Lietuvos matematikos rinkinys ISSN 0132-2818
Proc. of the Lithuanian Mathematical Society, Ser. A Vol. **57**, 2016
DOI: 10.15388/LMR.A.2016.04 pages 18–23

Radial basis function method modelling borehole heat transfer: the practical application*

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Abstract. In this work, the method of radial basis function (RBF) is used for the borehole heat exchanger transfers (BHET) problems. The RBF method is an efficient mesh free technique for the numerical solution of partial differential equations. The condition number of the system of linear equations (matrix **A**) is large it means that the ill-conditioning of matrix **A** makes the numerical solution unstable. Tikhonov regularization (TR) method are presented to solve such ill-conditioned systems. In this work, generalized cross-validation (GCV) method is carried out to determine the regularization parameter ξ for the TR method that minimizes the GCV function. The some practical results of numerical experiments are presented.

Keywords: borehole heat exchanger transfer (BHET), radial basis function (RBF), Tikhonov regularization (TR), generalized cross-validation (GCV).

1 Introduction

The traditional numerical techniques based on discretization of problem using polar or cylindrical grids or meshes has been used by Zeng et al. [6], Al-Khoury et al. [1] to model vertical BHT. In many geological situations when dealing with a heat conduction in the ground it is not always possible to specify the boundary conditions or the initial ground temperature. The main difficulty in the treatment of inverse problems is the unstability of their solution in the presence of noise in the measured data and are generally identified as ill-posed [2]. We will present a practical geology application with meshless numerical scheme, based on the radial basis function of the heat equation, in order to approximate the solution of a backward inverse heat conduction problem(BIHCP), the problem in which an unknown initial condition or/and temperature distribution in previous time will be determined. In the presented method, the A matrix is ill-conditioned, Tikhonov regularization (TR) method is applied in order to solve the problem. The generalized cross-validation (GCV) criterion has been assigned to adopt an optimal regularization parameter. The structure of the rest of this work is organized as follows: in Section 2, we represent the mathematical for-

* The authors would like to thank professor Mifodijus Sapagovas for his fruitful insights, guidance and support and associate professor dr. Svajūnas Sajavičius for the valuable and time consuming discussions, showing the applicability of meshless algorithms.

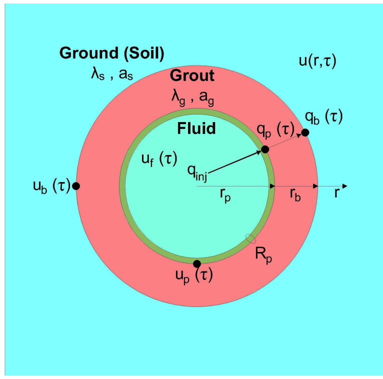

Fig. 1. BHET

mulation of the problem. The numerical study using Radial Basis Function method (RBF) are described in Section 3. Section 4 embraces Tikhonov regularization method with a rule for choosing an appropriate regularization parameter. In Section 5, we present the obtained numerical results of solving a problem. Section 6 ends in a brief conclusion and some suggestions.

2 Mathematical formulation of the problem

Let's denote $\mathbf{x} = (x, y)$ two-dimensional case where r^* could be the distance values from the interval $[r^*_{p-}, R]$. We define for different material layers and denote it as $k = f, p, g, s, b$ for fluid pipe grout, soil, borehole regions, special notation $-$, $+$ would be used for the left and right side of the boundary. The radial heat transfer model is considered approximating the U-tube on an equivalent-diameter pipe. The general input parameters of the model are: q_{inj} – a constant heat flux, a fluid thermal capacity C_f of the circulating fluid, the S_f intersection area of the circulating and in the equivalent-diameter pipe is kept equal to the U-tube, a R_p is accounted for both fluid and pipe resistances. The resulting radial heat transfer problem is shown in Fig. 1 carefully changing the general schema from [5]. For the heat transfer problem the temperature distribution $u(r^*, \tau)$ must satisfy the following radial heat conduction equation in both the grout and the soil regions

$$\frac{1}{\alpha_k(r^*)} \frac{\partial u_k}{\partial \tau} = \frac{\partial^2 u_k}{\partial^2 r^*} + \frac{1}{r^*} \frac{\partial u_k}{\partial r^*}. \tag{1}$$

The radial heat flux in the grout and the soil regions is

$$q(r^*, \tau) = 2\pi r^* \big(- \lambda(r^*) \big) \frac{\partial u_k}{\partial r^*} \quad \text{and} \quad q(r^*_{p+}, \tau) = q_{inj} - C_f S_f \frac{\partial u(r^*_{p-}, \tau)}{\partial \tau}.$$

The heat flux at the grout-soil interface is continuous and the boundary condition $r^* = r^*_b$ is

$$\lambda_{grout} \frac{\partial u}{\partial r^*}\bigg|_{r^*=r^*_{b-}} = \lambda_{soil} \frac{\partial u}{\partial r^*}\bigg|_{r^*=r^*_{b+}}, \qquad u(R, \tau) = \text{const}. \tag{2}$$

Table 1. Some radial basis functions.

RBF	Definition
Multiquadratic (MQ)	$\phi(\mathbf{x},\mathbf{x}_j)=\phi(r_j)=\sqrt{r_j^2+\epsilon_j^2}$
Inverse Multiquadratic (IMQ)	$\phi(\mathbf{x},\mathbf{x}_j)=\phi(r_j)=\sqrt{r_j^2+\epsilon_j^2}^{-1}$
Inverse Quadratic (IQ)	$\phi(\mathbf{x},\mathbf{x}_j)=\phi(r_j)=(r_j^2+\epsilon_j^2)^{-1}$
Gausian (GA)	$\phi(\mathbf{x},\mathbf{x}_j)=\phi(r_j)=\exp^{-(r_j^2+\epsilon_j^2)}$
Bassel RBF	$\phi(\mathbf{x},\mathbf{x}_j)=\phi(r_j)=J_0(k(\sqrt{r_j^2+\epsilon_j^2}))$

The boundary condition at the pipe-grout interface with the heat balance of the fluid in the pipe is

$$u\bigl(r_{p-}^*,\tau\bigr)-u\bigl(r_{p+}^*,\tau\bigr)=R_p\cdot q\bigl(r_{p+}^*,\tau\bigr), \tag{3}$$

here the thermal resistance R_p is for both fluid and pipe C_f, S_f are defined as known constants. The initial temperatures of the fluid in the pipe, the grout and the soil regions are taken as constants.

$$u\bigl(r_{p-}^*,0\bigr)=\text{const},\qquad u\bigl(r^*,0\bigr)=\text{const},\quad r^*\geqslant q\bigl(r_{p+}^*,\tau\bigr). \tag{4}$$

3 Numerical study

The $\phi_j(\mathbf{x})=\phi(r)$ where $\mathbf{x}=(x,y)$ and $r=\|\mathbf{x}-\mathbf{x}_j\|_2$ be the Euclidean distance between $\mathbf{x}_j$ and $\mathbf{x}$ are fixed point and arbitrary point in $\mathbb{R}^d$, respectively. *RBF*'s ϕ are radially symmetric about $\mathbf{x}_j$, e.g. depends on distance between $\mathbf{x}_j$ and $\mathbf{x}$. Some well known infinitely smooth *RBF*'s are listen in Table 1, where ϵ the empirically chosen shape parameter who plays an important role using RBF. The main idea of using the RBF is the approximation of solution $u(\mathbf{x})$ which could be expressed with linear combination of RBF Table 1.

$$u^n(\mathbf{x})=u(\mathbf{x},n\Delta\tau)=\sum_{j=1}^{N}c_j\phi(\mathbf{x},\mathbf{x}_j), \tag{5}$$

where $\mathbf{c}_j$ a unknown coefficients for $j=1,\dots,N$, to be determined by collocation technique. In the case of RBF method, the spatial domain and its boundary are represented by a set of scattered nodes coincide with the centers of RBFs. $\bar{\Omega}=\Omega\cap\Gamma$, where $\Omega=\Omega_{r_b^*}\cap\Omega_R$ and $\Gamma=\Gamma_{r_{p-}^*}\cap\Gamma_{r_b}\cap\Gamma_R$. Let denote the set of collocation nodes $\Xi=\{x_j,y_j\}_{j=1}^N\subset[r_{p-}^*,R]\times[0,\tau_{\max}]$, where the total number of nodes in each domain is equal to $N=N_{\Gamma_{r_{p-}^*}}+N_{\Gamma_{r_b^*}}+N_{\Gamma_R}+N_{\Omega_{r_b^*}}+N_{\Omega_R}$.

3.1 Discretization schema

The shape parameter ϵ of RBF could be be found by numerical experiments and the coefficients c_j^0 are determined the following equation (4)

$$\sum_{j=1}^{N}c_j^0\phi(\mathbf{x}_i,\mathbf{x}_j)=u_0(\mathbf{x}_i).$$

Initially, it is assumed that $c_j^n{=}c_j^{n-1}$ and the equations below can be solved for c_j^n. For internal domains the system of equations should be used

$$\frac{1}{\alpha(r^*)\Delta\tau}u^n(\mathbf{x}_i) - \sum_{j=1}^{N} c_j^n \frac{\partial^2\phi(\mathbf{x}_i,\mathbf{x}_j)}{\partial\mathbf{x}_i^2} = \frac{1}{\alpha(r^*)\Delta\tau}u^{n-1}(\mathbf{x}_i) - \sum_{j=1}^{N} c_j^{n-1} \frac{\partial^2\phi(\mathbf{x}_i,\mathbf{x}_j)}{\partial\mathbf{x}_i^2}.$$

The system of equations describe the boundary conditions below

$$\sum_{j=1}^{N} c_j^n \phi(\mathbf{x}_i,\mathbf{x}_j) = u_0(R, n\Delta\tau),$$

$$\sum_{j=1}^{N} c_j^n \frac{\partial\phi(r_{p+}^*,\mathbf{x}_j)}{\partial\mathbf{x}}\bigg|_{\mathbf{x}=r_{p+}^*} = \frac{q_{r_{p+}^*}}{2\pi r_{p+}^*(-\lambda(r^*))},$$

$$\lambda_s \sum_{j=1}^{N} c_j^n \frac{\partial\phi(r_{b+}^*,\mathbf{x}_j)}{\partial\mathbf{x}}\bigg|_{\mathbf{x}=r_{b+}^*} = \lambda_g \sum_{j=1}^{N} c_j^n \frac{\partial\phi(r_{b-}^*,\mathbf{x}_j)}{\partial\mathbf{x}}\bigg|_{\mathbf{x}=r_{b-}^*}$$

$$(1+K)\sum_{j=1}^{N} c_j^n \phi\big(r_{p-}^*,\mathbf{x}_j\big) - \sum_{j=1}^{N} c_j^n \phi\big(r_{p+}^*,\mathbf{x}_j\big) = R_p \cdot q_{inj} + K \cdot \sum_{j=1}^{N} c_j^{n-1}\phi\big(r_{p+}^*,\mathbf{x}_j\big),$$

where $K = \frac{C_f S_f R_p}{\Delta\tau}$. This system of equations can be written in the following matrix form as

$$\mathbf{A}\mathbf{c}^{(n)} = \mathbf{b}, \tag{6}$$

where $\mathbf{c} = (\mathbf{c}_1^n, \mathbf{c}_2^n, \dots, \mathbf{c}_N^n)$, $\forall n \in (0, \tau_{\max})$ and coefficients matrix $\mathbf{A}$ $\mathbf{a}_{ij}$ are often ill-conditioned for both direct and inverse heat transfer problems. The prediction of approximate solution $u(\mathbf{x}, n\Delta\tau)$ would be unstable especially than the input data contains noise. The shape parameter ϵ also controls the accuracy of the method. In order to get the accurate results the small value of shape parameter ϵ is required. We assure that the matrix $\mathbf{A}$ doesn't tend to be very ill-conditioned because of $\epsilon \to 0$. At the same time the method should be very accurate and not well-conditioned the uncertainty principle should be satisfied.

4 Regularization method and parameter

Before presenting the regularization method and regularization parameter, we introduce the singular value decomposition (SVD) of the coefficient matrix $\mathbf{A} = W\Sigma V^T$ where $W = [w_1, w_2, \dots, w_N] \in \mathbb{R}^{N\times N}$, $W^TW = I_N$ and $V = [\nu_1, \nu_2, \dots, \nu_N] \in \mathbb{R}^{N\times N}$, $V^TV = I_N$, where I_N denotes the N-th order identity matrix. The singular values of matrix $\mathbf{A}$ are the diagonal entries of $\Sigma = \operatorname{diag}(\sigma_1, \sigma_2, \dots, \sigma_N)$ which are non-negative and are arranged in increasing order $\sigma_1 \geqslant \sigma_2 \geqslant \dots \geqslant \sigma_N \geqslant 0$. Using SVD it's easy to get solution for the $\mathbf{c} = \Sigma_{j=1}^N \frac{w_j^T \mathbf{b}\nu_j}{\sigma_j}$.

The conventional L_2 condition of $\mathbf{A}$ is defined as $Cond(\mathbf{A}) = \frac{\sigma_1}{\sigma_N}$, in which σ_1 and σ_N are the largest and smallest singular values of $\mathbf{A}$. Solving ill-conditioned system because of the the large condition number of A which depends on the shape parameter ϵ and on the number of scattered nodes N. In practice [4] Matlab toolbox was used, the shape parameter must be adjusted with the number of interpolating points. TR method is mostly used by researchers. Since, small perturbation in initial data may produce a large amount of perturbation in the solution $b^* = b_i(1+\delta\,\mathrm{rand}(i))$, $i = 1,\dots,N$ where b_i is the exact data, $\mathrm{rand}(i)$ is a random number uniformly distributed in $[-1,1]$, and the magnitude δ displays the noise level of the measurement data. In this method, the TR solution $\mathbf{c}^\xi$ for the system of linear equations is explained as the solution of the following minimization problem: $\min(\|\mathbf{Ac}-\mathbf{b}\|_2^2 + \xi^2\|I_N\mathbf{c}\|_2^2)$ where ξ is called the regularization parameter.

In this work, we apply GCV [3] to obtain regularization parameter ξ. The optimal value of regularization parameter ξ minimizes the following (GCV) function:

$$GCV(\mathbf{c}) = \frac{\|\mathbf{A}\mathbf{c}^\xi - \mathbf{b}\|_2^2}{\mathrm{trace}(I_N - \mathbf{A}\mathbf{A}^I)^2} \quad \text{where } \mathbf{A}^I = \left(\mathbf{A}^T\mathbf{A} + \xi^2 I_N\right)^{-1}\mathbf{A}^T.$$

The regularization parameter ξ have to use for the indirect problems and GCV procedure used to solve such ill-conditioned or ill-posed problems. The regularized solution is shown by $\mathbf{c}^\xi = [\mathbf{c}_1^\xi, \mathbf{c}_2^\xi, \dots, \mathbf{c}_N^\xi]$, ξ is a minimizer of GCV. The approximate solution for mathematical problem is written as following

$$u^*(\mathbf{x}) = u(\mathbf{x}, n\Delta\tau) = \sum_{j=1}^N c_j^\xi \phi(\mathbf{x}, \mathbf{x}_j).$$

5 Numerical experiment

In this section the numerical results were validated and investigated the performance of RBF method. The numerical finite elements solution was implemented using the Comsol Multiphysics software. We define the target functions as the root mean square error, the relative root mean square error, the maximum absolute error.

$$\mathbf{RMSE}(u) = \sqrt{\frac{1}{N_t}\sum_{k=1}^{N_t}\left(u(\mathbf{x}) - u^*(\mathbf{x})\right)^2}, \qquad \mathbf{RRMSE}(u) = \frac{\sqrt{\sum_{k=1}^{N_t}(u(\mathbf{x}) - u^*(\mathbf{x}))^2}}{\sqrt{\sum_{k=1}^{N_t}(u(\mathbf{x}))^2}},$$

$$\mathbf{E}_\infty(u) = \max_{1\leqslant k\leqslant N_t} \left\|u(\mathbf{x}) - u^*(\mathbf{x})\right\|.$$

The following validation schema was implemented. First, the 8 different testing points on mesh were selected for monitoring purposes from the boundary conditions or at fixed distance from the borehole center. The points are r_{p-}, r_{p+}, r_{b-}, vr_b, vr_{b+} and others are fixed distance from the borehole center $r = r_c$. Second, the step value was selected for proper time step $\frac{\delta}{\tau} = 0.5$. Third, the different number of allocation points was generated on the boundary and internal domain of simulation area, in total from 100 to 1000. Forth, the distribution of allocation points could be regular, random, perturbed, showing the accuracy and quality of estimates. Fifth, one of the RBF would be selected as Multiquadratic (MQ), Inverse Multiquadratic (IMQ), Inverse Quadratic (IQ), Bessel RBF (BRBF). The testing results what RBF approximates the temperature function, also value of the parameter ϵ is should be valid. How we performed the simulation it is described in details below. The total number $N = 100, 400, 2500$ of allocation points was selected to get the optimal quality of results. The ϵ values of RBF parameter was selected from the interval $[0.1, 0.2]$.

The time step is selected as the most appropriate for simulation using the fraction like $\frac{\epsilon}{\Delta}$ = const and $\frac{\Delta\tau}{\Delta^2} = 100$. The duration of simulation should be no less than 100 h.

6 Conclusion

Borehole heat transfer problems can be solved by the radial basis collocation methods. Results shown the shape parameter of RBF affects the accuracy of the solution.The shape parameter value of RBF is suggested. Is shown that the accuracy of solutions is more accurate then the grid spacing and time step decrease. The u_j^n solutions of selected test points gave the same order errors using the finite elements and RBF methods. Acceptance of these methods depends on more testing and application to real data. The RBF method have potential to be acceptable numerical method solving transient and steady-state borehole heat transfer problems. The further steps of investigation of RBF method would be the application to real TRT data and three-dimensional BHE modelling solving transient and steady-state borehole heat transfer problems. The simulation of BHE transfer expands the knowledge abound surrounding ground and hydrogeological conditions on different layers in the three-dimensions.

References

[1] R. Al-Khoury, P.G. Bonnier and R.B.J. Brinkgreve. Efficient finite elements formulation for geothermal heating systems.part i:steady-state. *Int. J. Numer. Methods Eng.*, **63**(7):988–1013, 2005.

[2] O.M. Alifanov. *Inverse Heat Conduction Problems*. Springer, 1994.

[3] G.H. Golub, M. Heath and G. Wahba. Generalized cross validation as a method for choosing a good ridge parameter. *Technometrics*, **21**(2):215–223, 1979.

[4] P.C. Hansen. Regularization tools: a matlab package for analysis and solution of discrete ill-posed problems. *Numer. Algorithms*, **6**(1-2):1–35, 1994.

[5] S. Javed. *Thermal modelling and evaluation of borehole heat transfer*. Doctoral thesis, Lund University, Sweden, 2012.

[6] H. Zeng, N. Diao and Z. Fang. Heat transfer in vertical ground heat exchangers. *Int. J. Heat Mass Transfer*, **46**(23):4467–4473, 2003.

REZIUMĖ

Vertikalių kolektorių šilumos modeliavimas radialinėmis bazinėmis funkcijomis: praktininis taikymas

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Šiame tyrime yra taikomas radialinių bazinių funkcijų metodas (RBF) vertikaliems kolektoriams, analizuojant šilumos perdavimo problematiką. RBF metodas yra ekonomiškas betinklis metodas sprendžiant diferencialines lygtis. Lygčių sistemos (matrica $\mathbf{A}$) sąlyginis skaičius yra dažnai didelis, todėl matrica $\mathbf{A}$ yra silpnai sąlyginė, o gautas skaitinis sprendinys nestabilus. Tichonovo regularizacijos (TR) metodas yra taikomas šiai problemai eliminuoti, gautas apibendrintu abipusio palyginimo metodu (GCV) gaunamas regularizacijos parametras ξ TR metodui, minimizuojant GCV funkciją. Rezultatuose pateiktos kelios rekomendacijos.

Raktiniai žodžiai: vertikalus kolektorius (VK), radialinė bazinė funkcija (RBF), Tichonovo regularizacija (TR), apibendrintas abipusio palygimo metodas (GCV)