mathematics

Editorial

Preface to "Geometry and Topology with Applications"

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1. Motivation

Geometry is a very active research field in pure mathematics, with a history and tradition going back to the antiquity. One of its first goals was in connection with precise land measurements, and the work of Euclid has been fundamental for the systematic and abstract generalization of the concrete geometric concepts already known in that period. He created a model (nowadays called *Euclidean geometry*) that remained unsurpassed for hundreds of years.

More than a millennium later came Euler, Gauss, Lobachevsky, Riemann, Hilbert and Poincaré: their ideas led to the birth of works that brought together the various mathematical theories elaborated previously. New types of geometries were born (for instance, *non-Euclidean geometries*, *differential geometry*, *Riemannian geometry*, and *hyperbolic geometry*), and the interactions and applications with other branches of mathematics carried out to more profound, interesting and powerful developments.

These progresses have given rise to several new research fields in pure and applied mathematics (such as differential topology, complex analysis, algebraic geometry, the theory of general relativity, chaos theory, low-dimensional topology, geometric analysis, and algebraic topology) and deep problems and questions (such as the Riemann hypothesis, Hilbert's 23 problems, the n-body problem, or the notorious Poincaré conjecture).

In the second half of the recent century, geometry has experienced rapid growth thanks to its interactions with other areas of mathematics, such as analysis, algebra, and topology, as well as with applications, mostly in mathematical physics. However, recently it has also been used in statistics, graph theory, machine learning, information theory, and the study of complex networks.

Geometry is indeed a very broad subject. If we take in consideration all of its manifestations, then it can surely be regarded as one of the major areas of research in modern mathematics.

Geometry can be found almost everywhere, and geometric intuition can be used and exploited in many cases. With this approach, one can find a new perspective that introduces a geometric component, facilitating both pure research, visualization, and the proof writing.

This is just the leitmotif of the Special Issue "Geometry and Topology with Applications" in a broad sense.

Following this spirit, I will focus this brief exposition of geometry and topology on the two specific branches I prefer and know the most, and which seem to me to be very thorough: *geometric topology* and *geometric group theory*. Despite this, other research fields were also considered in this Special Issue.

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2. The Development of Geometric Topology

The importance of research in geometry stems in part from its position at the crossroads of many active fields in mathematics, such as topology, analysis, partial differential equations, Lie groups and group theory, in part from its close connection to theoretical physics and mechanics.

On the other hand, the so-called (general) *topology* (literally the study of "places and forms") as an independent research branch of pure mathematics goes back to Hausdorff (even though it originates in the work of Euler and Riemann) and then to Poincaré himself, who can actually be considered the true founder and father of modern algebraic topology.

More recently, two of the greatest geometers of the recent century, namely William P. Thurston and Mikhail L. Gromov, have contributed in a revolutionary way to make an epochal turning point to the study of both geometry and topology by considering their interactions and ties.

Thurston gave an immense boost to the development of hyperbolic geometry, low-dimensional topology, and geometric topology, while Gromov's work ranged from Riemannian geometry to differential topology, from group theory to graph theory, and from analysis to probability.

2.1. On Thurston's Work

William Thurston has been the dominant figure in the study of geometry and topology in three dimensions. In 1982, he was awarded the Fields Medal for his contribution in these fields [1–3].Thurston's main contribution is the venerable *geometrization conjecture* (which includes that of Poincaré's) [4]. It is a three-dimensional version of the Riemann uniformization theorem proved at the end of the 19th century for surfaces. Thurston described eight basic types of geometric objects, and he hypothesized that any three-dimensional space could be obtained as a union of components of this type. Since then, the interactions between geometry, topology, and analysis have become more dense, and the branch called *geometric topology* has been subjected to immense development. The geometrization conjecture was finally proved by Grigori Perelman in 2003, with methods from geometric analysis and partial differential equations [4].

To be more precise, the Riemann uniformization theorem says that a simply connected Riemann surface supports one of the three classical geometries (Euclidean, spherical, or hyperbolic). On the other hand, not every 3-manifold can support a single geometry. Thurston's conjecture states instead that every 3-manifold can be canonically decomposed into pieces, each of which supports one and only one specific geometric structure among the eight possible geometries of the third dimension (called Thurston's geometries).

Thurston proved the geometrization conjecture for a large class of 3-manifolds, called Haken manifolds. Shortly after, Richard Hamilton proved it for closed 3-manifolds with a metric of a positive Ricci curvature. He also provided a detailed program aimed to prove the full geometrization conjecture by means of the so-called Ricci flow with surgery (a certain partial differential equation for a Riemannian metric with singularities), which was efficiently carried out by Perelman in 2003 and completed, in the following years, by several other mathematicians who filled in the complete details of their arguments.

So, thanks to the work of these mathematicians, we have now both a proof of the Poincaré conjecture and precise knowledge of the world of closed 3-manifolds.

2.2. On Gromov's Work

After leaving the USSR, working in the USA, and then as a permanent professor at the IHES near Paris, the Russian–French mathematician Mikhail Gromov was awarded the Oswald Veblen Prize in Geometry in 1981, the Wolf Prize in Mathematics in 1993, and finally, in 2009, won the prestigious Abel Prize "for his revolutionary contributions to geometry".

Besides his profound contributions to differential geometry [5], symplectic geometry, algebraic topology and analysis [6,7], Gromov also initiated and developed a new deep theory, which correlates geometric and topological invariants of spaces (manifolds, simplicial complexes, or graphs) to properties of algebraic objects (discrete groups or algebras) [8,9]. In his work on this subject, there were so much ideas, new techniques, and methods that a new branch of mathematics, called the *geometric group theory*, originated after it, (or, at least, its establishment as a distinct area of modern mathematics). Gromov's theorem on groups of polynomial growth [10] still remains the best and the main result in this field.

In his work, Gromov introduced, in addition to a variety of theories and countless profound results, the h-principle, the theory of convex integration, the notion of almost-flat manifolds, the Gromov–Hausdorff metric together with the Gromov–Hausdorff distance, the notion of hyperbolic group, the theory of random groups, and the theory of pseudo-holomorphic curves.

Asymptotic geometry, hyperbolic groups, expander graphs, and the study of random groups and graphs [11] are today among the branches of mathematics that experienced a big expansion in recent years, also thanks to his applications in contemporary sciences, informatics, and in applied mathematics too.

During the recent decades, various works, especially those of Cannon, Serre, Stallings, Sela, Rips, and Thurston himself, introduced new techniques of combinatorial and computational nature for the study of groups and graphs [12] with applications to computer science, complexity theory, and the theory of formal languages.

3. Some Details About This Special Issue

The aim of this Special Issue, titled "Geometry and Topology with Applications", was to attract and present new interesting papers concerning geometry and/or topology, in a broad sense, with applications. In total, 33 manuscripts were submitted to be considered for publication, and only 12 were accepted. These papers were written by scientists working in prestigious universities or known research centers in France, Italy, Lithuania, Croatia, Korea, South Africa, Serbia, Uzbekistan, Saudi Arabia, United Arab Emirates, China, and Jordan.

Let us mention that, among the published articles, there are two interesting survey papers: one dealing with the topology and geometry of discrete groups and the other one with the open problems of the topology in the fourth dimension. These two reviews perform an excellent job in providing a background, context, and a very readable discussion on these topics. They are certainly recommended for researchers interested in low-dimensional topology and related questions in the geometric group theory.

4. Conclusions

We hope that the published works will have a positive impact on the international scientific community working in geometry, topology, group theory, and their applications, inspiring other researchers to further develop the topics addressed in this Special Issue.

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