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Bounds for the Clayton copula

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Abstract. We provide two upper bounds on the Clayton copula $C_\theta(u_1, \dots, u_n)$ if $\theta > 0$ and $n \geqslant 2$ and a lower bound in the case $\theta \in [-1, 0)$ and $n \geqslant 2$. The obtained bounds provide a nice probabilistic interpretation related to some negative dependence structures and also allow defining three new two-dimensional copulas, which tighten the classical Fréchet–Hoeffding bounds for the Clayton copula when $n = 2$.

Keywords: Archimedean copula, Clayton copula, copula bounds, negative dependence.

1 Introduction and main results

We consider one of many Archimedean copula families, the multivariate Clayton copula (also called Mardia–Takahashi–Clayton–Cook–Johnson copula; see, e.g. [15, Table 4.1, family (4.2.1)] or [7, Ex. 1.5] for the bivariate case and [15, Ex. 4.23] or [7, Sect. 4.6.1] for the n-variate case $n > 2$)

$$C_\theta(\mathbf{u}) = \left[\left(\sum_{i=1}^{n} u_i^{-\theta} - n + 1\right)_+\right]^{-1/\theta}, \quad \mathbf{u} = (u_1, \dots, u_n) \in [0, 1]^n,$$

where $a_+ := \max\{a, 0\}$. If $\theta > 0$, then the dimension can be any integer $n \geqslant 2$, and if $\theta \in [-1, 0)$, then $n \leqslant 1 - 1/\theta$ is only allowed (see, e.g. [12, Ex. 2.3]). By continuity, we let $C_0(\mathbf{u}) = \Pi(\mathbf{u}) := \prod_{i=1}^{n} u_i$. It is also assumed that $C_\theta(\mathbf{u}) = 0$ if $\theta > 0$ and $u_i = 0$ for at least one $i = 1, \dots, n$.

The Clayton copula is interesting as it can model various kinds of dependence, ranging from comonotonicity in the limit as $\theta \to \infty$, independence if $\theta \downarrow 0$ (also if $\theta \uparrow 0$) and countermonotonicity if $\theta = -1$ [7, p. 168]. This copula is often used in modelling when data shows asymmetry and lower tail dependence; see, e.g. [9, 16] and [17] in finance, [1] and [18] in insurance, [2] in multiple test theory, among many other applications. For some other facts on the role of Clayton copulas in the Archimedean copula families, see, e.g. [12] and [13].

The interest in obtaining sharper than the classical Fréchet–Hoeffding bounds (see, e.g. [15, p. 30]) for the Clayton copula is partially motivated by the investigations of Dindienė and Leipus [5] who wondered whether, given a sequence of random variables $X_1, X_2, \ldots$, such that for any integer $n \geqslant 1$ and $(x_1, \ldots, x_n) \in \mathbb{R}^n$, if

$$\mathbf{P}\left(\bigcap_{i=1}^{n}\{X_i \leqslant x_i\}\right) = C_\theta\bigl(F_1(x_1), \ldots, F_n(x_n)\bigr), \tag{1}$$

where $F_i(x_i) = \mathbf{P}(X_i \leqslant x_i)$, $i = 1, \ldots, n$, then there exists a $\kappa > 0$ such that

$$\mathbf{P}\left(\bigcap_{i=1}^{n}\{X_i > x_i\}\right) \leqslant \kappa \prod_{i=1}^{n} \mathbf{P}(X_i > x_i), \tag{2}$$

that is, random variables $X_1, X_2, \ldots$ are *upper extended negatively dependent* (or have the UEND property) (see Section 3.1). It is well known (see [15, Cor. 4.6.3]) that $C_\theta(\mathbf{u})$ is bounded from below (resp. above) by the independence copula Π if $\theta > 0$ (resp. $\theta \in [-1, 0)$):

$$\begin{aligned} C_\theta(\mathbf{u}) &\geqslant \Pi(\mathbf{u}) \quad \text{if } \theta > 0, \\ C_\theta(\mathbf{u}) &\leqslant \Pi(\mathbf{u}) \quad \text{if } \theta \in [-1, 0). \end{aligned} \tag{3}$$

In this paper, we provide two upper bounds on $C_\theta(\mathbf{u})$ if $\theta > 0$ and a lower bound in the case $\theta \in [-1, 0)$. The first bound (see Theorem 1 below) yields that random variables $X_1, X_2, \ldots$ satisfying (1) for any $n \geqslant 1$ are *pairwise UEND*, i.e. $\mathbf{P}(X_i > x_i, X_j > x_j) \leqslant (1 + \theta)\mathbf{P}(X_i > x_i)\mathbf{P}(X_j > x_j)$ for any $i \neq j$. However, the full UEND property requires further investigations, in particular, a sharper lower bound in Lemma 2 is needed.

Following Marshall et al. [10, p. xxvi], let us introduce some notations. For a vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $n \geqslant 2$, let $x_{[1]} \geqslant \cdots \geqslant x_{[n]}$ and $x_{(1)} \leqslant \cdots \leqslant x_{(n)}$ denote the components of x in decreasing and increasing order, respectively.

Our first result is the following theorem for the case $\theta > 0$:

Theorem 1. *Let $\theta > 0$ and $n \geqslant 2$. If $\mathbf{u} = (u_1, \ldots, u_n) \in [0, 1]^n$, then*

$$C_\theta(\mathbf{u}) \leqslant \theta(1 - u_{(1)} - u_{(2)}) + (1 + \theta)u_{(1)}u_{(2)}. \tag{4}$$

Remark 1. When $n = 2$ and $\theta > 0$, the above inequality is simply

$$\begin{aligned} C_\theta(u_1, u_2) &= \bigl(u_1^{-\theta} + u_2^{-\theta} - 1\bigr)^{-1/\theta} \\ &\leqslant \theta(1 - u_1 - u_2) + (1 + \theta)u_1u_2, \quad (u_1, u_2) \in [0, 1]^2, \end{aligned}$$

with no indication why the two smallest arguments $u_{(1)}$ and $u_{(2)}$ appear in the general case. This is essentially due to the upper bound in Lemma 2 (see also Remark 3 below).

An application of Gronwall's inequality allows obtaining another bound:

Theorem 2. *Let $\theta > 0$ and $n \geqslant 2$. If $\mathbf{u} = (u_1, \ldots, u_n) \in (0,1]^n$, then*

$$C_\theta(\mathbf{u}) \leqslant \Pi(\mathbf{u}) \exp\{\theta \ln u_{(1)} \ln u_{(2)}\}. \tag{5}$$

Note that neither $R_{1,\theta}(u_1, u_2) := \theta(1 - u_1 - u_2) + (1 + \theta)u_1 u_2$ nor $R_{2,\theta}(u_1, u_2) := u_1 u_2 \exp\{\theta \ln u_1 \ln u_2\}$ is a two-dimensional copula for any $\theta > 0$ as the former is not grounded, i.e. $\theta(1 - u_1) = R_{1,\theta}(u_1, 0) \not\equiv 0 \not\equiv R_{1,\theta}(0, u_2) = \theta(1 - u_2)$ whenever $(u_1, u_2) \in [0, 1)^2$ (see [15, (2.2.2a)]), and the latter is unbounded in a neighbourhood of the origin, e.g. if $x_m = y_m = \exp\{-m/\theta\}$, $m \geqslant 1$, then $R_{2,\theta}(x_m, y_m) = \exp\{m(m-2)/\theta\} \to +\infty$ as $m \to \infty$. Nevertheless, combining the obtained and Fréchet–Hoeffding upper bounds, we have

Proposition 1. *Let $\theta > 0$. The functions $T_{j,\theta} : [0,1]^2 \to [0,1]$, $j = 1, 2$, given by*

$$\begin{aligned} T_{j,\theta}(u_1, u_2) &:= \min\bigl\{M(u_1, u_2), R_{j,\theta}(u_1, u_2)\bigr\} \\ &= \begin{cases} R_{j,\theta}(u_1, u_2) & \text{if } \min\{u_1, u_2\} > \nu_j; \\ M(u_1, u_2) & \text{if } \min\{u_1, u_2\} \leqslant \nu_j, \end{cases} \end{aligned}$$

where $\nu_j = \theta/(1+\theta)\mathbf{1}_{\{j=1\}} + \mathrm{e}^{-1/\theta}\mathbf{1}_{\{j=2\}}$ and $M(u_1, u_2) := \min\{u_1, u_2\}$, are copulas.

On the other hand, in the case $\theta \in [-1, 0)$ we have

Theorem 3. *For any $\theta \in [-1, 0)$ and $2 \leqslant n \leqslant 1 - 1/\theta$, the following inequality holds*:

$$C_\theta(\mathbf{u}) \geqslant (-\theta)\left(\sum_{i=1}^{n} u_i - n + 1\right) + (1 + \theta)\Pi(\mathbf{u}), \quad \mathbf{u} = (u_1, \ldots, u_n) \in [0,1]^n. \tag{6}$$

Remark 2. When $n = 2$ and $\theta \in [-1, 0)$, inequality (6) becomes

$$C_\theta(u_1, u_2) \geqslant \theta(1 - u_1 - u_2) + (1 + \theta)u_1 u_2, \quad (u_1, u_2) \in [0,1]^2,$$

which is simply the reverse inequality discussed in Remark 1. So one may wonder why the reverse inequality of (4) is not featured in Theorem 3 when $n \geqslant 3$? Such an inequality simply fails already for $n = 3$ and, for example, $\theta = -1/4$. Indeed, by taking $\mathbf{u}_0 := (3/4, 3/4, 3/4)$, we get

$$C_{-1/4}(\mathbf{u}_0) = \bigl(3\sqrt[4]{3/4} - 2\bigr)^4 \approx 0.3931 \ngeqq \frac{1}{4}\left(2 \cdot \frac{3}{4} - 1\right) + \left(1 - \frac{1}{4}\right)\left(\frac{3}{4}\right)^2 = \frac{35}{64}.$$

Similar to the case when $\theta > 0$, the lower bound given by (6) is not a copula for any $n \geqslant 2$ as the right hand side is negative in a neighbourhood of the origin. Nevertheless, by enforcing the lower bound to stay nonnegative, i.e. by using the lower Fréchet–Hoeffding bound $W(u_1, u_2) := (u_1 + u_2 - 1)_+$ on the set where $R_{1,\theta}$ becomes negative, we recover a known result (see family (4.2.7) if $\theta \in (-1, 0]$ and family (4.2.1) if $\theta = -1$ in Table 4.1 of [15]; a simple reparametrization is needed in the first case)[1].

[1]We are grateful to prof. R.B. Nelsen for pointing this out at the Salzburg Workshop on Dependence Models & Copulas, organized on September 19–22, 2016, where the results of this paper were presented.

Proposition 2. *Let* $\theta \in [-1, 0]$. *The function* $T_{3,\theta} : [0,1]^2 \to [0,1]$ *given by*

$$T_{3,\theta}(u_1, u_2) := \max\bigl\{0, R_{1,\theta}(u_1, u_2)\bigr\}$$

is a copula (*in fact, Archimedean*).

The rest of the paper is organized as follows: Section 2 contains the proofs of the stated results. Section 3 provides the details of the connection of the obtained bounds and pairwise UEND property of a sequence of random variables joined by Clayton copula (see Section 3.1) and describes when the new bounds are superior to the classical Fréchet–Hoeffding bounds (see Section 3.2).

2 Proofs

In this section, the proofs of Theorems 1, 2 and 3 are split into several lemmas for easier readability. In particular, Lemmas 1, 2, and 3 provide ingredients for the proof of Theorem 1. The proof of Proposition 1 is given at the end.

Lemma 1. *If* $(u_1, u_2) \in (0, 1]^2$, *then* $u_{(1)} \ln u_1 \ln u_2 \leqslant (1 - u_1)(1 - u_2)$.

Proof. If $u_{(2)} = 1$ then the stated inequality trivially becomes equality. So assume $u_{(2)} < 1$. Using Karamata's inequality (see, e.g. [14, 3.6.15]), namely, $\ln x \leqslant (x - 1)/\sqrt{x}$ for all $x \geqslant 1$, we get $u_{(1)} \ln u_1 \ln u_2 \leqslant (1 - u_1)(1 - u_2)\sqrt{u_{(1)}/u_{(2)}} \leqslant (1 - u_1) \times (1 - u_2)$. □

Lemma 2. *For any* $\mathbf{x} = (x_1, \dots, x_n) \in (1, \infty)^n$, $n \geqslant 2$, *the following holds*:

$$0 < \frac{(\sum_{i=1}^n x_i - n + 1) \ln(\sum_{i=1}^n x_i - n + 1) - \sum_{i=1}^n x_i \ln x_i}{(\sum_{i=1}^n x_i - n + 1) \ln x_{[1]} \ln x_{[2]}} \leqslant 1. \tag{7}$$

Proof. The non-negativity of the numerator above and its strict positivity on $(1, +\infty)^n$ is a simple consequence of the majorization theory (see [10, Ch. 3]). Indeed, the function $\nu(x) = (1+x)\ln(1+x)$ is strictly convex on $I = (-1, \infty)$, hence $\eta(\mathbf{x}) := \sum_{i=1}^n \nu(x_i)$ is strictly Schur-convex on I^n (see [10, Prop. C.1a]), that is, $\eta(\hat{\mathbf{x}}) < \eta(\tilde{\mathbf{x}})$ if $\hat{\mathbf{x}}$ is majorized by $\tilde{\mathbf{x}}$, i.e.

$$\sum_{i=1}^k \hat{x}_{[i]} \leqslant \sum_{i=1}^k \tilde{x}_{[i]}, \quad k = 1, \dots, n-1, \quad \text{and} \quad \sum_{i=1}^n \hat{x}_{[i]} = \sum_{i=1}^n \tilde{x}_{[i]},$$

and $\hat{\mathbf{x}}$ is not a permutation of $\tilde{\mathbf{x}}$. Clearly, for any $\mathbf{x} = (x_1, \dots, x_n) \in (1, +\infty)^n$, $\hat{\mathbf{x}} := (x_1 - 1, \dots, x_n - 1)$ is majorized by $\tilde{\mathbf{x}} = (\sum_{i=1}^n (x_i - 1), 0, \dots, 0)$, which is not a permutation of $\hat{\mathbf{x}}$, hence

$$\eta(\hat{\mathbf{x}}) = \sum_{i=1}^n x_i \ln x_i < \eta(\tilde{\mathbf{x}}) = \left(\sum_{i=1}^n x_i - n + 1\right) \ln\left(\sum_{i=1}^n x_i - n + 1\right).$$

To prove the stated upper bound in (7), for any $\mathbf{x} = (x_1, \dots, x_n) \in [1, \infty)^n$, consider $f_n(\mathbf{x}) := g_n(\mathbf{x}) - (\sum_{i=1}^n x_i - n + 1) \ln x_{[1]} \ln x_{[2]}$, where $g_n(\mathbf{x}) := \eta(\tilde{\mathbf{x}}) - \eta(\hat{\mathbf{x}})$. We will show that $f_n(\mathbf{x}) \leqslant 0$ on $[1, \infty)^n$. Notice that, due to the symmetry of f_n, we can assume that $\mathbf{x} = (x_{[1]}, \dots, x_{[n]})$. Then

$$\frac{\partial f_n(\mathbf{x})}{\partial x_1} = \ln\Biggl(\sum_{i=1}^n x_i - n + 1\Biggr) - \ln x_1 - \ln x_1 \ln x_2 - \frac{\ln x_2}{x_1}\Biggl(\sum_{i=1}^n x_i - n + 1\Biggr)$$

and

$$\begin{aligned}
\frac{\partial^2 f_n(\mathbf{x})}{\partial x_1 \partial x_2} &= \Biggl(\sum_{i=1}^n x_i - n + 1\Biggr)^{-1} - \frac{\ln x_1}{x_2} - \frac{\ln x_2}{x_1} - \frac{\sum_{i=1}^n x_i - n + 1}{x_1 x_2} \\
&= -\frac{\ln x_1}{x_2} - \frac{\ln x_2}{x_1} - \frac{-x_1 x_2 + (\sum_{i=1}^n x_i - n + 1)^2}{x_1 x_2 \bigl(\sum_{i=1}^n x_i - n + 1\bigr)}.
\end{aligned} \tag{8}$$

Now, for $h_n(\mathbf{x}) := -x_1 x_2 + (\sum_{i=1}^n x_i - n + 1)^2$, $\mathbf{x} \in [1, \infty)^n$, we have

$$\frac{\partial h_n(\mathbf{x})}{\partial x_2} = -x_1 + 2\Biggl(\sum_{i=1}^n x_i - n + 1\Biggr) = x_1 + 2\sum_{i=2}^n (x_i - 1) \geqslant 1,$$

so that h_n is nondecreasing in $x_2 \in [1, \infty)$ and

$$\begin{aligned}
h_n(\mathbf{x}) &\geqslant h_n(x_1, 1, x_3, \dots, x_n) = -x_1 + \Biggl(x_1 + \sum_{i=3}^n (x_i - 1)\Biggr)^2 \\
&\geqslant -x_1 + x_1^2 \geqslant 0,
\end{aligned}$$

since all $x_i \geqslant 1$. Therefore, by (8), $\partial^2 f_n(\mathbf{x})/\partial x_1 \partial x_2 \leqslant 0$, implying that $\partial f_n/\partial x_1$ is nonincreasing in x_2, and so

$$\frac{\partial f_n(\mathbf{x})}{\partial x_1} \leqslant \frac{\partial f_n}{\partial x_1}(x_1, 1, x_3, \dots, x_n) = \frac{\partial f_n}{\partial x_1}(x_1, 1, \dots, 1) = 0,$$

since $x_2 = x_{[2]} = 1$ implies $x_2 = x_3 = \dots = x_n = 1$. This means that f_n is nonincreasing in x_1 and so $f_n(\mathbf{x}) \leqslant f_n(1, x_2, \dots, x_n) = f_n(1, \dots, 1) = 0$, since $x_1 = x_{[1]} = 1$ implies $x_1 = x_2 = \dots = x_n = 1$. $\square$

Remark 3. Some comments about the choice of $x_{[1]}$ and $x_{[2]}$ are in order. One can try, more generally, taking $(\sum_{i=1}^n x_i - n + 1) \ln x_{[m]} \ln x_{[k]}$ with $(m, k) \in \{(i, j)\colon i, j = 1, \dots, n\}$ in the denominator of the fraction in (7). Then since, clearly, $\ln x_{[m]} \ln x_{[k]} \leqslant \ln x_{[1]} \ln x_{[2]} \leqslant \ln^2 x_{[1]}$, for any $(m, k) \in \{(i, j)\colon i, j = 1, \dots, n\}$, $(m, k) \neq (1, 1)$, the upper bound in (7) would

- be false if $k \geqslant 3$, as on $(1, +\infty)^n$ the numerator of the fraction in (7) stays bounded and positive whereas the new considered denominator vanishes if $x_{[k]} \downarrow 1$ and $x_{[2]} > 1$ is kept fixed;

- be true, but inferior to the claim of Lemma 2, if $k = m = 1$;
- be false if $k = m = 2$ in general, e.g. if $n = 2$ and $(x_1, x_2) = (1.1, 1.5)$, then $g_2(1.1, 1.5) > 0.038966 > 0.014535 > 1.6(\ln 1.1)^2$.

Lemma 3. *Let $\theta > 0$. Then for any $\mathbf{u} = (u_1, \dots, u_n) \in (0,1)^n$, $n \geqslant 2$,*

$$0 \leqslant \frac{\partial}{\partial\theta} C_\theta(\mathbf{u}) \leqslant C_\theta(\mathbf{u}) \ln u_{(1)} \ln u_{(2)} \leqslant u_{(1)} \ln u_{(1)} \ln u_{(2)}. \tag{9}$$

Proof. Write $C_\theta(\mathbf{u}) = \exp\{H_\theta(\mathbf{u})\}$, so that

$$\frac{\partial}{\partial\theta} C_\theta(\mathbf{u}) = C_\theta(\mathbf{u}) \frac{\partial}{\partial\theta} H_\theta(\mathbf{u}), \tag{10}$$

where

$$\frac{\partial}{\partial\theta} H_\theta(\mathbf{u}) = \frac{g_n(u_1^{-\theta}, \dots, u_n^{-\theta})}{\theta^2 \left(\sum_{i=1}^n u_i^{-\theta} - n + 1\right)}$$

and $g_n(\mathbf{x}) := (\sum_{i=1}^n x_i - n + 1) \ln(\sum_{i=1}^n x_i - n + 1) - \sum_{i=1}^n x_i \ln x_i$. For $x_i := u_i^{-\theta} \in (1, \infty)$, $i = 1, \dots, n$, we have $x_{[1]} = u_{(1)}^{-\theta}$, $x_{[2]} = u_{(2)}^{-\theta}$ and $\theta^2 = (\ln x_{[1]} \ln x_{[2]}) / (\ln u_{(1)}^{-1} \ln u_{(2)}^{-1})$. Lemma 2 gives

$$0 \leqslant \frac{\partial}{\partial\theta} H_\theta(\mathbf{u}) = \frac{\ln u_{(1)}^{-1} \ln u_{(2)}^{-1} \, g_n(\mathbf{x})}{\left(\sum_{i=1}^n x_i - n + 1\right) \ln x_{[1]} \ln x_{[2]}} \leqslant \ln u_{(1)} \ln u_{(2)}. \tag{11}$$

Since $C_\theta(\mathbf{u})$ is a copula, the upper Fréchet–Hoeffding bound yields $C_\theta(\mathbf{u}) \leqslant u_{(1)}$, and the last inequality in (9) follows from (10) and (11). $\square$

Proof of Theorem 1. If $u_{(2)} = 1$, the stated inequality becomes an equality. So assume $u_{(2)} < 1$. It is known (see, e.g. [15, p. 115]) that $\lim_{\theta \downarrow 0} C_\theta(\mathbf{u}) = \prod_{i=1}^n u_i$. Thus it is enough to show that, for any $\epsilon \in (0, \theta)$,

$$C_\theta(\mathbf{u}) - C_\epsilon(\mathbf{u}) = \int_\epsilon^\theta \frac{\partial C_x(\mathbf{u})}{\partial x} \, \mathrm{d}x \leqslant (\theta - \epsilon)(1 - u_{(1)})(1 - u_{(2)}) \tag{12}$$

and then pass to the limit as $\epsilon \downarrow 0$. Due to Lemmas 3 and 1, (12) follows from

$$\int_\epsilon^\theta \left(\frac{\partial C_x(\mathbf{u})}{\partial x} - (1 - u_{(1)})(1 - u_{(2)}) \right) \mathrm{d}x \leqslant 0 \quad \forall \epsilon > 0. \qquad \square$$

Proof of Theorem 2. Inspecting equations (10)–(12), we see that

$$C_\theta(\mathbf{u}) \leqslant C_\epsilon(\mathbf{u}) + \ln u_{(1)} \ln u_{(2)} \int_\epsilon^\theta C_x(\mathbf{u}) \, \mathrm{d}x$$

for $\mathbf{u} \in [0,1]^n$, $\theta > 0$ and $\epsilon \in (0, \theta)$. Applying of Gronwall's inequality (see, e.g. [6]) to the function $\phi(x) := C_{x+\epsilon}(\mathbf{u})$, $x \in [0, \theta - \epsilon]$ yields $C_\theta(\mathbf{u}) \leqslant C_\epsilon(\mathbf{u}) \exp\{(\theta - \epsilon) \times \ln u_{(1)} \ln u_{(2)}\}$. Passing to the limit as $\epsilon \downarrow 0$, we obtain the claim of the theorem. $\square$

Proof of Theorem 3. If $\theta = -1$, the stated inequality is trivial. So we only need to consider the case $\theta \in (-1,0)$. For $\mathbf{u} = (u_1,\dots,u_n) \in [0,1]^n$, define the function

$$G_\theta^{(n)}(\mathbf{u}) := C_\theta(\mathbf{u}) + \theta\left(\sum_{i=1}^n u_i - n + 1\right) - (1+\theta)\Pi(\mathbf{u}),$$

which is obviously jointly continuous. We claim that $G_\theta^{(n)}(\mathbf{u}) \geqslant 0$ on the hypercube $[0,1]^n$, which we divide into two sets:

$$A := \left\{(u_1,\dots,u_n) \in [0,1]^n\colon \sum_{i=1}^n u_i^{-\theta} > n-1\right\} \quad \text{and} \quad B := [0,1]^n \setminus A.$$

Then

$$\begin{aligned}\frac{\partial G_\theta^{(n)}(\mathbf{u})}{\partial u_1} &= u_1^{-\theta-1}\left(\sum_{i=1}^n u_i^{-\theta} - n + 1\right)^{-1/\theta-1}\mathbf{1}_A(\mathbf{u}) + \theta - (1+\theta)\prod_{i=2}^n u_i \\ &= u_1^{-\theta-1}\big(C_\theta(\mathbf{u})\big)^{\theta+1}\mathbf{1}_A(\mathbf{u}) + \theta - (1+\theta)\prod_{i=2}^n u_i.\end{aligned}$$

On the set A, since $-1/\theta - 1 > 0$ and $C_\theta(\mathbf{u}) \leqslant \Pi(\mathbf{u})$ for $\theta \in (-1,0)$ and $\mathbf{u} \in [0,1]^n$, we have

$$\begin{aligned}\frac{\partial G_\theta^{(n)}(\mathbf{u})}{\partial u_1} &\leqslant \left(\prod_{i=2}^n u_i\right)^{1+\theta} + \theta - (1+\theta)\prod_{i=2}^n u_i \\ &\leqslant 1 + (1+\theta)\left(\prod_{i=1}^n u_i - 1\right) + \theta - (1+\theta)\prod_{i=1}^n u_i = 0\end{aligned}$$

by Bernoulli inequality. On the set B, $\partial G_\theta^{(n)}(\mathbf{u})/\partial u_1 \leqslant 0$ trivially. Hence, for each fixed $(u_2,\dots,u_n) \in [0,1]^{n-1}$, $G_\theta^{(n)}(\mathbf{u})$ (being continuous and piece-wise differentiable) is nonincreasing in u_1, which gives

$$G_\theta^{(n)}(\mathbf{u}) \geqslant G_\theta^{(n)}(1,u_2,\dots,u_n) \geqslant 0, \tag{13}$$

where the last inequality follows by induction. Indeed, for $n=2$, $G_\theta^{(2)}(1,u_2) = 0$ for any $u_2 \in [0,1]$. Suppose $G_\theta^{(n)}(1,u_2,\dots,u_n) \geqslant 0$ for any $n = 2,\dots,k-1$. When $n = k \geqslant 3$,

$$G_\theta^{(k)}(1,u_2,\dots,u_k) = G_\theta^{(k-1)}(u_2,\dots,u_k) \geqslant G_\theta^{(k-1)}(1,u_3,\dots,u_k) \geqslant 0$$

by the first inequality in (13) and induction hypothesis. □

Proof of Proposition 1. First observe that both $T_{1,\theta}$ and $T_{2,\theta}$ satisfy the required boundary conditions of a copula:

$$T_{j,\theta}(0,u_2) = T_{j,\theta}(u_1,0) = 0, \quad j = 1,2,$$

and

$$T_{1,\theta}(u_1, 1) = T_{2,\theta}(u_1, 1) = u_1, \qquad T_{1,\theta}(1, u_2) = T_{2,\theta}(1, u_2) = u_2.$$

It remains to show that both $T_{1,\theta}$ and $T_{2,\theta}$ are 2-increasing, i.e. for any rectangle $A_0 := [a_1, a_2] \times [b_1, b_2] \subset [0, 1]^2$ with $0 \leqslant a_1 \leqslant a_2 \leqslant 1$ and $0 \leqslant b_1 \leqslant b_2 \leqslant 1$,

$$\begin{aligned} V_{T_{j,\theta}}(A_0) &:= T_{j,\theta}(a_1, b_1) - T_{j,\theta}(a_2, b_1) - T_{j,\theta}(a_1, b_2) + T_{j,\theta}(a_2, b_2) \\ &\geqslant 0, \quad j = 1, 2. \end{aligned}$$

Split the square $[0, 1]^2$ into four non-overlapping (except for touching boundaries) squares:

$$\begin{aligned} &A_{1,j} := [0, 0] \times [\nu_j, \nu_j], \quad A_{2,j} := [\nu_j, 1] \times [0, \nu_j], \\ &A_{3,j} := [0, \nu_j] \times [\nu_j, 1], \quad A_{4,j} := [\nu_j, 1] \times [\nu_j, 1], \quad j = 1, 2, \end{aligned}$$

(for the choice of ν_j, see Section 3.2 (i) and (ii)). Then the intersections $A_0 \cap A_{i,j}$, $i = 1, 2, 3$; $j = 1, 2$, are again rectangles (possibly line segments or even empty) and

$$V_{T_{j,\theta}}(A_0) = V_{R_{j,\theta}}(A_0 \cap A_{j,4}) + \sum_{i=1}^{3} V_M(A_0 \cap A_{j,i}).$$

As M is a copula, $V_M(A_0 \cap A_{j,i}) \geqslant 0$ for each $i = 1, 2, 3$ and $j = 1, 2$. Moreover, if $A_0 \cap A_{j,4} = [c_{1,j}, c_{2,j}] \times [d_{1,j}, d_{2,j}] \neq \emptyset$, where $\nu_j \leqslant c_{1,j} \leqslant c_{2,j} \leqslant 1$ and $\nu_j \leqslant d_{1,j} \leqslant d_{2,j} \leqslant 1$, then also

$$V_{R_{1,\theta}}(A_0 \cap A_{1,4}) = (1 + \theta)(c_{2,1} - c_{1,1})(d_{2,1} - d_{1,1}) \geqslant 0 \tag{14}$$

and

$$V_{R_{2,\theta}}(A_0 \cap A_{2,4}) = z_\theta(c_{2,2}; d_{1,2}, d_{2,2}) - z_\theta(c_{1,2}; d_{1,2}, d_{2,2}) \geqslant 0,$$

since the function $z_\theta(x; a, b) := x b^{1+\theta \ln x} - x a^{1+\theta \ln x}$ for $\mathrm{e}^{-1/\theta} \leqslant a \leqslant b \leqslant 1$ and $x \in [\mathrm{e}^{-1/\theta}, 1]$ is nondecreasing in x, which follows from

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} z_\theta(x; a, b) &= b^{1+\theta \ln x}(1 + \theta \ln b) - a^{1+\theta \ln x}(1 + \theta \ln a) \\ &\geqslant (1 + \theta \ln a)\big(b^{1+\theta \ln x} - a^{1+\theta \ln a}\big) \geqslant 0. \end{aligned}$$

Hence both $T_{1,\theta}$ and $T_{2,\theta}$ are bivariate copulas. □

3 Discussion

In this section we discuss an application of the obtained bounds to certain negative dependence structures mentioned in Section 1 as well as give a comparison with the classical Fréchet–Hoeffding bounds.

3.1 Connection to UEND structures

We now discuss some of the applications of the obtained bounds. For $n = 2$, Theorems 1 and 3 yield an interesting probabilistic interpretation, related to certain dependence structures. More explicitly, suppose random variables X_1 and X_2 are distributed according to laws F_1 and F_2, respectively, and satisfy (1) with $n = 2$. Then, by (3) and (4), for $\theta > 0$,

$$\overline{F_1}(x)\overline{F_2}(y) \leqslant \mathbf{P}(X_1 > x,\, X_2 > y) \leqslant (1+\theta)\overline{F_1}(x)\overline{F_2}(y),$$

where $\overline{F_i} := 1 - F_i$, $i = 1, 2$, i.e. the variables X_1, X_2 are both *positively dependent* and *upper extended negatively dependent* (UEND) (see [8]). Similarly, if $-1 \leqslant \theta < 0$, then

$$(1+\theta)\overline{F_1}(x)\overline{F_2}(y) \leqslant \mathbf{P}(X_1 > x,\, X_2 > y) \leqslant \overline{F_1}(x)\overline{F_2}(y)$$

and, similarly, variables X_1, X_2 are both *upper extended positively dependent* (UEPD) and *negatively dependent*. Note that the mentioned UEND and UEPD properties are much easier to verify for classical Farley–Gumbel–Morgenstern, Frank or Ali–Mikhail–Haq copulas (see [5]).

Unfortunately, extending the UEND property for $\theta > 0$ (resp. UEPD for $\theta \in [-1, 0)$) to $n \geqslant 3$ random variables $X_1, \dots, X_n$ with mutual distribution function generated by the Clayton copula (see (1)) requires a sharper lower bound (resp. upper bound) on C_θ than provided by the independence copula Π. Indeed, e.g. for $n = 3$ and any $(x_1, \dots, x_n) \in \mathbb{R}^n$, we have, by Sklar's theorem and inclusion–exclusion principle,

$$\begin{aligned}\mathbf{P}\Biggl(\bigcap_{i=1}^{3}\{X_i > x_i\}\Biggr) &= -2 + \sum_{i=1}^{3}\overline{F_i}(x_i) + \sum_{1\leqslant i<j\leqslant 3} C_\theta\bigl(1-\overline{F_i}(x_i), 1-\overline{F_j}(x_j)\bigr)\\ &\quad - C_\theta\bigl(1-\overline{F_1}(x_1),\, 1-\overline{F_2}(x_2),\, 1-\overline{F_3}(x_3)\bigr).\end{aligned} \tag{15}$$

Now since Clayton copula is Archmedean and (3) holds, we can write

$$C_\theta(u_1, u_2, u_3) = C_\theta\bigl(u_1, C_\theta(u_2, u_3)\bigr) \geqslant u_1 C_\theta(u_2, u_3), \quad (u_1, u_2, u_3) \in [0,1]^3.$$

Using this and Theorem 1 for two of the three terms in the second sum of (15), we get

$$\begin{aligned}\mathbf{P}\Biggl(\bigcap_{i=1}^{3}\{X_i > x_i\}\Biggr) &\leqslant -2 + \sum_{i=1}^{3}\overline{F_i}(x_i) + C_\theta\bigl(1-\overline{F_2}(x_2),\, 1-\overline{F_3}(x_3)\bigr)\\ &\quad + \bigl(1-\overline{F_1}(x_1)\bigr)\bigl(1-\overline{F_2}(x_2)\bigr) + \theta\overline{F_1}(x_1)\overline{F_2}(x_2)\\ &\quad + \bigl(1-\overline{F_1}(x_1)\bigr)\bigl(1-\overline{F_3}(x_3)\bigr) + \theta\overline{F_1}(x_1)\overline{F_3}(x_3)\\ &\quad - \bigl(1-\overline{F_1}(x_1)\bigr)C_\theta\bigl(1-\overline{F_2}(x_2), 1-\overline{F_3}(x_3)\bigr)\\ &= (1+\theta)\overline{F_1}(x_1)\bigl(\overline{F_2}(x_2) + \overline{F_3}(x_3)\bigr)\\ &\quad + \overline{F_1}(x_1)\bigl(C_\theta\bigl(1-\overline{F_2}(x_2), 1-\overline{F_3}(x_3)\bigr) - 1\bigr)\\ &\leqslant \theta\overline{F_1}(x_1)\bigl(\overline{F_2}(x_2) + \overline{F_3}(x_3)\bigr) + (1+\theta)\prod_{i=1}^{3}\overline{F_i}(x_i),\end{aligned}$$

since, by Theorem 1,

$$C_\theta\bigl(1-\overline{F_2}(x_2),1-\overline{F_3}(x_3)\bigr)-1\leqslant(1+\theta)\overline{F_2}(x_2)\overline{F_3}(x_3)-\bigl(\overline{F_2}(x_2)+\overline{F_3}(x_3)\bigr).$$

Of course, by symmetry, we can replace the term $\overline{F_1}(x_1)(\overline{F_2}(x_2)+\overline{F_3}(x_3))$ of the last inequality by

$$\min\bigl\{\overline{F_1}(x_1)\bigl(\overline{F_2}(x_2)+\overline{F_3}(x_3)\bigr),\overline{F_2}(x_2)\bigl(\overline{F_1}(x_1)+\overline{F_3}(x_3)\bigr),\overline{F_3}(x_3)\bigl(\overline{F_1}(x_1)+\overline{F_1}(x_1)\bigr)\bigr\},$$

but it still dominates the product $\prod_{i=1}^{3}\overline{F_i}(x_i)$ when all $\overline{F_i}(x_i)$, $i=1,2,3$, are close to zero. A sharper upper bound on the joint survival function could be possible provided a better lower bound in Lemma 2 is obtained. This is left for future research.

Note that the extended negative dependence concept has been demonstrated to be important in proving limit theorems of probability theory such as the strong law of large numbers (see, e.g. [3, 11]), showing some max–sum equivalence properties for heavy-tailed distributions (see, e.g. [4, 5]) or obtaining precise large deviations (see [8]).

3.2 New bounds vs. Fréchet–Hoeffding bounds

In this section, we compare the new bounds given in (4), (5) and (6) with the classical Fréchet–Hoeffding bounds (see, e.g. [15, p. 30]), more precisely, with the upper bound $C_\theta(\mathbf{u})\leqslant M(\mathbf{u}):=u_{(1)}$ for $\theta>0$ and with the lower bound $C_\theta(\mathbf{u})\geqslant W(\mathbf{u}):=\max\{\sum_{i=1}^{n}u_i-n+1,0\}$ for $\theta\in[-1,0)$ and $\mathbf{u}=(u_1,\dots,u_n)\in[0,1]^n$. A typical plot of the Clayton copula (with $\theta=0.4$) is shown in Fig. 1(a). Figures 1(b) and 1(c) show its upper bounds $R_{1,\theta}$ and $R_{2,\theta}$, respectively, while Figs. 1(d) and 1(e) provide plots of the corresponding bounding copulas. To compare the improvement over classical Fréchet–Hoeffding bounds, Fig. 2 shows the plots of the differences $M-C_\theta$, $T_{1,\theta}-C_\theta$ and $T_{2,\theta}-C_\theta$, respectively, when $\theta=0.4$ while Fig. 3 provides similar graphs for $\theta=-0.4$. All graphs were produced using Maple computer algebra software by Maplesoft.

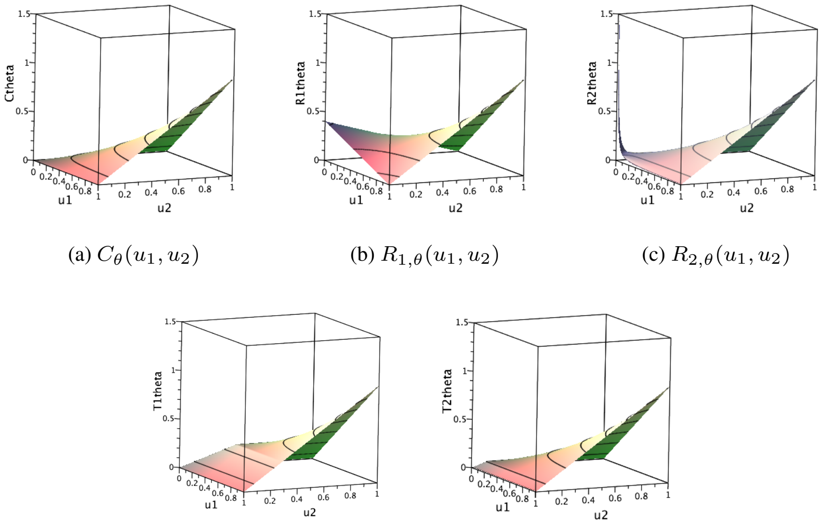
To make comparisons more concise, let

$$R_{1,\theta}(\mathbf{u}):=\theta(1-u_{(1)}-u_{(2)})+(1+\theta)u_{(1)}u_{(2)},\quad \mathbf{u}\in[0,1]^n,$$
$$R_{2,\theta}(\mathbf{u}):=\Pi(\mathbf{u})\exp\{\theta\ln u_{(1)}\ln u_{(2)}\},\quad \mathbf{u}\in(0,1]^n.$$

Then we have:

(i) Bound (4) for $u_{(2)}<1$ is sharper than $M(\mathbf{u})$ whenever $u_{(1)}\geqslant\theta/(1+\theta)$, i.e. $R_{1,\theta}(\mathbf{u})\leqslant u_{(1)}$ if and only if $u_{(1)}\geqslant\theta/(1+\theta)$ (see Fig. 1(d)). If $u_{(2)}=1$, bound (4) coincides with $M(\mathbf{u})$.

(ii) Bound (5) is superior to $M(\mathbf{u})$ for $u_{(1)}>0$ if and only if $\theta\ln u_{(1)}\ln u_{(2)}+\sum_{j=2}^{n}\ln u_{(j)}\leqslant 0$ (see Fig. 1(e)). The case $u_{(1)}=0$ is not considered in Theorem 2 since $R_{2,\theta}$ can be defined at $u_{(1)}=0$ only for $u_{(2)}\geqslant \mathrm{e}^{-1/\theta}$ by continuity as 0 if $u_{(2)}>\mathrm{e}^{-1/\theta}$ and as $\mathrm{e}^{-1/\theta}\prod_{j=3}^{n}u_{(j)}$ if $u_{(2)}=\mathrm{e}^{-1/\theta}$. In particular, for $u_{(2)}<1$, bound (5) is sharper whenever $u_{(1)}\geqslant\mathrm{e}^{-1/\theta}$. If $u_{(2)}=1$, bound (5) coincides with $M(\mathbf{u})$.



(a) $C_\theta(u_1, u_2)$ (b) $R_{1,\theta}(u_1, u_2)$ (c) $R_{2,\theta}(u_1, u_2)$

(d) $T_{1,\theta}(u_1, u_2)$ (e) $T_{2,\theta}(u_1, u_2)$

Figure 1. Plots of Clayton copula and its upper bounds ($\theta = 0.4$) together with new copulas obtained from them. Surfaces show contour lines.

(iii) Since $\mathrm{e}^{-1/\theta} \leqslant \theta/(1+\theta)$ for $\theta > 0$, we can clearly see that bound (5) is sharper than (4) in the range $u_{(1)} \in [\mathrm{e}^{-1/\theta}, \theta/(1+\theta))$. In fact, a bit more can be asserted. At least when $n = 2$, this interval extends to $[\mathrm{e}^{-1/\theta}, 1]$ (compare Figs. 1(d) and 1(e)). To see this, for $\mathbf{u} = (u_1, u_2)$ consider $Q_\theta(\mathbf{u}) := R_{1,\theta}(\mathbf{u}) - R_{2,\theta}(\mathbf{u}) = u_1 u_2 (1 + \theta(u_1^{-1} - 1)(u_2^{-1} - 1) - \exp\{\theta \ln u_1 \ln u_2\}) = u_1 u_2 h_\theta(u_1^{-1}, u_2^{-1})$, where

$$h_\theta(x, y) := 1 + \theta(x-1)(y-1) - \exp\{\theta \ln x \ln y\}, \quad x, y \in \bigl[1, \mathrm{e}^{1/\theta}\bigr].$$

Since

$$\frac{\partial^2 h_\theta}{\partial x^2}(x, y) = \frac{\theta \ln y \exp\{\theta \ln x \ln y\}}{x^2}(1 - \theta \ln y) \geqslant 0$$

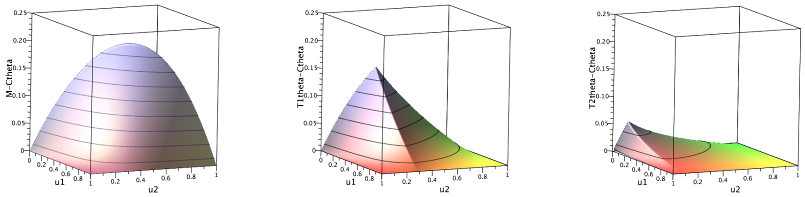
for $x, y \in [1, \mathrm{e}^{1/\theta}]$, the first partial derivative $\partial h_\theta / \partial x$ is nondecreasing in x and satisfies

$$\begin{aligned} \frac{\partial h_\theta}{\partial x}(x, y) &= \theta(y-1) - \frac{\theta \ln y \exp\{\theta \ln x \ln y\}}{x} \geqslant \frac{\partial h_\theta}{\partial x}(1, y) \\ &= \theta(y - 1 - \ln y) \geqslant 0. \end{aligned}$$

Hence, for any $y \in [1, \mathrm{e}^{1/\theta}]$,

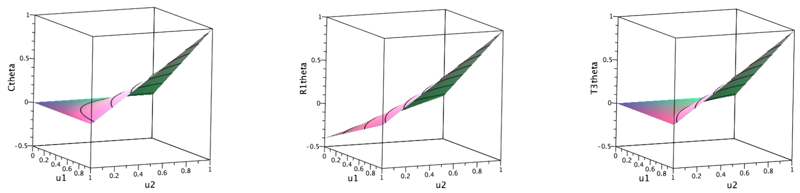
$$h_\theta(x, y) \geqslant h_\theta(1, y) = 0,$$

which yields $Q_\theta(u_1, u_2) \geqslant 0$ whenever $u_{(1)} \in [\mathrm{e}^{-1/\theta}, 1]$.

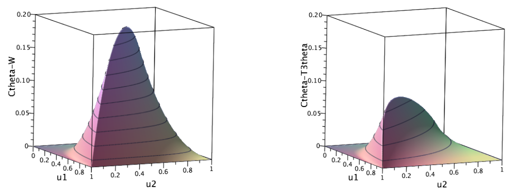


(a) $M(u_1, u_2) - C_\theta(u_1, u_2)$ (b) $T_{1,\theta}(u_1, u_2) - C_\theta(u_1, u_2)$ (c) $T_{2,\theta}(u_1, u_2) - C_\theta(u_1, u_2)$

Figure 2. Plots of differences between various upper bounds and Clayton copula ($\theta = 0.4$). Surfaces show contour lines.



(a) $C_\theta(u_1, u_2)$ (b) $R_{1,\theta}(u_1, u_2)$ (c) $T_{3,\theta}(u_1, u_2)$



(d) $C_\theta(u_1, u_2) - W(u_1, u_2)$ (e) $C_\theta(u_1, u_2) - T_{3,\theta}(u_1, u_2)$

Figure 3. Plots of Clayton copula and its lower bound ($\theta = -0.4$) together with the new copula $T_{3,\theta}(u_1, u_2)$ as well as the differences $C_\theta(u_1, u_2) - W(u_1, u_2)$ and $C_\theta(u_1, u_2) - T_{3,\theta}(u_1, u_2)$. Surfaces show contour lines.

(iv) If $\mathbf{u} = (u_1, \dots, u_n) \in [0, 1]^n$ satisfies $\sum_{i=1}^n u_i - n + 1 > 0$ (i.e. $W(\mathbf{u}) > 0$, in particular $u_{(1)} > 0$), then bound (6) is sharper than the lower Fréchet–Hoeffding bound (i.e. given by $W(\mathbf{u})$) by Weierstrass inequality (see, e.g. [14, 3.2.37(1)]):

$$\prod_{i=1}^n u_i = \prod_{i=1}^n \bigl(1 - (1 - u_i)\bigr) \geqslant 1 - \sum_{i=1}^n (1 - u_i) = \sum_{i=1}^n u_i - n + 1,$$

for any $u_i \in (0, 1]$, $i = 1, \dots, n$ (the inequality is sharp if at least two u_is are less than 1). On the other hand, if $\mathbf{u}$ is such that $W(\mathbf{u}) = 0$, then (6) is sharper if the right hand side of (6) is non-negative. When $\theta = -0.4$ and $n = 2$, this is illustrated in Fig. 3.

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