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ASYMPTOTIC DISTRIBUTIONS RELATED TO THE EWENS SAMPLING FORMULA

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ASIMPTOTINIAI PASISKIRSTYMAI EVENSO FORMULĖS ATŽVILGIU

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Notation

$\mathbf{N}$	the set of natural numbers
$\mathbf{R}$	the set of real numbers
$\mathbf{Z}_+$	the set of nonnegative integer numbers $\mathbf{N} \cup \{0\}$
$\mathbf{C}$	the set of complex numbers
$\mathbf{Z}_+^n$	the set of vectors $\bar{s} := (s_1, \dots, s_n)$, where $s_j \in \mathbf{Z}_+$ and $1 \le j \le n$
$\ell(\bar{s}) = 1s_1 + \cdots + ns_n$	the linear combinations defining the mapping $\ell : \mathbf{Z}_+^n \to \mathbf{Z}_+$
$\ell^{-1}(m) = \big\{\bar{s} \in \mathbf{Z}_+^n : \ \ell(\bar{s}) = m\big\}$	the pre-image, where $m \in \mathbf{Z}_+$
$\mathbf{S_n}$	the symmetric group of permutations acting on $n \ge 1$ letters
σ	a permutation in the symmetric group $\mathbf{S_n}$
$w = w(\sigma)$	the number of cycles of a permutation σ
$k_j(\sigma)$	the number of cycles of length j in a permutation σ
$\bar{k}(\sigma) = \big(k_1(\sigma), \dots, k_n(\sigma)\big)$	the cycle vector of a $\sigma \in \mathbf{S_n}$
$x^{(m)} = x(x+1)\cdots(x+m-1)$	rising factorial if $m \in \mathbf{N}$ and $x^{(0)} := 1$
$x_{(m)} = x(x-1)\cdots(x-m+1)$	falling factorial if $m \in \mathbf{N}$ and $x_{(0)} := 1$
$\psi_n(m) = \frac{n!}{\theta^{(n)}} \frac{\theta^{(m)}}{m!}$	the product of the binomial coefficients
$\nu_{n,\theta}(\cdot)$	the Ewens Probability Measure on $\mathbf{S}_n$, defined by $\nu_{n,\theta}(\{\sigma\}) = \theta^{w(\sigma)}/\theta^{(n)}$
$h(\sigma) = a_1 k_1(\sigma) + \cdots + a_n k_n(\sigma)$	a completely additive function, $h : \mathbf{S}_n \to \mathbf{R}$ for $a_{nj} \in \mathbf{R}$
	for brevity we denote $a_j := a_{nj}$
$h_n(\sigma) = a_{n1} k_1(\sigma) + \cdots + a_{nn} k_n(\sigma)$	a sequence of completely additive functions
$V_{n,\theta}(x) = \nu_{n,\theta}\big(h_n(\sigma) < x\big)$	the distribution function of $h_n(\sigma)$ with respect to $\nu_{n,\theta}$
$\mathbf{E}_{n,\theta} h_n(\sigma)$	the mean value of $h_n(\sigma)$ with respect to $\nu_{n,\theta}$
$Var_{n,\theta} h_n(\sigma)$	the variance of $h_n(\sigma)$ with respect to $\nu_{n,\theta}$
$\hat{\gamma}_{nr,\theta} = \mathbf{E}_{n,\theta} h_n(\sigma)_{(r)}$	the rth factorial moment
r.v.	random variable
ξ_j	the Poisson r.v. with parameter θ/j
$X_n = a_1\xi_1 + \cdots + a_n\xi_n$	the linear combination, where ξ_j, $1 \le j \le n$, are mutually independent
$\mathbf{E}X$	the mean value of a r.v. X defined on a probability space $\{\Omega, \mathcal{F}, P\}$
$\mathbf{E}X_{(r)}$	the rth factorial moment of a r.v. X
$L(X, P)$	the Lévy distance of a r.v. X from the set of constants
$P_{n,\theta}(\{\bar{s}\}) = \frac{n!}{\theta^{(n)}} \prod_{j \le n} \left(\frac{\theta}{j}\right)^{s_j} \frac{1}{s_j!}$	the Ewens Sampling Formula defining a measure on the subsets of $\ell^{-1}(n)$
$G(\bar{s}) = \prod_{j \le n} g_j(s_j)$	a multiplicative function $G : \mathbf{Z}_+^{\mathbf{n}} \to \mathbf{C}$
$\mathfrak{M}$	the set of multiplicative functions $G : \mathbf{Z}_+^{\mathbf{n}} \to \mathbf{C}$
$\mathfrak{M}_c$	the set of completely multiplicative functions $G : \mathbf{Z}_+^{\mathbf{n}} \to \mathbf{C}$, defined by
	$g_j(k) = g_j^k$, $g_j \in \mathbf{C}$, for $k \ge 0$ and $j \le n$; $0^0 := 1$
$\mathfrak{M}_s$	the set of strongly multiplicative functions $G : \mathbf{Z}_+^{\mathbf{n}} \to \mathbf{C}$, defined by
	$g_j(k) = g_j(1)$ for $k \ge 1$ and $j \le n$
$M_{n,\theta}(G)$	the mean value of a multiplicative function G with respect to $P_{n,\theta}$

$S(n,l)$	Stirling's number of the second kind
$s(n,l)$	Stirling's number of the first kind
$\lvert A\rvert$	cardinality of the set A
$\mathbf{1}\{\dots\}$	the indicator function
$a \wedge b = \min\{a,b\}$	if $a,b \in \mathbf{R}$
$a \vee b = \max\{a,b\}$	if $a,b \in \mathbf{R}$
$u^\circ = (1 \wedge \lvert u\rvert)\operatorname{sgn} u$	if $u \in \mathbf{R}$
$\ll$	equivalent to $O(\cdot)$
$a \asymp b$	equivalent to $a \ll b$ and $b \ll a$.

1 Introduction

1.1 Research problem

The weak convergence of distributions of additive functions defined on the symmetric group with respect to the Ewens probability measure is investigated in this thesis.

1.2 Actuality

The object of investigation and examined problems refer to probabilistic combinatorics, an important branch of contemporary mathematics. It has far-reaching applications in theoretical computer science, statistical physics, mathematical genetics, and other directions of sciences where large classes of combinatorial structures appear. Facing difficulties to show the existence of any unique object in a certain class, it is convenient to introduce a probability measure in it. Then it suffices to show that the probability of examined objects is positive. As a result, the conclusion is done and such an object exists. When the cardinality of a class is out of reach for computers, and exact methods to describe properties of individual elements fail, the only way to discover needed information about a "typical" object in the class is "to take an element at random" and describe it by the probability means. Nowadays probabilistic combinatorics offers general methods; the development of new ones is especially desirable, however.

Recent growing interest in random combinatorial structures is very notable. This, in particular, refers to decomposable structures. Among them, the most popular and important example is permutations. In this work, we analyze permutations which, by definition, are bijective mappings on a finite set into itself. All permutations of a finite set comprise the symmetric group. A permutation can be decomposed into cycles which gives the cycle structure vector hiding the most important properties. To discover them, one defines additive and multiplicative functions. If a permutation is taken at random, the mentioned functions become sums and product of dependent random variables (r.vs). In this regard, the objectives of the present work may be attributed to probability theory.

We now give two hints about possible applications. Firstly, we mention that particular additive functions are good approximations for the logarithm of group theoretical order of a permutation, which is important in algebra, especially, in the Galua theory. Secondly, in physics, some phenomena are simulated by random unitary matrices. The permutation matrices corresponding to a symmetric group are the simplest instances. The real and imaginary parts of logarithm of their characteristic polynomials are also additive functions. Moreover, many of the trace type functionals over their eigenvalues also fall within the scope of our theory. In this work, we deal with the asymptotic distributions of additive functions defined on the symmetric group as the order of the group tends to infinity. The group is endowed with the weighted probability called Ewens Probability Measure (EPM). The very motivation comes from the observation that a class of conjugate permutations can be identified to the common cycle vector of its representatives and the probability of this class coincides with the Ewens Sampling Formula (ESF) of a vector from the semi-lattice of vectors with non-negative coordinates. So, the probabilistic theory of permutations under the EPM is equivalent to that of the vectors under ESF. The latter formula was introduced in 1972 by W.J. Ewens to model the mutation in a population genetics. Nowadays this formula plays a crucial role in other branches of mathematical statistics. An advance in the theory of random permutations directly has its interpretation in these theories.

Finally, the actuality of the subject is supported by the mathematical interest to advance the very theory. The theory of value distribution of additive functions on the symmetric group has much in common with probabilistic number theory. Having started almost at the same time, the latter is a bit ahead. The task to fill up the existing gaps in probabilistic combinatorics is also very actual from mathematical point of view.

1.3 Aims and tasks

The main purpose of the thesis is to investigate the value distribution of additive functions defined on the symmetric group and to establish general conditions under which the distribution functions weakly converge to a limit law. In particular, we focus on the following tasks:

- To investigate weak convergence of distributions of completely additive functions, defined on the symmetric group with respect to the Ewens probability measure.
- To establish necessary and sufficient conditions for the number of cycles with restricted lengths under which it obeys a limit law.
- To obtain lower bounds for the mean values of multiplicative functions defined on the additive semigroup $\mathbf{Z}_+^n$ with respect to the Ewens Sampling Formula.
- To explore the class of possible limit distributions.
- To obtain the expressions of the power and factorial moments of additive functions defined on the symmetric group.

1.4 Methods

We apply general methods of probability theory, probabilistic combinatorics, and asymptotic theory of combinatorial structures. The proofs of the weak convergence of distributions of additive functions are mainly based on the formulae and properties of factorial moments. The concentration function and the tail probability estimates are also essential. The generating function method is the basic technical tool in many proofs. The methodology which has proved to be effective in probabilistic number theory is adopted in the present work. In particular, this let us to obtain lower bounds for the mean values of multiplicative functions, to examine the necessity of the convergence conditions.

1.5 Defended propositions

- Propositions on the weak convergence of distributions of completely additive functions defined on the symmetric group with respect to the Ewens probability measure.

- Obtained lower bounds for the mean values of multiplicative functions defined on the additive semigroup $\mathbf{Z}_+^n$ with respect to the Ewens Sampling Formula.
- The weak law of large numbers for completely additive functions.
- The range of asymptotic distributions and their instances for additive functions defined on the symmetric group.
- Formulae of the power and factorial moments of additive functions.

1.6 Novelty

All presented results are new. They extend, generalize and supplement the results on random permutations obtained so far by many authors. They fill up the existing gap in probabilistic combinatorics and correspond to the recent advancement achieved in analogues problems of probabilistic number theory. The obtained results have been approved in local and international conferences and exposed in our papers.

1.7 Approbation

Conferences:

- The 50th Conference of Lithuanian Mathematical Society, Vilnius (Lithuania), 2009, *"Approximation of the number of components of random structures by Poisson law"*.
- The 51st Conference of Lithuanian Mathematical Society, Šiauliai (Lithuania), 2010, *"Additive functions on the symmetric group and their factorial moments"*.
- The 10th Vilnius International conference on Probability Theory and Mathematical Statistics, Vilnius (Lithuania), 2010, *"Additive functions on the symmetric group and their factorial moments"*.
- The 52nd Conference of Lithuanian Mathematical Society, Vilnius (Lithuania), 2011, *"On additive functions defined on the symmetric group"*.
- 27th Journées Arithmétiques, Vilnius (Lithuania), 2011, *"On additive functions defined on the symmetric group"*.

The results of the thesis were presented at the seminars on Number and Probability Theory of the Department of Mathematics and Informatics of Vilnius University.

1.8 Principal publications

The main results of the thesis are published in the following papers:

- T. Kargina, Additive functions on permutations and the Ewens probability, *Šiauliai Mathematical Seminar*, 2(10), 33-41 (2007).

- T. Kargina, Asymptotic distributions of the number of restricted cycles in a random permutation, *Liet. matem. rink. LMD darbai*, 50, 420-425 (2009).
- T. Kargina, E. Manstavičius, Multiplicative functions on $\mathbf{Z}_+^n$ and the Ewens Sampling Formula, *RIMS Kôkyûroku Bessatsu*, B34, 137-151 (2012).
- T. Kargina, Additive functions on the Symmetric group and their factorial moments, *10th Vilnius International conference on Probability Theory and Mathematical Statistics*, Abstracts, p. 181, Vilnius (Lithuania), 2010.
- T. Kargina, E. Manstavičius, The law of large numbers with respect to Ewens probability, *Annales Univ. Sci. Budapest., Sect. Comp.*, 40, 2013 (13 pages, to appear).

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1.9 Historical overview and the state of-the-art in the field

Random permutations implicitly made their appearance in the first edition of Pierre - Rèmond de Montmort's *Essai d'Analyse sur les Jaeux de Hasard* published in Paris in 1708. Pierre's game is related to the number of fixed points of a random permutation:

If Pierre had a pack consisting of n cards all of a single suit, then Pierre would win if the permutation induced by shuffling the cards had at least one fixed point. What is his chance?

Another example, which the most often could be met in books on probability, is the so-called *Hat-Check Problem*:

n mathematicians drop off their hats at a restaurant before having a meal. After the meal their hats are returned at random. The question is what is the chance that no one gets back their own hat.

W. Feller [23], Chapter IV, gives a number of equivalent descriptions. It is also interesting to know the distribution of the number of mathematicians who get back their own hats, its mean, variance and so on. These questions can be formulated in terms of random permutations. To give some impression on the theory, we start with the main notation.

Let $\mathbf{S}_n$ denote the symmetric group of permutations σ acting on $n \geq 1$ letters. Each $\sigma \in \mathbf{S}_n$ has a unique representation (up to the order) by the product of independent cycles κ_i:

$$\sigma = \kappa_1 \cdots \kappa_w, \tag{1.1}$$

where $w = w(\sigma)$ denotes the number of cycles. Set $k_j(\sigma) \geq 0$ for the number of cycles in (1.1) of length j if $1 \leq j \leq n$ and $\bar{k}(\sigma) := (k_1(\sigma), \ldots, k_n(\sigma))$. Then

$$w = k_1(\sigma) + \cdots + k_n(\sigma).$$

The vector $\bar{k}(\sigma)$, called a *cycle vector* of the permutation σ, satisfies the relation

$$\ell(\bar{k}(\sigma)) := 1k_1(\sigma) + \cdots + nk_n(\sigma) = n.$$

The uniform probability measure on symmetric group $\mathbf{S}_n$ is defined by

$$\nu_n(\ldots) = (n!)^{-1}|\{\sigma \in \mathbf{S}_n : \ldots\}|.$$

Thus, in this notation, Pierre's success in the game coincides with the event $\{\sigma \in \mathbf{S}_n : k_1(\sigma) \geq 1\}$ and the solution to the Hat-Check problem is the frequency $\nu_n\big(k_1(\sigma) = 0\big)$. If n is large, an asymptotic analysis of the behavior of this and more involved probabilities becomes useful. Afterwards we assume that $n \to \infty$ without indicating this.

The first probabilistic results on random permutations were obtained in 1942 by V. Goncharov [27] (see also [28]). Apart from elementary approaches, V. Goncharov applied the generating function method. In particular, investigating the distribution of the cycle vector, he established the following relations:

$$\nu_n(k_j(\sigma) = s) = \frac{j^{-s}}{s!} \sum_{l=0}^{\lfloor n/j \rfloor - s} (-1)^l \frac{j^{-l}}{l!}, \quad j \in \mathbf{N},\, s \in \mathbf{Z}_+,\, js \leq n,$$

and

$$\nu_n\big(\bar{k}(\sigma)=\bar{s}\big)=P\big(\bar{\xi}=\bar{s}\big|\,\ell(\bar{\xi})=n\big),\quad \bar{s}\in\mathbf{Z}_+^n, \tag{1.2}$$

where $\xi_j,\ j\geq 1$, are mutually independent Poisson random variables (r.vs) defined in some probability space $(\Omega,\mathfrak{F},P)$ with parameters $\mathbf{E}\xi_j=1/j$ and $\bar{\xi}=(\xi_1,\dots,\xi_n)$. The latter is called the *conditioning relation.* It implies the fact that the process of cycle counts converges in distribution to a Poisson process on $\mathbf{N}$ with intensity j^{-1}, namely,

$$(k_1(\sigma),k_2(\sigma),\dots)\overset{d}{\Rightarrow}(\xi_1,\xi_2,\dots)$$

Analyzing the statistics $w(\sigma)$, he found asymptotical values for the mathematical expectation and the variance:

$$\mathbf{E}_n w(\sigma)=\log n+\gamma+o(1)$$

and

$$\mathbf{V}\mathrm{ar}_n w(\sigma)=\sqrt{\log n}-\left(\frac{\pi^2}{12}-\frac{\gamma}{2}\right)\frac{1}{\sqrt{\log n}}+o\left(\frac{1}{\sqrt{\log n}}\right),$$

where γ is Euler's constant. His central limit theorem gave birth to a new direction of combinatorics.

Goncharov's theorem. *We have*

$$\big(w(\sigma)-\log n\big)/\sqrt{\log n}\overset{d}{\Rightarrow}\mathcal{N}(0,1).$$

Here and in what follows $\mathcal{N}(0,1)$ *stands for the standard normal r.v.*

In the fifties of the last century, P. Erdős and P. Turán did the next step in developing the random permutation theory. Their target was the group theoretical order $\mathrm{Ord}(\sigma)$, $\sigma\in\mathbf{S}_n$, being the least natural number m such that σ^m is the identical permutation. We stress here that an approximation of $\log\mathrm{Ord}(\sigma)$ by the function

$$\sum_{j\leq n}(\log j)k_j(\sigma)$$

for almost all $\sigma\in\mathbf{S}_n$ plays a central role in their proofs. Appropriately normalized the latter function also obeys the normal law. To the historical account on the Erdős - Turán problem presented in the book [2] one must add the most important latest contribution belonging to V. Zacharovas ([87], [89]).

S.W. Golomb in [25] (and also collaborating with other authors [26]) found the mean value of the maximal cycle length in a random permutation. In 1966 L.A. Shepp and S.P. Lloyd [71] initiated investigations of the order statistics

$$j_1(\sigma)\geq j_2(\sigma)\geq\dots,$$

where $j_r(\sigma)$ is the rth length of a cycle appearing in σ. The authors established asymptotic formulas for the $\mathbf{E}_n j_r(\sigma)^m$ as well as that for the rth shortest cycle length, where $r=1,2,\cdots$. They also obtained the limiting distribution for the lengths. Together with the generating functions, they use Tauberian theorems. This direction stands a bit further from our interests; therefore, we just mention that nowadays [2] this is contained in the relation

$$n^{-1}\big(j_1(\sigma),j_2(\sigma),\dots\big)\overset{d}{\Rightarrow}L:=(L_1,L_2,\dots),$$

where L has the Poisson-Dirichlet distribution $PD(1)$ concentrated on the simplex $\{(x_1 \geq x_2 \geq \cdots) \in [0,1]^\infty : \ x_1 + x_2 + \cdots \leq 1\}$.

W. Feller [22] succeeded in defining a random permutation in a sufficiently rich abstract probability space. This became the start of the coupling method. The idea was taken into use by A. Rènyi [69]) who established Goncharov's theorem by using a Bernoulli representation of $k_j(\sigma)$, $j \leq n$, and applying the Lindeberg-Feller central limit theorem for independent r.vs.

Dealing with random permutations, V.F. Kolchin (1971, [35]; 1986, [36]) proposed to use a representation in terms of random allocations of particles into cells. In a book [38] there is found an approximation of the distribution of partial sum

$$a_1 k_1(\sigma) + \cdots + a_r k_r(\sigma), \tag{1.3}$$

by the sum $a_1\xi_1 + \cdots + a_r\xi_r$, where a_j, $j \leq r \ll n/\log n$, are arbitrary. Actually, this begins investigations of completely additive functions on permutations. On the other hand, the condition $r \ll n/\log n$ makes their result trivial from the point of view of contemporary theory, where $r = o(n)$ is achieved. The results obtained by Russian authors V.F. Kolchin, B.A. Sevastyanov, V.P. Chistyakov and others were summarized in the books ([36] - [38]). It were P. Diaconis and J.W. Pitman (1986, unpublished) and A.D. Barbour [6] who showed that the total variation distance between the distributions of the vectors $\bar{k}_r(\sigma) := (k_1(\sigma), \ldots, k_r(\sigma))$ and $\bar{\xi}_r = (\xi_1, \ldots, \xi_r)$ is

$$\rho_{TV}\big(\mathcal{L}(\bar{k}_r(\sigma)), \mathcal{L}(\bar{\xi}_r)\big) \ll \frac{r}{n}, \quad 1 \leq r \leq n.$$

This extends the above mentioned approximation of (1.3) up to the optimal value $r = o(n)$.

Several papers were devoted to the local limit laws. We just mention the pioneering paper by L. Moser and M. Wyman (1958, [64]), who dealt with the Stirling numbers of the first kind, namely, with the frequencies $|s(n,k)| = n!\nu_n(w(\sigma) = k)$. For a more systematic and deep treatment including asymptotic expansions and the large deviation probabilities in the local laws, we refer to H.-K. Hwang's papers (see, for instance, [32], [33]).

In 1996 E. Manstavičius [45] started to examine integral asymptotic laws of real valued additive functions. By definition, a function $h : \mathbf{S}_n \to \mathbf{R}$ defined by

$$h(\sigma) = \sum_{j=1}^{n} a_j k_j(\sigma), \quad \sigma \in \mathbf{S}_n, \tag{1.4}$$

where $a_j \in \mathbf{R}$ is called *a completely additive function.* The number of cycles $w(\sigma)$ is the simplest example of such functions. Allowing dependence on n, which will be indicated by the extra index n, we have sequences of functions. The problem explored in [45] can be formulated as follows:
Under which necessary and sufficient conditions the frequencies $\nu_n(h_n(\sigma) - A(n) < x)$ for some centralizing sequence $A(n)$ converge weakly to a limit distribution law.

E. Manstavičius [45] found general sufficient convergence conditions to infinitely divisible limit laws and presented instances when the limit laws lay outside this class. The later published book [2] contained a chapter dealing only with limit laws having finite the second moments. It is also notable, that a new analytic method adopting many ideas, which had been proved to be useful in probabilistic number theory, is developed in the paper [45]. Nowadays, we have a complete answer to the formulated problem in the case of the degenerate at the zero point limit law (see [59]) and if $h_n(\sigma) = h(\sigma)/\beta(n)$, where $\beta(n) \to \infty$ and is slowly oscillating at infinity (see [62]). In its full generality, the problem is still open.

The motivation to examine general additive functions is not just a desire to generalize the mentioned Goncharov, Erdős - Turán, Kolchin - Chistyakov theorems. In recent years, even a stronger need came from investigations of random permutation matrices. Let $M := M(\sigma) := \big(\mathbf{1}\{i = \sigma(j)\}\big)$, $1 \le i, j \le n$ and $\sigma \in \mathbf{S}$, be such a matrix taken uniformly, e.i. with the frequency $\nu_n(\{M\}) = \nu_n(\{\sigma\}) = 1/n!$,

$$Z_n(x;\sigma) := \det\big(I - xM(\sigma)\big) = \prod_{j\le n}(1 - x^j)^{k_j(\sigma)}, \tag{1.5}$$

be its characteristic polynomial, and let $e^{2\pi i\varphi_j(\sigma)}$, where $\varphi_j(\sigma) \in [0, 1)$ and $j \le n$ be its eigenvalues. The papers by K. Wieand ([84], [85]) and many other authors (see [29], [91]) or many preprints put in the AMS arXiv (see, for instance, [4] and [34] and the references therein) concern $\log|Z_n(x;\sigma)|$, $\Im \log Z_n(x;\sigma)$ or the linear statistics

$$\mathrm{Trf}(\sigma) := \sum_{j\le n} f\big(\varphi_j(\sigma)\big) = \sum_{j\le n} k_j(\sigma) \sum_{0\le s\le j-1} f\Big(\frac{s}{j}\Big),$$

where $f : [0, 1] \to \mathbf{R}$ is a sufficiently smooth function. The last relation, easily seen from (1.5), is present in [4]. A great portion of the newly announced results fall within the scope of the above formulated problem. Nevertheless, the authors seldom observe this and prefer to rediscover properties of the particular statistics.

The recent research of random permutation matrices confirmed the necessity to develop the value distribution theory of additive functions with respect to weighted measures defined in the symmetric group. The Ewens probability measure become the most popular among them. Our dissertation is completely devoted to this objective. Let us add some new definitions.

Let $\theta > 0$ be a fixed parameter. The Ewens probability measure (EPM) on the subsets of $\mathbf{S}_n$ is defined by

$$\nu_{n,\theta}(A) = \frac{1}{\theta^{(n)}} \sum_{\sigma\in A} \theta^{w(\sigma)}, \quad A \subset \mathbf{S}_n,$$

where $x^{(n)} := x(x + 1)\cdots(x + n - 1)$ and $x^{(0)} = 1$ denotes the increasing factorial. For the class of permutations with the common cycle vector $\bar{s}$, we have

$$\nu_{n,\theta}(\bar{k}(\sigma) = \bar{s}) := \mathbf{1}\big\{\ell(\bar{s}) = n\big\} \frac{n!}{\theta^{(n)}} \prod_{j=1}^{n} \Big(\frac{\theta}{j}\Big)^{s_j} \frac{1}{s_j!}. \tag{1.6}$$

We stress that the quantity on the right-hand side is the probability ascribed by J.W. Ewens (1972, [20]) to a vector from the set $\ell^{-1}(n) := \{\bar{s} \in \mathbf{Z}_+^n : \ \ell(\bar{s}) = n\}$. In other words, he defined the distribution:

$$P_{n,\theta}(\{\bar{s}\}) := \frac{n!}{\theta^{(n)}} \prod_{j=1}^{n} \Big(\frac{\theta}{j}\Big)^{s_j} \frac{1}{s_j!}, \quad \bar{s} \in \ell^{-1}(n), \tag{1.7}$$

which nowadays is known as the Ewens Sampling Formula (ESF). He introduced it in the context of population genetics to model mutation of a population. The parameter θ then served for the mutation rate. The ESF relates the random permutation theory under the EPM with statistical problems. For a comprehensive account of the latter we refer to book [21].

The connection highly motivated the increasing interest to the value distribution problems in $\mathbf{S}_n$ with respect to the EPM. Check that, similar to the particular case $\theta = 1$, we have [2]

$$(k_1(\sigma), \ldots, k_n(\sigma), 0, \ldots) \overset{\nu_{n,\theta}}{\Rightarrow} (\xi_1, \ldots, \xi_n, \xi_{n+1}, \ldots).$$

Here and since now $\xi_j, j \geq 1$, are independent Poisson r.vs. given on some probability space $\{\Omega, F, P\}$ with $\mathbf{E}\xi_j = \theta/j$. The conditioning relation (1.2) and the total variation distance estimate remain to hold.

We also observe that an easy combinatorial argument (see [2]) gives the distribution of the cycle vector and the coincidence:

$$\nu_{n,\theta}\big(\bar{k}(\sigma) = \bar{s}\big) = P_{n,\theta}(\{\bar{s}\}),$$

if $\bar{s} \in \Omega(n)$. Thus, dealing with statistics of random permutations expressed via $\bar{k}(\sigma)$, we may examine corresponding statistics of random vectors $\bar{s} \in \Omega(n) = \ell^{-1}(n)$ taken with probabilities (1.7).

The importance of distributions with respect to the EPM on permutations increased after they had been employed to approximate distributions defined on more general decomposable structures belonging to the so-called logarithmic class (see [2]). Distribution of additive functions with respect to EPM was investigated by analytic and probabilistic methods. For instance, deep asymptotic analysis of the convergence rate in the central limit theorem was done by V. Zacharovas ([87], [89], [90]). J. Norkūnienė developed strong convergence concept by proving the laws of iterated logarithm including their functional Strassen versions ([65] - [68]).

Let us discuss the results containing necessary and sufficient convergence conditions. The first milestone was reached by G.J. Babu and E. Manstavičius [7] in functional limit theorems. They caught an idea to model the Brownian motion (denoted by W) by means of the truncated additive functions going back even to probabilistic number theory (see [40], [43], [44]). Before them, the number-of-cycles function $w(\sigma)$ was used by J.M. DeLaurentis and B.G. Pittel [16] in the case $\theta = 1$ and, for arbitrary θ by J.C. Hansen [30], P. Donnelly *et al.* [17], and R. Arratia and S. Tavaré [3]. We now formulate the result from [7] for general additive functions. Define

$$H_n := H_n(\sigma, t) = \frac{1}{B(n)}\bigg(\sum_{j \leq y(t)} h_j(k_j(\sigma)) - A(y(t))\bigg),$$

where

$$A(u) := \sum_{j \leq u} \frac{a(j)}{j}\theta\,, \qquad B^2(u) := \sum_{j \leq u} \frac{a(j)^2}{j}\theta\,,$$

$h_j(1) =: a_j$, and

$$y(t) := y_n(t) = \max\{u \leq n :\ B^2(u) \leq tB^2(n)\}, \quad t \in [0,1].$$

Consider the weak convergence (denoted also by $\Rightarrow$) of the process H_n in the space $\mathbf{D}[0,1]$ equipped with the supremum norm. Equivalently, one could also examine a linearized version of the process H_n and deal only with elements of the space $\mathbf{C}[0,1]$.

Theorem BM. *Let $h(\sigma)$ be a real additive function, $h_j(1) = a(j)$, $B(n) \to \infty$. For the weak convergence*

$$\nu_{n,\theta} \cdot H_n^{-1} \Rightarrow W$$

to hold it is necessary and sufficient that, for each $\varepsilon > 0$,

$$\frac{1}{B^2(n)} \sum_{j \le n} \frac{a(j)^2}{j} \mathbf{1}\{|a(j)| \ge \varepsilon B(n)\} = o(1). \tag{1.8}$$

Unfortunately, the Lindeberg-Feller type condition (1.8) is not necessary for the one-dimensional limit result $\nu_{n,\theta} \cdot H_n(\cdot, 1)^{-1} \Rightarrow \mathcal{N}(0,1)$. The further investigations carried out by these authors established necessary and sufficient conditions for the convergence in $\mathbf{D}[0,1]$ equipped with Skorokhod's topology to processes with independent increments including stable processes (see [8] - [10] or [15] and the references therein). The paper [52] solves the problem for sequences of additive functions if $\theta = 1$. In this case, the sufficiency part sometimes intersects with our results but our approach is different. Let us stress that the convergence of distributions of processes $H_n(\sigma, t)$, $0 \le t \le 1$, contains much more information than that of $H_n(\sigma, 1)$ and this has been exploited in proving the necessity of conditions.

The second idea to prove necessity of the convergence conditions also came from probabilistic number theory. It was J. Šiaulys who, in a series of papers (see [73] - [78]) dealing with number-theoretic functions, observed that convergence of power or factorial moments hide the needed piece of information. The idea was further developed by him jointly with G. Stepanauskas in [79] - [83]. E. Manstavičius succeeded in adopting this for random permutations taken with equal probabilities ([55], [56], [59]). The main purpose of our work is to develop this approach in $\mathbf{S}_n$ under the EPM.

1.10 Main results of the thesis

In the first two parts of dissertation we have laid out the results which are more auxiliary for our further proofs and main problems tackling. The first section is devoted to the moments. General expressions and estimates of power and factorial moments of completely and strongly additive functions with $a_j \in \mathbf{R}$ are obtained in it. We present here only main formulae of the moments to make an impression of their complexity. Let $\mathbf{E}_{n,\theta}$ stand for expectation with respect to $\nu_{n,\theta}$. Further, where this is not specified, $n, r_1, r_2, ... \in \mathbf{N}$ and $1 \leq j_1, j_2, \ldots \leq n$. We have

$$\mathbf{E}_{n,\theta} h(\sigma)^k = \hat{\beta}_{nk,\theta} := \sum_{u=1}^{k} \theta^u \sum_{r_1+\cdots+r_u=k} \binom{k-1}{r_1-1} \cdots \binom{k-r_1-\cdots-r_{u-1}-1}{r_u-1} \times \sum_{j_1+\cdots+j_u \leq n} \frac{a_{j_1}^{r_1} \cdots a_{j_u}^{r_u}}{j_1 \cdots j_u} \psi_n(n-j_1-\cdots-j_u), \tag{1.9}$$

where

$$\psi_n(l) := \frac{n!}{\theta^{(n)}} \frac{\theta^{(l)}}{l!}.$$

Analogically, an expression of factorial moment $\hat{\gamma}_{nk,\theta} := \mathbf{E}_{n,\theta} h(\sigma)_{(k)}$ is obtained. The formula contains just $a_{j_1(r_1)} \cdots a_{j_u(r_u)}$ instead of the product $a_{j_1}^{r_1} \cdots a_{j_u}^{r_u}$. Here and afterwards $x_{(m)} := x(x-1)\ldots(x-m+1)$, denotes the falling factorial. So, if $a_j \in \{0,1\}$ for $j \leq n$, then

$$\hat{\gamma}_{nk,\theta} := \theta^k \sum_{j_1 \leq n}^{*} \frac{1}{j_1} \cdots \sum_{j_k \leq n}^{*} \frac{1}{j_k} \mathbf{1}\{j_1+\cdots+j_k \leq n\} \psi_n(n-j_1-\cdots-j_k), \tag{1.10}$$

where $k \in \mathbf{N}$ and the $(*)$ over the sums replaces the condition $a_j = 1$.

The purpose of the second thesis section is to obtain lower bounds for mean values of the multiplicative functions defined on additive semigroup $\mathbf{Z}_+^n$ with respect to the ESF. They imply useful estimates of probabilities of random permutations missing some cycles. The results are analogues to that obtained by P. Erdős, I.Z. Ruzsa [18], and K. Alladi [1] for the number theoretical functions. So, implementing the already mentioned idea, instead of permutations, we now deal with random vectors. The main advantage of such imbedding is the fact that $\mathbf{Z}_+^n$ has an additive semigroup structure as well as the partial order defined by $\bar{s} = (s_1, \ldots, s_n) \leq \bar{t} = (t_1, \ldots, t_n)$ meaning that $s_j \leq t_j$ for each $1 \leq j \leq n$. Moreover, we may exploit the geometry of semi-lattice $\mathbf{Z}_+^n$, say, by introducing the orthogonality of $\bar{s}, \bar{t} \in \mathbf{Z}_+^n$, denoted by $\bar{s} \perp \bar{t}$ and meaning that $s_1 t_1 + \cdots + s_n t_n = 0$. In this way, we come closer to probabilistic number theory dealing with random numbers taken from the multiplicative semigroup $\mathbf{N}$ (see [40]) having the partial order defined by division. The semigroup structures and the partial orders in $\mathbf{Z}_+^n$ and $\mathbf{N}$ play the crucial role in developing parallel theories. Developing this, we obtained the lower estimates of the mean values of multiplicative functions related to the so-called small sieve problem in number theory.

Let us recall necessary definitions. A mapping $G : \mathbf{Z}_+^n \to \mathbf{C}$, $G(\bar{0}) = 1$, is called a *multiplicative function* if $G(\bar{s}+\bar{t}) = G(\bar{s})G(\bar{t})$ for every pair $\bar{s}, \bar{t} \in \mathbf{Z}_+^n$ such that $\bar{s} \perp \bar{t}$. If $\bar{e}_j := (0, \ldots, 1, \ldots, 0)$, where the only 1 stands at the jth place, then the multiplicative function G has the decomposition

$$G(\bar{k}) = \prod_{j \leq n} G(k_j \bar{e}_j) =: \prod_{j \leq n} g_j(k_j).$$

Conversely, given a complex two-dimensional array $\{g_j(k)\}$, $1 \leq j \leq n, k \geq 0$, satisfying the condition $g_j(0) \equiv 1$, by the last equality, we can define a multiplicative function. If $g_j(k) = g_j(1) =: g_j$ for all $k \geq 1$ and $j \leq n$, the function G is called *strongly* multiplicative and, similarly, if $g_j(k) = g_j^k$and $0^0 := 1$, then G is called *completely* multiplicative. Denote, respectively, by $\mathfrak{M}$, $\mathfrak{M}_s$, and $\mathfrak{M}_c$ the sets of just introduced multiplicative functions. Observe that if $G \in \mathfrak{M}_c$ and $g_j \in \{0,1\}$, then $G \in \mathfrak{M}_s$ and, conversely, the latter together with $g_j \in \{0,1\}$ implies $G \in \mathfrak{M}_c$. The multiplicative function

$$\Pi(\bar{k}) := \prod_{j \leq n} \left(\frac{\theta}{j}\right)^{k_j} \frac{1}{k_j},$$

depending on θ, plays a special role in the sequel.

If $G \in \mathfrak{M}$, then its mean value with respect to $P_{n,\theta}$ is

$$\begin{aligned} M_{n,\theta}(G) &:= \sum_{\bar{k} \in \Omega(n)} G(\bar{k}) P_{n,\theta}(\bar{k}) = \frac{n!}{\theta^{(n)}} \sum_{\bar{k} \in \Omega(n)} \prod_{j \leq n} \left(\frac{\theta}{j}\right)^{k_j} \frac{g_j(k_j)}{k_j!} \\ &= \frac{n!}{\theta^{(n)}} [x^n] Z_\theta(x; G), \end{aligned} \tag{1.11}$$

where

$$Z_\theta(x; G) = \prod_{j \geq 1} \left(1 + \sum_{r \geq 1} \left(\frac{\theta}{j}\right)^r \frac{g_j(r)}{r!} x^{jr}\right)$$

and $[x^n]Z(x)$ denotes the nth coefficient of the formal power series $Z(x)$. We also assume that $M_{0,\theta}(G) \equiv 1$ for every $G \in \mathfrak{M}$.

Our interest is in estimates of $M_{n,\theta}(G)$ holding uniformly in G belonging to some subclass of $G \in \mathfrak{M}$. If $G \in \mathfrak{M}_c$ and $0 < \theta^- \leq g_j \leq \theta^+ < \infty$ for all $j \leq n$, then according to the E. Manstavičius paper [51], we have an asymptotic

$$M_{n,1}(G) \asymp \exp\left\{\sum_{j \leq n} \frac{g_j - 1}{j}\right\}, \quad n \geq 1. \tag{1.12}$$

The involved constants in (1.12) depend on θ^- and θ^+. Afterwards, the constants in these symbols will be dependent at most on θ.

We observe that, if $G(\bar{k})$ takes the zero value rather often, the lower estimation of $M_{n,1}(G)$ becomes rather involved and, in general, the lower bound as it is stated in (1.12) is false. A satisfactory result was achieved by E. Manstavičius only for $G(\bar{k}) \in \{0,1\}$ (see [48] and [50]). Our results extended them.

We started from an easier problem of estimation the averaged mean values

$$\widetilde{M}_{n,\theta}(G) := \frac{1}{\Gamma_{n,\theta}} \sum_{0 \leq m \leq n} \frac{\theta^{(m)}}{m!} M_{m,\theta}(G), \tag{1.13}$$

where

$$\Gamma_{n,\theta} := \sum_{0 \leq m \leq n} \frac{\theta^{(m)}}{m!} = \frac{n^\theta}{\Gamma(\theta+1)} \Big(1 + O\Big(\frac{1}{n}\Big)\Big), \quad n \geq 1, \quad \theta^{(0)} = 1.$$

Theorem 1.1. *Let $\theta > 0$ and $G \in \mathfrak{M}_s$ be defined via sequence $0 \leq g_j \leq 1$ where $1 \leq j \leq n$. Then*

$$\widetilde{M}_{n,\theta}(G) \asymp \exp\left\{\theta \sum_{j \leq n} \frac{g_j - 1}{j}\right\}.$$

The main result of our investigation of the multiplicative function mean values is the following theorem.

Theorem 1.2. *Let $\theta \geq 1$ and $G \in \mathfrak{M}_s$ be defined via sequence $0 \leq g_j \leq 1$ where $1 \leq j \leq n$. If*

$$\sum_{j \leq n} \frac{1-g_j}{j} \leq K \tag{1.14}$$

for some $K > 0$, then there exist positive constants c_0 and c together with a function $\mathcal{N}: \mathbf{R}_+ \to \mathbf{N}$ such that

$$\Upsilon(K) := \inf\{M_{n,\theta}(G) : n \geq \mathcal{N}(K)\} \geq c_0 \exp\{-e^{cK}\}. \tag{1.15}$$

In the following section of our dissertation we deal with the case when $a_j \in \{0,1\}$. Then the functions $h_n(\sigma)$ defined via such a_j express the number of cycles with restricted lengths. In this case, we proved an exhaustive result.

Theorem 1.3. *Let $h_n(\sigma)$ be a sequence of completely additive functions with $a_j \in \{0,1\}$ and $\theta > 0$. The frequencies $V_{n,\theta}(x) := \nu_{n,\theta}(h_n(\sigma) < x)$ converge weakly to a limit law if and only if there exist finite limits*

$$\lim_{n\to\infty} \hat{\gamma}_{nm,\theta} =: \hat{\gamma}_{m,\theta} \tag{1.16}$$

for every $m \in \mathbf{N}$. Moreover, if (1.16) is satisfied, the characteristic function of the limit distribution is

$$1 + \sum_{m=1}^{\infty} \frac{\hat{\gamma}_{m,\theta}}{m!} \left(\mathrm{e}^{it} - 1\right)^m, \quad t \in \mathbf{R}.$$

General results in the case when $\theta = 1$ were obtained by E. Manstavičius in papers [45], [55], [56]. The author analyzes the convergence to non-degenerate laws including the Poisson one. In particular, he established necessary and sufficient conditions under which $V_{n,1}(x)$ converges. It is worth to note that sufficient conditions for the existence of an infinitesimal limit law for $V_{n,1}(x)$ follow also from the functional limit theorem for partial sum processes defined via sequences of truncated additive functions (see [52]). If $\theta = 1$, our approach also yields the moments of the limit law.

The methodology applied in the discussed problem goes back to probabilistic number theory. It is due to J. Šiaulys (see papers ([77], [78] and others) who examined the general weak convergence conditions for analogues distributions of values of number-theoretic strongly additive functions.

In the fifth section, we establish a general weak law of large numbers without taking any *a fortiori* assumption on the sequence a_{nj}. We find necessary and sufficient conditions under which the frequencies $V_{n,\theta}(x)$ converge to the degenerated at the point zero limit law.

Then we recall the concept of the Lévy distance of the r.v. $h(\cdot)$ from the set of constants

$$L(h; \nu_{n,\theta}) := \inf\left\{\varepsilon + \nu_{n,\theta}(|h(\sigma) - a| \geq \varepsilon) : \ a \in \mathbf{R}, \varepsilon > 0\right\}.$$

Let $u \vee v := \max\{u, v\}$, $u \wedge v := \min\{u, v\}$ and as earlier $u^\circ := 1 \wedge |u| \operatorname{sgn} u$ if $u, v \in \mathbf{R}$,

$$U_n(h, \lambda) := \sum_{j \leq n} \frac{\theta}{j} (a_{nj} - \lambda j)^{\circ 2} \psi_n(n-j)$$

and $U_n(h) = \min\{U_n(h,\lambda)\colon\ \lambda \in \mathbf{R}\}$. In the sequel, $\ll$ is used as an analog of $\mathrm{O}(\cdot)$, moreover, dependence on θ in the involved constants is allowed.

Theorem 1.4. *If $\theta \geq 1$ and $h(\sigma)$ is a completely additive function, then*

$$L(h;\nu_{n,\theta}) \leq 2\big(1 \wedge (2U_n(h))^{1/3}\big)$$

and

$$U_n(h) \ll (1/n) \vee L(h;\nu_{n,\theta})$$

for all $n \geq 1$.

We now give the answer to the above question in the case of the degenerate limit distribution modifying a bit the conditions of the theorem.

Corollary 1.1. *Let $\theta \geq 1$ and $h_n(\sigma)$ be completely additive functions on $\mathbf{S}_n$ defined via $\{a_j\}$, $j \leq n$, in* (1.4)*. The distributions $\nu_{n,\theta}\big(h(\sigma) - A(n) < x\big)$ converge to the degenerate law at the point zero if and only if*

$$\sum_{j<n} \frac{(a_j - \lambda j)^{\circ 2}}{j} \psi_n(n-j) = \mathrm{o}(1)$$

for some $\lambda = \lambda_n \in \mathbf{R}$ and

$$A(n) = n\lambda + \sum_{\substack{j<n \\ |a_j - \lambda j|<1}} \frac{\theta(a_j - \lambda j)}{j} \psi_n(n-j) + \mathrm{o}(1).$$

In 2005, professor E. Manstavičius, motivated by the impressive result of Ruzsa (see [70]) in probabilistic number theory, proved these assertions for $\theta = 1$. We succeeded to generalize them, unfortunately, for $\theta \geq 1$ only. We do believe that the first claim of Theorem 1.4 can be extended to general additive functions if $\theta > 0$. For this, it would be sufficient to adopt technical ideas going back also to a number-theoretic paper by A. Biró and T. Szamuely [14]. On the other hand, there exists an indirect possibility to obtain the upper estimates based upon the inequality

$$\nu_{n,\theta}\big(|h(\sigma) - a| \geq u\big) \ll P^{1\wedge\theta} P\big(|X_1 + \cdots + X_n \geq u/3\big) + \mathbf{1}\{\theta < 1\} n^{-\theta},$$

where $a \in \mathbf{R}$ and $u \geq 0$ are arbitrary, proved by E. Manstavičius jointly with G.J. Babu [7]. In the case $\theta < 1$, an appropriate estimate for these independent r.vs yields the upper bound for $L(h, \nu_{n,\theta})$. Denote

$$\widetilde{U}_n(h,\lambda) := \min_{\lambda \in \mathbf{R}} \sum_{jk \leq n} \frac{(h_j(k) - \lambda jk)^{\circ 2}}{j^k k!}, \quad \theta < 1.$$

Theorem 1.5. *Let $\theta < 1$ and $h(\sigma)$ be an additive function on $\mathbf{S}_n$. We have*

$$L(h;\nu_n) \ll \widetilde{U}_n^{\theta/(2\theta+1)}(h) + n^{-\theta}. \tag{1.17}$$

In theorems 1.3 and 1.5, one can observe some relation between limit theorems for additive functions on the symmetric group $\mathbf{S}_n$ and those for sums of independent r.s. Obtaining necessary and sufficient conditions for the weak convergence of $V_{n,\theta}(x)$ to a non-degenerate limit law, will likely be more difficult; nevertheless, we demonstrate some achievement in this direction in Section 6 of our thesis. We now examine the weak convergence of distributions of completely additive functions with $a_j \in \mathbf{Z}$ to the Poisson limit law.

Let introduce some notation before. Set

$$a_j(m) = \begin{cases} a_j & if\ \ 0 \le a_j \le m, \\ m & if\ \ a_j > m, \\ 0 & if\ \ a_j < 0 \end{cases} \tag{1.18}$$

and denote truncated completely additive functions

$$h(\sigma; m) := \sum_{j=1}^{n} a_j(m) k_j(\sigma).$$

The principal result of this section is the following theorem.

Theorem 1.6. *Let $h_n(\sigma)$ be a sequence of completely additive functions with $\{a_j\} \in \mathbf{Z}$, $j \le n$ and $\theta \ge 1$. The frequencies $V_{n,\theta}(x)$ converge weakly to the Poisson limit law with parameter $\mu > 0$ if and only if*

$$(i) \sum_{\substack{j \le n \\ a_j \le -1}} \frac{\theta}{j} \psi_n(n-j) = o(1), \tag{1.19}$$

$$(ii) \lim_{m\to\infty} \limsup_{n\to\infty} \mathbf{E}_n h_n(\sigma; m)_{(l)} = \lim_{m\to\infty} \liminf_{n\to\infty} \mathbf{E}_n h_n(\sigma; m)_{(l)} = \mu^l, \tag{1.20}$$

for each fixed $l \in \mathbf{N}$.

The first attempt to investigate the case when the Poisson limit law with parameter $\mu > 0$ appears in $V_{n,1}(x)$ was made by E. Manstavičius. Based on a few ideas originated in probabilistic number theory, especially upon those proposed by J. Šiaulys ([73] - [76]), he discovered an unexpected phenomenon (see Theorem 3, [59]), that the influence of long cycles must be negligible and this implies that counting the cycles with lengths in $[\varepsilon n, n]$ we can not obtain the Poisson law if $a_{nj} \in \{0, 1\}$. If a_j are unbounded, the situation is different. In this regard, we include an example generalizing that of E. Manstavičius given in his paper [52].

Proposition 1. *Let $\theta \ge 1$, $\mu \le -\log(1 - v_\theta(1))$, where*

$$v_\theta(x) := \theta \int_{1/2}^{x} (1-u)^{\theta-1} \frac{du}{u}.$$

Introduce the sequence $1/2 = d_0 < d_1 < \cdots$ by

$$v_\theta(d_m) = e^{-\mu} \sum_{k=1}^{m} \frac{\mu^k}{k!}, \quad m \in \mathbf{N}$$

and set $a_j = m$ if $nd_{m-1} < j \le nd_m$ and $a_j = 0$ otherwise. If $h_n(\sigma)$ is a sequence of completely additive functions defined via these a_j, then it posses the Poisson limit law with parameter μ.

As we have mentioned earlier, the flow of new papers and even announcements of already known results in the field does not stop. We present the next instant related to quasi-Poisson distribution.

M. Lugo in his paper [42], analyzing the case for $\theta = 1$, rediscovers (the results is contained in [55]) the limiting distribution of the number of cycles of length between γn and δn in a permutation of order n chosen uniformly at random, for constants γ, δ such that $1/(k+1) \le \gamma < \delta \le 1/k$ for $k \in \mathbf{Z}$. This distribution is supported on $\{0, 1, \ldots, k\}$ and has k moments equal to those of a Poisson distribution with parameter $\log \delta/\gamma$. In case of the Ewens distribution with $\theta \neq 1$, M. Lugo raised the following conjecture.

Conjecture 15[42]. *The expected number of cycles of length in* $[\gamma n, \delta n]$ *of a permutation of order* n *chosen from the Ewens distribution approaches*

$$\lambda = \int_\gamma^\delta \frac{1}{x}(1-x)^{\theta-1}dx$$

as $n \to \infty$*. Furthermore, in the case where* $1/(k+1) \le \gamma < \delta < 1/k$ *for some positive integer* k*, the distribution of the number of cycles converges in distribution to quasi-Poisson*(k, λ).

Recall, that a r.v. X has a quasi-Poisson (r, λ), $r \in \mathbf{N}$, $\lambda \in (0, 1]$, distribution if $\mathbf{E}X_{(k)} = \lambda^k$ for $k = 0, 1, \ldots, r$ and X is supported on $\{0, 1, \ldots, r\}$.

According to our calculations in subsection 7.2, the announced formula for the mean value lacks the factor θ on the right-hand side. It is much more important, that the limit distribution in the conjecture is mistaken if $\theta \neq 1$. It does exist but is not a quasi-Poisson.

The last section of our dissertation is devoted to exploring of other distributions, apart those we mentioned above, which can appear as limits in Theorem 1.3. We reckon just a few cases. The factorial moments of a limit distribution satisfy the inequality $\hat{\gamma}_{k,\theta} \le \hat{\gamma}_{1,\theta}^k$, for each $k \in \mathbf{N}$. By this reason, a mixture of Poisson distribution, binomial and geometrical do not belong to the limiting class. Actually, we owe to J. Šiaulys and G. Stepanauskas [82] who has observed and applied this simple criteria in number theory.

2 Moments

In this section, we present exact and asymptotic formulae of power and factorial moments for a completely additive function $h(\sigma)$ defined in (1.4) via $a_j \in \mathbf{R}$, $1 \le j \le n$, with respect to the Ewens probability measure $\nu_n := \nu_{n,\theta}$, where $\theta > 0$. The relevant expectation will be denoted by $\mathbf{E}_n$. Similarly, in the estimates below, the dependence on θ is allowed but not additionally indicated. In the sequel, $a \ll b$ is an analog of $a = O(b)$, the symbol $\asymp$ means that $a \ll b$ and $b \ll a$, where $a, b \in \mathbf{R}$.

Throughout the thesis, we will often apply the following well known asymptotic formulas:

$$\frac{\theta^{(n)}}{n!} = \frac{n^{\theta-1}}{\Gamma(\theta)}\left(1 + O\left(\frac{1}{n}\right)\right), \quad n \in \mathbf{N}, \tag{2.1}$$

and

$$\psi_n(n-m) := \frac{n!}{\theta^{(n)}} \frac{\theta^{(n-m)}}{(n-m)!} \asymp \left(1 - \frac{m}{n+1}\right)^{\theta-1}, \tag{2.2}$$

where $0 \le m \le n$.

We will need the following formal equality for the mean value with respect to $\nu_{n,1}$ of a completely multiplicative function

$$f(\sigma) := \prod_{j \le n} b_j^{k_j(\sigma)}, \quad b_j \in \mathbf{C},\ 0^0 := 1,$$

Lemma 2.1. *We have the following formal power series equality*

$$\sum_{n=0}^{\infty} \frac{1}{n!}\left\{ \sum_{\sigma \in \mathbf{S}_n} f(\sigma)\right\} y^n = \exp\left\{ \sum_{j=1}^{\infty} \frac{b_j}{j} y^j \right\}. \tag{2.3}$$

It is worth to recall the *proof.* By Cauchy's formula, if $\bar{s} = (s_1, \ldots, s_n) \in \mathbf{Z}_+^n$ and $\ell(\bar{s}) = n$, then

$$\left|\{\sigma \in \mathbf{S}_n : \ \bar{k}(\sigma) = \bar{s}\}\right| = n! \prod_{j \le n} \left(\frac{1}{j}\right)^{s_j} \frac{1}{s_j!}.$$

Hence grouping over classes of permutations with the common cycle vector $\bar{s}$, we obtain

$$\sum_{\sigma \in \mathbf{S}_n} f(\sigma) = n! \sum_{\ell(\bar{s})=n} \prod_{j=1}^{n} \left(\frac{b_j}{j}\right)^{s_j} \frac{1}{s_j!}.$$

Here and afterwards in such sums, the summation is extended over vectors $\bar{s}$ satisfying the equality $\ell(\bar{s}) = n$. To verify (2.3), it suffices now to compare the coefficients in it at y^n and apply the last equality.

The lemma is proved.

2.1 Power moments

In this subsection we present exact expressions of the moments of completely additive functions $h(\sigma)$ with $a_j \in \mathbf{R}$.

Denote

$$\begin{aligned}\hat{\beta}_{nk} := \quad & \sum_{u=1}^{k} \theta^u \sum_{r_1+\cdots+r_u=k} \binom{k-1}{r_1-1} \cdots \binom{k-r_1-\cdots-r_{u-1}-1}{r_u-1} \\ & \times \sum_{j_1+\cdots+j_u \le n} \frac{a_{j_1}^{r_1} \cdots a_{j_u}^{r_u}}{j_1 \cdots j_u} \psi_n(n-j_1-\cdots-j_k). \qquad (2.4)\end{aligned}$$

Theorem 2.1. *For a completely additive function $h(\sigma) = a_1k_1(\sigma)+\cdots+a_nk_n(\sigma)$ and every $k \in \mathbf{N}$, we have*

$$\mathbf{E}_n h(\sigma)^k = \hat{\beta}_{nk}, \qquad (2.5)$$

where $\theta > 0$.

Proof. We first prove a recurrence relation for $\beta_{nk} := \big(\theta^{(n)}/n!\big)\hat{\beta}_{nk}$. Set $\beta_{n0} = \theta^{(n)}/n!$ if $n \ge 0$. Moreover, let $\beta_{0k} = 0$ if $k \ge 1$. Further, let $\varphi_0(z) = 1$ and

$$\varphi_n(z) = \frac{\theta^{(n)}}{n!} \mathbf{E}_{n,\theta} \mathrm{e}^{zh(\sigma)}$$

for $z \in \mathbf{C}$. Thus, $\varphi_n^{(k)}(z)|_{z=0} = \beta_{nk}$.

We have

$$\varphi_n(z) = \sum_{\sigma \in \mathbf{S}_n} \prod_{j=1}^{n} \left(\frac{\theta \mathrm{e}^{za_j}}{j}\right)^{k_j} \frac{1}{k_j!}, \quad z \in \mathbf{C}.$$

This leads to the following formal series equalities from Lemma 2.1:

$$\sum_{n\ge0} \varphi_n(z) w^n = \exp\left\{\theta \sum_{j\ge1} \frac{\mathrm{e}^{za_j}}{j} w^j\right\}$$

and

$$\sum_{n\ge0} \varphi_n'(z) w^n \quad = \quad \theta \sum_{m\ge0} \varphi_m(z) w^m \cdot \sum_{j\ge1} \frac{a_j \mathrm{e}^{za_j}}{j} w^j = \theta \sum_{n\ge0} \left(\sum_{j\le n} \varphi_{n-j}(z) \frac{a_j \mathrm{e}^{za_j}}{j}\right) w^n.$$

Hence

$$\varphi_n'(z) = \theta \sum_{j\le n} \varphi_{n-j}(z) \frac{a_j \mathrm{e}^{za_j}}{j}.$$

Taking the derivatives with respect to z of the $(k-1)$th order, we arrive at

$$\varphi_n^{(k)}(z) = \theta \sum_{j\le n} \sum_{l=0}^{k-1} \binom{k-1}{l} \frac{a_j^{l+1} e^{za_j}}{j} \varphi_{n-j}^{(k-1-l)}(z).$$

Consequently,

$$\beta_{nk} = \theta \sum_{r=1}^{k-1} \binom{k-1}{r-1} \sum_{j\le n} \frac{a_j^r}{j} \beta_{n-j,k-r} + \theta \sum_{j\le n} \frac{a_j^k}{j} \frac{\theta^{(n-j)}}{(n-j)!}. \qquad (2.6)$$

We now apply the mathematical induction. A direct application of (2.4), multiplied by $\theta^{(n)}/n!$ with $k=1$ yields

$$\beta_{n1}=\theta\sum_{j\le n}\frac{a_j}{j}\frac{\theta^{(n-j)}}{(n-j)!}.$$

Assume that the induction hypothesis (2.4) holds for $\beta_{n-j,k-r}$ if $k-r\ge 1$. Applying this formula we use the summation indexes $r_2,\dots$ and $j_2,\dots$ leaving r_1 and j_1 for the summation in (2.6) with respect to r and j. So, inserting the assumption into (2.6), we obtain

$$\begin{aligned}\beta_{nk}&=\theta\sum_{r_1=1}^{k-1}\binom{k-1}{r_1-1}\sum_{j_1\le n}\frac{a_{j_1}^{r_1}}{j_1}\\ &\times\sum_{u=2}^{k-r_1+1}\theta^{u-1}\sum_{r_2+\cdots+r_u=k-r_1}\binom{k-r_1-1}{r_2-1}\cdots\binom{k-r_1-\cdots-r_{u-1}-1}{r_u-1}\\ &\times\sum_{j_2+\cdots+j_u\le n-j_1}\frac{a_{j_2}^{r_2}\cdots a_{j_u}^{r_u}}{j_2\cdots j_u}\frac{\theta^{(n-j_1-\cdots-j_u)}}{(n-j_1-\cdots-j_u)!}+\theta\sum_{j=1}^{n}\frac{a_j^k}{j}\frac{\theta^{(n-j)}}{(n-j)!}.\end{aligned}$$

Interchanging the summation, we arrive at

$$\begin{aligned}\beta_{nk}&=\sum_{u=2}^{k}\theta^{u}\sum_{r_1+\cdots+r_u=k}\binom{k-1}{r_1-1}\cdots\binom{k-r_1-\cdots-r_{u-1}-1}{r_u-1}\\ &\times\sum_{j_1+\cdots+j_u\le n}\frac{a_{j_1}^{r_1}\cdots a_{j_u}^{r_u}}{j_1\cdots j_u}\frac{\theta^{(n-j_1-\cdots-j_u)}}{(n-j_1-\cdots-j_u)!}+\theta\sum_{j=1}^{n}\frac{a_j^k}{j}\frac{\theta^{(n-j)}}{(n-j)!}.\end{aligned}$$

The last sum equals the summand corresponding to $u=1$ in the previous sum over u. Joining them together and multiplying by $\theta^{(n)}/n!$, we obtain (2.5).

The theorem is proved.

A general formula for power moments of an additive function

$$t(\sigma):=\sum_{j=1}^{n}t_j(k_j(\sigma)),$$

where $t_j(k)\in\mathbf{R}$ and $t_j(0)\equiv 0$ for $1\le j\le n$ and $k\ge 0$ is rather involved. To demonstrate the complexity, we include also the case of strongly additive functions, which are defined as $t(\sigma)=\sum_{j=1}^{n}a_j\mathbf{1}\{k_j(\sigma)\ge 1\}$, where $a_j\in\mathbf{R}$. Let us denote:

$$\begin{aligned}\alpha_{nl}&:=\frac{n!}{\theta^{(n)}}\sum_{r=1}^{l}\frac{l!}{r!}\sum_{\substack{s_1,\dots,s_r\ge 1\\ s_1+\cdots+s_r=l}}\frac{1}{s_1!\dots s_r!}\sum_{\substack{j_1,\dots,j_r\le n\\ j_\varsigma\ne j_\tau,1\le\varsigma\ne\tau\le r}}a_{nj_1}^{s_1}\dots a_{nj_r}^{s_r}\\ &\times\prod_{i=1}^{r}\frac{\theta}{j_i}\sum_{\substack{1z_1+\cdots+(n-\ell(\bar e))z_{n-\ell(\bar e)}=n-\ell(\bar e)\\ z_j\ge 0,1\le j\le n-\ell(\bar e)}}\prod_{i\le n-\ell(\bar e)}\left(\frac{\theta}{i}\right)^{z_i}\frac{1}{z_i!}\prod_{\substack{i=1\\ j_i\le n-\ell(\bar e)}}^{r}\frac{1}{z_i+1},\end{aligned}\tag{2.7}$$

where the vector $\bar e=(e_j,\dots,e_n)$ is defined

$$e_j=\begin{cases}1 & if\ \ j=j_1,\dots,j_r,\\ 0 & otherwise\end{cases}\tag{2.8}$$

Lemma 2.2. *Let $t(\sigma)$ be a strongly additive function, $\theta>0$, and let α_{nl} be defined above. Then*

$$\mathbf{E}_n t(\sigma)^l=\alpha_{nl},\tag{2.9}$$

for all $n\in\mathbf{N}$.

Proof.

$$\mathbf{E}_n t(\sigma)^l = \mathbf{E}_n\Big(\sum_{j=1}^{n} a_j \mathbf{1}\{k_j(\sigma)\ge 1\}\Big)^l = \sum_{j_1,\dots,j_l\le n} a_{j_1}\dots a_{j_l}$$
$$\times \mathbf{E}_n\big(\mathbf{1}\{k_{j_1}(\sigma)\ge 1\}\dots \mathbf{1}\{k_{j_l}(\sigma)\ge 1\}\big)$$
$$= \sum_{r=1}^{l}\frac{l!}{r!}\cdot \sum_{\substack{s_1,\dots,s_r\ge 1\\ s_1+\dots+s_r=l}}\frac{1}{s_1!\dots s_r!}\cdot \sum_{j_1\le n} a_{j_1}^{s_1}\sum_{\substack{j_2\le n\\ j_2\ne j_1}} a_{j_2}^{s_2}\dots \sum_{\substack{j_r\le n\\ j_r\ne j_1,\dots j_{r-1}}} a_{nj_r}^{s_r}$$
$$\times \mathbf{E}_n\big(\mathbf{1}\{k_{j_1}(\sigma)\ge 1\}\cdots \mathbf{1}\{k_{j_r}(\sigma)\ge 1\}\big).$$

To calculate $\mathbf{E}_n\big(\mathbf{1}\{k_{j_1}(\sigma)\ge 1\}\dots \mathbf{1}\{k_{j_r}(\sigma)\ge 1\}\big)$ with pairwise different $j_1,\cdots,j_r$, we use the vector $\bar e=(e_1,\dots,e_n)$, defined in (2.8).

If the vector $\bar k(\sigma)$ is a sum $\bar k(\sigma)=\bar e+\bar q(\sigma)$, then by the conditioning relation

$$\mathbf{E}_n\big(\mathbf{1}\{k_{j_1}(\sigma)\ge 1\}\dots \mathbf{1}\{k_{j_r}(\sigma)\ge 1\}\big) = \nu_n(\{\sigma\in \mathbf{S}_n : k_{j_1}(\sigma),\dots,k_{j_r}(\sigma)\ge 1\})$$
$$= \nu_n\big(\{\sigma\in\mathbf{S}_n, \ell(\bar k(\sigma))=\ell(\bar e)+\ell(\bar q(\sigma))\}\big) = P(\ell(\bar\xi)=\ell(\bar e)+\ell(\bar\eta)|\ell(\bar\xi)=n),$$

where $\bar\eta=(\eta_1,\dots,\eta_n)$ is the random vector with independent coordinates such that $\eta_i=\xi_i$ for $i\ne j_1,\dots,j_r$ and $P(\eta_i=k)=P(\xi_i=k-1)$.

Earlier we obtained, that

$$p(n):=P\Big(\ell(\bar\xi)=n\Big)=\exp\Big\{-\sum_{i\le n}\frac{\theta}{i}\Big\}\frac{\theta^{(n)}}{n!}.$$

So

$$P(\ell(\bar\xi)=\ell(\bar e)+\ell(\bar\eta)|\ell(\bar\xi)=n) = p(n)^{-1}\cdot P(\ell(\bar\xi)=l(\bar e)+\ell(\bar\eta), \ell(\bar\xi)=n)$$
$$= p(n)^{-1}\cdot P(\ell(\eta)=n-\ell(\bar e)) =: p(n)^{-1}\cdot P'.$$

Next, if $u:=n-l(\bar e)$,

$$P'=\prod_{u<i\le n}P(\eta_i=0)P\Big(\sum_{i\le u} i\eta_i=u\Big)=$$
$$\exp\Big\{-\theta\sum_{u<i\le n}\frac{1}{i}\Big\}\cdot P\Big(\sum_{i\le u} i\eta_i=u\Big) =: \exp\Big\{-\theta\sum_{u<i\le n}\frac{1}{i}\Big\}\cdot P''.$$

Now

$$P''=\sum_{\substack{z_1,\dots,z_u\ge 0\\ 1z_1+\dots+uz_u=u}}\prod_{i\le n}P(\eta_i=z_i)$$
$$= \sum_{\substack{z_1,\dots,z_u\ge 0\\ 1z_1+\dots+uz_u=u}} \exp\Big\{-\theta\sum_{i\le u}\frac{1}{i}\Big\}\prod_{\substack{i=1\\ j_i\le u}}^{r}\Big(\frac{\theta}{j_i}\Big)^{z_i+1}\frac{1}{(z_i+1)!}\cdot \prod_{\substack{i\le u\\ i\ne j_1,\dots,j_r}}\Big(\frac{\theta}{i}\Big)^{z_i}\frac{1}{z_i!}.$$

Continuing we obtain

$$\nu_n(\{\sigma\in\mathbf{S}_n : k_{j_1}(\sigma),\dots,k_{j_r}(\sigma)\ge 1\}) = p(n)^{-1}\exp\Big\{-\theta\sum_{i\le u}\frac{1}{i}\Big\}\prod_{i=1}^{r}\frac{\theta}{j_i}$$
$$\times \sum_{\substack{1z_1+\dots+uz_u=u\\ z_j\ge 0, 1\le j\le u}}\prod_{i\le u}\Big(\frac{\theta}{i}\Big)^{z_i}\frac{1}{z_i!}\cdot\prod_{\substack{i=1\\ j_i\le u}}^{r}\frac{1}{z_i+1}$$
$$= \frac{n!}{\theta^{(n)}}\cdot\prod_{i=1}^{r}\frac{\theta}{j_i}\cdot \sum_{\substack{1z_1+\dots+uz_u=u\\ z_j\ge 0, 1\le j\le u}}\prod_{i\le u}\Big(\frac{\theta}{i}\Big)^{z_i}\frac{1}{z_i!}\cdot\prod_{\substack{i=1\\ j_i\le u}}^{r}\frac{1}{z_i+1}.$$

So the lth moment equals

$$\mathbf{E}_n t(\sigma)^l = \frac{n!}{\theta^{(n)}}\sum_{r=1}^{l}\frac{l!}{r!}\sum_{\substack{s_1,\dots,s_r\geq 1\\ s_1+\dots+s_r=l}}\frac{1}{s_1!\dots s_r!}\sum_{\substack{j_1,\dots,j_r\leq n\\ j_\varsigma\neq j_\tau,1\leq\varsigma,\tau\leq r}} a_{j_1}^{s_1}\dots a_{j_r}^{s_r}$$

$$\times\prod_{i=1}^{r}\frac{\theta}{j_i}\sum_{\substack{1z_1+\dots+(n-\ell(\bar{e}))z_{n-\ell(\bar{e})}=n-\ell(\bar{e})\\ z_j\geq 0,1\leq j\leq n-\ell(\bar{e})}}\prod_{i\leq n-l(\bar{e})}\left(\frac{\theta}{i}\right)^{z_i}\frac{1}{z_i!}\cdot\prod_{\substack{i=1\\ j_i\leq n-\ell(\bar{e})}}^{r}\frac{1}{z_i+1}.$$

The Lemma is proved.

Further, in the following sections, we will need an estimate of the variance.

$$\mathbf{V}\mathrm{ar}_n h(\sigma) = \mathbf{E}_n h(\sigma)^2 - \big(\mathbf{E}_n h(\sigma)\big)^2$$

with respect to the Ewens probability measure ν_n.

Lemma 2.3. *If $\theta \geq 1$, then*

$$\mathbf{V}\mathrm{ar}_n h(\sigma) \leq 2\theta\sum_{j\leq n}\frac{a_j^2}{j}\psi_n(n-j) =: 2B_n^2(h). \tag{2.10}$$

Proof. Let x^+ denote the nonnegative part of $x\in\mathbf{R}$ and $x^- := x^+ - x$. The sequences $\{a_j^+\}$ and $\{a_j^-\}$, $1\leq j\leq n$, give the splitting $h(\sigma) = h^+(\sigma) - h^-(\sigma)$, where $h^\pm(\sigma)$ are the completely additive functions defined via $a_j^\pm$ respectively. Thus, by virtue of $(x+y)^2\leq 2x^2+2y^2$, it suffices to prove that $\mathbf{V}\mathrm{ar}_n h(\sigma)\leq B_n^2(h)$ in the case $a_j\geq 0$ for all $j\leq n$.

Lemma 2.4 yields

$$\mathbf{E}_n k_j(\sigma) = \frac{\theta a_j}{j}\psi_n(n-j), \qquad \mathbf{E}_n h(\sigma) = \theta\sum_{j\leq n}\frac{a_j}{j}\psi_n(n-j), \tag{2.11}$$

and

$$\begin{aligned}\mathbf{E}_n h(\sigma)^2 &= \sum_{i,j\leq n} a_i a_j \mathbf{E}_n\big(k_i(\sigma)k_j(\sigma)\big)\\ &= \sum_{j\leq n} a_j^2\Big(\mathbf{E}_n k_j(\sigma) + \mathbf{E}_n k_j(\sigma)_{(2)}\Big) + \theta^2\sum_{\substack{i+j\leq n\\ i\neq j}}\frac{a_i a_j}{ij}\psi_n(n-i-j)\\ &= B_n^2(h) + \sum_{i+j\leq n}\frac{a_i a_j}{ij}\psi_n(n-i-j).\end{aligned}$$

Hence

$$\begin{aligned}\mathbf{V}\mathrm{ar}_n h(\sigma) &= B_n^2(h) + \sum_{i+j\leq n}\frac{a_i a_j}{ij}\Big(\psi_n(n-i-j) - \psi_n(n-i)\psi_n(n-j)\Big)\\ &\quad - \sum_{\substack{i+j>n\\ i,j\leq n}}\frac{a_i a_j}{ij}\psi_n(n-i)\psi_n(n-j)\end{aligned}$$

for $\theta > 0$.

If $\theta\geq 1$, we have $\psi_n(n-i-j)\leq\psi_n(n-i)\psi_n(n-j)$. Recalling that $a_j\geq 0$, $j\leq n$, we can omit negative terms and obtain the desired claim $\mathbf{V}\mathrm{ar}_n h(\sigma)\leq B_n^2(h)$.

The lemma is proved.

Remark. It is worth to recall two results showing the quality of the constant in (2.10). Denote

$$\tau_n(\theta) = \sup\big\{\mathbf{V}\mathrm{ar}_n h(\sigma)/B_n^2(h):\ h(\sigma)\not\equiv 0\big\}.$$

We have that $\tau_n(1) = 3/2 + \mathrm{O}(n^{-1})$ and $\tau_n(2) = 4/3 + \mathrm{O}(n^{-1})$ (see [57] and [63]).

2.2 Factorial moments

In this subsection we present exact expressions of the factorial moments of completely additive functions $h(\sigma)$ with $a_j \in \mathbf{R}$. Particular attention is spared to the case with $a_j \in \{0,1\}$ and their approximations. We start with a lemma for mixed factorial moments of $k_i(\sigma)$ where $1 \le i \le r \le n$.

Lemma 2.4. *For* $(j_1, \cdots, j_r) \in \mathbf{Z}_+^r$, $l = 1j_1 + \cdots + rj_r$ *and* $1 \le r \le n$,

$$\mathbf{E}_n\Big\{\prod_{i=1}^r k_{i(j_i)}(\sigma)\Big\} = \psi_n(n-l)\mathbf{1}\{l \le n\}\prod_{i=1}^r \Big(\frac{\theta}{i}\Big)^{j_i}. \tag{2.12}$$

Proof. This formula was established by Watterson (1974). See (5.6) on page 96 in [2].

Note in particular that, in (2.12),

$$\prod_{i=1}^r \Big(\frac{\theta}{i}\Big)^{j_i} = \mathbf{E}\prod_{i=1}^r (\xi_i)_{(j_i)},$$

where ξ_i is a Poisson r.v. with $\mathbf{E}\xi_i = \theta/i, \quad 1 \le i \le r$.

In some applications, a general expression of factorial moments with $a_j \in \mathbf{R}$ is more convenient than that for power moments presented in Theorem 2.1. Denote

$$\begin{aligned}\hat{\gamma}_{nk} \quad &:= \quad \sum_{u=1}^k \theta^u \sum_{r_1+\cdots+r_u=k} \binom{k-1}{r_1-1}\cdots\binom{k-r_1-\cdots-r_{u-1}-1}{r_u-1} \\ &\quad\quad \times \sum_{j_1+\cdots+j_u\le n} \frac{a_{j_1(r_1)}\cdots a_{j_u(r_u)}}{j_1\cdots j_u}\psi_n(n-j_1-\cdots-j_k)\end{aligned} \tag{2.13}$$

and

$$\gamma_{nk} := \frac{\theta^{(n)}}{n!}\hat{\gamma}_{nk}. \tag{2.14}$$

Theorem 2.2. *For a completely additive function* $h(\sigma) = a_1k_1(\sigma)+\cdots+a_nk_n(\sigma)$ *and every* $k \in \mathbf{N}$, *we have*

$$\mathbf{E}_n h(\sigma)_{(k)} = \hat{\gamma}_{nk}, \tag{2.15}$$

where $\theta > 0$.

Proof. Repeat the previous argument for

$$\tau_n(z) := \frac{\theta^{(n)}}{n!}\mathbf{E}_n z^{h_n(\sigma)}$$

instead of $\varphi_n(z)$ (see Theorem 2.1). We omit the further details but present an instant instead.

Corollary 2.1. *Let* $\theta > 0$, $k, n \in \mathbf{N}$ *and* $a_j \in \{0,1\}$, *then for a completely additive function* $h(\sigma)$, *we have*

$$\mathbf{E}_n h(\sigma)_{(k)} := \theta^k \sum_{j_1\le n}^{*} \frac{1}{j_1}\cdots\sum_{j_k\le n}^{*}\frac{1}{j_k}\mathbf{1}\{j_1+\cdots+j_k \le n\}\psi_n(n-j_1-\cdots-j_k). \tag{2.16}$$

Proof. Simplify $\hat{\gamma}_{nk}$, using $a_j \in \{0,1\}$.

Lemma 2.5. *Let $\theta \geq 1$, $h(\sigma)$ be a completely additive function with $a_j \in \{0,1\}$ and let $\hat{\gamma}_{nm}$ be defined in (2.13) above. Then*

$$\hat{\gamma}_{nm} \leq \hat{\gamma}_{n1}^m. \tag{2.17}$$

Proof. Firstly we observe that

$$\begin{aligned}
&\mathbf{1}\{j_1 + \cdots + j_r + j_{r+1} + \cdots + j_k \leq n\} \\
&\leq \mathbf{1}\{j_1 + \ldots + j_r \leq n\}\mathbf{1}\{j_{r+1} + \cdots + j_k \leq n\},
\end{aligned}$$

where $1 \leq r \leq k-1$ and $k \geq 2$. Having in mind that

$$\psi_n(r) := \prod_{k=r+1}^{n} \Big(1 + \frac{\theta - 1}{k}\Big)^{-1} \leq \psi_n(r)\psi_n(k-r) \tag{2.18}$$

for $\theta \geq 1$, we obtain

$$\hat{\gamma}_{nk} \leq \hat{\gamma}_{nr}\hat{\gamma}_{n,k-r}$$

for each $1 \leq r \leq k-1$ and $k \geq 2$. Hence the estimate in (2.17) follows.

The lemma is proved.

Now we find the main asymptotical terms of the factorial moments of $h(\sigma)$ with $a_j \in \{0,1\}$ as $n \to \infty$.

Lemma 2.6. *If $\theta > 0$, $k, n \in \mathbf{N}$ and $a_j \in \{0,1\}$, then*

$$\begin{aligned}
R_{nk} &:= \mathbf{E}_n h_n(\sigma)_{(k)} - \theta^k \sum_{j_1 \leq n}^{*} \frac{1}{j_1} \cdots \sum_{j_k \leq n}^{*} \frac{\mathbf{1}\{j_1 + \cdots + j_k < n\}}{j_k} \Big(1 - \frac{j_1 + \cdots + j_k}{n}\Big)^{\theta - 1} \\
&\ll_k n^{-(1 \wedge \theta)}(1 + \log^k n),
\end{aligned}$$

where $1 \wedge \theta := \min\{1, \theta\}$.

Proof. It suffices to deal with the case if $\theta \neq 1$ and n is sufficiently large. Separating the terms with $j_1 + \cdots + j_k = n$ and using (2.1) twice, we obtain

$$\begin{aligned}
R_{nk} &\ll \sum_{j_1 \leq n} \frac{1}{j_1} \cdots \sum_{j_k \leq n} \frac{\mathbf{1}\{j_1 + \cdots + j_k < n\}}{j_k\big(n - (j_1 + \cdots + j_k)\big)} \Big(1 - \frac{j_1 + \cdots + j_k}{n}\Big)^{\theta - 1} \\
&\quad + n^{1-\theta} \sum_{j_1 \leq n} \frac{1}{j_1} \cdots \sum_{j_k \leq n} \frac{\mathbf{1}\{j_1 + \cdots + j_k = n\}}{j_k} =: R'_{nk} + r_{nk}.
\end{aligned} \tag{2.19}$$

Now, in the sums of second remainder, at least one $j_i \geq n/k$, $1 \leq i \leq k$. Hence

$$\begin{aligned}
r_{nk} &\leq \frac{k^2}{n^\theta} \sum_{j_1 \leq n} \frac{1}{j_1} \cdots \sum_{j_{k-1} \leq n} \frac{\mathbf{1}\{j_1 + \cdots + j_{k-1} \leq n - n/k\}}{j_{k-1}} \\
&\leq \frac{k^2}{n^\theta} \Big(\sum_{j \leq n} \frac{1}{j}\Big)^{k-1} \ll_k \frac{\log^{k-1} n}{n^\theta}
\end{aligned}$$

for every $k \geq 1$. This is better than the expected order.

For brevity, introduce temporarily the notation $J = j_1 + \cdots + j_k$ and $j = j_{k+1}$. We will apply the mathematical induction for either of the sums in the splitting

$$\begin{aligned} R'_{n,k+1} &\ll \sum_{j_1 \le n} \frac{1}{j_1} \cdots \sum_{j_k \le n} \frac{\mathbf{1}\{J < n\}}{j_k} \sum_{j \le (n-J)/2} \frac{1}{j} \frac{1}{(n-J)-j} \\ &+ \sum_{j_1 \le n} \frac{1}{j_1} \cdots \sum_{j_k \le n} \frac{\mathbf{1}\{J < n\}}{j_k} \sum_{(n-J)/2 < j < n-J} \frac{1}{j} \frac{1}{(n-J)-j} \Big(1 - \frac{J+j}{n}\Big)^{\theta-1} \\ &=: \widetilde{R}'_{n,k+1} + \widetilde{R}''_{n,k+1}. \end{aligned}$$

Now,

$$\begin{aligned} \widetilde{R}'_{n1} + \widetilde{R}''_{n1} &= \sum_{j \le n/2} \frac{1}{j} \frac{1}{n-j} + \sum_{n/2 < j < n} \frac{1}{j} \frac{1}{n-j} \Big(1 - \frac{j}{n}\Big)^{\theta-1} \\ &\ll \frac{\log n}{n} + \frac{1}{n^\theta} \sum_{n/2 < j < n} (n-j)^{\theta-2} \ll \frac{\log n}{n} + \frac{1}{n^{1 \wedge \theta}} \ll \frac{\log n}{n^{1 \wedge \theta}} \end{aligned}$$

Assuming that $\widetilde{R}'_{nk} \ll_k (\log^k n)/n$, we have

$$\widetilde{R}'_{n,k+1} \ll \widetilde{R}'_{nk} \log n \ll_k (\log^{k+1} n)/n$$

in either of the cases $\theta < 1$ or $\theta > 1$. Further, if $\theta > 1$, $(1 - (J+j)/n)^{\theta-1} \le 1$; therefore, the same induction argument can be applied to obtain

$$\widetilde{R}''_{n,k+1} \ll \widetilde{R}''_{nk} \log n \ll_k (\log^{k+1} n)/n.$$

If $\theta < 1$, it remains to consider the term $\widetilde{R}''_{n,k+1}$. Now

$$\begin{aligned} \widetilde{R}''_{n,k+1} &\ll \sum_{j_1 \le n} \frac{1}{j_1} \cdots \sum_{j_k \le n} \frac{\mathbf{1}\{J < n\}}{j_k(n-J)} \sum_{(n-J)/2 < j < n-J} \Big(1 - \frac{J+j}{n}\Big)^{\theta-1} \frac{1}{(n-J)-j} \\ &\ll_k \frac{1}{n^{\theta-1}} \sum_{j_1 \le n} \frac{1}{j_1} \cdots \sum_{j_k \le n} \frac{\mathbf{1}\{J < n\}}{j_k(n-J)} \sum_{1 \le s < n} s^{\theta-2} \ll \frac{1}{n^{\theta-1}} \cdot \frac{\log^k n}{n} = \frac{\log^k n}{n^\theta} \end{aligned}$$

since the last sum is bounded and the remaining iterated sum has been estimated before dealing with R'_{nk} in the case $\theta > 1$.

Collecting all the estimates we complete the proof of the lemma.

So, based on obtained approximation in Lemma 2.6, we can rewrite an expression of the factorial moments (2.16) in a form:

$$\hat{\gamma}_{nk} = \theta^k \sideset{}{^*}\sum_{\substack{j_i \le n \\ 1 \le i \le k}} \frac{\mathbf{1}\{j_1 + \cdots + j_k \le n\}}{j_1 \cdots j_k} \left(1 - \frac{j_1 + \cdots + j_k}{n+1}\right)^{\theta-1} + O_k\left(\frac{\log^k n}{n}\right), \tag{2.20}$$

if $n \ge 2$.

3 Multiplicative functions on $\mathbf{Z}_+^n$

In this part of the dissertation we deal with multiplicative functions $G : \mathbf{Z}_+^n \to \mathbf{C}$ and their mean values. We assume for simplicity that $G \in \mathfrak{M}_s$. Further in Corollaries 3.1 - 3.2 we demonstrate that an extension to general multiplicative functions can be achieved by some convolution argument.

3.1 Mean values of multiplicative functions on $\mathbf{Z}_+^n$

3.1.1 Averaged mean values

In the following theorem, we have estimated the quantity $\widetilde{M}_{n,\theta}(G)$, which is just the mean value of G with respect to the measure defined via $\widetilde{P}_{n,\theta}(\{\bar{s}\}) = \Pi(\bar{s})/\Gamma_{n,\theta}$ and supported by the set $\bar{s} \in \mathbf{Z}_+^n : 0 \leq \ell(\bar{s}) \leq n$. To check this, it suffices to observe that $s_j = 0$ if $\ell(\bar{s}) < j \leq n$ and apply an appropriate combinatorial identity.

Theorem 1.1. *Let $\theta > 0$ and $G \in \mathfrak{M}_s$ be defined via sequence $0 \leq g_j \leq 1$ where $1 \leq j \leq n$. Then*

$$\widetilde{M}_{n,\theta}(G) \asymp \exp\Big\{\theta \sum_{j \leq n} \frac{g_j - 1}{j}\Big\}.$$

Proof. Estimating from above, we examine an arbitrary $G \in \mathfrak{M}$ such that $0 \leq g_j(k) \leq 1$ for $j, k \geq 1$ with $g_j := g_j(1)$. Applying (1.11), we obtain

$$\widetilde{M}_{n,\theta}(G) = \frac{1}{\Gamma_{n,\theta}} \sum_{0 \leq m \leq n} [x^m] Z(x; G) \leq \frac{1}{\Gamma_{n,\theta}} \prod_{j \leq n} \Big(1 + \sum_{r \geq 1} \Big(\frac{\theta}{j}\Big)^r \frac{g_j(r)}{r!}\Big).$$

Since $0 \leq G(\bar{s}) \leq 1$, the infinite product

$$\prod_{j \geq 1} \Big(1 + \sum_{r \geq 1} \Big(\frac{\theta}{j}\Big)^r \frac{g_j(r)}{r!}\Big) e^{-\theta g_j / j}$$

converges uniformly in G. Thus the previous estimate implies

$$\widetilde{M}_{n,\theta}(G) \ll \frac{1}{\Gamma_{n,\theta}} \exp\Big\{\theta \sum_{j \leq n} \frac{g_j}{j}\Big\} \ll \exp\Big\{\theta \sum_{j \leq n} \frac{g_j - 1}{j}\Big\}$$

as claimed.

To obtain the desired lower estimate, we define the Möbius function $\mu(\bar{k})$ on $\mathbf{Z}_+^n$ related to the partial order defined by $\bar{s} \leq \bar{k}$ meaning that $s_i \leq k_i$ for each $i \leq n$. If $\bar{k} = (k_1, \ldots, k_n) \in \mathbf{Z}_+^n$, then we set

$$\mu(\bar{k}) = \prod_{j \leq n} \mu_j(k_j), \qquad \text{where} \quad \mu_j(k) = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k \geq 2, \\ -1 & \text{if } k = 1. \end{cases}$$

Given $G \in \mathfrak{M}_s$, we introduce its dual function $G^* \in \mathfrak{M}_s$ defined by $g_j^* = 1 - g_j$ for each $1 \leq j \leq n$. Then

$$G^*(\bar{k}) = \sum_{\bar{t} \leq \bar{k}} \mu(\bar{t}) G(\bar{t}) = \prod_{\substack{j \leq n \\ k_j \geq 1}} (1 - g_j), \qquad G(\bar{k}) = \sum_{\bar{t} \leq \bar{k}} \mu(\bar{t}) G^*(\bar{t}).$$

By (1.11),

$$M_{m,\theta}(\mu^2)=\frac{m!}{\theta^{(m)}}[x^m]\prod_{j\geq 1}\left(1+\frac{\theta x^j}{j}\right).$$

Hence, as it has been shown in [51], $M_m(\mu^2)\asymp 1$ for $m\geq 0$. This implies $\widetilde{M}_n(\mu^2)\asymp 1$ for $n\geq 1$.

If $\ell(\bar{k})=m\leq n$ and $\mu^2(\bar{k})=1$, then $\bar{t}\leq \bar{k}$ implies $\bar{t}\perp \bar{k}-\bar{t}=:\bar{s}$. Hence

$$\sum_{\bar{t}\leq\bar{k}} G(\bar{t})G^*(\bar{k}-\bar{t})=\prod_{\substack{j\leq n\\ k_j=1}}(g_j+g_j^*)=1$$

and

$$\Pi_n(\bar{k})=\Pi_n(\bar{t}+\bar{s})=\Pi_n(\bar{t})\Pi_n(\bar{s}).$$

Consequently,

$$\begin{aligned}
1 \quad &\ll \quad \widetilde{M}_{n,\theta}(\mu^2)=\frac{1}{\Gamma_{n,\theta}}\sum_{\ell(\bar{k})\leq n}\mu^2(\bar{k})\Pi(\bar{k})\sum_{\bar{t}\leq\bar{k}}G(\bar{t})G^*(\bar{k}-\bar{t})\\
&\leq \quad \frac{1}{\Gamma_{n,\theta}}\sum_{\ell(\bar{t})\leq n}G(\bar{t})\Pi(\bar{t})\times\sum_{\ell(\bar{s})\leq n}\mu^2(\bar{s})G^*(\bar{s})\Pi(\bar{s})\\
&= \quad \widetilde{M}_{n,\theta}(G)\Gamma_{n,\theta}\widetilde{M}_{n,\theta}(G^*\mu^2)\ll\widetilde{M}_{n,\theta}(G)\exp\left\{\theta\sum_{j\leq n}\frac{1-g_j}{j}\right\}.
\end{aligned}$$

In the last step we applied already proved upper estimate.

The theorem is proved.

3.1.2 Lower estimate of mean values

The main result of multiplicative functions defined in (1.11) is the next theorem. It is very important and often used as auxiliary in proofs of other theorems.

Theorem 1.2. *Let $\theta\geq 1$ and $G\in\mathfrak{M}_s$ be defined via sequence $0\leq g_j\leq 1$ where $1\leq j\leq n$. If*

$$\sum_{j\leq n}\frac{1-g_j}{j}\leq K \tag{3.1}$$

for some $K>0$, then there exist absolute constants c_0 and c together with a function $\mathcal{N}:\mathbf{R}_+\to\mathbf{N}$ such that

$$\Upsilon(K):=\inf\{M_{n,\theta}(G):n\geq\mathcal{N}(K)\}\geq c_0\exp\{-\exp^{cK}\}. \tag{3.2}$$

Before starting the proof of the theorem, we make a remark, that the lower bound for n, that is, the use of $n\geq\mathcal{N}(K)$ in (3.2) is unavoidable. Without such a bound, given $K\geq 1$, one can assure condition (3.1) for some function $G\in\mathfrak{M}_s$ such that $g_j=0$ for each $1\leq j\leq e^{K-1}$. Then $M_n(G)=0$ for each $1\leq n\leq e^{K-1}$.

Proof. Let the truncated strongly multiplicative function G_r be defined from $G\in\mathfrak{M}_s$ by setting $g_j=1$ for each $r\leq j\leq n$, where $1\leq r\leq n+1$. Then $G_{n+1}=G$, and $G_1(\bar{k})\equiv 1$. Apart from the vectors $\bar{e}_j$, we introduce $\bar{e}^r=(1,\ldots,1,0,\ldots 0)\in\mathbf{Z}_+^n$, where the zeroes start at the r-th,

$r \geq 1$, place. By $\bar{k} \wedge \bar{t}$ we denote the vector with the coordinates $\min\{k_j, t_j\}$ for all $1 \leq j \leq n$. Observe that

$$G_r(\bar{k}) = \sum_{\bar{t} \leq \bar{k} \wedge \bar{e}^r} \mu(\bar{t}) G^*(\bar{t}).$$

If $n/2 < j \leq n$ and $\bar{k} = \bar{e}_j + \bar{t} \in \Omega(n)$, then $\bar{t} \in \Omega(n-j)$, $\bar{e}_j \perp \bar{t}$, and

$$P_{n,\theta}(\bar{k}) = \frac{n!}{\theta^{(n)}} \Pi(\bar{e}_j) \Pi(\bar{t}) = \frac{n!}{\theta^{(n)}} \frac{\theta}{j} \Pi(\bar{t}). \tag{3.3}$$

Now, if $1/n \leq \delta < 1/2$ is arbitrary and $r = m := [(1-\delta)n]$, then, summing over a part of vectors, we obtain

$$\begin{aligned}
M_{n,\theta}(G_m) &\geq \sum_{\substack{\ell(\bar{e}_j + \bar{t}) = n \\ m \leq j \leq n}} G_m(\bar{e}_j + \bar{t}) P_{n,\theta}(\bar{e}_j + \bar{t}) \\
&= \frac{n!}{\theta^{(n)}} \sum_{m \leq j \leq n} \frac{\theta}{j} \sum_{\ell(\bar{t}) = n-j} G(\bar{t}) \Pi(\bar{t}) \\
&\geq \frac{\theta}{n} \frac{n!}{\theta^{(n)}} \Gamma_{[\delta n],\theta}\, \widetilde{M}_{[\delta n],\theta}(G) \\
&\geq c_1 \delta^\theta \exp\{-\theta K\},
\end{aligned} \tag{3.4}$$

where $K > 0$ is as in (3.1). In the last step we used $\theta^{(n)}/n! \ll n^{\theta-1}$ if $n \geq 1$ and Theorem 1.1. Recalling our agreement that $M_{0,\theta}(G_m) = 1$, we observe that (3.4) also holds for $0 \leq \delta < 1/n$.

The next identity is crucial in the forthcoming induction argument. We have

$$M_{n,\theta}(G_r) = M_{n,\theta}(G) + \theta \sum_{r \leq j \leq n} \frac{g_j^*}{j} \psi_n(n-j) M_{n-j,\theta}(G_j) \tag{3.5}$$

for an arbitrary $G \in \mathfrak{M}_s$ and $n/2 < r \leq n$. Indeed, the formal identity

$$1 - \prod_{j=1}^{s} (1 - \alpha_j) = \sum_{j=1}^{s} \alpha_j \prod_{i=1}^{j-1} (1 - \alpha_i), \quad s \geq 1,$$

implies

$$\begin{aligned}
G_r(\bar{k}) - G(\bar{k}) &= G_r(\bar{k}) \Bigg(1 - \prod_{\substack{r \leq j \leq n \\ k_j = 1}} g_j\Bigg) = G_r(\bar{k}) \Bigg(1 - \prod_{\substack{r \leq j \leq n \\ k_j = 1}} (1 - g_j^*)\Bigg) \\
&= G_r(\bar{k}) \sum_{\substack{r \leq j \leq n \\ k_j = 1}} g_j^* \prod_{\substack{r \leq i \leq j-1 \\ k_i = 1}} (1 - g_i^*) = \sum_{\substack{r \leq j \leq n \\ k_j = 1}} g_j^* G_j(\bar{k})
\end{aligned}$$

for each $\bar{k} \in \Omega(n)$. By virtue of (3.3), we obtain the mean value of the last sum:

$$\begin{aligned}
\sum_{r \leq j \leq n} g_j^* \sum_{\bar{k} \in \Omega(n)} P_{n,\theta}(\bar{k}) \mathbf{1}\{k_j = 1\} G_j(\bar{k}) &= \frac{\theta n!}{\theta^{(n)}} \sum_{r \leq j \leq n} \frac{g_j^*}{j} \sum_{\bar{t} \in \Omega(n-j)} G_j(\bar{t}) \Pi(\bar{t}) \\
&= \theta \sum_{r \leq j \leq n} \frac{g_j^*}{j} \psi_n(n-j) M_{n-j,\theta}(G_j).
\end{aligned} \tag{3.6}$$

Combining this with the equality above, we complete the proof of (3.5).

Up to the end of the proof of Theorem 1.2, we fix the notation $m = [(1-\delta)n]$, where $\delta = e^{-K-C}$ and $C \geq 1$ is a constant to be chosen later. For $\theta \geq 1$, we have

$$\psi_n(n-j) \leq 1. \tag{3.7}$$

Hence and from (4.2) and (3.4) we obtain

$$\begin{aligned} M_{n,\theta}(G) &\geq M_{n,\theta}(G_m) - \theta \sum_{m\leq j\leq n} \frac{g_j^*}{j} M_{n-j,\theta}(G_j) \\ &\geq c_1 e^{-\theta C} e^{-2\theta K} - \theta \sum_{m\leq j\leq n} \frac{g_j^*}{j} \geq \alpha(C) e^{-2\theta K}, \end{aligned} \tag{3.8}$$

where $\alpha(C) := (c_1/2)e^{-\theta C}$, provided that

$$\lambda := \sum_{m\leq j\leq n} \frac{g_j^*}{j} \leq (\alpha(C)/\theta)e^{-2\theta K}.$$

If $c \geq 2\theta$, the bound (3.8) for all $K > 0$ is better than that given in Theorem 1.2 with $\mathcal{N}(K) \equiv 2$ and $c_0 \leq \alpha(C)/\theta$.

In what follows, we assume that $\lambda \geq (\alpha(C)/\theta)e^{-2\theta K}$. We will bound $\Upsilon(K)$ from below applying the real type induction on K. To verify the initial step, we argue as in obtaining (3.5). We firstly notice that

$$\sum_{\bar{k}\in\Omega(n)} P_{n,\theta}(\bar{k})\mathbf{1}\{k_j \geq 1\} \leq \sum_{\bar{k}\in\Omega(n)} k_j P_{n,\theta}(\bar{k}) = \psi_n(n-j)\frac{\theta}{j} \leq \frac{\theta}{j}\,,$$

where the first moment formula found in [2] (p. 96, (5.6)) and inequality (3.7) are used. Using this, we obtain

$$\begin{aligned} M_{n,\theta}(G) &= \sum_{\bar{k}\in\Omega(n)} P_{n,\theta}(\bar{k}) \prod_{\substack{j\leq n\\ k_j\geq 1}} \big(1-(1-g_j)\big) \\ &= \sum_{\bar{k}\in\Omega(n)} P_{n,\theta}(\bar{k}) \Big(1 - \sum_{\substack{j\leq n\\ k_j\geq 1}} g_j^* \prod_{\substack{i\leq j-1\\ k_i\geq 1}} g_i\Big) \\ &\geq 1 - \sum_{j\leq n} g_j^* \sum_{\bar{k}\in\Omega(n)} P_{n,\theta}(\bar{k})\mathbf{1}\{k_j\geq 1\} \\ &\geq 1 - \theta \sum_{j\leq n} \frac{g_j^*}{j} \geq 1 - \theta K \end{aligned}$$

for $n \geq 1$. If $\theta K \leq 1/2$, this is better than the desired estimate (3.2) with any $c > 0$ and $c_0 \leq 1$.

Let $\theta K > 1/2$ and $n \geq 1/\delta$. We further examine the set Ω' of vectors $\bar{k} \in \Omega(n)$ having a coordinate $k_j \geq 1$ for some $\delta n \leq j \leq n/2$. The indicator function of this set is

$$\mathbf{1}\{\bar{k}\in\Omega'\} = \max\big\{\mathbf{1}\{\bar{k}:\ k_j\geq 1\}:\ \delta n\leq j\leq n/2\big\}.$$

By virtue of $\ell(\bar{k}) = n$, the equality $\mathbf{1}\{\bar{k}:\ k_j \geq 1\} = 1$ holds for at most $1/\delta$ of $j \in [\delta n, n/2]$. Hence

$$\mathbf{1}\{\bar{k}\in\Omega'\} \geq \delta \sum_{\delta n\leq j\leq n/2} \mathbf{1}\{\bar{k}:\ k_j\geq 1\}.$$

If $\bar{k} \in \Omega'$, then $\bar{k} = \bar{e}_j + \bar{t}$, where $\bar{t} \in \Omega(n-j)$. Moreover,

$$G(\bar{k}) = g_j \prod_{\substack{i\leq n-j\\ i\neq j}} g_i^{\mathbf{1}\{t_i\geq 1\}} \geq g_j G(\bar{t}).$$

Similarly, due to $n \geq j(t_j+1) \geq \delta n(t_j+1)$, we have $t_j + 1 \leq 1/\delta$ and

$$P_{n,\theta}(\bar{e}_j+\bar{t}) = \frac{\theta}{j(t_j+1)}\psi_n(n-j)\, P_{n-j,\theta}(\bar{t}) \geq \frac{c_2\delta}{j} P_{n-j,\theta}(\bar{t})$$

for $\delta n \le j \le n/2$. Here we have applied the estimate $\theta^{(r)}/r! \asymp r^{\theta-1}$ if $r \in \mathbf{N}$. Hence

$$\begin{aligned}
M_{n,\theta}(G) &\ge \sum_{\bar{k}\in\Omega(n)} G(\bar{k})P_{n,\theta}(\bar{k})\mathbf{1}\{\bar{k}\in\Omega'\} \\
&\ge \delta \sum_{\delta n\le j\le n/2} g_j \sum_{\bar{t}\in\Omega(n-j)} G(\bar{t})P_{n,\theta}(\bar{e}_j+\bar{t}) \\
&\ge c_2\delta^2 \sum_{\delta n\le j\le n/2} \frac{g_j}{j} \sum_{\bar{t}\in\Omega(n-j)} G(\bar{t})P_{n-j,\theta}(\bar{t}) \\
&= c_2\delta^2 \sum_{\delta n\le j\le n/2} \frac{g_j}{j} M_{n-j,\theta}(G). \qquad (3.9)
\end{aligned}$$

We now assume that the claim of Theorem 1.2 is proved for $K-\Delta =: K-(\alpha(C)/\theta)e^{-2\theta K}$, that is,

$$\Upsilon(K-\Delta) \ge c_0 \exp\{-e^{c(K-\Delta)}\} \qquad (3.10)$$

and $\mathcal{N}(K-\Delta)$ is found in the latter. Here $c \ge 2\theta$ and $0 < c_0 \le \min\{1, \alpha(C)/\theta\}$ are constants. The task now is to extend this lower estimate for K and define $\mathcal{N}(K)$. We apply (3.10) for the mean values on the right-hand side of (3.9).

If $\delta n \le j \le n/2$, then

$$\sum_{i\le n-j} \frac{g_i^*}{i} \le K - \sum_{m<i\le n} \frac{g_i^*}{i} = K-\lambda \le K-\Delta$$

by our earlier agreement on λ and the definition of m. Set

$$\mathcal{N}(x) = \max\{e^{K+C}, 2\mathcal{N}(K-\Delta)\}$$

for $K-\Delta < x \le K$. If $n \ge \mathcal{N}(K)$ and $\delta n \le j \le n/2$, then $n-j \ge \mathcal{N}(K-\Delta)$. Hence, by (3.10)

$$M_{n-j,\theta}(G) \ge \Upsilon(K-\Delta) \ge c_0\exp\{-e^{c(K-\Delta)}\}.$$

Consequently, (3.9) implies

$$\begin{aligned}
M_{n,\theta}(G) &\ge c_0c_2\delta^2\exp\{-e^{c(K-\Delta)}\}\cdot \sum_{\delta n\le j\le n/2} \frac{1-g_j^*}{j} \\
&\ge c_0c_2\delta^2\exp\{-e^{c(K-\Delta)}\}\Big(-\log(2\delta) - \frac{C_1}{\delta n} - K\Big) \\
&\ge c_0c_2\delta^2\exp\{-e^{c(K-\Delta)}\}\big(C - \log 2 - C_1\big)
\end{aligned}$$

where $C_1 > 0$ is an absolute constant. The choice of $C = (\log 2 + C_1 + 1)/c_2$ is at our disposal. It gives

$$M_{n,\theta}(G) \ge c_0\delta^2\exp\{-e^{c(K-\Delta)}\}.$$

Now, if

$$e^{-2K-2C}\exp\{-e^{c(K-\Delta)}\} \ge \exp\{-e^{cK}\} \qquad (3.11)$$

for all $K \ge 1/(2\theta)$ and for some sufficiently large $c \ge 2\theta$, from the last inequality, we obtain the desired estimate (3.2) with this very c.

Inequality (3.11) is equivalent to

$$e^{cK}\left(1-e^{-c\Delta}\right)\geq 2K+2C.$$

Assuming that $c\geq\theta\alpha(C)^{-1}$, we see that the last inequality follows from

$$e^{cK}\left(1-e^{-e^{-2\theta K}}\right)\geq 2K+2C.$$

Furthermore, due to $xe^{-x}\leq 1-e^{-x}$ for $x\geq 0$, this is implied by

$$e^{(c-2\theta)K}e^{-e^{-2\theta K}}\geq 2K+2C$$

and, further, by

$$e^{(c-2\theta)K}e^{-1}=e^{2K+2C}\exp\{(c-2\theta-2)K-2C-1\}\geq 2K+2C.$$

It is evident that the last inequality holds for all $K\geq 1/(2\theta)$ if $(c-2\theta-2)/(2\theta)\geq 2C+1$. Therefore, to assure this and validity of the previous cases, it suffices to chose

$$c=\max\{\theta\alpha(C)^{-1},2+4\theta(C+1)\}.$$

The theorem is proved.

3.1.3 Lemmata

To prove the corollaries stated below, we need some auxiliary lemmas.

Lemma 3.1. *Assume that*

$$\chi_j(z)=\sum_{n\geq 2}c_{nj}z^n,\quad j\geq 1,$$

are entire functions satisfying $|c_{nj}|\leq C_2^n/n!$ for all $j\geq 1$ and $n\geq 0$, where $C_2>0$ is a constant. Then

$$[z^k]\prod_{j\geq 1}\left(1+\chi_j(z^j/j)\right)\leq\frac{C_3}{k^2},\quad k\geq 1,$$

where C_3 is a positive constant depending on C_2 only.

Proof. This is essentially Lemma 6 from [33], where the case of $\chi_j(z)$ not depending on j has been examined. The proof in more general case goes by the repetition of the same argument.

Lemma 3.2. *Let $F(\bar{k})$ be a complex valued multiplicative function defined via $f_j(s)$ such that $|f_j(s)|\leq 1$ for all $j\geq 1$ and $s\geq 1$. Define the completely multiplicative function $G(\bar{k})$ by setting $g_j=f_j(1)$, $j\geq 1$. If $Z(z;F)$ and $Z(z;G)$ are the corresponding generating functions, then*

$$[z^k]H(z)=[z^k]\big(Z(z;F)/Z(z;G)\big)\ll k^{-2},\quad k\geq 1.\tag{3.12}$$

Proof. We write

$$H(z)=\prod_{j\geq 1} e^{-\theta g_j z^j/j}\left(1+\sum_{s\geq 1}\frac{\theta^s f_j(s)}{j^s s!}z^{sj}\right)=:\prod_{j\geq 1}\big(1+\chi_j(z^j/j)\big).$$

Here

$$\chi_j(z)=\sum_{n\geq 2}\frac{\theta^n z^n}{n!}\sum_{\substack{r+s=n\\ r,s\geq 0}}\binom{n}{r}(-g_j)^r f_j(s)=:\sum_{n\geq 2}c_{nj}z^n$$

are entire functions. Moreover, $|c_{nj}|\leq (2\theta)^n/n!$. By Lemma 3.1, this implies (3.12).

The lemma is proved.

Lemma 3.3. *Let $G\in\mathfrak{M}_c$ be as in Theorem* 1.2 *and $2\leq T\leq\sqrt{n}$ be arbitrary. Then there exist positive constants c_3 and $R_1(K)$ such that*

$$M_{n,\theta}(G)=\frac{\Gamma(\theta)}{2\pi i}\int_{1-iT}^{1+iT}\frac{e^z}{z^\theta}\exp\left\{\theta\sum_{j\leq n}\frac{g_j-1}{j}\,e^{-zj/n}\right\}\mathrm{d}z+\mathrm{O}\Big(R_1(K)T^{-c_3}\Big).$$

Proof. This is a corollary of Proposition in [11]. Checking its proof, one could find an expression of $R_1(K)$.

Lemma 3.4. *Suppose $G\in\mathfrak{M}_c$ be as in Theorem* 1.2*. Then*

$$M_{m,\theta}(G)-M_{n,\theta}(G)\ll n^{-c_4}R_2(K)\tag{3.13}$$

uniformly in $n-\sqrt{n}\leq m\leq n$. Here $R_2(K)=\max\{R_1(K),e^{\theta K}\}$.

Proof. We apply twice the integral representation given in the last lemma and compare the integrands. Let $z=1+it$, $t\in\mathbf{R}$, $|t|\leq T$, and $2\leq T\leq\sqrt{n}$, then

$$\begin{aligned}\sum_{j\leq n}\frac{g_j-1}{j}\,e^{-zj/n}-\sum_{j\leq m}\frac{g_j-1}{j}\,e^{-zj/m}\quad&\ll\quad\frac{1}{\sqrt{n}}+\sum_{j\leq m}\frac{1}{j}\Big|1-e^{-zj(n-m)/mn}\Big|\\ &\ll\quad\frac{T\log n}{\sqrt{n}}\end{aligned}$$

for $n-\sqrt{n}\leq m\leq n$. If $T\leq n^{1/3}$, this and Lemma 3.3 imply

$$M_{m,\theta}(G)-M_{n,\theta}(G)\ll\frac{T(\log T)\log n}{\sqrt{n}}\exp\left\{\theta\sum_{j\leq n}\frac{1-g_j}{j}\right\}+R_1(K)T^{-c_3}.$$

Now we chose $T=n^{1/4}$ to complete the proof of (3.13).

The lemma is proved.

3.1.4 Lower probability estimates

Let $J\subset\{1,\ldots,n\}$ and $\Omega(n;J)=\{\bar{k}\in\Omega(n):\ k_j=0\ \forall\, j\in J\}$.

Corollary 3.1. *Let $\theta\geq 1$, $K>0$, and J be such that*

$$\sum_{j\in J}\frac{1}{j}\leq K<\infty.\tag{3.14}$$

Then

$$P_{n,\theta}\big(\Omega(n;J)\big)\geq c_0\exp\big\{-e^{cK}\big\}$$

for $n\geq\mathcal{N}(K)$. Here c, c_0, and $\mathcal{N}(K)$ are the same as in Theorem 1.2.

Proof. Apply Theorem 1.2 for the strongly multiplicative indicator function $G(\bar{k})$ defined via $g_j = 0$ if $j \in J$ and $g_j = 1$ otherwise.

The next corollary involves two types of sifting (one with respect to the indexes and another with respect to the value of coordinates) of the vectors from $\Omega(n)$. We also observe that in the Corollary 3.2, the indicator function of the examined event is not strongly multiplicative. Its proof is based on the convolution argument combined with auxiliary lemmas, presented above.

Corollary 3.2. *Let $\theta \geq 1$, $K > 0$, and J be as in Corollary 3.1. Denote $I = \{1, \ldots, n\} \setminus J$. Then there exists a positive constant $R(K)$ such that*

$$P_{n,\theta}\big(\bar{k} \in \Omega(n; J) : \; k_i \leq 1 \; \forall \, i \in I\big) \geq R(K), \tag{3.15}$$

provided that $n \geq \mathcal{N}_1(K)$ is sufficiently large.

Proof. The indicator of the event in (3.15) is the multiplicative function $F(\bar{k})$ defined by

$$f_j(k) = \begin{cases} 0 & \text{if } j \in J, \\ 0 & \text{if } j \in I \text{ and } k \geq 2, \\ 1 & \text{otherwise.} \end{cases}$$

Introduce also the multiplicative indicator function $G \in \mathfrak{M}_c \cap \mathfrak{M}_s$ so that $g_j = f_j(1)$ where $j \leq n$. The corresponding generating functions satisfy the following relation

$$Z(z; F) = Z(z; G) H(z),$$

where, by Lemma 3.2, $h_k := [z^k]H(z) \ll k^{-2}$ for $k \geq 1$.

Applying Lemma 3.4, we obtain

$$\begin{aligned} M_{n,\theta}(F) &= \Big(\sum_{k \leq \sqrt{n}} + \sum_{\sqrt{n} < k \leq n} \Big) h_k M_{n-k,\theta}(G) \\ &= \big(M_{n,\theta}(G) + \mathrm{O}\big(n^{-c_4} R_2(K)\big)\big) \sum_{k \leq \sqrt{n}} h_k + \mathrm{O}\Big(\sum_{\sqrt{n} < k \leq n} \frac{1}{k^2} \Big) \\ &= H(1) M_{n,\theta}(G) + \mathrm{O}\big(n^{-c_4} R_2(K)\big) + \mathrm{O}(n^{-1/2}). \end{aligned}$$

By definition,

$$H(1) = \prod_{j \in I} e^{-1/j} \Big(1 + \frac{1}{j}\Big) \geq \frac{2}{e} \prod_{j \geq 2} \Big(1 - \frac{1}{j^2}\Big) = \frac{1}{e}.$$

Inserting this and the estimate obtained in Corollary 3.1 into the previous inequality, we complete the proof.

Corollary 3.2 is proved.

4 Cycles with restricted lengths

In this section we answer the question about general conditions under which the distributions $V_n(x) := V_{n,\theta}(x)$ converge weakly to a limit distribution law with respect to EPM. We observe, that under the condition $a_{nj} \in \{0,1\}$ the additive function $h(\sigma)$ is just the number of cycles with restricted lengths of a random permutation $\sigma \in \mathbf{S}_n$. Further, the dependence on θ is allowed but not additionally indicated.

4.1 Lemmata

The following lemma concerns the concentration function

$$Q_n(u) := \sup \big\{ \nu_n\big(|h(\sigma) - x| < u\big) : \ x \in \mathbf{R} \big\}, \quad u \geq 0.$$

Denote

$$D_n(u) := \min_{\lambda} D_n(u;\lambda) := \min \bigg\{ \sum_{j \leq n} \frac{u^2 \wedge (a_j - \lambda j)^2}{j} \ : \ \lambda \in \mathbf{R} \bigg\}.$$

Lemma 4.1. *For arbitrary $\theta > 0$, we have*

$$Q_n(u) \ll u\big(D_n(u)\big)^{-1/2}. \tag{4.1}$$

Proof. In case, when $\theta = 1$, the lemma is proved in [53]. To prove the inequality, we use

$$\frac{1}{n!} \bigg| \sum_{\sigma \in \mathbf{S}_n} f(\sigma) \bigg| \leq C_1 \exp \bigg\{ - C \min_{|u| \leq \pi} \sum_{j=1}^{n} \frac{1 - \Re(b(j) a^{iuj})}{j} \bigg\},$$

where

$$f(\sigma) = \prod_{j=1}^{n} b(j)^{k_j(\sigma)}, \quad b(j) = e^{2\pi i h_j(1) t},$$

$t \in \mathbf{R}$, $\quad 1 \leq j \leq n$ and $0^0 := 1$.

For $\theta > 0$, we have [49]

$$\frac{1}{\theta^{(n)}} \sum_{\sigma \in \mathbf{S}_n} e^{ith_n(\sigma)} \theta^{w(\sigma)} \leq C_2 \exp \bigg\{ C \min_{|u| \leq \pi} \sum_{j=1}^{n} \frac{1 - \Re(b(j) a^{iuj})}{j} \bigg\},$$

where $C > 0$ depends mostly on θ. Thus, in the general case, further estimation of the concentration function remains the same.

The lemma is proved.

The previous lemma is used to obtain lower estimates of the further needed frequencies. Let $J \subset \{j : j \leq n\}$ be an arbitrary nonempty set, maybe, depending on n, and $\overline{J} = \{j : j \leq n\} \setminus J$.

Lemma 4.2. *Let $\theta \geq 1$, $K > 0$, and J be such that*

$$\sum_{j \in J} \frac{1}{j} \leq K < \infty. \tag{4.2}$$

Denote

$$\mu_n(K) = \inf_J \nu_n(k_j(\sigma) = 0 \quad \forall j \in J),$$

where the infimum is taken over J satisfying (4.2)*. For a sufficiently large $n_0(K)$, there exists a positive constant $c(K)$, depending at most on θ and K, such that*

$$\mu_n(K) \geq c(K)$$

if $n \geq n_0(K)$.

Proof. The claim is Corollary 3.1 of Theorem 1.2 (see section 3.).

Lemma 4.3. *Let J satisfy* (4.2)*, $n_0(K)$ be as in Lemma 4.2, and*

$$I \subset \big((J \times \{1\}) \cup \{(j,k) \in \mathbf{N}^2 : k \geq 2\}\big) \cap \{(j,k) \in \mathbf{N}^2, jk \leq n - n_0(K)\}$$

be an arbitrary subset of ordered pairs of natural numbers. Define

$$\widetilde{S_n} := \bigcup_{(j,k)\in I} S_n^{j,k},$$

where

$$S_n^{j,k} := \Big\{\sigma \in S_n : k_j(\sigma) = k,\ k_i(\sigma) = 0\ \forall i \in J \setminus \{j\},\ k_l(\sigma) \leq 1\ \forall l \in \overline{J} \setminus \{j\}\Big\}.$$

Then

$$\nu_n(\widetilde{S_n}) \gg_K \sum_{(j,k)\in I} \Big(\frac{\theta}{j}\Big)^k \frac{1}{k!}\Big(1 - \frac{jk}{n}\Big)^{\theta-1}, \tag{4.3}$$

provided that $n \geq 2n_0(K)$.

Proof. We use (1.2) to obtain

$$\nu_n(S_n^{j,k}) = P\Big(\xi_j = k,\ \xi_i = 0\ \forall i \in J \setminus \{j\},\ \xi_l \leq 1\ \forall l \in \overline{J} \setminus \{j\}\Big|\, \ell(\bar{\xi}) = n\Big).$$

Set $Q(m) = P\big(\ell(\bar{\xi}) = m\big)$ for $0 \leq m \leq n$. Then, if $j \in J$,

$$Q(n)\nu_n(S_n^{j,k}) = \frac{1}{k!}\bigg(\frac{\theta}{j}\bigg)^k P\bigg(\xi_i = 0\ \forall i \in J,\ \xi_l \leq 1\ \forall l \in \overline{J},\ \sum_{\substack{i \notin J \\ i \leq n}} i\xi_i = n - jk\bigg). \tag{4.4}$$

Denote $J_m := J \cap [1; m]$ and $\overline{J}_m := \{j : j \leq m\} \setminus J_m$ for $0 \leq m \leq n$. Observing that $\ell(\bar{\xi}) = n - jk$ implies $\xi_i = 0$ for each $n - jk < i \leq n$, we obtain

$$\begin{aligned}
Q(n)\nu_n(S_n^{jk}) &= \Big(\frac{\theta}{j}\Big)^k \frac{1}{k!} \exp\Big\{-\theta \sum_{n-jk<i\leq n} \frac{1}{i}\Big\} \\
&\quad \times P\bigg(\xi_i = 0\ \forall i \in J_{n-jk},\ \xi_l \leq 1\ \forall l \in \overline{J}_{n-jk},\ \sum_{i\leq n-jk} i\xi_i = n - jk\bigg) \\
&= \Big(\frac{\theta}{j}\Big)^k \frac{1}{k!} \exp\Big\{-\theta \sum_{n-jk<i\leq n} \frac{1}{i}\Big\} Q(n-jk) \\
&\quad \times \nu_{n-jk}\big(k_i(\sigma) = 0\ \forall i \in J_{n-jk},\ \xi_l \leq 1\ \forall l \in \overline{J}_{n-jk}\big)
\end{aligned}$$

Here we again use (1.2). By Cauchy's equality,

$$\begin{aligned}
Q(n) &= P\big(\ell(\bar{\xi}) = n\big) = \sum_{\substack{s_1,\dots,s_n \geq 0 \\ \ell(\bar{s})=n}} \prod_{i\leq n} e^{-\theta/i}\Big(\frac{\theta}{i}\Big)^{s_i} \frac{1}{s_i!} \\
&= \frac{\theta^{(n)}}{n!} \exp\Big\{-\sum_{i\leq n} \frac{\theta}{i}\Big\}.
\end{aligned}$$

Inserting this into the previous equality, using (1.2) and Lemma 4.2, we obtain

$$\nu_n(S_n^{jk}) \geq c(K)\Big(\frac{\theta}{j}\Big)^k \frac{1}{k!}\frac{\theta^{(n-jk)}}{(n-jk)!}\frac{n!}{\theta^{(n)}} \gg_K \Big(\frac{\theta}{j}\Big)^k \frac{1}{k!}\Big(1-\frac{jk}{n}\Big)^{\theta-1} \tag{4.5}$$

provided that $n - jk \geq n_0(K)$.

If $j \in \overline{J}$, instead of (4.4), we have

$$Q(n)\nu_n(S_n^{j,k}) = \Big(\frac{\theta}{j}\Big)^k \frac{1}{k!} P\Big(\xi_i = 0 \; \forall i \in J \cup \{j\},\; \xi_l \leq 1 \; \forall l \in \overline{J} \setminus \{j\},\; \sum_{\substack{i \notin J \\ i \leq n}} i\xi_i = n - jk\Big).$$

Repeating the previous argument, from this we obtain (4.5) with $c(K+1)$ instead of $c(K)$.

The sets S_n^j for $j \in I$ are pairwise disjoint, summing up (4.5) over $(j,k) \in I$, we complete the proof of the lemma.

Proposition 4.1. *Assume that $T_n(x) \Rightarrow F(x)$, where $F(x)$ is a distribution function having a positive jump at a point $x = x_0$. Then*

$$\sum_{j \leq n} \frac{\mathbf{1}\{a_j \neq 0\}}{j} \leq C_F < \infty, \tag{4.6}$$

where C_F is a positive constant depending on F.

Proof. Since the limit law has an atom, we obtain a lower estimate of the concentration function $Q_n(u) \geq c > 0$ for every $u > 0$ if n is sufficiently large. Now applying Lemma 4.1, we have $D_n(u, \lambda) \ll u^2$ which includes (4.6) with $a_j - \lambda j$, where $\lambda = \lambda_n \in \mathbf{R}$, instead of a_j. If $J := \{j \leq n : a_j \neq \lambda j\}$ and

$$h_n(\sigma) = \lambda \ell(\bar{k}(\sigma)) + \sum_{j \in J}(a_j - \lambda j)k_j(\sigma) =: \lambda n + \hat{h}_n(\sigma),$$

then, by Lemma 4.2,

$$\nu_n(h_n(\sigma) = \lambda n) = \nu_n(\hat{h}_n(\sigma) = 0) \geq \nu_n\big(k_j(\sigma) = 0 \; \forall j \in J\big) \geq c_1 > 0.$$

Hence if $\lambda n \to \infty$ for some subsequence of $n \to \infty$, at least c_1 of the probability mass disappears at infinity. This contradicts to the assumption of theorem. Hence $\lambda \ll n^{-1}$. Now the inequality $(x+y)^2 \leq 2x^2 + 2y^2$, $x, y \in \mathbf{R}$ implies

$$D_n(1,0) \leq 2D_n(1,\lambda) + 2\sum_{j \leq n} \frac{1 \wedge (\lambda j)^2}{j} \ll 1 + \lambda^2 n^2 \ll 1.$$

The latter estimate contains (4.6).

The proposition is proved.

4.2 Convergence of factorial moments

Theorem 1.3. *Let $h_n(\sigma)$ be a sequence of completely additive functions with $a_j \in \{0,1\}$ and $\theta > 0$. The frequencies $V_n(x)$ converge weakly to a limit law if and only if there exist finite limits*

$$\lim_{n \to \infty} \hat{\gamma}_{nm} =: \hat{\gamma}_m \tag{4.7}$$

for all $m \in \mathbf{N}$. Moreover, if (4.7) is satisfied, the characteristic function of the limit distribution is

$$1 + \sum_{m=1}^{\infty} \frac{\hat{\gamma}_m}{m!}\big(\mathrm{e}^{it} - 1\big)^m, \quad t \in \mathbf{R}.$$

Proof. Sufficiency. Condition (4.7) of the theorem implies

$$\hat{\gamma}_{n1} \le C < \infty,$$

if $n \ge 1$, where $C > 0$ depends on θ only. Further we use (4.7) and the expression

$$\mathbf{E}_n e^{ith_n(\sigma)} = 1 + \sum_{m=1}^{L} \frac{\hat{\gamma}_{nm}}{m!}(e^{it}-1)^m + O\bigg(\frac{\hat{\gamma}_{n,L+1}}{(L+1)!}|e^{it}-1|^{L+1}\bigg). \tag{4.8}$$

To estimate the reminders, we now prove that

$$\hat{\gamma}_{nm} \le C_1 \hat{\gamma}_{n,m-1},$$

where $C_1 > 0$ does not depend on $m \ge 1$.

Let recall the formula of the factorial moments $\hat{\gamma}_{nm}$ with $a_j \in \{0,1\}$

$$\hat{\gamma}_{nm} = \theta^m \sum_{j_1 \le n}^{*} \frac{1}{j_1} \cdots \sum_{j_m \le n}^{*} \frac{1}{j_m} \mathbf{1}\{j_1 + \cdots + j_m \le n\} \psi_n(n - j_1 - \cdots - j_m).$$

Denote $J := j_1 + \ldots + j_{m-1}$ and majorise the most inner sum over $j_m =: j$, so using $c\theta^{(n)}/n! \le (1+n)^{\theta-1} \le C_2 \theta^{(n)}/n!$ we obtain

$$\begin{aligned}
\sum_{j \le n-J}^{*} \frac{1}{j} \frac{\theta^{(n-J-j)}}{(n-J-j)!} \quad &= \quad \theta \sum_{j \le n-J}^{*} \frac{1}{j} \psi_n(n-j) \frac{\psi_{n-j}(n)}{\theta} \frac{\theta^{(n-J-j)}}{(n-J-j)!} \\
&\le \quad \frac{C_2}{\theta c^2} \hat{\gamma}_{n1} \ll \frac{C_2 C}{c^2} =: C_1
\end{aligned}$$

By induction $\hat{\gamma}_{nm} \ll C_3^m$, where $C_3 = \max\{C, C_1\}$. So applying the latter evaluation we obtain that

$$\mathbf{E}_n e^{ith_n(\sigma)} = 1 + \sum_{m=1}^{L} \frac{\hat{\gamma}_m}{m!}(e^{it}-1)^m + O\bigg(\frac{{C_3}^L}{(L+1)!}\bigg) + o_L(1),$$

where either of the estimates is uniform in $t \in \mathbf{R}$ and the second one depends on $L \ge 1$. Taking now $n \to \infty$ and $L \to \infty$, we complete the proof of convergence of the characteristic function and find the formula of its limit. This implies the weak convergence of $V_n(x)$.

Necessity. Let $V_n(x)$ converges weakly to a limit distribution $P(\phi < x)$, where ϕ is a random variable taking values in the set $\mathbf{Z}_+$. Using the proposition 4.1, we

$$\hat{\gamma}_{n1} \ll \sum_{j \le n}^{*} \frac{1}{j}(1 - \frac{j}{n+1})^{\theta-1} \ll 1,$$

for $n \ge 1$.

Then applying that $\hat{\gamma}_{nm} \ll {C_3}^m$, we obtain

$$\sup_n \mathbf{E}_n h_n(\sigma)_{(m)} \ll C_4^m$$

for every $m \ge 1$ and with some positive constant depending on θ.

From the weak convergence of frequencies $V_n(x)$ we obtain convergence of factorial moments.

Theorem is proved.

4.3 Laws with a finite support

Corollary 4.1. *Let $m \geq 2$ be fixed, $\theta > 0$ and $h_n(\sigma)$ be a sequence of completely additive functions with $a_j \in \{0,1\}$. The frequencies $V_n(x)$ converge weakly to a limit distribution $F_\phi(u)$ with a finite support $\{0,1,2,\dots,m-1\}$, where ϕ is a r.v. if and only if*

$$\lim_{n\to\infty} \sideset{}{^*}\sum_{j\leq n/m} \frac{1}{j} = 0, \tag{4.9}$$

$$\begin{aligned}\lim_{n\to\infty} \hat{\gamma}_{nk} &= \lim_{n\to\infty} \theta^k \sideset{}{^*}\sum_{n/m<j_1\leq n} \frac{1}{j_1} \cdots \sideset{}{^*}\sum_{n/m<j_k\leq n} \frac{1}{j_k} \mathbf{1}\{j_1+\cdots+j_k \leq n\} \\ &\times \Big(1 - \frac{j_1+\cdots+j_k}{n+1}\Big)^{\theta-1} = \hat{\gamma}_k \end{aligned} \tag{4.10}$$

for each $1 \leq k \leq m$. Moreover, the characteristic function of the limit distribution $F_\phi(u)$ has the form

$$1 + \sum_{k=1}^{m-1} \frac{\hat{\gamma}_k}{k!} \big(\mathrm{e}^{it}-1\big)^k, \quad t \in \mathbf{R}$$

Proof. Necessity. Let $V_n(x)$ converge weakly to a limit distribution $F_\phi(u)$ of a random variable ϕ. According to the Theorem 1.3,

$$\lim_{n\to\infty} \hat{\gamma}_{nk} = \hat{\gamma}_k$$

exist for each $k \geq 0$ and are equal to the factorial moments of $F_\phi(u)$. Since the support is finite, $\hat{\gamma}_k = 0$, if $k \geq m$. Let us prove condition (4.9) of the corollary. From the formula of the mth factorial moment (2.20), we have

$$0 = \lim_{n\to\infty} \theta^m \sideset{}{^*}\sum_{\substack{j_i\leq n\\ 1\leq i\leq m}} \frac{\mathbf{1}\{j_1+\cdots+j_m \leq n\}}{j_1\cdots j_m} \Big(1 - \frac{j_1+\ldots+j_m}{n+1}\Big)^{\theta-1}.$$

Let $0 < \varepsilon < 1/m$ and take $j_r \leq n(1-\varepsilon)/m$ for each $1 \leq r \leq m$, then

$$\Big(1 - \frac{j_1+\ldots+j_m}{n+1}\Big)^{1-\theta} \geq \begin{cases} \varepsilon^{\theta-1}, & if \ \ \theta \geq 1, \\ 1, & if \ \ \theta < 1. \end{cases}$$

Then

$$\Big(\theta \sideset{}{^*}\sum_{j\leq n(1-\varepsilon)/m} \frac{1}{j}\Big)^m \times \min\{1, \varepsilon^{\theta-1}\} = o(1),$$

if $n \to \infty$.

On the other hand,

$$\sum_{n(1-\varepsilon)/m<j\leq n/m} \frac{1}{j} \ll \log \frac{1}{1-\varepsilon} \ll \varepsilon.$$

Consequently,

$$\sideset{}{^*}\sum_{j\leq n/m} \frac{1}{j} \ll o_\varepsilon(1) + \varepsilon$$

for each $0 < \varepsilon < \frac{1}{m}$. So this implies condition (4.9) of the corollary.

For proving the second condition (4.10), it is sufficient to apply the mathematical induction. It is obvious that $1/(2m) < 1 - j/(n+1) \leq 1$ if $j \leq n/m$; therefore by (4.9),

$$\hat{\gamma}_{n1} = \theta \sum_{j \leq n}^{*} \frac{1}{j}\left(1 - \frac{j}{n+1}\right)^{\theta-1} + o(1) = \sum_{n/m < j \leq n} \frac{1}{j}\left(1 - \frac{j}{n+1}\right)^{\theta-1} + o(1).$$

Now we prove that, if equality (4.10) is true for any k, $1 \leq k \leq m-1$, so it is true for $k+1 \leq m-1$.

The argument is seen in the case $k = 2$. The summation region for j_1 and j_2 in $\hat{\gamma}_{n2}$ is the "triangle" $\{j_1, j_2 :\ j_i + j_2 \leq n\}$. Condition (4.9) takes out the square $\{j_1, j_2 \leq \varepsilon n := n/m\}$ from it. The remaining region to be cut of consists of two symmetric triangles, one of which is

$$\{j_1, j_2 :\ j_i \leq \varepsilon n, \varepsilon n < j_2 \leq n, j_i + j_2 \leq n\}.$$

For $\theta \geq 1$, we have

$$\begin{aligned} &\sum_{j \leq \varepsilon n}^{*} \frac{1}{j} \sum_{\varepsilon n < i \leq n-j}^{*} \frac{1}{i}\left(1 - \frac{i+j}{n+1}\right)^{\theta-1} \ll \sum_{j \leq \varepsilon n}^{*} \frac{1}{j} \log \frac{n-j}{\varepsilon n} \\ &\ll_{\varepsilon} \sum_{j \leq \varepsilon n}^{*} \frac{1}{j} = o_{\varepsilon}(1). \end{aligned}$$

In the case $\theta < 1$, it suffices to show that the inner sum of the latter expression is bounded:

$$\sum_{\varepsilon n < i \leq n-j}^{*} \frac{1}{i}\left(1 - \frac{i+j}{n+1}\right)^{\theta-1} \leq \frac{1}{n^{\theta-1}} \sum_{\varepsilon n < i \leq n-j}^{*} \frac{1}{i}(n+1-i-j)^{\theta-1} \leq \frac{1}{\varepsilon n^{-\theta}} \sum_{1 \leq k \leq n} k^{\theta-1} \ll_{\varepsilon} 1.$$

In this way, we can eliminate the summation over $j_1, \ldots, j_{m-1}$ one of which is larger than εn. Then the factorial moments attain the form given in the Corollary. We omit the remaining details.

Sufficiency. From the equalities (4.9) and (4.10) follows (4.7) for each fixed $l = 1, 2, \ldots, m-1$. So to complete the proof of sufficiency of the Corollary 4.1, we just apply the Theorem 1.3.

5 Weak law of large numbers

In this section we answer the question about necessary and sufficient conditions for the weak law of large numbers if the parameter is not less than one. Firstly we present some observations.

In virtue of

$$\frac{1}{2}\sum_{s_1,\ldots,s_r\geq 0}\big|\nu_n\big(k_1(\sigma)=s_1,\ldots,k_r(\sigma)=s_r\big)-P(\xi_1=s_1,\ldots,\xi_r=s_r)\big|=o(1),\quad if\quad r=o(n)$$

one could expect that the conditions are close to that for the sums of independent r.vs $X_j:=a_j\xi_j$, $j\leq n$. The instance of $\lambda\ell(\bar{k}(\sigma))\equiv\lambda n$, with an arbitrary sequence $\lambda:=\lambda_n\in\mathbf{R}$ shows that this is not the case, however. This sequence of functions obeys the degenerated limit law at the point zero if centralized by λn, while the corresponding sum of X_j does not in general. This shows that an additive function can have a deterministic summand to be extracted in the first step of the problem solving. If we are successful in doing this, the difference

$$h(\sigma)-\lambda\ell(\bar{k}(\sigma))=\sum_{j=1}^{n}(a_j-\lambda j)k_j(\sigma),$$

demonstrates closer behavior in some stochastic sense to the sums of independent r.vs $(a_j-\lambda j)\xi_j$, $j\leq n$. That have been established to be true if permutations are taken with equal probabilities or even according to a generalized Ewens measure, provided that the influence of long cycles is negligible (see [62], [15]). If the latter does not hold and $\theta\neq 1$, more bias appears. As it is seen in the below formulated result, this gives an extra factor $(1-j/n)^{\theta-1}$ in the conditions. A quantitative result is demonstrated in the Theorem 1.4 of this section. The proof of this theorem is based upon the number theoretical ideas originated by I.Z.Ruzsa in [70] and also adopted in probabilistic combinatorics by E.Manstavičius in the case $\theta=1$ (see [59]).

Further the dependence on θ is allowed but not additionally indicated.

5.1 Lemmata

Lemma 5.1. *Let $\theta\geq 1$, ξ_j, $1\leq j\leq n$ be independent Poisson r.vs. with $\mathbf{E}\xi_j=\theta/j$, $h(\sigma)$ be an additive function, $b\in\mathbf{R}$, and $u\geq 0$. Then*

$$\nu_n(|h(\sigma)-b|\geq u)\ll P\bigg(\Big|\sum_{j=1}^{n}h_j(\xi_j)-b\Big|\geq u/3\bigg).\tag{5.1}$$

Proof. See [47].

Applying (5.1) and formula $\mathbf{E}|X|^l=l\int_0^\infty|u|^{l-1}P(|X|\geq u)du$ twice, then using Rosenthal's inequality for power moments of $X_n-\mathbf{E}X_n$, we obtain the following result.

Corollary 5.1. *Let $\theta\geq 1$. For arbitrary $l\geq 2$, we have*

$$\mathbf{E}_n|h_n(\sigma)-A(n;h)|^l\ll_l\bigg(\sum_{jk\leq n}\Big(\frac{\theta}{j}\Big)^k\frac{h_j^2(k)}{k!}\bigg)^{l/2}+\sum_{jk\leq n}\Big(\frac{\theta}{j}\Big)^k\frac{|h_j(k)|^l}{k!},$$

where $A(n;h)$ is either of the sums

$$A_1(n;h):=\sum_{jk\leq n}\Big(\frac{\theta}{j}\Big)^k\frac{h_j(k)}{k!},\quad A_2(n;h):=\sum_{j\leq n}\Big(\frac{\theta}{j}\Big)h_j(1).$$

5.2 Estimate of Lévy distance

Let us recall some notation. The Lévy distance of the r.v. $h(\cdot)$ from the set of constants

$$L(h;\nu_n) := \inf\big\{\varepsilon + \nu_n(|h(\sigma)-a|\geq\varepsilon)\colon\ a\in\mathbf{R}, \varepsilon>0\big\}.$$

Let $u\vee v := \max\{u,v\}$, $u\wedge v := \min\{u,v\}$ and as earlier $u^\circ := 1\wedge|u|\operatorname{sgn} u$ if $u,v\in\mathbf{R}$,

$$U_n(h,\lambda) := \sum_{j\leq n}\frac{\theta}{j}\big(a_j-\lambda j\big)^{\circ 2}\psi_n(n-j)$$

and $U_n(h) = \min\{U_n(h,\lambda)\colon\ \lambda\in\mathbf{R}\}$. In the sequel, $\ll$ is used as an analog of $\mathrm{O}(\cdot)$, moreover, dependence on θ in the involved constants is allowed.

Theorem 1.4. *If $\theta\geq 1$ and $h(\sigma)$ is a completely additive function, then*

$$L(h;\nu_n)\leq 2\big(1\wedge(2U_n(h))^{1/3}\big)$$

and

$$U_n(h)\ll (1/n)\vee L(h;\nu_n)$$

for all $n\geq 1$.

Proof.

The upper estimate. Recall that $\ell(\bar{k}(\sigma)) = n$ for each $\sigma\in\mathbf{S}_n$. Hence $L(h;\nu_n) = L(h-\lambda\ell;\nu_n)$ for every $\lambda\in\mathbf{R}$. Without loss of generality, we further assume that $\lambda=0$ and set $U_n(h,0) = U_n(h) =: \delta$. If $\delta=0$, then $a_j=0$ for each $j\leq n$ and $L(h,\nu_n)=0$. If $\delta\geq 1/2$, the trivial upper bound in Theorem 1.4 holds. It remains the case with $0<\delta<1/2$.

Define

$$h'(\sigma) = \sum_{j\leq n} a_j\mathbf{1}\{|a_j|<1\}k_j(\sigma), \qquad h''(\sigma) = h(\sigma)-h'(\sigma).$$

Observe that, by virtue of (2.11),

$$\begin{aligned}
\nu_n\big(h''(\sigma)\neq 0\big) &\leq \sum_{j\leq n}\mathbf{1}\{|a_j|\geq 1\}\nu_n\big(k_j(\sigma)\geq 1\big)\\
&\leq \sum_{j\leq n}\mathbf{1}\{|a_j|\geq 1\}\mathbf{E}_n k_j(\sigma)\\
&\leq \sum_{j\leq n}\mathbf{1}\{|a_j|\geq 1\}\frac{\theta}{j}\psi_n(n-j).
\end{aligned}\tag{5.2}$$

Lemma 2.3 implies

$$\nu_n\big(\big|h'(\sigma)-\mathbf{E}_n h'(\sigma)\big|\geq\varepsilon\big)\leq 2\varepsilon^{-2}B_n^2(h').$$

Now,

$$\begin{aligned}
&\nu_n\big(\big|h(\sigma)-\mathbf{E}_n h'(\sigma)\big|\geq\varepsilon\big)\leq\nu_n\big(\big|h'(\sigma)-\mathbf{E}_n h'(\sigma)\big|\geq\varepsilon\big)+\nu_n\big(h''(\sigma)\neq 0\big)\\
\leq\ & 2\varepsilon^{-2}B_n^2(h')+\sum_{j\leq n}\mathbf{1}\{|a_j|\geq 1\}\frac{\theta}{j}\psi_n(n-j)\leq 2\varepsilon^{-2}U_n(h).
\end{aligned}$$

For $\varepsilon = (2\delta)^{1/3}$ we achieve the minimum of $\varepsilon + 2\varepsilon^{-2}\delta$. Thus, $L(h;\nu_n) \le 2(2\delta)^{1/3}$ in the case $0<\delta<1/2$. Recalling the previous trivial bound we complete the upper estimation in Theorem.

The lower estimate. If $L(h;\nu_n) \ge c > 0$ for some constant c, the task is trivial. Now, let $\delta := 2L(h;\nu_n) < c$ for a constant $c < 1/2$ to be chosen later. We have that

$$\nu_n(|h(\sigma)-a| \ge \delta) \le \delta$$

for some $a \in \mathbf{R}$ and

$$Q_n(\delta) \ge \nu_n(|h(\sigma)-a|<\delta) \ge 1-\delta \ge 1/2.$$

Hence, by Lemma 4.1,

$$\sum_{j\le n} \frac{a_j^2}{j}\mathbf{1}\{|a_j|<\delta\} \le C\delta^2, \qquad \sum_{j\le n} \frac{\mathbf{1}\{|a_j|\ge\delta\}}{j} \le C. \tag{5.3}$$

Here we would have used $a_j(\lambda)$ instead of $a_j = a_j(0)$ for some $\lambda \in \mathbf{R}$. Justifying this simplification, we recall that $L(h;\nu_n) = L(h-\lambda n;\nu_n) = \delta/2$ for every λ; therefore, we could further deal with the shifted function $h(\sigma,\lambda)$. Thus, taking $\lambda = 0$ had no effect on the generality. Afterwards, having in mind that $\psi_n(n-j) \le 1$ if $\theta \ge 1$, we will include this quantity as a factor of the summands in (5.3).

Set $\hat{a}_j = a_j$ if $|a_j|<\delta$ and $\hat{a}_j = 0$ otherwise, and denote $\check{a}_j = a_j - \hat{a}_j$ for $j \le n$. Further, define, as in (6.2), two completely additive functions $\hat{h}(\sigma)$ and $\check{h}(\sigma)$ via $\hat{a}_j$ and $\check{a}_j$ respectively.

We now use Lemma 4.2 with $J = \{j \le n : |\check{a}_j| \ge \delta\}$, $K = C$, and

$$I = \big\{1 \le j \le n - n_0(C), |\check{a}_j| \ge \sqrt{\delta}\big\},$$

where $n > 2n_0(C)$. If $\widetilde{S}_n$ is defined as in Lemma 4.2, then

$$\nu_n(\widetilde{S}_n) \ge c_1 \sum_{j\le n-n_0(C)} \frac{1}{j}\psi_n(n-j)\mathbf{1}\{|a_j| \ge \sqrt{\delta}\} =: c_1\alpha.$$

The completion of this sum over $n - n_0(C) < j \le n$ would contribute not more than C_2/n with some $C_2 > 1$ for $n \ge 2n_0(C)$. Hence if $\alpha \le M\delta$, where $M \ge 1$ is arbitrary, then taking into account the first estimate in (5.3) with $\sqrt{\delta}$ instead of δ, we had the desired claim in the form

$$U_n(h) \le \theta(C\delta + M\delta + C_2 n^{-1}) \ll n^{-1} \vee \delta.$$

Since now we assume that $\alpha \ge M\delta$, where $M > c_1^{-1}$ is a constant to be chosen later. This gives $\nu_n(\widetilde{S}_n) \ge c_1 M\delta$. Further we examine the values of the additive functions when $\sigma \in \widetilde{S}_n$. If $\sigma \in S_n^j$, then $\check{h}(\sigma) = a_j$, where $|a_j| \ge \sqrt{\delta}$. So, $|\check{h}(\sigma)| \ge \sqrt{\delta}$ for each $\sigma \in \widetilde{S}_n$. Hence, if $\sigma \in \widetilde{S}_n$ and $|h(\sigma)-a|<\delta$, then $|\hat{h}(\sigma)-a| \ge \sqrt{\delta}-\delta$ and

$$\begin{aligned} \nu_n\big(|\hat{h}(\sigma)-a| \ge \sqrt{\delta}-\delta\big) &\ge \nu_n\big(\sigma \in \widetilde{S}_n\big) - \nu_n\big(|h(\sigma)-a| \ge \delta\big) \\ &\ge (c_1 M - 1)\delta. \end{aligned} \tag{5.4}$$

Denote

$$\widehat{S}_n = \big\{\sigma \in \mathbf{S}_n :\ k_j(\sigma) = 0\ \forall\, j \in J\big\}.$$

By Lemma 4.2, we also have $\nu_n(\widehat{S}_n) \ge c_2 > 0$ if $n > n_0(C)$.

Hence and the fact that $h(\sigma) = \hat{h}(\sigma)$ if $\sigma \in \widehat{S}_n$ we obtain

$$\begin{aligned} \nu_n(|\hat{h}(\sigma) - a| < \delta) &\geq \nu_n(\sigma \in \widehat{S}_n : \ |h(\sigma) - a| < \delta) \\ &\geq c_2 - \nu_n(|h(\sigma) - a| \geq \delta) \geq c_2 - \delta \geq c_2/2 \end{aligned} \tag{5.5}$$

if $\delta < c \leq c_2/2$, where, as we have agreed, the choice of c is at our disposition.

It is known (see, e.g. [59]) that, for a real random variable X, we have that $\mathbf{V}\mathrm{ar}X \geq 1/2p_1p_2d^2$ if $P(X \in A) \geq p_1$, $P(X \in B) \geq p_2$, and $d = \inf\{|x - y| : \ x \in A, y \in B\}$, where $A, B \subset \mathbf{R}$. This, (5.4), and (5.5) yields

$$\mathbf{V}\mathrm{ar}_n\hat{h} \geq (1/4)(c_1M - 1)c_2\delta(\sqrt{\delta} - 2\delta)^2 \geq (1/16)(c_1M - 1)c_2\delta^2$$

if $\delta < c < 1/16$ and $n \geq 2n_0(C)$.

On the other hand, by Lemma 2.3 and (5.3), we have

$$\mathbf{V}\mathrm{ar}_n\hat{h} \leq 2B_n^2(\hat{h}) \leq 2\theta C\delta^2$$

which contradicts to the previous inequality if M and n are sufficiently large. Consequently, the estimate $U_n(h) \ll n^{-1} \vee \delta$ is proved for $n > 2n_0(C)$. For $1 \leq n \leq 2n_0(C)$, it is trivial.

The theorem is proved.

5.3 Necessary and sufficient conditions

Corollary 1.1. *Let $\theta \geq 1$ and $h(\sigma)$ be completely additive functions on $\mathbf{S}_n$ defined via $\{a_j\}$, $j \leq n$, in* (6.2)*. The distributions $\nu_n\big(h(\sigma) - A(n) < x\big)$ converge to the degenerate distribution at the point zero if and only if*

$$\sum_{j<n} \frac{(a_j - \lambda j)^{\circ 2}}{j} \psi_n(n-j) = \mathrm{o}(1)$$

for some $\lambda = \lambda_n \in \mathbf{R}$ and

$$A(n) = n\lambda + \sum_{\substack{j<n \\ |a_j|<1}} \theta \frac{a_j - \lambda j}{j} \psi_n(n-j) + \mathrm{o}(1).$$

Proof. It suffices to apply the well known equality $\psi_n(n-j) = (1 - j/n)^{\theta-1}\big(1 + \mathrm{O}((n-j)^{-1})\big)$ if $0 \leq j \leq n-1$ and the fact that, in the weak law of large numbers, the centralizing sequence $A(n)$ is uniquely determined up to an error $\mathrm{o}(1)$.

As it has been yet mentioned the first claim of the Theorem 1.4 we can extend to general additive functions if $\theta > 0$. The lower estimate, if $\theta < 1$, raises much more difficulties. To overcome them, one needs effective lower estimates for the mean values of multiplicative functions defined on the symmetric group. The approach applied in the results of Theorem 1.1 - Theorem 1.2 of the present note is of little help.

5.4 Estimate of Lévy distance in the case $\theta < 1$

In this subsection, we analyze Lévy distance for general additive functions

$$h(\sigma) := \sum_{j\le n} h_j\big(k_j(\sigma)\big), \quad h_j(k) \in \mathbf{R},\ h_j(0) = 0.$$

Define

$$h_j(k,\lambda) = h_j(k) - \lambda jk, \quad \lambda \in \mathbf{R}.$$

Denote

$$\widetilde{U}_n(h,\lambda) := \min_{\lambda} \sum_{jk\le n} \frac{h_j(k,\lambda)^{\circ 2}}{j^k k!}, \quad \theta < 1.$$

For convenience, we introduce sums of independent r.vs

$$Y_n = \sum_{j\le n} h_j(\xi_j,\lambda), \quad \widetilde{Y}_n = \sum_{j\le n} \hat{h}_j(\xi_j,\lambda),$$

where

$$\hat{x} = \begin{cases} x & if\ \ |x| < 1, \\ 0 & if\ \ |x| \ge 1. \end{cases}$$

Theorem 1.5. *Let $\theta < 1$ and $h(\sigma)$ be an additive function on $\mathbf{S}_n$. We have*

$$L(h;\nu_n) \ll \widetilde{U}_n^{\theta/(2\theta+1)}(h) + n^{-\theta}. \tag{5.6}$$

Proof. Using G.J. Babu and E. Manstavičius result [7], we estimate

$$\begin{aligned} L(h;\nu_n) &\ll \inf\left\{\varepsilon + P^\theta\left(|Y_n - a| \ge \frac{\varepsilon}{3}\right) + n^{-\theta} : a \in \mathbf{R}, \varepsilon > 0\right\} \\ &= \inf\left\{\varepsilon + P^\theta\left(|Y_n - \widetilde{Y}_n + \widetilde{Y}_n - a| \ge \frac{\varepsilon}{3}\right) + n^{-\theta} : a \in \mathbf{R}, \varepsilon > 0\right\} \\ &\ll \inf\left\{\varepsilon + P^\theta\left(|Y_n - \widetilde{Y}_n| \ge \frac{\varepsilon}{6}\right) + n^{-\theta} : \varepsilon > 0\right\} \\ &+ \inf\left\{\varepsilon + P^\theta\left(|\widetilde{Y}_n - a| \ge \frac{\varepsilon}{6}\right) + n^{-\theta} : a \in \mathbf{R}, \varepsilon > 0\right\} \end{aligned}$$

Let estimate separately the probability:

$$P\left(|Y_n - \widetilde{Y}_n| \ge \frac{\varepsilon}{6}\right) \tag{5.7}$$

According to the definition of $\hat{x}$, the function $\hat{h}_j(\xi_j,\lambda)$ is equal to the function $h_j(\xi_j,\lambda)$, if $|h_j(\xi_j,\lambda)| < 1, 1 \le j \le n$. Then the probability (5.7):

$$\begin{aligned} P\left(|Y_n - \widetilde{Y}_n| \ge \frac{\varepsilon}{6}\right) &\le \sum_{j\le n} P(|h_j(\xi_j,\lambda)| \ge 1) \\ &= \sum_{j\le n} \sum_{\substack{1\le k\le n \\ |h_j(k,\lambda)|\ge 1}} e^{-\theta/j}\left(\frac{\theta}{j}\right)^k \frac{1}{k!} \le \sum_{\substack{jk\le n \\ |h_j(k,\lambda)|\ge 1}} \left(\frac{\theta}{j}\right)^k \frac{1}{k!} \le \widetilde{U}_n(h), \end{aligned}$$

so

$$P\left(|Y_n - \widetilde{Y}_n| \ge \frac{\varepsilon}{6}\right) \le \widetilde{U}_n(h). \tag{5.8}$$

Coming back to the estimation of the distance $L(h;\nu_n)$ and using (5.8), we obtain:

$$L(h;\nu_n) \ll \inf\left\{\varepsilon + P^\theta\left(|\widetilde{Y}_n - a| \geq \frac{\varepsilon}{6}\right) + n^{-\theta} : a \in \mathbf{R}, \varepsilon > 0\right\} + \widetilde{U}_n^\theta(h).$$

Applying the Chebyshov lemma and using the sum of averages $\sum_{j\leq n}\mathbf{E}\hat{h}_j(\xi_j,\lambda)$ instead of centralizing constants $a \in \mathbf{R}$, we obtain

$$\begin{aligned}
\inf\{\varepsilon &+ P^\theta\left(|\widetilde{Y}_n - a| \geq \frac{\varepsilon}{6}\right) + n^{-\theta} : a \in \mathbf{R}, \varepsilon > 0\} + \widetilde{U}_n^\theta(h) \\
&= \inf\left\{\varepsilon + P^\theta\left(\left(\widetilde{Y}_n - a\right)^2 \geq \frac{\varepsilon^2}{36}\right) + n^{-\theta} : a \in \mathbf{R}, \varepsilon > 0\right\} + \widetilde{U}_n^\theta(h) \\
&\leq \inf\left\{\varepsilon + \left[\frac{36}{\varepsilon^2}\sum_{j\leq n}\mathbf{E}\left(\hat{h}_j(\xi_j,\lambda) - \mathbf{E}\hat{h}_j(\xi_j,\lambda)\right)^2\right]^\theta : \varepsilon > 0\right\} \\
&+ n^{-\theta} + \widetilde{U}_n^\theta(h) \\
&\leq \inf\left\{\varepsilon + \frac{\widetilde{U}_n^\theta(h)}{\varepsilon^{2\theta}} : \varepsilon > 0\right\} + n^{-\theta} + \widetilde{U}_n^\theta(h).
\end{aligned}$$

Let us find such ε under which $\left\{\varepsilon + \frac{\widetilde{U}_n^\theta(h)}{\varepsilon^{2\theta}} : \varepsilon > 0\right\}$ earns the minimum value. It is possible, if

$$\begin{aligned}
\varepsilon &= \frac{\widetilde{U}_n^\theta(h)}{\varepsilon^{2\theta}}, \\
\varepsilon &= \widetilde{U}_n^{\theta/(2\theta+1)}(h).
\end{aligned}$$

Entering such ε to the latter inequality, we obtain

$$\begin{aligned}
\inf\{\varepsilon &+ \frac{\widetilde{U}_n^\theta(h)}{\varepsilon^{2\theta}} : \varepsilon > 0\} + n^{-\theta} + \widetilde{U}_n^\theta(h) \\
&= \widetilde{U}_n^{\theta/(2\theta+1)}(h) + \widetilde{U}_n^{\frac{2\theta^2+\theta-2\theta^2}{2\theta+1}}(h) + n^{-\theta} + \widetilde{U}_n^\theta(h) \\
&\ll \widetilde{U}_n^{\theta/(2\theta+1)}(h) + \widetilde{U}_n^\theta(h).
\end{aligned}$$

So we have that

$$L(h_n;\nu_n) \ll \widetilde{U}_n^{\theta/(2\theta+1)}(h) + \widetilde{U}_n^\theta(h) + n^{-\theta}.$$

Consequently,

$$L(h_n;\nu_n) \ll \widetilde{U}_n^{\theta/(2\theta+1)}(h) + n^{-\theta}.$$

The theorem is proved.

6 Weak convergence to the Poisson law

In this section, we will demonstrate the weak convergence of the distributions of completely additive functions $h_n(\sigma)$ with $a_j \in \mathbf{Z}$ to the Poisson limit law. The establishing necessary and sufficient conditions for arbitrary function are likely more difficult than in case of degenerate limit law. This was the main reason to examine integer valued functions. In what follows, the dependence on θ is allowed but not additionally indicated and as earlier $h(\sigma; m)$ denotes the truncated additive functions.

6.1 Necessary and sufficient conditions

Theorem 1.6. *Let $h_n(\sigma)$ be a sequence of completely additive functions with $a_j \in \mathbf{Z}$, $j \leq n$, and $\theta \geq 1$. The frequencies $V_n(x)$ converge weakly to the Poisson limit law with parameter $\mu > 0$ if and only if*

$$(i) \sum_{\substack{j \leq n \\ a_j \leq -1}} \frac{\theta}{j} \psi_n(n-j) = o(1),$$

$$(ii) \lim_{m\to\infty} \limsup_{n\to\infty} \mathbf{E}_n h(\sigma;m)_{(l)} = \lim_{m\to\infty} \liminf_{n\to\infty} \mathbf{E}_n h(\sigma;m)_{(l)} = \mu^l,$$

for each fixed $l \in \mathbf{N}$.

Proof. Necessity. **I.** Firstly we will prove estimate (i) of the theorem. Set

$$J^- := \{j : j \leq n, a_j \leq -1\},$$

$$J^+ := \{j : j \leq n, a_j \geq 1\}$$

and $J = J^- + J^+$. Define

$$S_n^j = \{\sigma \in \mathbf{S}_n : k_j(\sigma) = 1, k_i(\sigma) = 0 \quad \forall i \in J \setminus \{j\}\}$$

and note that $h(\sigma) = a_j \leq -1$ for all $\sigma \in S_n^j$ with $j \in J^-$. We observe, that according to Proposition 4.1, we have

$$\sum_{\substack{j \leq n \\ a_j \neq 0}} \frac{\theta}{j} \leq K(\mu) < \infty. \tag{6.1}$$

Now we can apply Lemma 4.3 for $n \geq n_0(K)$ with $K = K(\mu)$ and so obtain

$$o(1) = \nu_n(h(\sigma) \leq -1) \geq \nu_n\Big(\bigcup_{\substack{j \leq n - n_0(K) \\ j \in J^-}} S_n^j\Big) \gg c(K) \sum_{\substack{j \leq n - n_0 \\ j \in J^-}} \frac{1}{j}\Big(1 - \frac{j}{n+1}\Big)^{\theta - 1}.$$

Since the sum over $n - n_0(K) \leq j \leq n$ of $\frac{1}{j}\Big(1 - \frac{j}{n+1}\Big)^{\theta-1}$ is negligible, this implies condition (i) of the theorem.

We finish this part by a remark. By Theorem 2.2 and condition (i), for each $\varepsilon > 0$,

$$\nu_n\Big(\Big|h(\sigma) - \sum_{j \leq n} a_j^+ k_j(\sigma)\Big| \geq \varepsilon\Big) \leq \sum_{\substack{j \leq n \\ a_j \leq -1}} \nu_n(k_j(\sigma) \geq 1)$$

$$\leq \sum_{\substack{j \leq n \\ a_j \leq -1}} \mathbf{E}_n k_j(\sigma) = \sum_{\substack{j \leq n \\ a_j \leq -1}} \frac{\theta}{j} \psi_n(n-j) = o(1), \tag{6.2}$$

we see that the weak convergence of the distributions $V_n(x)$ to the Poisson limit law is also satisfied for the nonnegative functions $h(\sigma)$ with $a_j^+, 1 \le j \le n$. So, further, this allows us consider only the case, where $a_j \ge 0$ for all $j \le n$.

II. In this step we prove that

$$\limsup_{n\to\infty} \sum_{\substack{j\le n\\ a_j\ge m}} \frac{\theta}{j}\Big(1-\frac{j}{n}\Big)^{\theta-1} \ll \frac{\mu^m}{m!}, \quad m \in \mathbf{N}. \tag{6.3}$$

Use Lemma 4.3 for $I = \{1\} \times \{j \le n, a_j \ge m\}$ and

$$\widetilde{S}_n = \bigcup_{\substack{j\le n-n_0(K)\\ j\in I}} \Big\{\sigma \in \mathbf{S}_n : k_j(\sigma) = 1, k_i(\sigma) = 0 \quad \forall i \in J \setminus \{j\}\Big\},$$

where $n_0(K) \in \mathbf{N}$ is sufficiently large. So applying Lemma 4.3 we obtain

$$\nu_n(h(\sigma) \ge m) \ge \nu_n(\tilde{S}_n) \gg c(K) \sum_{\substack{j\le n-n_0(K)\\ a_j\ge m}} \frac{1}{j}\Big(1-\frac{j}{n+1}\Big)^{\theta-1}.$$

Adding the negligible sum of $\frac{1}{j}\Big(1-\frac{j}{n+1}\Big)^{\theta-1}$ over $[n-n_0, n]$ and using

$$e^{\mu}\frac{\mu^k}{k!} + o(1) = \nu_n(h(\sigma) = k), \tag{6.4}$$

we complete the proof of (6.3).

III. Observe from the weak convergence of the frequencies $V_n(x)$ to Poisson limit law, it follows that

$$\mathbf{E}_n h(\sigma; m)_{(l)} \ll_l m^l, \quad l \in \mathbf{N}. \tag{6.5}$$

Indeed, according to the definition and (6.3), we have

$$\theta \sum_{\substack{j\le n\\ a_j\ge 1}} \frac{a_j(m)}{j} \le m \sum_{\substack{j\le n\\ a_j\ge 1}} \frac{\theta}{j} \ll m,$$

$$\theta \sum_{\substack{j\le n\\ a_j\ge 1}} \frac{a_j^2(m)}{j} \le m^2 \sum_{\substack{j\le n\\ a_j\ge 1}} \frac{\theta}{j} \ll m^2.$$

Consequently, (6.5) follows from the Corollary of Lemma 5.1, where $k = 1$ and instead of h, we have $h(\sigma; m)$.

IV. In the last step, we will prove that

$$\lim_{m\to\infty} \limsup_{n\to\infty} \mathbf{E}_n h(\sigma; m)_{(l)} = \lim_{m\to\infty} \liminf_{n\to\infty} \mathbf{E}_n h(\sigma; m)_{(l)} = \mu^l.$$

Split

$$\mathbf{E}_n h(\sigma; m)_{(l)} = \sum_{\sigma\in\mathbf{S}_n} \Big(\mathbf{1}\{h(\sigma; m) \le m-1\} + \mathbf{1}\{m \le h(\sigma; m) \le m^{l+2}\}$$
$$+\mathbf{1}\{h(\sigma; m) > m^{l+2}\}\Big) h(\sigma; m)_{(l)} \cdot \frac{\theta^{w(\sigma)}}{\theta^{(n)}} =: R_1(l, n, m) + R_2(l, n, m) + R_3(l, n, m).$$

Let $\rho_{m,l}(n)$ be some remainder term which approaches to zero for any fixed l and $\theta \ge 1$ when $n \to \infty$ and, then $m \to \infty$.

In the sum $R_1(l,n,m)$, we have $h(\sigma;m)=h(\sigma)$, thus having in mind (6.4),

$$R_1(l,n,m)=\sum_{k=1}^{m-1}k_{(l)}\nu_n(h(\sigma)=k)=\sum_{k=1}^{m-1}k_{(l)}e^{-\mu}\frac{\mu^k}{k!}+\rho_{m,l}(n)$$
$$=\quad \mu^l-\sum_{k\geq m}e^{-\mu}\frac{\mu^k}{k!}+\rho_{m,l}(n)=\mu^l+O\biggl(\frac{\mu^m}{m!}\biggr)+\rho_{m,l}(n).$$

So

$$\lim_{m\to\infty}\limsup_{n\to\infty}R_1(l,n,m)=\lim_{m\to\infty}\liminf_{n\to\infty}R_1(l,n,m)=\mu^l. \tag{6.6}$$

Analogically, if $m\geq 2l$,

$$R_2(l,n,m)\leq\sum_{k=m}^{m^{l+2}}k_{(l)}\nu_n(h(\sigma)\geq k)\leq\sum_{k=m}^{m^{l+2}}\frac{\mu^{l+2}}{(k-l)(k-l-1)}+\rho_{m,l}(n)$$
$$\ll\frac{1}{m}+\rho_{m,l}(n).$$

We obtain

$$\lim_{m\to\infty}\limsup_{n\to\infty}R_2(l,n,m)=\lim_{m\to\infty}\liminf_{n\to\infty}R_2(l,n,m)=0. \tag{6.7}$$

At the end, having in mind (6.5), for $m\geq k$,

$$R_3(l,n,m)\leq\frac{1}{m^{l+2}-l}\mathbf{E}_nh(\sigma;m)_{(l+1)}\ll m^{-1}.$$

Consequently,

$$\lim_{m\to\infty}\limsup_{n\to\infty}R_3(l,n,m)=\lim_{m\to\infty}\liminf_{n\to\infty}R_3(l,n,m)=0. \tag{6.8}$$

The expected condition (ii) follows from the expressions (6.6)-(6.8).

Sufficiency. By condition (i) and estimate (6.2), it suffices to consider nonnegative functions $h_n(\sigma)$. Using the Taylor expansion, we have that

$$\mathbf{E}_ne^{ith(\sigma;m)}=1+\sum_{l=1}^{L}\frac{\mathbf{E}_nh(\sigma;m)_{(l)}}{l!}(e^{it}-1)^l+O\biggl(\frac{\mathbf{E}_nh(\sigma;m)_{(L+1)}}{(L+1)!}|e^{it}-1|^{L+1}\biggr),$$

uniformly for $|t|\leq T$ for each fixed $T>0$ and $L\in\mathbf{N}$. According to the condition (ii), we obtain

$$\mathbf{E}_ne^{ith(\sigma;m)}=1+\sum_{l=1}^{L}\frac{\mu^l}{l!}(e^{it}-1)^l+O\biggl(\frac{2^L\mu^L}{(L+1)!}\biggr)+\rho_{m,L}(n). \tag{6.9}$$

In other words,

$$\lim_{m\to\infty}\limsup_{n\to\infty}\biggl|\mathbf{E}_ne^{ith(\sigma;m)}-\sum_{l=0}^{L}\frac{\mu^l}{l!}(e^{it}-1)^l\biggr|\ll\frac{2^L\mu^L}{(L+1)!}$$

for every $L\geq 1$. On the other hand, since

$$\limsup_{n\to\infty}\sum_{\substack{j\leq n\\ a_j>m}}\frac{\theta}{j}\biggl(1-\frac{j}{n}\biggr)^{\theta-1}\leq\frac{1}{m}\lim_{r\to\infty}\limsup_{n\to\infty}\sum_{j\leq n}\frac{\theta a_j(r)}{j}\biggl(1-\frac{j}{n}\biggr)^{\theta-1}=\frac{\mu}{m},$$

if $m\geq 1$, then

$$\mathbf{E}_n|e^{ith(\sigma;m)}-e^{ith_n(\sigma)}|\ll\nu_n(h(\sigma)\neq h(\sigma;m))\leq\nu_n(\exists j:a_j>m,\quad k_j(\sigma)\geq 1)$$
$$\leq\quad\sum_{\substack{j\leq n\\ a_j>m}}\nu_n(k_j(\sigma)\geq 1)\leq\sum_{\substack{j\leq n\\ a_j>m}}\frac{\theta}{j}\psi_n(n-j)=\rho_m(n).$$

The last two approximations imply

$$\limsup_{n\to\infty}\left|\mathbf{E}_n e^{ith_n(\sigma)} - \sum_{l=0}^{L}\frac{\mu^l(e^{it}-1)^l}{l!}\right| \ll \frac{2^L\mu^L}{(L+1)!}.$$

It remains to take $L\to\infty$.

The theorem is proved.

6.2 Corollaries

Corollary 6.1. *Let $h_n(\sigma)$ be a sequence of completely additive functions defined in (1.4) with $a_j \in \{0,1\}$, $\quad 1\le j\le n$. The frequencies $V_n(x)$ converge weakly to the Poisson limit law with parameter $\mu>0$ if and only if*

$$\sum_{j\le\varepsilon n}^{*}\frac{\theta}{j}=\mu+o(1) \tag{6.10}$$

and, for each fixed $0<\varepsilon<1$

$$\sum_{\varepsilon n<j\le n}^{*}\frac{\theta}{j}\psi_n(n-j)=o(1). \tag{6.11}$$

Proof. Sufficiency. We only need to check the condition (ii) of the theorem. In other words, we have to establish the convergence of the factorial moments $\hat\gamma_{nl}$. From conditions (6.10) and (6.11) we see that the expression of the factorial moments

$$\hat\gamma_{nl}=\theta^l\sum_{j_1,\dots,j_l}^{*}\frac{\mathbf{1}\{j_1+\dots+j_l\le n\}}{j_1\cdots j_l}\left(1-\frac{j_1+\dots+j_l}{n+1}\right)^{\theta-1}+O\left(\frac{\log^l n}{n}\right)$$

we can change by

$$\theta^l\sum_{j_1,\dots,j_l\le\varepsilon n}^{*}\frac{\mathbf{1}\{j_1+\dots+j_l\le n\}}{j_1\dots j_l}\left(1-\frac{j_1+\dots+j_l}{n+1}\right)^{\theta-1}+o(1)=\mu^l+o(1).$$

Then

$$\hat\gamma_{nl}=\mu^l+o(1).$$

Necessity. According to the condition (ii) of the theorem, we have

$$\hat\gamma_{n1}=\sum_{j\le n}^{*}\frac{\theta}{j}\left(1-\frac{j}{n+1}\right)^{\theta-1}+O\left(\frac{\log n}{n}\right)=\mu+o(1).$$

Moreover,

$$\hat\gamma_{nl}=\mu^l+o(1), \tag{6.12}$$

for any $l\ge1$. Consequently, having in mind the inequality (2.18), we have

$$o(1)=\hat\gamma_{n1}^l-\hat\gamma_{nl}=\theta^l\sum_{j_1,\dots,j_l\le n}^{*}\frac{\mathbf{1}\{j_1+\dots+j_l\le n\}}{j_1\dots j_l}\Big[\psi_n(n-j_1)\dots\psi_n(n-j_l)-\psi_n(n-j_1-\dots-j_l)\Big]$$
$$+\theta^l\sum_{j_1,\dots,j_l\le n}^{*}\frac{\mathbf{1}\{j_1+\dots+j_l>n\}}{j_1\dots j_l}\psi_n(n-j_1)\dots\psi_n(n-j_l)\ge\theta^l\left(\sum_{n/l<j\le n}^{*}\frac{1}{j}\psi_n(n-j)\right)^l,$$

for any $l \geq 1$ and taking $0 < \varepsilon < 1$, we have

$$\sum_{\varepsilon n < j \leq n}^{*} \frac{1}{j} \psi_n(n-j) = o(1).$$

So we obtain (6.11).

The Corollary is proved.

Corollary 6.2. *Let $h_n(\sigma)$ be completely additive functions with $a_j \in \mathbf{Z}$, $1 \leq j \leq n$ that*

$$\sum_{\substack{j \leq n/2 \\ a_j \neq 0}} \frac{\theta}{j} \psi_n(n-j) = o(1) \tag{6.13}$$

The frequencies $V_n(x)$ converge weakly to the Poisson limit law with parameter $\mu > 0$ if and only if the condition (i) of the Theorem 1.6 is satisfied and

$$\lim_{m \to \infty} \limsup_{n \to \infty} \sum_{\substack{n/2 < j \leq n \\ a_j > m}} \frac{\theta}{j} \psi_n(n-j) = 0 \tag{6.14}$$

and, for each $k \geq 1$,

$$\sum_{\substack{n/2 < j \leq n \\ a_j = k}} \frac{\theta}{j} \psi_n(n-j) = \frac{\mu^k}{k!} e^{-\mu} + o(1). \tag{6.15}$$

Proof. Necessity. From Theorem 1.6 we have (i) and also having in mind Proposition 4.1, we can analyze only the case, when $a_j \geq 0$. Let, as above, $j, j_1, \ldots \leq n$ and $s, l, r_1, \ldots, r_s \in \mathbf{N}$. According to the expression of factorial moments (2.13), we have

$$\begin{aligned} \mathbf{E}_n h(\sigma; m)_{(l)} &= \sum_{u=1}^{l} \theta^u \sum_{r_1 + \cdots + r_u = l} \binom{l-1}{r_1 - 1} \cdots \binom{l - r_1 - \cdots - r_{u-1} - 1}{r_u - 1} \\ &\times \sum_{j_1 + \cdots + j_u \leq n} \frac{a_{j_1}(m)_{(r_1)} \cdots a_{j_u}(m)_{(r_u)}}{j_1 \cdots j_u} \psi_n(n - j_1 - \cdots - j_l) \end{aligned} \tag{6.16}$$

The condition (6.13) allow us to assume that $a_j = 0$ for any $j \leq n/2$, so in the equality (6.16) we analyze the case, when $j_i \in (n/2; n]$. Consequently, the condition (ii) in Theorem 1.6 could be rewritten in such a form:

$$\begin{aligned} &\lim_{m \to \infty} \limsup_{n \to \infty} \sum_{n/2 < j \leq n} \theta \frac{a_j(m)_{(l)}}{j} \left(1 - \frac{j}{n+1}\right)^{\theta - 1} \\ &= \lim_{m \to \infty} \liminf_{n \to \infty} \sum_{n/2 < j \leq n} \theta \frac{a_j(m)_{(l)}}{j} \left(1 - \frac{j}{n+1}\right)^{\theta - 1} = \mu^l, \end{aligned}$$

for $l \geq 1$. On the other hand,

$$\begin{aligned} &\limsup_{n \to \infty} \sum_{\substack{j \leq n \\ a_j > m}} \frac{\theta}{j} \left(1 - \frac{j}{n+1}\right)^{\theta - 1} \\ &\leq \frac{1}{m} \lim_{r \to \infty} \limsup_{n \to \infty} \sum_{n/2 < j \leq n} \theta \frac{a_j(r)}{j} \left(1 - \frac{j}{n+1}\right)^{\theta - 1} = \frac{\mu}{m} \end{aligned} \tag{6.17}$$

We also have

$$\lim_{m \to \infty} \limsup_{n \to \infty} \left| \sum_{n/2 < j \leq n} (e^{ita_j(m)} - 1) \frac{\theta}{j} \left(1 - \frac{j}{n+1}\right)^{\theta - 1} - \exp\{\mu(e^{it} - 1)\} - 1 \right| = 0$$

uniformly for $|t| \le T$ for each fixed $T > 0$.

Further,

$$\Big| \sum_{n/2<j\le n} (e^{ita_j(m)} - 1)\frac{\theta}{j}\left(1 - \frac{j}{n+1}\right)^{\theta-1} - \sum_{n/2<j\le n} (e^{ita_j} - 1)\frac{\theta}{j}\left(1 - \frac{j}{n+1}\right)^{\theta-1} \Big|$$
$$\ll \sum_{\substack{j\le n \\ a_j>m}} \frac{1}{j}\left(1 - \frac{j}{n+1}\right)^{\theta-1}.$$

Now, from the two last estimations, having in mind (6.17), we have

$$\sum_{n/2<j\le n} (e^{ita_j} - 1)\frac{\theta}{j}\left(1 - \frac{j}{n+1}\right)^{\theta-1} = \exp\{\mu(e^{it} - 1)\} - 1 + o(1)$$
$$= \sum_{k=1}^{\infty} \frac{\mu^k}{k!}(e^{it} - 1)^k + o(1).$$

Integrating this equality multiplied by e^{-itk} on the interval $[-\pi, \pi]$, we obtain (6.15).

Sufficiency. By (6.14) it suffices to prove that

$$\lim_{m\to\infty} \limsup_{n\to infty} \Big| \sum_{\substack{n/2<j\le n \\ a_j\le m}} (e^{ita_j} - 1)\frac{\theta}{j}\left(1 - \frac{j}{n+1}\right)^{\theta-1} - 1 - \exp\{\mu(e^i t - 1)\}\Big| = 0.$$

This clearly follows from (6.15).

The Corollary is proved.

7 Examples

Here we present some examples of distributions which occur as limit laws for $V_n(x)$. Let $\mathfrak{L}$ be the class of such laws.

In the following subsection, we examine a case when Poisson law appears as limit distribution for the frequencies $V_n(x)$ of completely additive functions $h_n(\sigma)$ with $a_j \in \mathbf{Z}_+$ and cycles' lengths $j \in J \subset (n/2, n]$, where $J \subset \{j : j \le n\}$. In particular, there will be established an area of parameter $\mu > 0$, under which $h_n(\sigma)$ posses the Poisson limit law Π_μ.

At the beginning, we present the following lemma for the moment generating function.

Lemma 7.1. *If $a_j \neq 0$ for $j \in J \subset (n/2, n]$ and $a_j = 0$ elsewhere, then the moment generating function*

$$\begin{aligned}
\mathbf{E}_n z^{h(\sigma)} &= 1 + \theta \sum_{j \in J} \frac{z^{a_j} - 1}{j} \psi_n(n-j) \\
&= 1 + \theta \sum_{j \in J} \frac{z^{a_j} - 1}{j} \Big(1 - \frac{j}{n}\Big)^{\theta - 1} + o(1), \quad |z| \le 1.
\end{aligned}$$

Proof. We obtain from the formula (2.3):

$$\sum_{n \ge 0} \varphi_n(z) w^n = \exp\left\{\theta \sum_{j \ge 1} \frac{z^{a_j}}{j} w^j\right\}$$

that

$$\mathbf{E}_n z^{h(\sigma)} = \frac{n!}{\theta^{(n)}} [w^n] \left(\frac{1}{(1-w)^\theta} \exp\left\{\theta \sum_{j \in J} \frac{z^{a_j} - 1}{j} w^j \right\} \right),$$

where $[w^m] f(w)$ means the mth coefficient of a power series $f(w)$. Expanding the exponential function we easily find the nth one.

To obtain the asymptotical formula, it suffices to approximate $\psi_n(n-j)$.

The lemma is proved.

In further calculations of moments and factorial moments, we need a tool for the remainder terms estimate. The following inequality will serve us.

Koksma - Hlawka inequality. If $f : [0,1] \to \mathbf{R}$ is differentiable and $|f'(t)|$ is integrable, then

$$\left| \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right) - \int_0^1 f(t)dt \right| \le \frac{1}{n} \int_0^1 |f'(t)| dt. \tag{7.1}$$

This inequality is very convenient in cases, if we have monotonic function f or its derivative is bounded. In the first case, the derivative is of the same sign, so the integral on the right-hand side of the inequality (7.1) does not exceed $|f(1) - f(0)|$.

Checking the fact of weak convergence of the distributions $V_n(x)$ with $a_j \in \{0, 1\}$, we must have in mind inequality (2.17) $\hat{\gamma}_{nl} \le \hat{\gamma}_{n1}^l$ from Lemma 2.5.

7.1 The Poisson distribution on long cycles

At the beginning, we need an easy calculus exercise.

Lemma 7.2. *Let $\theta \geq 1$ and $x \in [1/2, 1]$. The function*

$$v_\theta(x) := \theta \int_{1/2}^{x} (1-u)^{\theta-1} \frac{du}{u}$$

is strictly increasing in x and $0 = v_\theta(1/2) < v_\theta(1) \leq v_1(1) = \log 2 < 1$.

Proof. Apply calculus.

Proposition 1. *Let $\mu \leq -\log(1 - v_\theta(1))$, where*

$$v_\theta(x) := \theta \int_{1/2}^{x} (1-u)^{\theta-1} \frac{du}{u}.$$

Introduce the sequence $1/2 = d_0 < d_1 < \cdots$ by

$$v_\theta(d_m) = e^{-\mu} \sum_{k=1}^{m} \frac{\mu^k}{k!}, \quad m \in \mathbf{N}.$$

and set $a_j = m$ if $nd_{m-1} < j \leq nd_m$ and $a_j = 0$ otherwise. If $h_n(\sigma)$ is a sequence of completely additive functions defined these a_j, then it posses the Poisson limit law with parameter μ.

Proof. Lemma 7.2 assures that the sequence d_m is correctly defined. Indeed, the values on the right-hand side

$$e^{-\mu} \sum_{k=1}^{m} \frac{\mu^k}{k!} < 1 - e^{-\mu} \leq v_\theta(1).$$

Now the claim is evident.

Using Lemma 7.1 and definition of a_j, we calculate the characteristic function

$$\begin{aligned}
\mathbf{E}_n e^{ith_n(\sigma)} &= 1 + \theta \sum_{n/2<j\leq n} \frac{e^{ita_j}-1}{j} \Big(1 - \frac{j}{n}\Big)^{\theta-1} + o(1) \\
&= 1 + \sum_{m=1}^{\infty} (e^{itm}-1) \sum_{nd_{m-1}<j\leq nd_m} \frac{\theta}{j} \Big(1 - \frac{j}{n}\Big)^{\theta-1} + o(1),
\end{aligned}$$

where $t \in \mathbf{R}$. For the inner sum, we apply the Koksma-Hlawka approximation formula (7.1) and, having in mind uniform in n convergence, proceed

$$\begin{aligned}
\mathbf{E}_n e^{ith_n(\sigma)} &= 1 + \sum_{m=1}^{\infty} (e^{itm}-1)[v_\theta(d_m) - v_\theta(d_{m-1}) \\
&\qquad\qquad + O\big((d_m - d_{m-1})n^{-1}\big)] + o(1) \\
&= 1 + \sum_{m=1}^{\infty} (e^{itm}-1) \frac{e^{-\mu}\mu^m}{m!} + O\Big(\frac{1}{n} \sum_{m=1}^{\infty} (d_m - d_{m-1})\Big) + o(1) \\
&= \exp\big\{\mu(e^{it}-1)\big\} + o(1)
\end{aligned}$$

uniformly in $t \in \mathbf{R}$.

The proposition is proved.

7.2 The quasi-Poisson distribution

Here we examine the quasi-Poisson distribution in relation with the proposed conjecture of M. Lugo (see [42]). For reader's convenience we quote it.

Conjecture 15 *The expected number of cycles of length in $[\gamma n, \delta n]$ of a permutation of n chosen from the Ewens distribution approaches*

$$\lambda = \int_\gamma^\delta \frac{1}{x}(1-x)^{\theta-1}dx \tag{7.2}$$

as $n \to \infty$. Furthermore, in the case where $1/(k+1) \le \gamma < \delta < 1/k$ for some positive integer k, the distribution of the number of cycles converges in distribution to quasi-Poisson(k, λ).

Now we will show that it is actually mistaken.

Proposition 2. *M. Lugo conjecture 15[42] is false.*

Proof. Define the sequence of sets of natural numbers

$$J_n = \mathbf{N} \cap \Big(n/3, n/2\Big]$$

and take $a_j = 1$ if $j \in J_n$ and $a_j = 0$ elsewhere. The factorial moments of the additive function defined via these a_j are equal to

$$\hat\gamma_{n1} = \sum_{j \in J_n} \frac{\theta}{j}\psi_n(n-j) = \theta \int_{1/3}^{1/2} (1-u)^{\theta-1}\frac{du}{u} + O\Big(\frac{\log n}{n}\Big), \quad n \ge 2.$$

$$\hat\gamma_{n2} = \sum_{i,j\in J_n} \mathbf{1}\{i+j \le n\}\frac{1}{ij}\psi_n(n-i-j) = \theta^2 \sum_{n/3<i\le n/2} \frac{1}{i} \sum_{n/3<j\le n/2} \frac{1}{j}\psi(n-i-j).$$

The inner sum of the latter equality, having in mind Koksma-Hlawka approximation, equals

$$\sum_{n/3<j\le n/2} \frac{1}{j}\Big(1 - \frac{i}{n+1} - \frac{j}{n+1}\Big)^{\theta-1} = \int_{1/3}^{1/2} \frac{1}{u}\Big(1 - \frac{i}{n+1} - u\Big)^{\theta-1} du + O\Big(\frac{1}{n}\Big).$$

Using Lemma 7.1, we continue the calculations of $\hat\gamma_{n2}$:

$$\hat\gamma_{n2} = \theta^2 \int_{1/3}^{1/2} \frac{1}{uv}(1-u-v)^{\theta-1}dudv + O\Big(\frac{\log^2 n}{n}\Big), \quad n \ge 2.$$

Analogically, we obtain

$$\hat\gamma_{n1}^2 = \theta^2 \int_{1/3}^{1/2} \frac{1}{uv}(1-u)^{\theta-1}(1-v)^{\theta-1}dudv + O\Big(\frac{\log n}{n}\Big).$$

Checking the inequality (2.17), we compare the integrands of the last two equalities and see that $(1-u)(1-v) > 1-u-v$, i.e.

$$\hat\gamma_{n2} < \hat\gamma_{n1}^2.$$

So, we conclude that the first claim of M. Lugo conjecture, mentioned above, is not true because of a lack of the parameter θ on the right-hand side of the formula. The second claim is also mistaken if $\theta \ne 1$, i.e. according to the Theorem 1.3 and the latter checking, the limit distribution really exists, but it is not the quasi-Poisson.

The proposition is proved.

7.3 Bernoulli distribution

We now demonstrate that the Bernoulli distribution $\mathrm{Be}(p)$, if the parameter p is small enough, can appear as a limit if $\theta \geq 1$ and $a_j \in \{0, 1\}$.

Check that $\hat{\gamma}_1 = p$ and $\hat{\gamma}_l = 0$ if $l \geq 2$. Let $v_\theta(x)$ be the function defined in Proposition 1 on $[1/2, 1]$.

Proposition 3. *For every $p \leq v_\theta(1)$, find α such that $v_\theta(\alpha) = p$ and let $h_n(\sigma)$ be the sequence of additive functions defined via*

$$a_j = \begin{cases} 1 & if\ \ n/2 < j \leq \alpha n, \\ 0 & otherwise. \end{cases}$$

Then the limit distribution of $h_n(\sigma)$ is $\mathrm{Be}(p)$.

Proof. Check that $1/2 < \alpha \leq 1$ and by the Koksma-Hlawka approximation

$$\hat{\gamma}_{n1} = \theta \sum_{n/2 < j \leq \alpha n} \frac{1}{j} \psi_n(n - j) = v_\theta(\alpha) + o(1) = p + o(1).$$

On the other hand, $\hat{\gamma}_{nl} = o(1)$ for every $l \geq 2$.

The proposition now follows from Theorem 1.3.

7.4 Binomial distribution

Let Z be a random variable distributed according to the binomial distribution $\mathcal{B}(p, M)$ with parameters $p \in (0, 1)$ and $M \in \mathbf{N}$, that is,

$$P(Z = k) = \binom{M}{k} p^k (1 - p)^{M-k}, \quad k \in \{0, 1, \ldots, M\}.$$

The factorial moments are equal $\mathbf{E}Z_{(l)} = M_{(l)} p^l$ if $l = 1, 2, \ldots, M$ and $\mathbf{E}Z_{(l)} = 0$ if $l = M + 1, M + 2, \ldots$.

We now present a construction of additive functions obeying a binomial asymptotic distribution. For simplicity, we confine ourselves to a particular case.

Proposition 4. *If $\theta = 1$ and*

$$0 < p \leq (\log 2)/\sqrt{2} = 0.490...,$$

then the distribution $\mathcal{B}(p, 2) \in \mathfrak{L}$.

Proof. It is important to note that

$$2\mathbf{E}Z_{(2)} = (\mathbf{E}Z)^2 \tag{7.3}$$

if $M = 2$.

The idea of our construction goes back to G.Stepanauskas and J.Šiaulys' number-theoretical paper [83]. Let $0 < \alpha \leq \log 2$ and $0 < \beta \leq \log(3/2)$ be temporary parameters. Define the sequence of sets of natural numbers

$$J_n = \mathbf{N} \cap \Big((n/3, (n/3)e^{\alpha}] \cup (2n/3, (2n/3)e^{\beta}] \Big)$$

and take $a_j = 1$ if $j \in J_n$ and $a_j = 0$ elsewhere. The factorial moments of the additive function defined via these a_j are asymptotically equal to

$$\hat{\gamma}_{n1} = \sum_{j \in J_n} \frac{1}{j} = \alpha + \beta + o(1) \leq \log 3 + o(1),$$

$$\hat{\gamma}_{n2} = \sum_{i,j \in J_n} \mathbf{1}\{i + j \leq n\} \frac{1}{ij} = \bigg(\sum_{n/3 < j \leq (n/3)e^{\alpha}} \frac{1}{j} \bigg)^2 = \alpha^2 + o(1),$$

and $\gamma_{nl} = 0$ if $l \geq 3$. By virtue of (7.3), we have to require that

$$2\alpha^2 = (\alpha + \beta)^2.$$

Hence

$$\alpha = (\sqrt{2} + 1)\beta.$$

Given p satisfying the condition of Proposition, we can choose β and, consequently, α so that

$$2p = \alpha + \beta = (\sqrt{2} + 2)\beta \leq \log 3.$$

Now, taking $\beta = (2 - \sqrt{2})p$ and $\alpha = p\sqrt{2}$, due to the condition on p, we have finished.

The proposition is proved.

7.5 Outside the class of limit laws

Let $\overline{\mathfrak{L}}$ be the class of limit laws which do not occur as limit laws for distributions $V_n(x)$. Again, we exploit the observation due to J.Šiaulys and G. Stepanauskas [82] concerning the relations between the first two factorial moments (see Lemma 2.5).

7.5.1 Mixture of the Poisson distribution

We begin with a mixed Poisson distribution $\Pi_{\lambda,\tau}$, where $0 < \beta < 1$ and $\lambda, \tau > 0$. Let Y have such a distribution, then its factorial moments

$$\mathbf{E}Y_{(l)} = \beta\lambda^l + (1 - \beta)\tau^l, \quad l = 1, 2, \ldots$$

Proposition 5. *If $\lambda \neq \tau$, then $\Pi_{\lambda,\tau} \in \overline{\mathfrak{L}}$.*

Proof. If $\Pi_{\lambda,\tau} \in \mathfrak{L}$, then by Lemma 2.5, we have

$$\mathbf{E}Y_{(2)} \leq \big(\mathbf{E}Y\big)^2.$$

Solving this, we obtain that the equality $\lambda = \tau$ must be satisfied. The contradiction proves the proposition.

7.5.2 Geometrical distribution

Let G be a r.v. having the geometrical distribution Γ_β:

$$P(G = b) = (1-\beta)\beta^b, \quad b = 0, 1, 2, \ldots,$$

where the parameter $\beta \in (0,1)$, and the factorial moments

$$\mathbf{E}G_{(l)} = l!\left(\frac{\beta}{1-\beta}\right)^l.$$

Proposition 6. *If* $\beta \in (0,1)$, *then* $\Gamma_\beta \in \overline{\mathfrak{L}}$.

Proof is trivial.

8 Conclusions

- Treatment of the moments of additive functions defined on the symmetric group endowed with the Ewens probability measure is available via the conditional moments of sums of independent random variables.

- Sharp estimates of the mean values of multiplicative functions defined on the additive semigroup $\mathbf{Z}_+^n$ with respect to the Ewens Sampling Formula can be obtained by the small sieve method cultivated in probabilistic number theory.

- For the number of cycles with restricted lengths of a random permutation, the weak convergence of distributions under the Ewens probability measure is equivalent to the convergence of all factorial moments.

- The weak convergence of distributions of integer-valued additive functions to the Poisson law is equivalent to the convergence of all factorial moments of an appropriately truncated function.

- The Poisson, Bernoulli, quasi-Poisson distributions belong to the class of limiting distributions for additive functions, while the non-degenerate mixed Poisson, geometric, and binomial do not.

- The methodology developed in probabilistic number theory can be adopted in the theory of random permutations.

9 Reziumė

Disertacijoje nagrinėjamos atsitiktinių keitinių problemos yra priskirtinos tikimybinei kombinatorikai. Gauti rezultatai aprašo visiškai adityviųjų funkcijų, apibrėžtų simetrinëje grupėje, reikšmių asimptotinius skirstinius Evenso tikimybinio mato atžvilgiu, kai grupės eilė neaprėžtai didėja. Išvestos adityviųjų funkcijų laipsninių ir faktorialinių momentų formulės. Funkcijų, išreiškiančių atsitiktinio keitinio ciklų su bet kokiais apribojimais skaičius, atveju rastos būtinos ir pakankamos ribinių tikimybinių dėsnių egzistavimo sàlygos. Išsamiai išnagrinėtas konvergavimas į Puasono, quasi-Puasono, Bernulio, binominio ir kitus skirstinius, sukoncentruotus sveikųjų neneigiamų skaičių aibėje. Rezultatai apibendrinti sveikareikšmių visiškai adityviųjų funkcijų klasėje. Darbe įrodytas bendras silpnasis didžiųjų skaičių dėsnis, rastos būtinos ir pakankamos adityviųjų funkcijų sekų pasiskirstymo funkcijų konvergavimo į išsigimusį nuliniame taške dėsnį egzistavimo sąlygos.

Sprendžiamos problemos yra susietos su tikimybiniais vektorių, turinčių sveikąsias neneigiamas koordinates, uždaviniais. Adicinėje tokių vektorių pusgrupėje išnagrinėti multiplikatyviųjų funkcijų vidurkiai tikimybinio mato, vadinamo Ewenso atrankos formule, atžvilgiu. Gauti tikslūs viršutinieji ir apatinieji įverčiai. Iš jų išplaukia svarbios atsitiktinių keitinių tikimybių savybės.

Disertacijoje plėtojami faktorialinių momentų ir kiti kombinatoriniai bei tikimybiniai metodai.

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