambda_6 - \frac{13}{14}\lambda_7\right| < M, \tag{A.6a}$$

$$\left|\lambda_2 + \frac{53}{78}\lambda_5 + \frac{119}{66}\lambda_6 - \frac{1}{2}\lambda_7\right| < M, \tag{A.6b}$$

$$\left|\lambda_2 + \frac{363}{182}\lambda_5 - \frac{1}{22}\lambda_6 - \frac{1}{14}\lambda_7\right| < M, \tag{A.6c}$$

$$\left|\lambda_2 + \frac{7}{78}\lambda_5 + \frac{49}{66}\lambda_6 + \frac{7}{6}\lambda_7\right| < M, \tag{A.6d}$$

$$\left|\lambda_2 - \frac{121}{42}\lambda_5 + \frac{1}{6}\lambda_6 + \frac{17}{14}\lambda_7\right| < M, \tag{A.6e}$$

$$\left|\lambda_2 + \frac{103}{66}\lambda_5 - \frac{101}{66}\lambda_6 + \frac{23}{42}\lambda_7\right| < M. \tag{A.6f}$$

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