VILNIUS UNIVERSITY

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SOLUTION OF A TWO-DIMENSIONAL PARABOLIC EQUATION WITH AN INTEGRAL CONDITION BY THE FINITE-DIFFERENCE METHOD

Summary of Doctoral Dissertation

Physical Sciences, Mathematics (01 P)

Vilnius 2013

Doctoral dissertation was prepared at the Institute of Mathematics and Informatics of Vilnius University in 2008–2012.

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The dissertation will be defended at the public meeting of the Council of Scientific Field of Mathematics in the auditorium number 203 at the Institute of Mathematics and Informatics of Vilnius University, at 1p.m. on the 17^{th} April, 2013.
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The summary of the doctoral dissertation was distributed on the 15^{th} of March, 2013. A copy of the doctoral dissertation is available for review at the Library of Vilnius University.

VILNIAUS UNIVERSITETAS

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DVIMATĖS PARABOLINĖS LYGTIES SU INTEGRALINE SĄLYGA SPRENDIMAS BAIGTINIŲ SKIRTUMŲ METODU

Daktaro disertacijos santrauka

Fiziniai mokslai, matematika (01 P)

Vilnius 2013

Disertacija rengta 2008–2012 metais Vilniaus universiteto Matematikos ir Informatikos institute.

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Disertacija bus ginama viešame Matematikos mokslo krypties tarybos posėdyje 2013 m. balandžio mėn. 17 d. 13 val. Vilniaus universiteto Matematikos ir Informatikos instituto 203 auditorijoje.
Adresas: Akademijos g. 4, LT-08663 Vilnius, Lietuva.

Disertacijos santrauka išsiuntinėta 2013 m kovo 15 d.
Disertaciją galima peržiūrėti Vilniaus universiteto bibliotekoje.

Introduction

Statement of problem

The dissertation deals with a finite difference method for solving two-dimensional parabolic equations with one or several nonlocal conditions. Algorithms have been constructed for solving parabolic equations with one or several nonlocal conditions and the results with different parameters in nonlocal conditions have been considered. Eigenvalue problems for a differential and difference operator with a nonlocal condition are analyzed. The finite difference method to solve an elliptic equation with a nonlocal condition has been examined as well.

Topicality of the problem

Even more frequently mathematical models for solving practical problems are made up using differential equations and these problems became ever more complicated of late. One of indicators of these complication is nonlocal boundary conditions.

The term of nonlocal boundary conditions was first used by N. I. Ionkin in 1977 [20]. He also was one of the first to begin a systematic investigation of parabolic equations with nonlocal conditions. Nonlocal conditions emerge in a problem when the functions or derivative values at boundary points are connected with its values within the domain or, to put it simply, when there is no way to directly measure the problem under consideration at the boundary of the domain. In 1963, J. R. Cannon [5] formulated a one-dimensional parabolic problem with an integral condition which now is called a nonlocal problem.

A. V. Bitsadze and A. A. Samarski [2] have aroused a great interest in problems with nonlocal conditions by writing a work in which they considered an elliptic two-dimensional problem with nonlocal boundary conditions. This paper induced a wider interest in such problems, and the first works of this type were [12, 17, 32, 39, 44]. The problems of this type were widely analyzed for solving elliptic equations [18, 19, 38, 51] and parabolic equations [3, 7, 10, 13. 21, 28, 30, 40, 52] as well as other hyperbolic or pseudo-parabolic equations.

There is a great interest in the problems of this type in Lithuania as well. The first to begin analyzing such problems were M. Sapagovas and R. Čiegis [38, 43, 44]. At present, differential problems with nonlocal conditions are investigated in

Lithuanian by a large group of scientists: R. Čiegis, R. Čiupaila, F.Ivanauskoas J. Jachimavičienė, Ž. Jokšienė(Jesevičiūtė), S. Pečiulytė, S. Roman, M.Sapagovas, S. Sajavičius, A. Štikonas, O. Štikonienė, N. Tumanova and others [1, 4, 8, 11, 22, 23, 24, 25, 26, 27, 34, 35, 36, 37, 41, 42, 47, 48, 49, 50]. The problem area of these problems in rather a wide range of research and therefore differential problems with nonlocal conditions are in fact not quite exhaustively investigated as yet.

In the dissertation, solution of a two-dimensional parabolic equation with nonlocal integral boundary conditions by difference methods is analyzed. the results of other mathematicians, published in scientific literature, are discussed below.

Y. Lin, S. Xu and H-M. Yin in [31] analyzed explicit and implicit difference schemes for a differential problem with a nonlocal integral condition at the contour points of the considered domain. The basic premise is that the kernel of the nonlocal integral condition satisfies the set constraints. The authors have proved in the work that both schemes - explicit and implicit - retain the maximum principle properties and monotonicity of the solution. The fully implicit scheme is strictly monotonic thereto. On analogous problem with a slightly simplified nonlocal condition is analyzed in the dissertation (Chapter 1): to solve this problem we used the method of alternating direction, which cannot be found in the paper [31]. Besides, considerably less constraints on the kernel $K(x, y, \xi, \eta)$ were applied in the dissertation.

In [45] M. Sapagovas, G. Kairytė, O. Štikonienė ir A. Štikonas explore an implicit method of alternating direction for a two-dimensional parabolic equation in a rectangular domain with the Bisadze-Samarski nonlocal condition. In this work, sufficient stability conditions have been obtained when solving the problem by the Peaceman-Rachford method of alternating direction. The authors have proved that the parameter values in a nonlocal boundary condition have a great influence on the stability of the alternating direction method.

In Chapter 1 of the dissertation, solution of a two-dimensional parabolic equation in a rectangular domain is considered by an analogous method of alternating directions, where an integral condition is on given one rectangular boundary. Using the alternating direction method, two one dimensional difference problems are solved. The first solution is achieved with the classical boundary conditions, so its solution is found applying the Thomas algorithm. The second difference problem should be solved with a nonlocal boundary condition that links the solution at the contour point with its values in the whole two-dimensional domain, therefore this problem is solved by applying the Thomas algorithm twice.

The method, where the Thomas algorithm is applied twice, was first used by R. Čiegis in [8], to solve a problem when there in an unknown function in one boundary condition - one to that reason a nonlocal integral condition is formulated. In Chapter 1 of the dissertation , this R. Čiegis methodology is generalized to an nonlocal condition. Due to this nonlocal condition we have to solve in addition a system of small-order linear algebraic equations.

In 2002, Y. Wang wrote an article [51] in which he analyzes an elliptic equation with nonlocal boundary conditions. In this work, the corresponding eigenvalue problem has been considered, the existence of the solution to a system of difference equations proved, and an error of the solution to linear and slightly nonlinear equations has been analyzed.

In Chapter 2 of the thesis, solution of a two-dimensional parabolic equation in a rectangular domain is explored when a nonlocal integral boundary condition is given at all the rectangular contour points. With a view to solve such a problem, the method of alternating directions is applied and two one-dimensional difference problems with nonlocal conditions are solved. When solving the obtained one-dimensional problem, a linear system of equations is obtained from nonlocal conditions, and solved by the Gauss elimination method.

There are some papers with more complex integral conditions. In [33] C. V. Pao presents several iterative methods for solving nonlinear one-dimensional reaction-diffusion equations with nonlocal integral boundary conditions. The author has proved the existence of solution and explored the behavior of the solution depending an initial and boundary conditions.

In [9] R. Čiegis, A. Štikonas, O. Štikonienė, and O. Suboč analyze the solution of a one-dimensional parabolic problem with nonlocal integral boundary conditions. Sufficient condition for the existence and uniqueness of the solution have been found as well as that where the solution is non-negative and stable in a uniform norm. The research methods and results oh this article are closer to that used in the dissertation, than that used by other authors when investigating two-dimensional parabolic equations.

In [46] M. Sapagovas, T. Meškauskas, and F. Ivanauskas study the spectrum of a difference operator that corresponds to a differential eigenvalue problem using two nonlocal integral boundary conditions. The authors have defined that the stability of difference schemes depends on nonlocal conditions.

In Chapters 3 and 4 of the dissertation, eigenvalue problems for a differential operator with an integral condition and for the corresponding difference operator with a nonlocal boundary condition are examined. The research results of the difference operator spectrum in Chapter 4 are connected with the stability of the difference scheme considered in Chapter 1.

In various processes of science there appear nonclassical differential problems in which nonlocal integral conditions are additional conditions. As a rule, in the problems of this type there is an unknown function of argument t in a parabolic equation that should be found together with the solution of the equation. These problems are called inverse parabolic problems. Such problems are widely considered in mathematics literature. They are also considered by Lithuanian mathematicians [29]. Some problems mean for a parabolic equation with a nonlocal integral condition, can also be attributed ti inverse parabolic problems, only the unknown additional function may be not in the equation, but in a boundary condition.

The problem of this type was first explored by R. Cannon, Y. Lin,and A.Matheson in [6]. They have analyzed solution of a two-dimensional diffusion equation by the finite difference method, when there is an unknown function $\mu(t)$ in one boundary condition of the problem, and it should be found. Here the authors also present a difference method the results of its practical solution have shown that the error of the solution converges to zero with a decrease in the step h. Other authors analyze such problems, too. Locally one-dimensional difference method and the alternating direction method were investigated in the works [7, 8] by R. Čiegis and in [14, 15, 16] by M. Dehghan.

An analogous problem to solve an elliptic equation with an additional integral condition and an unknown parameter in a boundary condition has been explored by a finite difference method in the last Chapter (Chapter 5) of the dissertation. The essence of algorithm for finding the solution, presented here, is that a simpler problem is solved twice: the first time we freely take the constant λ_1 instead of an unknown parameter, and the second time λ_2 is chosen, such a methodology is also applied when solving an analogous parabolic problem.

Research object

The research object of the dissertation is a parabolic differential equation with nonlocal integral conditions, difference schemes of these problems, and algorithms for finding solutions.

Aim and tasks of the work

The aim of the thesis is to explore the finite difference method for solving a two-dimensional parabolic equation in a rectangular domain with an integral boundary condition, to comprise an algorithm, based on the alternating direction method for realizing difference schemes, to consider the stability of the algorithm, and to carry out a numerical experiment. To this end, the following problems were solved:

- to consider the algorithm for solving a two-dimensional parabolic problem with one nonlocal boundary condition, when there is a duoble integral in the nonlocal condition;
- to study the algorithm for solving a two-dimensional parabolic equation in a rectangular domain when several nonlocal integral conditions are given on the boundaries of the domain considered;
- to explore the spectrum structure of a system of one-dimensional differential equations with a nonlocal integral condition;
- to analyze the spectrum structure of a system of one-dimensional difference equations with a nonlocal integral boundary condition, using these results to research the stability of a difference scheme;
- to investigate the algorithm for solving a two-dimensional elliptic problem in case one boundary condition is implicit and, because of that, an additional nonlocal integral condition is introduced; to apply this algorithm to an analogous parabolic problem.

Methodology of research

Difference methods are applied in the dissertation for solving two-dimensional differential problems, using the locally one-dimensional or alternating direction method. To explore the stability of a difference scheme, the methodology for researching the spectrum structure of a difference operator with a nonlocal condition is employed. The numerical experiment is performed using the Matlab package.

Scientific novelty

A difference algorithm has been considered in the dissertation for solving a two-dimensional parabolic problem with nonlocal integral boundary conditions. The results of the work supplement the results, obtained by other scientists so far, when analyzing the problem to find the solutions to these problems and can also be used for solving more complicated nonclassical differential problems.

The algorithm for solving a two-dimensional parabolic equation with one nonlocal integral condition by a difference method is presented here which is applied in the sequel of the work (after some corrections) to solve an analogous problem, but with several nonlocal boundary conditions.

Many scientists solve differential problems with nonlocal boundary conditions by difference methods, but only in several works one can find algorithms meant for solving problems with integral boundary conditions when there is a double integral in a nonlocal boundary condition.

In the dissertation, with a view to define the stability of the considered differential problem, solved by the difference method, we examine the spectrum structure of a difference operator with a nonlocal condition of an eigenvalue problem.

A new algorithm to find the solution to the problem of elliptic and parabolic equations, in case there is an unknown function or parameter in one of the boundary conditions of the problem, which causes the appearance of an additional nonlocal condition in the statement of the problem.

Practical value of the results

The results, obtained in the dissertation, can be used in solving multi-dimensional problems of this type or problems with complex boundary conditions. Mathematical algorithms for solving problems of this type are of importance in solving practical problems of biochemistry, ecology, physics, medicine and other scientific areas.

Defended propositions

- The algorithm for finding a solution to a differential parabolic problem with nonlocal integral conditions using the difference method.
- The spectrum structure of one-dimensional differential and difference operators with nonlocal conditions.
- The stability of difference schemes of the differential problem under consideration.

The scope of the scientific work

The doctoral thesis consists of the introduction, five chapters, conclusions, the list of references, and that of author's publications. The total scope of the dissertation is 92 pages and 15 tables. The results of the doctoral dissertation are presented in 4 publications. The results were also presented at 4 national and 5 international conferences. The language of the doctoral dissertation is Lithuanian.

Chapter 1. The method of alternating direction for a two-dimensional parabolic equation with a nonlocal integral condition

In Chapter 1, a two-dimensional parabolic problem in a rectangular domain is formulated in there is a nonlocal integral condition at the points of one rectangle side:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x,y,t), \quad x,y \in \Omega, \quad 0 < t \leqslant T \tag{1}$$

$$u(0,y,t) = \mu_1(y,t), \quad u(1,y,t) = \mu_2(y,t), \tag{2}$$

$$u(x,1,t) = \mu_3(x,t), \tag{3}$$

$$u(x,0,t) = \gamma(x) \iint\limits_{\Omega} u(\xi,\eta,t)\mathrm{d}\xi\mathrm{d}\eta + \mu_4(x,t), \quad x \in \Gamma_1, \tag{4}$$

$$u(x,y,0) = \varphi(x,y). \tag{5}$$

Where $\Omega = \{0 < x, y < 1\}$, $\Gamma_1 = \{y = 0,\ 0 \leqslant x \leqslant 1\}$, and $t \in (0,T]$.

Condition (4) is a special case of the condition

$$u(x,y,t) = \iint\limits_{\Omega} K(x,y,\xi,\eta)u(\xi,\eta,t)\mathrm{d}\xi\mathrm{d}\eta, \quad x,y \in \partial\Omega. \tag{6}$$

A numerical algorithm for solving this problem is presented, the gist of which is that, after applying Thomas algorithm twice, a system of small-order linear equations is formulated, which allows us to find the looking values of he solution at boundary points.

In accordance with the basic idea of this method, two one-dimensional difference problems are written separately:

$$\frac{u_{ij}^{n+\frac{1}{2}} - u_{ij}^{n}}{\frac{\tau}{2}} = \Lambda_1 u_{ij}^{n+\frac{1}{2}} + \Lambda_2 u_{ij}^{n} + f_{ij}^{n+\frac{1}{2}}, \quad i,j = \overline{1, N-1}, \tag{7}$$

$$u_{0j}^{n+\frac{1}{2}} = \mu_{1_j}^{n+\frac{1}{2}}, \tag{8}$$

$$u_{Nj}^{n+\frac{1}{2}} = \mu_{2_j}^{n+\frac{1}{2}}, \tag{9}$$

and

$$\frac{u_{ij}^{n+1} - u_{ij}^{n+\frac{1}{2}}}{\frac{\tau}{2}} = \Lambda_1 u_{ij}^{n+\frac{1}{2}} + \Lambda_2 u_{ij}^{n+1} + f_{ij}^{n+1}, \quad i,j = \overline{1, N-1}, \tag{10}$$

$$u_{iN}^{n+1} = \mu_{3_i}^{n+1}, \tag{11}$$

$$u_{i0}^{n+1} = h^2 \gamma_i \sum_{k=1}^{N-1} \sum_{j=0}^{N-1} \rho_i u_{kj}^{n+1} + g_{i0}^{n+1} + \mu_{4_i}^{n+1}, \quad i = \overline{1, N-1}. \tag{12}$$

where

$$\rho_i = \begin{cases} 1, & i \neq 0, N, \\ 1/2, & i = 0, N, \end{cases}$$

$$\Lambda_1 u_{ij}^n = \frac{u_{i-1,j}^n - 2u_{ij}^n + u_{i+1,j}^n}{h^2},$$
$$\Lambda_2 u_{ij}^n = \frac{u_{i,j-1}^n - 2u_{ij}^n + u_{i,j+1}^n}{h^2}.$$

Formula (12) is a trapezoid rule to calculate approximately a duoble integral.

The algorithm is presented in this chapter how to find the u_{ij}^{n+1} value, as the values u_{ij}^n of the earlier layer are known.

The first part of the algorithm - problem (7)-(9) is realized by means of the classical Thomas algorithm. In the second part of the algorithm, we have to solve system (10)-(12), which, due to condition (12), cannot be solved separately with one fixed value of index i. Therefore this system (10)-(12) is solved using a modified Thomas algorithm, described in [8]. According to the modified Thomas algorithm, the solution to each fixed value $i = 1, 2, \ldots, N-1$ of system (10)-(12) is written in the following shape

$$u_{ij}^{n+1} = \tilde{\alpha}_j u_{ij-1}^{n+1} + \tilde{\beta}_{ij}^{n+1}, \quad j = \overline{1, N-1}. \tag{13}$$

Further theoretically we use the Thomas algorithm once more for each $i = 1, 2, \ldots, N-1$ only u_{ij}^{n+1} is searched in the following way

$$u_{ij}^{n+1} = \alpha_j u_{i0}^{n+1} + \beta_{ij}^{n+1}. \quad j = 0, 1, \ldots, N, \tag{14}$$

where $\alpha_0 = 1$, $\beta_{i0}^{n+1} = 0$. Then

$$\alpha_j = \tilde{\alpha}_j \alpha_{j-1}, \quad j = 1, 2, \ldots, N, \tag{15}$$

$$\beta_{ij}^{n+1} = \tilde{\alpha}_j \beta_{ij-1}^{n+1} + \tilde{\beta}_{ij}^{n+1}, \quad i = 1, 2, \ldots, N-1; \quad j = 1, 2, \ldots, N. \tag{16}$$

Insisting that (14) would satisfy nonlocal condition (12), we obtain

$$u_{i0}^{n+1} = h^2 \gamma_i \sum_{k=1}^{N-1} \sum_{j=0}^{N-1} \rho_i (\alpha_j u_{k0}^{n+1} + \beta_{kj}^{n+1}) + g_{i0}^{n+1} + \mu_{4i}^{n+1}. \tag{17}$$

Now we rewrite the system of equation as follows

$$A u_0^{n+1} = F, \tag{18}$$

where

$$A = \begin{pmatrix} (1 - h\gamma_1 \alpha) & -h\gamma_1 \alpha & \ldots & -h\gamma_1 \alpha \\ -h\gamma_2 \alpha & (1 - h\gamma_2 \alpha) & \ldots & -h\gamma_2 \alpha \\ \ldots & \ldots & \ldots & \ldots \\ -h\gamma_{N-1} \alpha & -h\gamma_{N-1} \alpha & \ldots & (1 - h\gamma_{N-1} \alpha) \end{pmatrix},$$

u_0^{n+1} is the $(N-1)$-order vector $u_0^{n+1} = \{u_{i0}^{n+1}\}$,

$$\alpha = h \sum_{j=0}^{N-1} \rho_j \alpha_j = h \left(\frac{\alpha_0}{2} + \sum_{j=1}^{N-1} \alpha_j \right). \tag{19}$$

Thus, in order to realize the second part of the algorithm of the alternating direction method, i. e. to solve the system of equations (10)-(12) with a nonlocal condition, first of all it is needed to find the coefficients $\tilde{\alpha}_j$, $\tilde{\beta}_{ij}^{n+1}$ by the Thomas algorithm and then to calculate the coefficients α_j, β_{ij}^{n+1}, and finally, to solve system (18). The main properties of the system of equations (18) are expressed in three lemmas.

Lemma 1.1. *The estimate*

$$0 < \alpha < \frac{1}{2} \tag{20}$$

is always true.

Lemma 1.2. *If* $-\infty < \gamma(x) < 2$*, then the determinant of system* (18) *is a positive number.*

Lemma 1.3. *If* $|\gamma(x)| \leqslant 2$*, then matrix* A *is diagonally dominant.*

Chapter 2. Solution of a two-dimensional parabolic equation with two nonlocal conditions by a finite difference method

In Chapter 2 of the dissertation, the results of Chapter 1 are applied to solve a more complicated differential problem. The specific feature of the differential problem considered here is that in two nonlocal conditions the values of solutions at contour points are associated with the integral of the solution in the whole domain Ω. Let us solve a boundary value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x,y,t), \quad x,y \in \Omega, \quad 0 < t \leqslant T \tag{21}$$

$$u(x,y,0) = \varphi(x,y), \quad x,y \in \Omega, \tag{22}$$

$$u(0,y,t) = \mu_1(y,t), \quad y \in \Omega, \quad 0 < t \leqslant T \tag{23}$$

$$u(1,y,t) = \mu_2(y,t), \quad y \in \Omega, \quad 0 < t \leqslant T \tag{24}$$

$$u(x,0,t) = \iint\limits_{\Omega} \gamma_3(x,\xi) u(\xi,\eta,t) \mathrm{d}\xi \mathrm{d}\eta + \mu_3(x,t), \quad x \in \Omega, \quad 0 < t \leqslant T, \tag{25}$$

$$u(x,1,t) = \iint\limits_{\Omega} \gamma_4(x,\xi) u(\xi,\eta,t) \mathrm{d}\xi \mathrm{d}\eta + \mu_4(x,t), \quad x \in \Omega, \quad 0 < t \leqslant T, \tag{26}$$

where $\Omega = \{0 \leqslant x, y \leqslant 1\}$.

The method of alternating direction is applied to differential problem (21)-(26). In this way we need to solve two systems of difference equations. First we solve a one-dimensional problem with the following boundary conditions

$$\frac{u_{ij}^{n+\frac{1}{2}} - u_{ij}^{n}}{\frac{\tau}{2}} = \Lambda_1 u_{ij}^{n+\frac{1}{2}} + \Lambda_2 u_{ij}^{n} + f_{ij}^{n+\frac{1}{2}}, \quad i,j = \overline{1, N-1}, \tag{27}$$

$$u_{0j}^{n+\frac{1}{2}} = \mu_{1_j}^{n+\frac{1}{2}}, \quad j = \overline{1, N-1}, \tag{28}$$

$$u_{Nj}^{n+\frac{1}{2}} = \mu_{2_j}^{n+\frac{1}{2}}, \quad j = \overline{1, N-1}. \tag{29}$$

It is solved using the Thomas algorithm. Afterwards we need to find a solution in the $(n+1)$ time layer from the following difference problem with nonlocal boundary conditions:

$$\frac{u_{ij}^{n+1} - u_{ij}^{n+\frac{1}{2}}}{\frac{\tau}{2}} = \Lambda_1 u_{ij}^{n+\frac{1}{2}} + \Lambda_2 u_{ij}^{n+1} + f_{ij}^{n+1}, \quad i, j = \overline{1, N-1}, \tag{30}$$

$$u_{i0}^{n+1} = h^2 \sum_{k=1}^{N-1} \sum_{l=0}^{N} \gamma_{3_{ik}} \rho_{kl} u_{kl}^{n+1} + g_{1_{i0}}^{n+1} + \mu_{3_i}^{n+1}, \quad i = \overline{1, N-1}, \tag{31}$$

$$u_{iN}^{n+1} = h^2 \sum_{k=1}^{N-1} \sum_{l=0}^{N} \gamma_{4_{ik}} \rho_{kl} u_{kl}^{n+1} + g_{2_{iN}}^{n+1} + \mu_{4_i}^{n+1}, \quad i = \overline{1, N-1} \tag{32}$$

Let us rewrite system (30) in a shorter way

$$a u_{ij-1}^{n+1} - c u_{ij}^{n+1} + b u_{ij+1}^{n+1} = F_{ij}^{n+1}, \tag{33}$$

where $a = \tau/2h^2$, $b = \tau/2h^2$, $c = 1 + \tau/h^2$, and $c > a + b$. The solution to system (33) is found from

$$u_{ij}^{n+1} = c_1^{n+1} {u_{ij}^{(1)}}^{n+1} + c_2^{n+1} {u_{ij}^{(2)}}^{n+1} + {u_{ij}^{(0)}}^{n+1}, \quad j = 0, 1, 2, \dots, N, \tag{34}$$

where $u_{ij}^{(1)}$ and $u_{ij}^{(2)}$ are two solutions of homogeneous systems (30) and $u_{ij}^{(0)}$ is the solution of the system of nonhomogeneous equations.

The variables c_1^{n+1} and c_2^{n+1} should be chosen so that nonlocal conditions were true. Thus, $c_1^{n+1} \equiv u_{i0}^{n+1}$ and $c_2^{n+1} \equiv u_{iN}^{n+1}$.

By inserting the expression of this solution into nonlocal conditions, we obtain a system of $2(N-1)$ linear algebraic equations

$$Au = F. \tag{35}$$

Matrix A is of the following shape

$$A = \begin{pmatrix} (1-\gamma_{3_{1,1}}\alpha) & \dots & -\gamma_{3_{1,N-1}}\alpha & -\gamma_{3_{1,1}}\beta & \dots & -\gamma_{3_{1,N-1}}\beta \\ -\gamma_{3_{2,1}}\alpha & \dots & -\gamma_{3_{2,N-1}}\alpha & -\gamma_{3_{2,1}}\beta & \dots & -\gamma_{3_{2,N-1}}\beta \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -\gamma_{3_{N-1,1}}\alpha & \dots & (1-\gamma_{3_{N-1,N-1}}\alpha) & -\gamma_{3_{N-1,1}}\beta & \dots & -\gamma_{3_{N-1,N-1}}\beta \\ -\gamma_{4_{1,1}}\alpha & \dots & -\gamma_{4_{1,N-1}}\alpha & (1-\gamma_{4_{1,1}}\beta) & \dots & -\gamma_{4_{1,N-1}}\beta \\ -\gamma_{4_{2,1}}\alpha & \dots & -\gamma_{4_{2,N-1}}\alpha & -\gamma_{4_{2,1}}\beta & \dots & -\gamma_{4_{2,N-1}}\beta \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -\gamma_{4_{N-1,1}}\alpha & \dots & -\gamma_{4_{N-1,N-1}}\alpha & -\gamma_{4_{N-1,1}}\beta & \dots & (1-\gamma_{4_{N-1,N-1}}\beta) \end{pmatrix},$$

where

$$\alpha = h^2 \sum_{l=0}^{N} \rho_l u_l^{(1)}, \quad \beta = h^2 \sum_{l=0}^{N} \rho_l u_l^{(2)}.$$

The Gaussian elimination method is applied with a view to solve a system of linear equations and to get the solutions u_{k0}^{n+1} and u_{kN}^{n+1} $(k = \overline{1, N-1})$. These solutions are substituted into equation (34) and thus we obtain he solution in the $(n+1)$ time layer.

This difference method is easily applied to solve a two-dimensional parabolic problem, if the nonlocal integral conditions

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x,y,t), \quad (x,y) \in \Omega, 0 < t \leqslant T,$$

$$u(x,y,t) = \iint\limits_{\Omega} K(x,y,\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta + \mu(x,y,t), \quad (x,y) \in \partial\Omega, \quad 0 < t \leqslant T.$$

are given on the whole contour of a rectangular domain. Only when solving this problem by the method of alternating direction, we obtain two one-dimensional difference problems with nonlocal boundary conditions, therefore the method, analyzed in this chapter, has to applied twice and to solve two linear systems of equation.

Chapter 3. Eigenvalue problem for a differential operator with an integral condition

In Chapters 3 and 4 we consider the stability of the previously used difference schemes. To this end, it is necessary to explore the spectrum structure of the corresponding difference operator. At first, we explore the spectrum of a differential operator, and afterwards we analogously investigate the spectrum of a difference operator.

Let us analyze an eigenvalue problem for the system

$$\frac{d^2 u_i}{dy^2} + \lambda u_i = 0, \qquad i = 1, 2, \ldots, N-1, \tag{36}$$

with boundary conditions

$$u_i(0) = 0, \tag{37}$$

$$u_i(1) = \gamma_i h \sum_{k=1}^{N-1} \int\limits_0^1 u_k(y)\mathrm{d}y, \tag{38}$$

where $hN = 1$. In this problem λ is an unknown parameter, and $u_i(x)$ are unknown function. This problem is associated with a two-dimensional parabolic equation with one integral boundary condition.

The aim of this chapter of the dissertation is to analyze the structure of the differential operator with an integral condition. Thus, three theorems are proved.

Theorem 3.1. *The eigenvalue* $\lambda = 0$ *of problem* (36)-(38) *appears if and only if*

$$\frac{1}{N}\sum_{i=1}^{N-1} = 2. \tag{39}$$

Theorem 3.2. *Problem* (36)-(38) *has a negative eigenvalue if and only if*

$$\frac{1}{N}\sum_{i=1}^{N-1}\gamma_i > 2.$$

Theorem 3.3. *With all the values* γ_i *problem* (36)-(38) *has* $N-2$ *multiple positive eigenvalues independent of* γ_i *and very many positive eigenvalues depending on* γ_i *.*

We have defined that one zero eigenvalue can be dependent on γ_i or one negative eigenvalue and extremely many positive eigenvalues as well as $N-2$ multiple positive eigenvalues do not depend on γ_i values.

Chapter 4. Stability of difference schemes for a two-dimensional parabolic equation with integral conditions

We research the spectrum structure of the operator of a difference problem

$$\frac{u_{ij}^{n+1} - u_{ij}^{n+\frac{1}{2}}}{\frac{\tau}{2}} = \Lambda_2 u_{ij}^{n+1} + \frac{1}{2} f_{ij}^{n+1}, \quad i,j = \overline{1, N-1} \tag{40}$$

$$u_{i0}^{n+1} = \mu_{3i}^{n+1}, \tag{41}$$

$$u_{iN}^{n+1} = \gamma_i h^2 \sum_{k=1}^{N-1}\left(\frac{u_{kN}^{n+1}}{2} + \sum_{j=1}^{N-1} u_{kj}^{n+1}\right) + g_{iN}^{n+1} + \mu_{4i}^{n+1}, \tag{42}$$

where

$$\Lambda_2 u_{ij} = \frac{u_{i,j-1}^{n+1} - 2u_{ij}^{n+1} + u_{i,j+1}^{n+1}}{h^2}.$$

However, here one boundary condition is of the first type, while the second boundary condition is a nonlocal condition that relates the values of the solution in quest in all the columns, i. e. with all the values i, j.

System (40)–(42) can be expressed in a vector shape

$$\frac{u^{n+1} - u^{n+\frac{1}{2}}}{\tau/2} = -Au^{n+1} + \overline{f}^{n+1}, \tag{43}$$

where A is $(N-1)^2$ -order matrix defined by the expressions of right-hand sides of the system. Now we can write system (40) in the following form:

$$u^{n+1} = Su^{n+\frac{1}{2}} + \frac{\tau}{2} S\overline{f}^{n+1}, \tag{44}$$

where

$$S = \left(E + \frac{\tau}{2}A\right)^{-1}. \quad (45)$$

Lemma 4.1. *The eigenvalue problem*

$$Au = \lambda u \quad (46)$$

of matrix A is equivalent to the following difference eigenvalue problem

$$\Lambda_2 u_{ij} + \lambda u_{ij} = 0, \quad i, j = 1, 2, \ldots, N-1, \quad (47)$$

$$u_{i0} = 0, \quad (48)$$

$$u_{iN} = \gamma_i h^2 \sum_{k=1}^{N-1} \left(\frac{u_{kN}}{2} + \sum_{j=1}^{N-1} u_{kj} \right). \quad (49)$$

We find the eigenvalues of problem (47)–(49) and prove the theorems below.

Theorem 4.1. *The number $\lambda = 0$ is the eigenvalue of matrix A if*

$$\overline{\gamma} = 2. \quad (50)$$

Theorem 4.2. *Matrix A has one negative eigenvalue, if $\bar{\gamma} > 2$ and $h < \dfrac{2}{\bar{\gamma}}$. It is equal to*

$$\lambda = -\frac{4}{h^2} \sinh^2 \frac{\beta h}{2}, \quad \beta > 0; \quad (51)$$

here β is the single positive root of the equation

$$\tanh \frac{\beta}{2} = \frac{2}{\bar{\gamma} h} \tanh \frac{\beta h}{2}. \quad (52)$$

Theorem 4.3. *If $\bar{\gamma} < 2$, then all eigenvalues of matrix A are real and positive; a part of eigenvalues depends on $\bar{\gamma}$, the other part does not depend on $\bar{\gamma}$.*

In the proof of Theorem 4.3, we have defined that the eigenvalues λ satisfy the condition $|1 - \lambda h^2/2| < 1$, i. e. the eigenvalues for which the inequality $\lambda < 4/h^2$ holds. There are no other eigenvalues in this case. However, if we refuse the condition $\bar{\gamma} < 2$, then, in the presence of some assumptions, there can exist a positive eigenvalue which satisfies the condition $\lambda \geqslant 4/h^2$.

The most important conclusion obtained in this chapter of the dissertation is that the stability of a difference scheme depends not on the absolute value of $\gamma(x)$ (i. e. not on $|\gamma(x)|$), but on the average value $\gamma(x)$ defined in a certain way. Namely, the difference scheme of (40)-(42) is stable if

$$\bar{\gamma} = h \sum_{i=1}^{N-1} < 2.$$

Chapter 5. Solution of a two-dimensional elliptic equation with a nonlocal condition

The problem of boundary values for a two-dimensional elliptic equation in a rectangular domain with an integral condition is considered here. So, the following boundary problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y), \quad (x,y) \in D = \{0 < x, y < 1\}, \tag{53}$$

$$u(0,y) = g_0(y), \quad u(1,y) = g_1(y), \quad u(x,1) = h_1(x), \tag{54}$$

$$u(x,0) = \mu h_0(x), \tag{55}$$

$$\iint\limits_D u(x,y)\mathrm{d}x\mathrm{d}y = m, \tag{56}$$

is solved, where $u(x,y)$ is the function in quest, and μ is an unknown number..

In this chapter, a variant of the finite difference method is described and grounded, that is based on a simple and well known idea. The solution of a linear differential equation or of a system of difference equations with a nonlocal condition can be found by reducing problem (53)–(56) into two problems with the usual (classical) boundary conditions.

We solve problem (53)-(56) by the finite difference method. We comprise such a system of difference equations in which are no indicates i, j:

$$\Lambda U = f, \quad (i,j) \in \Omega_h, \tag{57}$$

$$U = g, \quad (i,j) \in \Gamma_h, \tag{58}$$

$$U = \mu h_0, \quad (i,j) \in \Gamma_h^1, \tag{59}$$

$$V_h(U) = m, \tag{60}$$

where μ is an unknown constant.

The gist of solution algorithm, presented in this chapter of the dissertation, is that the solution to system (57)–(60) can be found by solving twice an analogous system with the classical boundary conditions. Assume

$$U = C_1 U_1 + C_2 U_2, \tag{61}$$

where $U_s, s = 1, 2$ are the solutions to the following difference systems:

$$\Lambda U_s = f, \quad (i, j) \in \Omega_h, \tag{62}$$

$$U_s = g, \quad (i, j) \in \Gamma_h, \tag{63}$$

$$U_s = \mu_s h_0, \quad (i, j) \in \Gamma_h^1, \tag{64}$$

where μ_1, μ_2 are two freely chosen fixed numbers. We find C_1, C_2, μ_h from the equations below:

$$C_1 = \frac{V_h(U_2) - m}{V_h(U_2) - V_h(U_1)}, \tag{65}$$

$$C_2 = \frac{m - V_h(U_1)}{V_h(U_2) - V_h(U_1)}. \tag{66}$$

We shall indicate the sufficient conditions, when the constants C_1 and C_2 can be calculated single-valued.

Theorem 5.1. *Suppose that* $h_{0,i} \geq 0, i = \overline{0, N}$ *and* $h_{0,i} \not\equiv 0$. *If* $\mu_1 < \mu_2$, *then* $V_h(U_2) - V_h(U_1) > 0$.

Using formula (61), we can find the solution not only to difference problem (57)-(60), but also to differential problem (53)-(56). Thus, we get that the solution of problem (53)-(56) belongs to the class of functions like that of the solution to problem (53)-(55) with a fixed μ.

Let us estimate the error of the solution U_{ij} to system (57)-(60) of difference equations. Denote

$$z_{h,ij} = u(x_i, y_j) - U_{ij}, \tag{67}$$

$$\eta = \mu - \mu_h \tag{68}$$

where $u(x_i, y_j)$ and U_{ij} are the solutions to differential problem (53)-(56) and to difference problem (57)-(60) respectively. z_h and η are the solutions of the following difference problem:

$$\Lambda z_h = R(h), \quad (i, j) \in \Omega_h, \tag{69}$$

$$z_h = 0, \quad (i, j) \in \Gamma_h, \tag{70}$$

$$z_h = \eta h_0, \quad (i, j) \in \Gamma_h^1, \tag{71}$$

$$V_h(z_h) = r(h). \tag{72}$$

where $R(h)$, $r(h)$ are errors of the differential equation and integral approximation respectively.

Theorem 5.2. *If the conditions of Theorem* 5.1 *and approximation estimates* $R(h) = O(h^2)$, $r(h) = O(h^2)$ *are valid for the function* $h_0(x)$, *then the estimates*

$$|\eta| \leq C_3 h^2, \quad \|z_h\|_C \leq C_4 h^2. \tag{73}$$

are true with all small enough values h.

General conclusions

1. The algorithm is presented for solving a two-dimensional parabolic equation with a nonlocal integral condition, given at the points of one side of a rectangular domain, by the method of alternating direction.

2. This algorithm was modified and generalizated to the case, where the nonlocal integral condition is given at the points of two sides of a rectangle.

3. The eigenvalue problem of an operator of the system of differential equations with integral conditions as well as the eigenvalue problem of a respective difference operator have been explored. On the basis of these researches the stability of the method of alternating directions has been considered.

4. A new method was made up for solving a two dimensional elliptic equation with one implicit boundary condition in case an additional integral condition is given. The gist of the method is that a one-dimensional elliptic problem with the classical boundary conditions is solved twice after selecting two new constants. The method is easily applicable to a multi-dimensional problem as well.

List of Published Works on the Topic of the Dissertation

[A1] R. Čiupaila, K. Jakubėlienė and M. Sapagovas, Dvimatės elipsinės lygties su nelokaliąja sąlyga sprendimas, *Matematikos ir Informatikos Institutas, Preprintas*, **2008-40**:12p., 2008.

[A2] R. Čiupaila, K. Jakubėlienė and M. Sapagovas, Solution of two-dimensional eliptic equation with nonlocal conditions, *Proceedings of International Conference Differential equations and their applications, Kaunas, Technologija 2009*, 92-98p., 2009.

[A3] K. Jakubėlienė, Solution of two-dimensional parabolic equation with nonlocal integral condition, *LMD darbai*, **52**:285-290, 2011.

[A4] M. Sapagovas and K. Jakubėlienė, Alternating direction method for two-dimensional parabolic equation with nonlocal integral condition, *Nonlinear Analysis: Modelling and control* **17**(1):91-98, 2012.

[A5] K. Jakubėlienė, Eigenvaliue problem for differential operator with integral condition, *LMD darbai.* 53. 2012.

[A6] M. Sapagovas and K. Jakubėlienė, On the stability of difference schemes for a two-dimensional parabolic equation with integral conditions, *Lith. Math J.,* (in press).

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DVIMATĖS PARABOLINĖS LYGTIES SU INTEGRALINE SĄLYGA SPRENDIMAS BAIGTINIŲ SKIRTUMŲ METODU

Problemos formulavimas

Disertacijoje išnagrinėtas baigtinių skirtumų metodai dvimačių parabolinių lygčių su viena ar keliomis nelokaliosiomis sąlygomis sprendimui. Sudaryti algoritmai parabolinėms lygtims su viena ar keliomis nelokaliosiomis sąlygomis spręsti, bei išnagrinėti rezultatai su skirtingais parametrais nelokaliosiose sąlygose. Analizuojami tikrinių reikšmių uždaviniai diferencialiniam ir skirtuminiam operatoriui su nelokaliąja sąlyga. Taip pat išnagrinėtas baigtinių skirtumų metodas elipsinei lygčiai su nelokaliąja sąlyga spręsti.

Darbo aktualumas

Vis dažniau matematiniai modeliai praktiniams uždaviniams spręsti yra sudaromi naudojantis diferencialinėmis lygtimis ir pastaruoju metu tokie uždaviniai tampa vis sudėtingesni. Vienas šio sudėtingumo rodikliu yra nelokaliosios kraštinės sąlygos.

Nelokaliųjų kraštinių sąlygų terminą vienas pirmųjų panaudojo N. I. Ionkinas 1977 metais [20]. Jis taip pat buvo vienas pirmųjų, pradėjęs sistemingai nagrinėti parabolines lygtis su nelokaliosiomis sąlygomis. 1963 metais J. R. Cannon [5] suformulavo vienamatį parabolinį uždavinį su integraline sąlyga, kuris dabar ir yra vadinamas nelokaliuoju uždaviniu.

Didelis susidomėjimas tokio tipo uždaviniais yra ir Lietuvoje. Pirmieji tokias problemas pradėjo nagrinėti M. Sapagovas ir R. Čiegis [38, 43, 44]. Dabartiniu metu Lietuvoje diferencialiniai uždaviniai su nelokaliosiomis sąlygomis yra nagrinėjami nemažos grupės mokslininkų: R. Čiegio, R. Čiupailos, F.Ivanausko, J. Jachimavičienės, Ž. Jokšienės(Jesevičiūtės), S. Pečiulytės, S. Roman, M.Sapagovo, S. Sajavičiaus, A. Štikono, O. Štikonienės, N. Tumanovos ir kitų [1, 4, 8, 11, 22, 23, 24, 25, 26, 27, 34, 35, 36, 37, 41, 42, 47, 48, 49, 50]. Tokių uždavinių problematika yra gana plati tyrinėjimo sritis ir dėl to diferencialiniai uždaviniai su nelokaliosiomis sąlygomis dar tikrai nėra visiškai išsamiai išnagrinėti.

Šioje disertacijoje nagrinėjamas dvimatės parabolinės lygties su nelokaliosiomis integralinėmis kraštinėmis sąlygomis sprendimas skirtuminiais metodais.

Y. Lin, S. Xu ir H-M. Yin straipsnyje [31] analizavo išreikštines ir neišreikštines skirtumines schemas diferencialiniam uždaviniui su nelokalia intergaline sąlyga ant nagrinėjamos srities kontūro taškuose. Disertacijoje yra nagrinėjamas analogiškas uždavinys (pirmame skyriuje) su šiek- tiek supaprastina nelokaliąja sąlyga; šio uždavinio sprendimui naudojamas kintamųjų krypčių metodas, ko nebuvo straipsnyje [31]. Be to, disertacijoje branduoliui $K(x, y, \xi, \eta)$ taikomi žymiai mažesni apribojimai.

Pirmajame disertacijos skyriuje kintamųjų krypčių metodu nagrinėjamas dvimatės parabolinės lygties stačiakampėje srityje sprendimas, kada ant vieno stačiakampio krašto yra duota integralinė sąlyga. Naudojant kintamųjų krypčių metodą, sprendžiami du vienmačiai skirtuminiai uždaviniai. Pirmasis sprendinys gaunamas su klasikinėmis kraštinėmis sąlygomis, taigi jo sprendinys randamas pritaikius perkelties algoritmą. Antrajį skirtuminį uždavinį, reikia spręsti su nelokaliąja kraštine sąlyga, susiejančia sprendinį kontūriniame taške su sprendinio reikšmėmis visoje dvimatėje srityje, todėl šis uždavinys sprendžiamas taikant du kartus perkelties algoritmą.

Metodika, kai du kartus taikomas perkelties algoritmas pirmą kartą buvo panaudota R. Čiegio darbe [8], kuriame yra sprendžiamas uždavinys kada vienoje kraštinėje sąlygoje yra nežinoma funkcija, dėl šios priežasties suformulojama ir nelokalioji integralinė sąlyga. Disertacijoje pirmame skyriuje apibendrinta ši R. Čiegio metodika nelokaliajai sąlygai. Dėl šios nelokaliosios sąlygos tenka papildomai spręsti neaukštos eilės tiesinių algebrinių lygčių sistemą.

Disertacijos antrame skyriuje analizuojamas sprendimas dvimatės parabolinės lygties stačiakampėje srityje, kai nelokalioji integralinė kraštinė sąlyga duota visuose stačiakampio kontūro taškuose. Norint išspręsti tokį uždavinį yra taikomas kintamųjų krypčių metodas ir sprendžiami du vienmačiai skirtuminiai uždaviniai su nelokaliosiomis sąlygomis. Sprendžiant gautajį vienmatį uždavinį yra sudaroma tiesinė, lygčių sistema, gaunama iš nelokaliųjų sąlygų, kuri sprendžiama Gauso eliminavimo metodu.

Trečiajame ir ketvirtajame disertacijos skyriuose yra analizuojami tikrinių reikšmių uždaviniai diferencialiniam operatoriui su integraline sąlyga ir atitinkamai skirtuminiam opreatoriui su nelokaliąja kraštine sąlyga. Ketvirtajame skyriuje skirtuminio operatoriaus spektro tyrimo rezultatai susiejami su pirmajame skyriuje nagrinėjamos skirtuminės schemos stabilumu.

Įvairiuose procesuose moksle atsiranda neklasikinių diferencialinių uždavinių, kuriuose nelokaliosios integralinės sąlygos yra papildomos sąlygos.

Paprastai tokio tipo uždaviniuose, parabolinėje lygtyje yra nežinoma argumento t funkcija, kuri turi būti rasta, kartu su lygties sprendiniu. Tokie uždaviniai yra vadinami atvirkštiniais paraboliniais uždaviniais. Šie uždaviniai plačiai nagrinėjami matematinėje literatūroje, juos nagrinėja taip pat ir Lietuvos matematikai [29]. Kai kurie uždaviniai, nagrinėjami parabolinei lygčiai su nelokaliąja integraline sąlyga irgi gali būti priskirti atvirkštiniams paraboliniams uždaviniams, tik nežinoma papildoma funkcija gali būti ne lygtyje, o kraštinėje sąlygoje.

Tokio tipo uždavinys pirmą kartą buvo išnagrinėtas R. Cannon, Y. Lin, A.Matheson straipsnyje [6]. Jie išanalizavo dvimatės difuzijos lygties sprendimą baigtinių skirtumų metodu, kai uždavinio vienoje kraštinėje sąlygoje yra nežinoma funkcija $\mu(t)$, kurią taip pat reikia surasti. Dėl šios priežasties yra formuluojama papildoma nelokalioji sąlyga. Autoriai darbe pateikia ir skirtuminį metodą, kurio praktinio sprendimo rezultatai parodė, kad sprendinio paklaida konverguoja į nulį mažėjan žingsniui h. Tokie uždaviniai nagrinėjami ir kitų autorių. Lokaliai vienmatis skirtuminis metodas bei kintamųjų krypčių metodas nagrinėtas R. Čiegio [7, 8] ir M. Dehghan [14, 15, 16] darbuose.

Analogiškas uždavinys elipsinei lygčiai su papildoma integraline sąlyga ir nežinomu parametru kraštinėje sąlygoje yra išnagrinėtas baigtinių skirtumų metodu paskutiniame penktajame disertacijos skyriuje. Disertacijoje pateikiamo sprendinio radimo algoritmo esmė ta, kad du kartus sprendžiamas paprastesnis uždavinys, pirmą kartą vietoje nežinomo parametro laisvai imant konstantą λ_1, o antrą kartą λ_2. Ši metodika yra pritaikoma ir sprendžiant analogišką parabolinį uždavinį.

Tyrimo objektas

Disertacijos tyrimo objektas yra parabolinė diferencialinė lygtis su nelokaliosiomis integralinėmis sąlygomis, tokių uždavinių skirtuminės schemos, algoritmai sprendiniams rasti.

Darbo tikslas ir uždaviniai

Disertacijos tikslas – išnagrinėti baigtinių skirtumų metodą dvimatei parabolinei lygčiai stačiakampėje srityje su integraline kraštine sąlyga, sudaryti skirtuminių schemų realizavimo algoritmą kintamųjų krypčių metodo pagrindu, išnagrinėti šio algoritmo stabilumą, atlikti skaitinį eksperimentą. Siekiant numatyto tikslo buvo sprendžiami šie uždaviniai:

- Išnagrinėti dvimačio parabolinio uždavinio su viena nelokaliąja kraštine sąlyga sprendimo algoritmą, kai nelokaliojoje sąlygoje yra dvilypis integralas.
- Išnagrinėti dvimatės parabolinės lygties stečiakampėje srityje sprendimo algoritmą, kada nagrinėjamos srities kraštuose yra duotos kelios nelokaliosios integralinės sąlygos.
- Ištirti vienmačių diferencialinių lygčių sistemos su nelokaliąja integraline sąlyga spektro struktūrą.
- Išanalizuoti vienmačių skirtuminių lygčių sistemos su nelokaliąja integraline kraštine sąlyga spektro struktūrą, panaudojant šiuos rezultatus skirtuminės schemos stabilumui tirti.
- Išanalizuoti dvimačio elipsinio uždavinio sprendimo algoritmą, kada viena kraštinė sąlyga yra neišreikštinė ir dėl šios priežasties įvesta papildoma nelokalioji integralinė sąlyga. Pritaikyti šį algoritmą analogiškam paraboliniam uždaviniui.

Tyrimo metodika

Disertacijoje taikomi skirtuminiai metodai dvimačių diferencialinių uždavinių sprendimui, naudojant kintamųjų krypčių arba lokaliai vienmatį metodą. Skirtuminės schemos stabilumui nagrinėti naudojama skirtuminio operatoriaus su nelokaliąja sąlyga spektro struktūros tyrimo metodika. Skaitinis eksperimentas atliktas naudojant Matlab paketą.

Darbo mokslinis naujumas ir jo reikšmė

Disertacijoje išnagrinėtas dvimačio parabolinio uždavinio su nelokaliosiomis integralinėmis kraštinėmis sąlygomis skirtuminis (skaitinis) sprendimo algoritmas. Atlikto darbo rezultatai papildo iki šiol kitų mokslininkų gautus rezultatus analizuojant tokių uždavinių sprendinių radimo problemą, bei gali būti panaudoti sprendžiant ir sudėtingesnius neklasikinius diferencialinius uždavinius.

Pateiktas dvimatės parabolinės lygties su viena nelokaliąja integraline sąlyga sprendimo skirtuminiu metodu algoritmas, kuris toliau darbe yra pritaikomas (pakoregavus) sprendžiant analogišką uždavinį tik jau su keliomis nelokaliosiomis kraštinėmis sąlygomis

Daugelis mokslininkų sprendžia skirtuminiais metodais diferencialinius uždavinius su nelokaliosiomis kraštinėmis sąlygomis. Tik nedaugelyje darbų aptinkama algoritmų, skirtų spręsti uždavinius su integralinėmis kraštinėmis sąlygomis, kai nelokaliojoje kraštinėje sąlygoje yra dvilypis integralas.

Šioje disertacijoje siekiant išanalizuoti nagrinėjamo diferencialinio uždavinio sprendžiamo skirtuminiu metodu stabilumą yra ištiriama skirtuminio operatoriaus su nelokaliąja sąlyga tikrinių reikšmių uždavinio spektro struktūra.

Pateiktas naujas uždavinio sprendinio radimo algoritmas elipsinei ir parabolinei lygtims, kai uždavinio vienoje iš kraštinių sąlygų yra nežinoma funkcija ar parametras ir dėl to uždavinio formulavime atsiranda papildoma nelokalioji sąlyga.

Darbo rezultatų praktinė reikšmė

Disertacijoje gauti rezultatai gali būti panaudoti sprendžiant daugiamačius tokio tipo uždavinius arba uždavinius su sudėtingomis kraštinėmis sąlygomis. Matematiniai tokio tipo uždavinių sprendimo algoritmai svarbūs sprendžiant biochemijos, ekologijos, fizikos, medicinos ir kitų mokslo sričių praktinius uždavinius.

Ginamieji teiginiai

- Diferencialinio parabolinio uždavinio su nelokaliosiomis integralinėmis sąlygomis sprendinio radimo algoritmas sprendžiant baigtinių skirtumų metodu.
- Vienmačių diferencialinių ir skirtuminių operatorių su nelokaliosiomis sąlygomis spektro struktūra.
- Nagrinėjamo diferencialinio uždavinio skirtuminių schemų stabilumas.

Disertacijos struktūra

Disertaciją sudaro įvadas, penki skyriai, išvados, literatūros sąrašas ir autorės publikacijų sąrašas disertacijos tema. Bendra disertacijos apimtis – 92 puslapiai, 15 lentelių. Disertacijos rezultatai paskelbti 5 publikacijose.

Šia tema skaityti 9 pranešimai mokslinėse konferencijose.

Šios disertacijos *pirmame skyriuje* suformuluotas dvimatis parabolinis uždavinys stačiakampėje srityje, kada vienos stačiakampio kraštinės taškuose yra nelokalioji integralinė sąlyga.Pateikiamas skaitinis šio uždavinio sprendimo algoritmas.

Antrajame skyriuje pritaikyti pirmojo skyriaus rezultatai, spręsti sudėtingesniam diferencialiniam uždaviniui, kai nagrinėjamos srities visuose kontūro taškuose yra duota nelokalioji integralinė kraštinė sąlyga. Pateikta išsami šio uždavinio sprendimo algoritmo analizė, kada pritaikius kintamųjų krypčių metodą yra sprendžiamos paprastesnės skirtuminės lygtys su nelokaliosiomis sąlygomis.

Trečiame skyriuje suformuluotas tikrinių reikšmių uždavinys diferencialinių lygčių sistemai su nelokaliąjaintegraline sąlyga. Tiriama šio diferencialinio operatoriaus spektro struktūra.

Ketvirtame skyriuje nagrinėjamas analogiškas tikrinių reikšmių uždavinys kaip ir ketvirtąjame disertacijos skyriuje tik jau skirtuminiu atveju. Ištiriama skirtuminio operatoriaus spektro stukrtūra ir skirtuminės schemos stabilumas, priklausomai nuo nelokaliosios sąlygos,

Penktajame skyriuje suformuluotas dvimatis elipsinis uždavinys su nelokaliąja sąlyga.Suformuluotas skaitinis šio diferencialinio uždavinio sprendimo algoritmas, kuris pritaikytas sprendžiant analogišką diferencialinį uždavinį nestacionariu atveju.

Bendrosios išvados

1. Išnagrinėtas dvimatės parabolinės lygties su viena nelokaliąja integraline sąlyga sprendimas kintamųjų krypčių metodu, kurio esmė - du kartus pritaikyti perkelties metodą bei spręsti papildomą neaukštos eilės tiesinių lygčių sistemą. Nustatyta kokią įtaką metodo stabilumui daro nelokalioji sąlyga.

2. Ištirtas baigtinių skirtumų metodas dvimatei parabolinei lygčiai su keliomis nelokaliosiomis integralinėmis sąlygomis nagrinėjamos srities kontūro taškuose.

3. Ištirta diferencialinių lygčių sistemos su nelokaliąja integraline sąlyga spektro struktūra, nustatyta kokią įtaką daro nelokalioji sąlyga šio uždavinio spektro struktūrai, atskirai imant, nustatyta, kada atsiranda neigiama tikrinė reikšmė.

4. Išnagrinėta skirtuminių lygčių sistemos su viena nelokaliąja integraline sąlyga spektro struktūra. Ji gali būti pritaikyta tiriant dvimačio parabolinio uždavinio su

nelokaliąja integraline sąlyga stabilumo sąlygas.

5. Sudarytas algoritmas dvimatei elipsinei lygčiai su viena neišreikštine kraštine sąlyga, bei integraline sąlyga. Šio algoritmo esmė yra ta, kad du kartus sprendžiamas skirtuminis elipsinis uždavinys su klasikinėmis kraštinėmis sąlygomis laisvai parinkus dvi naujas konstantas. Šis metodas gali būti apibendrintas ir daugiamačiam uždaviniui.

Trumpos žinios apie autorę

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DVIMATĖS PARABOLINĖS LYGTIES SU INTEGRALINE SĄLYGA SPRENDIMAS BAIGTINIŲ SKIRTUMŲ METODU

Daktaro disertacijos santrauka

Fiziniai mokslai (P 000),
Matematika (01 P)

Kristina Jakubėlienė

SOLUTION OF A TWO-DIMENSIONAL PARABOLIC EQUATION WITH AN INTEGRAL CONDITION BY THE FINITE-DIFFERENCE METHOD

Summary of Doctoral Dissertation

Physical sciences (P 000),
Mathematics (01 P)