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The rate of convergence of the Hurst index estimate for a stochastic differential equation*

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Abstract. We consider an estimator of the Hurst parameter of stochastic differential equation with respect to a fractional Brownian motion and establish the rate of convergence of this estimator to the true value of H when the diameter of partition of observation interval tends to zero.

Keywords: fractional Brownian motion, stochastic differential equation, second-order quadratic variations, estimates of Hurst parameter, rate of convergence.

1 Introduction

Consider a stochastic differential equation

$$X_t = \xi + \int_0^t f(s, X_s)\,\mathrm{d}s + \int_0^t g(s, X_s)\,\mathrm{d}B_s^H, \quad t \in [0;T], \tag{1}$$

where $T > 0$ is fixed, $(B_t^H)_{t\in[0;T]}$ is a fBm with the Hurst index $1/2 < H < 1$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$, ξ is an initial r.v., $f, g : [0;T] \times \mathbb{R} \to \mathbb{R}$ are measurable functions.

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Such equations are very frequently met in different applications. The list, though being far from complete, includes the following fractional versions of well-known models (see [6–8, 10, 12, 14, 15] and references therein) with corresponding fields of applications given in the brackets:

- Verhulst equation $X_t = \xi + \int_0^t \lambda X_s - X_s^2 \, \mathrm{d}s + \sigma \int_0^t X_s \, \mathrm{d}B_s^H$ (demography, biology);
- Ornstein–Uhlenbeck equation $X_t = X_0 - \lambda \int_0^t X_s \, \mathrm{d}s + \sigma B_t^H$ (physics, finance, networking);
- Landau–Ginzburg equation $X_t = \xi + \int_0^t \lambda X_s - X_s^3 \, \mathrm{d}s + \sigma \int_0^t X_s \, \mathrm{d}B_s^H$ (physics);
- Black–Scholes equation $X_t = \xi + \lambda \int_0^t X_s \, \mathrm{d}s + \sigma \int_0^t X_s \, \mathrm{d}B_s^H$ (finance);
- Fractional Brownian Traffic equation $X_t = at + \sigma B_t^H$ (networking).

It is therefore clear that an area of applications is very wide and there are many results devoted to estimation problems in models of this type. On the other hand, to our best knowledge there are no a lot of monographs treating subject in a systematic way. A recent one to mention is that of C. Berzin, A. Latour and J.R. León (see [1]). Moreover, most results devoted to estimation problems deal with construction of estimators and investigation of usual asymptotic properties such as consistency and normality. Our goal is different. We assume that one knows a discrete set $\{X_{kT/(2n)},\ k = 0, 1, \dots, 2n\}$ of observations of $(X_t)_{t\in[0;T]}$ and consider an estimator of H based on the second-order increments $\Delta^{(2)} X_{kT/(2n)} = X_{kT/(2n)} - 2X_{(k-1)T/(2n)} + X_{(k-2)T/(2n)}$, $k = 2, 3, \dots, 2n$, which is known, in most cases, to possess the properties mentioned above, and establish the rate of convergence of the estimator the true value of H.

The same problem was treated in [9]. Present paper improves results of [9] in two directions. First of all, equation (1) is more general than that of [9]. Secondly, the order of the rate of convergence given here is sharper.

The paper is organized in the following way. In Section 2, we present the main result of the paper and compare it to that of [9]. Section 3 is devoted to several auxiliary facts needed for the proofs. Section 4 contains the proof of the main result together with several auxiliary statements grounding the main result.

2 Main result

2.1 Statement

Before proceeding to the statement of the main results, we provide several comments regarding the solution of (1).

The conditions ensuring existence and uniqueness of $(X_t)_{t\in[0;T]}$, which satisfies (1) were established in [13] (see also [11]). We assume them to hold. However, note that considering particular models, one can relax or even drop some of them. For the sake of convenience, we restate the result of [13] in a one dimensional form which applies to our setting and herewith puts the assumptions made. Note that constants $K_{f,N}$, $K_{g,N}$ on the right-hand side of bounds below may depend on ω. If this is the case, the corresponding relations are assumed to hold with probability 1. Here and further on $C^\lambda([0;T];\mathbb{R})$, $\lambda \in (0;1]$, stands for a space of Hölder continuous functions equipped with a norm

$$\|f\|_\lambda := \|f\|_\infty + \sup_{0\leqslant s<t\leqslant T} \frac{|f(t)-f(s)|}{(t-s)^\lambda}, \qquad \|f\|_\infty = \sup_{t\in[0;T]} |f(t)|.$$

Theorem 1. (*See* [13, *Thm.* 2.1].) *Let the following continuity constraints on f and g hold*:

- (c1) *For all $x, y \in \mathbb{R}$, $\sup_{t\in[0;T]} |g(t,x) - g(t,y)| \leqslant K_{g,0}|x-y|$ (uniform Lipschitz continuity in x)*;
- (c2) *$g(s,x)$ is differentiable in x*;
- (c3) *For all $N>0$, there exist $\delta\in(1/H-1;1]$ and $K_{g,N}$ such that $\sup_{t\in[0;T]}|g'_x(t,x) - g'_x(t,y)| \leqslant K_{g,N}|x-y|^\delta$ for all $x, y \in [-N;N]$ (local uniform Hölder continuity in x)*;
- (c4) *For all $t, s \in [0;T]$, there exists $\beta \in (1-H;1]$ such that $\sup_{x\in\mathbb{R}}(|g(s,x) - g(t,x)| + |g'_x(s,x) - g'_x(t,x)|) \leqslant K_{g,0}|t-s|^\beta$ (uniform Hölder continuity in t)*;
- (c5) *For all $N > 0$, there exists $K_{f,N}$ such that $\sup_{t\in[0;T]} |f(t,x) - f(t,y)| \leqslant K_{f,N}|x-y|$ for all $x, y \in [-N;N]$ (local uniform Lipschitz continuity in x)*;
- (c6) *For $p \geqslant 1/\kappa$, where $\kappa \in (1-H; \min\{\beta, \delta/(1+\delta)\})$, there exists $f_0 \in \mathcal{L}^p([0;T];\mathbb{R})$ and $K_{f,0}$ such that $|f(t,x)| \leqslant K_{f,0}|x| + f_0(t)$ for all $x \in \mathbb{R}$ and $t \in [0;T]$.*

Then there exists unique solution of (1) *having property $X_\cdot(\omega) \in C^{1-\kappa}([0;T];\mathbb{R})$ a.s.*

Remark. In the statement above, we have omitted condition (H3) appearing in the original statement of Theorem 2.1 of [13]. This is due to the fact that the latter condition is used in the second part of the Theorem 2.1 devoted to boundedness of moments of norm of $(X_t)_{t\in[0;T]}$ and is irrelevant in our context.

In what follows, we add two additional constraints to the set of those imposed by the Theorem 1. First of all, we assume that f satisfies analog of (c4) with the same $\beta \in (1-H;1]$. To be more precise, we assume

- (c7) For all $t, s \in [0;T]$, $\sup_{x\in\mathbb{R}} |f(s,x) - f(t,x)| \leqslant K_{f,0}|t-s|^\beta$ with β given in (c4) (uniform Hölder continuity in t).

Secondly, we assume that $\int_0^T g^2(t,X_t)\,\mathrm{d}t > 0$ a.s.

Our main result is given below[1].

Theorem 2. *Suppose that all conditions of Theorem* 1 *hold. Let $\theta = \min\{1-\kappa, \beta\}$,*

$$\widehat{H}_n = \frac{1}{2} - \frac{1}{2\ln 2} \ln \frac{\sum_{k=2}^{2n} (\Delta^{(2)} X_{kT/(2n)})^2}{\sum_{k=2}^{n} (\Delta^{(2)} X_{kT/n})^2},$$

[1]Symbols O_ω, o_ω being used in a stochastic context should be understood in the usual sense. The only difference, as compared to deterministic versions, is that validity of the corresponding relationships holds with probability one. Subscript ω indicates possible dependence on ω.

$$\Delta^{(2)}X_{kT/n} = X_{kT/n} - 2X_{(k-1)T/n} + X_{(k-2)T/n}, \quad k = 2, \ldots, n,$$
$$\Delta^{(2)}X_{kT/(2n)} = X_{kT/(2n)} - 2X_{(k-1)T/(2n)} + X_{(k-2)T/(2n)}, \quad k = 2, 3, \ldots, 2n.$$

Then $\widehat{H}_n = H + O_\omega((\ln n/n)^{\theta/2})$.

In the rest of the paper, we retain notions of coefficients β, δ, κ, θ reserved for the quantities introduced in the theorems above.

2.2 Comparison with a result of paper [9]

We have already mentioned in the introduction that the same problem was treated in [9]. The authors considered equation

$$X_t = \xi + \int_0^t f(X_s)\,\mathrm{d}s + \int_0^t g(X_s)\,\mathrm{d}B_s^H, \quad t \in [0;T],$$

and the same statistic $\widehat{H}_n$ as given in the Theorem 2. Under assumptions that $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz, $g : \mathbb{R} \to \mathbb{R}$ is differentiable with bounded derivative $g' \in C^\alpha(\mathbb{R};\mathbb{R})$ for some $\alpha \in (H^{-1} - 1; 1]$ and that $\int_0^T g^2(X_t)\,\mathrm{d}t \geqslant c_0 > 0$ a.s., they have proved relationship

$$\widehat{H}_n = H + O_\omega\left(\frac{\ln^{1/4+\gamma} n}{n^{1/4}}\right),$$

where γ can take any positive value but is assumed to be fixed.

Specializing our result to their case, we see that:

- Omitting an argument of time in functions f, g yields almost the same set of restrictions required for an existence and uniqueness of solution[2];
- $\int_0^T g^2(X_t)\,\mathrm{d}t \geqslant c_0 > 0$ a.s. is replaced by $\int_0^T g^2(X_t)\mathrm{d}t > 0$ a.s.;
- Suppose that β, κ, δ satisfies the assumptions of Theorem 1. Set $\beta = \delta = 1$. Then $\theta \in (1/2, H)$. Since $H > 1/2$, taking $\theta = 1/2 + \varepsilon$ with $0 < \varepsilon < H - 1/2$, we get $(\ln n/n)^{1/4+\varepsilon/2}$.

3 Auxiliary facts

The proof of Theorem 2 is preceded by proofs of several technical statements. To make all exposition easier to follow, we introduce some notions and remind several known facts used in the sequel.

- In what follows, λ_1 stands for restriction of the Lebesgue measure on an interval $[0; 1]$, i.e. for all $A \in \mathcal{B}(\mathbb{R})$, $\lambda_1(A) = \lambda(A \cap [0; 1])$, where $\mathcal{B}(\mathbb{R})$ denotes the Borel σ-field on the line $\mathbb{R}$ equipped with a standard metric function $d_1(x, y) = |x - y|$, $x, y \in \mathbb{R}$.

[2]We say *almost the same* because in our case Theorem 1 does not impose boundedness of g' and therefore is a bit less restrictive.

- Let $\mathcal{W}_p([a;b])$ denotes the class of functions on $[a;b]$ with bounded p-variation (for details on p-variation, consult [5]) and $V_p(h;[a;b])$ stands for corresponding variation. For each $r \in \mathcal{W}_q$ and $h \in \mathcal{W}_p$ with $p,q \in (0,\infty)$, $1/p + 1/q > 1$, an integral $\int_a^b r\,\mathrm{d}h$ exists as the Riemann–Stieltjes integral provided r and h have no common discontinuities. In such a case, the Love–Young inequality

$$\left| \int\limits_a^b r\,\mathrm{d}h - r(y)\bigl[h(b) - h(a)\bigr] \right| \leqslant C_{p,q} V_q\bigl(r;[a;b]\bigr) V_p\bigl(h;[a;b]\bigr) \tag{2}$$

holds for all $y \in [a;b]$, where $C_{p,q} = \zeta(p^{-1} + q^{-1})$ and $\zeta(s) = \sum_{n\geqslant 1} n^{-s}$.
- fBm $(B_t^H)_{t\geqslant 0}$ is a centered Gaussian process with a covariance function given by

$$\mathbf{E} B_t^H B_s^H = \frac{1}{2}\bigl(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\bigr).$$

The fBm has the following properties:

- For each $H \in (0;1)$, almost all sample paths of $(B_t^H)_{t\in[0;T]}$ are locally Hölder of order strictly less than H. In other words, for any fixed $0 < \gamma < H$ and any fixed $T > 0$, there exists a nonnegative a.s. finite r.v. $G_{\gamma,T}$ such that

$$\sup_{0\leqslant s\neq t\leqslant T} \frac{|B_t^H - B_s^H|}{|t-s|^\gamma} \leqslant G_{\gamma,T} \quad \text{a.s.} \tag{3}$$

- Squared second-order increments of $(B_t^H)_{t\in[0;T]}$ satisfy some uniform LLN the precise statement of which is given by the theorem below.

Theorem 3. (*See* [9, *Thm.* 3].) *For any* $t \in [0;T]$, *define*[3] $r_{nt} = [nt/T]$, $\rho_{nt} = Tr_{nt}/n$. *Then a.s.*

$$\sup_{t\in[0;T]} \bigl|V_{nt}^{(2)} - \rho_{nt}\bigr| = O_\omega\bigl(n^{-1/2}\ln^{1/2} n\bigr), \tag{4}$$

where

$$V_{nt}^{(2)} = \frac{n^{2H-1}}{T^{2H-1}(4-2^{2H})} \sum_{k=2}^{r_{nt}} \bigl(\Delta^{(2)} B_{kT/n}^H\bigr)^2,$$

$$\Delta^{(2)} B_{kT/n}^H = B_{kT/n}^H - 2B_{(k-1)T/n}^H + B_{(k-2)T/n}^H.$$

4 Proofs

As already mentioned previously, the proof of the main theorem is preceded by several technical statements, which are given below.

[3] $[x]$ denotes an integer part of $x \in \mathbb{R}$.

Lemma 1. *Let* $-\infty < a < b < \infty$, $p, q \in (0; \infty)$, $h, r : [a; b] \to \mathbb{R}$, $\varepsilon > 0$, $x \in (a; b)$ *be such that*:

- $1/p + 1/q > 1$;
- $h \in C^{1/p}([a; b]; \mathbb{R})$, $r \in C^{1/q}([a; b]; \mathbb{R})$;
- $x \pm \varepsilon \in [a; b]$.

Then

$$\int_{x}^{x+\varepsilon} r \, \mathrm{d}h - \int_{x-\varepsilon}^{x} r \, \mathrm{d}h = r(x)\Delta^{(2)} h_{x,\varepsilon} + \theta_{x,\varepsilon}\psi(\varepsilon), \tag{5}$$

where $\Delta^{(2)} h_{x,\varepsilon} = h(x+\varepsilon) - 2h(x) + h(x-\varepsilon)$, $\theta_{x,\varepsilon} \in [-1; 1]$ *and* $\psi(\varepsilon) = O(\varepsilon^{1/p+1/q})$, $\varepsilon \to 0 + 0$.

Proof. By the Love Young inequality and Hölder continuity of h, r,

$$\begin{aligned}\left| \int_{x}^{x+\varepsilon} r \, \mathrm{d}h - r(x)\bigl(h(x+\varepsilon) - h(x)\bigr) \right| &\leqslant C_{p,q} V_q\bigl(r; [x; x+\varepsilon]\bigr) V_p\bigl(h; [x; x+\varepsilon]\bigr) \\ &\leqslant C_{p,q} K_r K_h \varepsilon^{1/p+1/q},\end{aligned}$$

where K_h, K_r are such that $\sup_{a \leqslant s < t \leqslant b} |r(t) - r(s)| \leqslant K_r (t-s)^{1/q}$, $\sup_{a \leqslant s < t \leqslant b} |h(t) - h(s)| \leqslant K_h (t-s)^{1/p}$. Therefore

$$\int_{x}^{x+\varepsilon} r \, \mathrm{d}h = r(x)\bigl(h(x+\varepsilon) - h(x)\bigr) + \theta^{+}_{x,\varepsilon} C_{p,q} K_r K_h \varepsilon^{1/p+1/q}$$

with some $\theta^{+}_{x,\varepsilon} \in [-1; 1]$. Using the same argument,

$$\int_{x-\varepsilon}^{x} r \, \mathrm{d}h = r(x)\bigl(h(x) - h(x-\varepsilon)\bigr) + \theta^{-}_{x,\varepsilon} C_{p,q} K_r K_h \varepsilon^{1/p+1/q}, \quad \theta^{-}_{x,\varepsilon} \in [-1; 1].$$

Setting $\theta_{x,\varepsilon} = (\theta^{+}_{x,\varepsilon} - \theta^{-}_{x,\varepsilon})/2$, $\psi(\varepsilon) = 2C_{p,q} K_r K_h \varepsilon^{1/p+1/q}$, one obtains (5). □

Lemma 2. *For any fixed* $\gamma \in (0; H)$,

$$\begin{aligned}\Delta^{(2)} X_{kT/n} &= X_{kT/n} - 2X_{(k-1)T/n} + X_{(k-2)T/n} \\ &= g\left(\frac{k-1}{n}T, X_{(k-1)T/n}\right)\Delta^{(2)} B^{H}_{kT/n} + O_\omega\left(\left(\frac{1}{n}\right)^{\theta+\gamma}\right).\end{aligned} \tag{6}$$

Proof. Fix $\gamma \in (0; H)$ and $\omega \in \{\omega\colon X_\cdot(\omega) \in C^{1-\kappa}([0; T]; \mathbb{R})\} \cap \{\omega\colon B^{H}_{\cdot}(\omega) \in C^{\gamma}([0; T]; \mathbb{R})\}$. Let $N = N(\omega) = \sup_{t \in [0;T]} |X_t|$. Then, since $\theta = \min\{1-\kappa, \beta\}$,

$f(\cdot, X_\cdot), g(\cdot, X_\cdot) \in C^\theta([0;T];\mathbb{R})$. Indeed, by (c5), (c7),

$$\begin{aligned}\bigl|f(s,X_s)-f(t,X_t)\bigr| &\leqslant \bigl|f(s,X_s)-f(t,X_s)\bigr| + \bigl|f(t,X_s)-f(t,X_t)\bigr| \\ &\leqslant K_{f,0}|s-t|^\beta + K_{f,N}|X_s-X_t| \\ &\leqslant K_{f,0}|s-t|^\beta + K_{f,N}K_{X_\cdot}|s-t|^{1-\kappa} \\ &\leqslant |s-t|^\theta\bigl(K_{f,0}T^{\beta-\theta}+K_{f,N}K_{X_\cdot}T^{1-\kappa-\theta}\bigr)\end{aligned}$$

for $s,t \in [0;T]$ and $K_{X_\cdot} = \sup_{0\leqslant s<t\leqslant T}|X_t-X_s|/(t-s)^{1-\kappa}$. Hence, the claim holds for f. The case of g is handled in the same way.

Next, note that

$$\begin{aligned}\Delta^{(2)}X_{kT/n} &= \left(\int\limits_{(k-1)T/n}^{kT/n} f(t,X_t)\,\mathrm{d}t - \int\limits_{(k-2)T/n}^{(k-1)T/n} f(t,X_t)\,\mathrm{d}t\right) \\ &\quad + \left(\int\limits_{(k-1)T/n}^{Tk/n} g(t,X_t)\,\mathrm{d}B_t^H - \int\limits_{(k-2)T/n}^{T(k-1)/n} g(t,X_t)\mathrm{d}B_t^H\right), \quad k=2,\dots,n.\end{aligned}$$

Then take $x=(k-1)T/n$, $\varepsilon = T/n$ and apply Lemma 1 to differences in the brackets to conclude that

$$\begin{aligned}\Delta^{(2)}X_{kT/n} &= g\left(\frac{k-1}{n}T, X_{(k-1)T/n}\right)\Delta^{(2)}B^H_{kT/n} \\ &\quad + O_\omega\left(\left(\frac{1}{n}\right)^{\theta+1}\right) + O_\omega\left(\left(\frac{1}{n}\right)^{\theta+\gamma}\right).\end{aligned}$$ □

Lemma 3. *Let $\alpha \in (0;1]$, and let $h : \Omega\times[0;T] \to \mathbb{R}$ be a random function, which is Hölder continuous of order α, i.e. for almost each $\omega \in \Omega$,*

$$\bigl|h_s(\omega)-h_t(\omega)\bigr| \leqslant K_h(\omega)|s-t|^\alpha$$

with some a.s. finite and positive r.v. K_h. Then

$$\left(\frac{n}{T}\right)^{2H-1}\sum_{k=2}^n h_{kT/n}\bigl(\Delta^{(2)}B_{kT/n}\bigr)^2 = \bigl(4-2^{2H}\bigr)\int\limits_0^T h_t\mathrm{d}t + O_\omega\bigl(n^{-\alpha/2}\ln^{\alpha/2} n\bigr). \quad (7)$$

Proof. For clarity, sake we split the proof into three steps.

Step 1. Let $\widetilde{\Omega} = [0;1] = I_1$, $\mathcal{B}_1 = \mathcal{B}(\mathbb{R})\cap[0;1] = \{A\cap[0;1]\colon A\in\mathcal{B}(\mathbb{R})\}$, $\widetilde{\mathbf{P}} = \lambda_1$ and L_1 denotes a set of r.vs. on $(\widetilde{\Omega}, \mathcal{B}_1, \widetilde{\mathbf{P}})$ supported on I_1, i.e.

$$L_1 = \bigl\{Z\colon \widetilde{\Omega}\to I_1 \bigm| Z \text{ is } (\mathcal{B}_1,\mathcal{B}_1) \text{ measurable}\bigr\}.$$

For each $\tau \in (0;1]$, define a metric d_τ on I_1 as follows: $d_\tau(x,y) = |x-y|^\tau$. Then any d_τ induces the same topology on I_1 and corresponding Borel σ-fields coincide with $\mathcal{B}_1$. Therefore it does not matter whether we treat I_1 as a metric space (I_1, d_α) or as a metric space (I_1, d_1). In each case, the set L_1 remains the same.

Let M_1 denotes the set of probability measures on $\mathcal{B}(\mathbb{R})$ corresponding to r.vs. of L_1, i.e.

$$M_1 = \bigl\{\mu\colon \mathcal{B}(\mathbb{R}) \to [0;1] \bigm| \exists Z \in L_1\colon \mathbf{P}_Z = \widetilde{\mathbf{P}}(Z \in \cdot) = \mu\bigr\}.$$

Define on M_1 two Wasserstein metrics:

$$d_{W_\tau}(\mu,\nu) = \inf_{Y\sim\mu,\, Z\sim\nu} \mathbf{E} d_\tau(Y,Z) = \inf_{Y\sim\mu,\, Z\sim\nu} \mathbf{E}|Y-Z|^\tau, \quad \tau \in \{\alpha, 1\}.$$

Take arbitrary $Y, Z \in L_1$ such that $Y \sim \mu$, $Z \sim \nu$. By Jensen's inequality, $\mathbf{E}|Y-Z|^\alpha \leqslant (\mathbf{E}|Y-Z|)^\alpha$. Thus, $d_{W_\alpha}(\mu,\nu) = \inf_{\widetilde{Y}\sim\mu, \widetilde{Z}\sim\nu} \mathbf{E}|\widetilde{Y}-\widetilde{Z}|^\alpha \leqslant (\mathbf{E}|Y-Z|)^\alpha$. Consequently, $d_{W_\alpha} \leqslant (d_{W_1})^\alpha$.

Step 2. Let $V_{nt}^{(2)}$, $t \in [0;T]$, be the same as in Theorem 3. Denote

$$p_{nk} = \left(\frac{n}{T}\right)^{2H-1} \frac{(\Delta^{(2)} B_{kT/n}^H)^2}{(4-2^{2H}) V_{nT}^{(2)}}, \quad P_{nk} = \sum_{j=2}^k p_{nj} = \frac{V_{kT}^{(2)}}{V_{nT}^{(2)}}, \quad k = 2, \ldots, n,$$

and

$$\mu_n(A) = \sum_{k=2}^n p_{nk} \delta_{k/n}(A) \quad \text{for } A \in \mathcal{B}(\mathbb{R}), \tag{8}$$

where δ_a denotes the Dirac measure, i.e. for each measurable set A, $\delta_a(A) = \mathbf{1}_A(a)$. Then a.s. μ_n is a discrete measure from M_1. Let F_n, F be distribution functions corresponding to the measures μ_n, λ_1 accordingly. For definiteness, here and further on we use right-continuous versions. By (8),

$$F_n(x) = \begin{cases} 0, & x < \frac{2}{n}; \\ P_{nk}, & x \in [\frac{k}{n}; \frac{k+1}{n}),\ k = 2, \ldots, n-1; \\ 1, & x \geqslant 1. \end{cases}$$

Since $F(x) = (x \wedge 1)\mathbf{1}_{(0;\infty)}(x)$, it follows that

$$\bigl|F_n(x) - F(x)\bigr| = \begin{cases} x, & x \in [0; 2/n); \\ 0, & x \notin [0;1); \\ |x - P_{nk}|, & x \in [\frac{k}{n}; \frac{k+1}{n}),\ k = 2, \ldots, n-1. \end{cases}$$

Equality (4) implies relationship $V_{nt}^{(2)} = t + O_\omega(n^{-1/2} \ln^{1/2} n)$, since denoting by $\{x\} \in [0;1)$ a fractional part of $x \in \mathbb{R}^+$ one has

$$\rho_{nt} = \frac{\frac{t}{T}n - \{\frac{t}{T}n\}}{n} T = t + O\left(\frac{1}{n}\right).$$

Consequently, for all $x \in [k/n; (k+1)/n)$, $k = 2, \dots, n-1$,

$$\begin{aligned}\bigl|F(x) - F_n(x)\bigr| = |x - P_{nk}| &= \left| x - \frac{\frac{k}{n}T + O_\omega((\frac{\ln n}{n})^{1/2})}{T + O_\omega((\frac{\ln n}{n})^{1/2})} \right| \\ &= \left| x - \frac{\frac{k}{n} + O_\omega((\frac{\ln n}{n})^{1/2})}{1 + O_\omega((\frac{\ln n}{n})^{1/2})} \right| = \left| \frac{x - \frac{k}{n} + O_\omega((\frac{\ln n}{n})^{1/2})}{1 + O_\omega((\frac{\ln n}{n})^{1/2})} \right| \\ &= O_\omega\biggl(\biggl(\frac{\ln n}{n} \biggr)^{1/2} \biggr),\end{aligned}$$

and $d_K(\mu_n, \lambda_1) = \sup_x |F_n(x) - F(x)| = O_\omega(n^{-1/2} \ln^{1/2} n)$, where d_K denotes the Kolmogorov metric on the set of probability measures on $\mathcal{B}(\mathbb{R})$.

Step 3. Let $\varphi(t) = h_{tT}/(T^\alpha K_h)$, $t \in [0; 1]$. Retaining notations introduced in the previous steps,

$$\begin{aligned}&\biggl(\frac{n}{T}\biggr)^{2H-1} \sum_{k=2}^{n} h_{kT/n} \bigl(\Delta^{(2)} B_{kT/n}\bigr)^2 \\ &\quad = \bigl(4 - 2^{2H}\bigr) V_{nT}^{(2)} T^\alpha K_h \sum_{k=2}^{n} \varphi\biggl(\frac{k}{n}\biggr) p_{nk} \\ &\quad = \bigl(\bigl(4 - 2^{2H}\bigr)T + O_\omega\bigl(n^{-1/2} \ln^{1/2} n\bigr)\bigr) T^\alpha K_h \int_{I_1} \varphi \, \mathrm{d}\mu_n \\ &\quad = \bigl(4 - 2^{2H}\bigr) T^{1+\alpha} K_h \int_{I_1} \varphi \, \mathrm{d}\mu_n + O_\omega\bigl(n^{-1/2} \ln^{1/2} n\bigr).\end{aligned} \tag{9}$$

By Kantorovich duality theorem (see [4, p. 421]),

$$\forall \mu, \nu \in M_1, \quad d_{W_\alpha}(\mu, \nu) = \sup_{\psi \in C_1^\alpha} \left| \int_{I_1} \psi \, \mathrm{d}\mu - \int_{I_1} \psi \, \mathrm{d}\nu \right|,$$

where $C_1^\alpha = \{\psi\colon I_1 \to \mathbb{R} \mid |\psi(s) - \psi(t)| \leqslant d_\alpha(s,t) = |s-t|^\alpha\}$. Since $\varphi \in C_1^\alpha$,

$$\left| \int_{I_1} \varphi \, \mathrm{d}\mu_n - \int_{I_1} \varphi \, \mathrm{d}\lambda_1 \right| \leqslant d_{W_\alpha}(\mu_n, \lambda_1). \tag{10}$$

Now, recall that there is another explicit formula for d_{W_1} on M_1 (see [3, p. 271]):

$$d_{W_1}(\mu, \nu) = \int_0^1 \bigl|F_\mu(x) - F_\nu(x)\bigr| \, \mathrm{d}x.$$

Thus, results of the previous steps yield

$$d_{W_\alpha}(\mu_n, \lambda_1) \leqslant d_{W_1}^\alpha(\mu_n, \lambda_1) \leqslant \biggl(\int_{I_1} d_K(\mu_n, \lambda_1) \, \mathrm{d}x \biggr)^\alpha = O_\omega\bigl(n^{-\alpha/2} \ln^{\alpha/2} n\bigr), \tag{11}$$

and from (9)–(11) it follows

$$\begin{aligned}
&\left(\frac{n}{T}\right)^{2H-1}\sum_{k=2}^{n} h_{kT/n}\bigl(\Delta^{(2)}B^H_{kT/n}\bigr)^2\\
&\quad=\bigl(4-2^{2H}\bigr)T^{1+\alpha}K_h\int\limits_{I_1}\varphi\,\mathrm{d}\lambda_1+O_\omega\bigl(n^{-\alpha/2}\ln^{\alpha/2}n\bigr)\\
&\quad=\bigl(4-2^{2H}\bigr)T\int\limits_0^1 h_{tT}\,\mathrm{d}t+O_\omega\bigl(n^{-\alpha/2}\ln^{\alpha/2}n\bigr)\\
&\quad=\bigl(4-2^{2H}\bigr)\int\limits_0^T h_t\,\mathrm{d}t+O\bigl(n^{-\alpha/2}\ln^{\alpha/2}n\bigr). \qquad\square
\end{aligned}$$

Proof of Theorem 2. Fix $\gamma\in(0;H)$, which satisfies $\gamma+\theta/4>H$. It was already shown[4] that $g(\cdot,X_\cdot)\in C^\theta([0;T];\mathbb{R})$. Therefore $g^2(\cdot,X_\cdot)\in C^\theta([0;T];\mathbb{R})$. Since $\int_0^T g^2(t,X_t)\,\mathrm{d}t>0$ a.s. and $B^H_\cdot\in C^\gamma([0;T];\mathbb{R})$ a.s., Lemmas 2, 3 yield

$$\begin{aligned}
&\left(\frac{n}{T}\right)^{2H-1}\sum_{k=2}^{n}\bigl(\Delta^{(2)}X_{kT/n}\bigr)^2\\
&\quad=\left(\frac{n}{T}\right)^{2H-1}\sum_{k=2}^{n} g^2\left(\frac{k}{n}T,X_{\frac{k}{n}T}\right)\bigl(\Delta^{(2)}B^H_{kT/n}\bigr)^2+O_\omega\left(\left(\frac{1}{n}\right)^{\theta-2(H-\gamma)}\right)\\
&\quad=\bigl(4-2^{2H}\bigr)\int\limits_0^T g^2(t,X_t)\,\mathrm{d}t+O_\omega\left(\left(\frac{\ln n}{n}\right)^{\theta/2}\right)\\
&\quad=\bigl(4-2^{2H}\bigr)\int\limits_0^T g^2(t,X_t)\,\mathrm{d}t\left(1+O_\omega\left(\left(\frac{\ln n}{n}\right)^{\theta/2}\right)\right).
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\left(\frac{2n}{T}\right)^{2H-1}\sum_{k=2}^{2n}\bigl(\Delta^{(2)}X_{kT/(2n)}\bigr)^2\\
&\quad=\bigl(4-2^{2H}\bigr)\int\limits_0^T g^2(t,X_t)\,\mathrm{d}t\left(1+O_\omega\left(\left(\frac{\ln n}{n}\right)^{\theta/2}\right)\right),
\end{aligned}$$

[4]See proof of Lemma 2.

and by Maclaurin's expansion,

$$\begin{aligned}
&\ln \frac{\sum_{k=2}^{2n}(\Delta^{(2)}X_{kT/(2n)})^2}{\sum_{k=2}^{n}(\Delta^{(2)}X_{kT/n})^2} \\
&\quad = \ln \frac{2^{-(2H-1)}(4-2^{2H})\int_0^T g^2(t,X_t)\,\mathrm{d}t[1+O_\omega((\frac{\ln n}{n})^{\theta/2})]}{(4-2^{2H})\int_0^T g^2(t,X_t)\,\mathrm{d}t[1+O_\omega((\frac{\ln n}{n})^{\theta/2})]} \\
&\quad = (2H-1)\ln 2^{-1} + \ln \frac{1+O_\omega((\frac{\ln n}{n})^{\theta/2})}{1+O_\omega((\frac{\ln n}{n})^{\theta/2})} \\
&\quad = (2H-1)\ln 2^{-1} + \ln\biggl(1+O_\omega\biggl(\biggl(\frac{\ln n}{n}\biggr)^{\theta/2}\biggr)\biggr) \\
&\quad = (2H-1)\ln 2^{-1} + O_\omega\biggl(\biggl(\frac{\ln n}{n}\biggr)^{\theta/2}\biggr).
\end{aligned}$$

□

References

1. C. Berzin, A. Latour, J.R. León, *Inference on the Hurst Parameter and the Variance of Diffusions Driven by Fractional Brownian Motion*, Lect. Notes Stat., Vol. 216, Springer, 2014.

2. F. Biagini, Y. Hu, B. Øksendal, T. Zhang, *Stochastic Calculus for Fractional Brownian Motion and Applications*, Probability and Its Applications, Springer, 2008.

3. M.M. Deza, E. Deza, *Encyclopedia of Distances,* 3rd ed., Springer, 2014.

4. R.M. Dudley, *Real Analysis and Probability*, Cambridge University Press, 2004.

5. R.M. Dudley, R. Norvaiša, *Concrete Functional Calculus*, Springer Monographs in Mathematics, Springer, New York, 2011.

6. C. Grimm, G. Schlüchtermann, *IP-Traffic Theory and Performance*, Springer Series on Signals and Communication Technology, Springer, 2008.

7. O.C. Ibe, *Elements of Random Walk and Diffusion Processes,* 1st ed., Wiley Series in Operations Research and Management Science, Wiley, Hoboken, NJ, 2013.

8. K. Kubilius, D. Melichov, On comparison of the estimators of the Hurst index of the solutions of stochastic differential equations driven by the fractional Brownian motion, *Informatica*, **22**(1):97–114, 2011.

9. K. Kubilius, Y. Mishura, The rate of convergence of estimate for Hurst index of fractional Brownian motion involved into stochastic differential equation, *Stochastic Processes Appl.*, **122**(11):3718–3739, 2012.

10. Y. Mishura, *Stochastic Calculus for Fractional Brownian Motion and Related Processes*, Lect. Notes Math., Vol. 1929, Springer, 2008.

11. Y. Mishura, G. Shevchenko, Mixed stochastic differential equations with long-range dependence: Existence, uniqueness and convergence of solutions, *Comput. Math. Appl.*, **64**(10):3217–3227, 2012.

12. D. Nualart, *The Malliavin Calculus and Related Topics,* 2nd ed., Probability and Its Applications, Springer, 2006.

13. D. Nualart, A. Răşcanu, Differential equations driven by fractional Brownian motion, *Collect. Math.*, **53**(1):55–81, 2002.

14. A.N. Shiryaev, *Essentials of Stochastic Finance: Facts, Models, Theory,* 1st ed., Adv. Ser. Stat. Sci. Appl. Probab., Vol. 3, World Scientific, Singapore, 1999.

15. P. Zhang, W.-L. Xiao, X.-L. Zhang, P.-Q. Niu, Parameter identification for fractional Ornstein–Uhlenbeck processes based on discrete observation, *Econ. Model.*, **36**:198–203, 2014.