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DECAY RATE OF THE STERILE NEUTRINO IN GRIMUS NEUFELD MODEL

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1 Introduction

In this paper, we calculate the decay rate of the sterile neutrino. The calculation has two parts: the three-particle decay process wherein a sterile neutrino transitions into three neutrinos, and a two-particle decay process where a sterile neutrino decays into a neutrino and a photon The Lagrangian that is used for this calculation is the Grimus Neufeld Model Lagrangian. We will show the formulation in the next section.

We will not use the whole Lagrangian since we only need the parts that can contribute to our decay rates. Therefore, the first part of both our calculations is writing the interaction Hamiltonians from the Lagrangian of the model. During the calculations, we also use a Majorana representation to manipulate the spinors, index summations and other simplifications that will aid us to get the total decay rate which is calculated by the Fermi's Golden Rule that is shown below.

$$\Gamma = \frac{S}{2\hbar m_1} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 - p_2 - \ldots - p_n) \times \prod_{j=2}^{n} 2\pi\delta(p_j^2 - m_j^2 c^2)\theta(p_j^0)\frac{d^4 p_j}{(2\pi)^4} \tag{1}$$

where m_i is the mass of the i^{th} particle and p_i is its four-momentum. S is a statistical factor that corrects for double counting when there are identical particles in the final state. That factor is 1 in our calculation since there will be no identical particles in the final state. $\mathcal{M}$ is called the amplitude. The amplitude is the dynamics of the process which is a function of momenta of our particles.

To find the decay rate of the sterile neutrino we are going to have to find this amplitude $\mathcal{M}$. For that, in the first sections of both of the chapters, the quantum operators will act on the fields of the interaction Hamiltonians, so we can write the spinors and the polarization vectors that is a part of the amplitude. After this step in the first chapter, we will integrate our equation over the momenta to write the amplitude and in the second part we will use different methods to do this step. Once we have our amplitude we will get the average of the spins and write the $|\mathcal{M}|^2$ that is the part of the decay rate formula. And after that, using the averaged spins, we will get the traces of the matrix elements and evaluate them to we will be able to show our matrix element as a function of different momenta. And then, the final part will be integrating that over the phase space with approximations of the minimum and maximum values of the energy or momenta as a boundary conditions of the integrals and find the decay rate of the sterile neutrino.

1.1 The Grimus Neufeld Model Lagrangian

The Lagrangian of the Grimus Neufeld Model is an extension of the Standard Model by a single heavy neutrino and a second Higgs doublet. The Lagrangian of the model is shown below.

$$\begin{aligned}\mathcal{L}_{GN} = \mathcal{L}_{SM'} - \big(\bar{\ell}_j \Phi_\alpha (Y_E^{(\alpha)})_{jk} E_k + \bar{\ell}_j \tilde{\Phi}_\alpha (Y_N^{(\alpha)})_j N_R + h.c\big) \\ + \big(\frac{1}{2}\bar{N}_R M_R C \bar{N}_R^\top + h.c.\big) + \big(D_\mu \Phi_\alpha\big)^\dagger \big(D^\mu \Phi_\alpha\big) \\ - Y_{\alpha b}\big(\Phi_\alpha \Phi_b\big) - \frac{1}{2} Z_{\alpha bcd}\big(\Phi_\alpha \Phi_b\big)\big(\Phi_c \Phi_d\big)\end{aligned} \tag{2}$$

Its components are $\mathcal{L}_{SM'}$ which is the Standard Model Lagrangian without Higgs, followed by the Higgs doublets with the electron Yukawa coupling $Y_E^{(\alpha)}$ where $\bar{\ell}_j$ left-chiral lepton doublet of generation j and E_k right-chiral charged lepton singlet of generation k, and then a lepton with a^{th} adjoint Higgs doublet $\tilde{\Phi}_\alpha$ with neutrino Yukawa coupling $Y_N^{(\alpha)}$ and their hermitian conjugates.

The following term on the second line of the equation is the N_R heavy Majorana singlet, M_R Majorana mass term of the heavy singlet, C the charge-conjugation Dirac matrix and their hermitian conjugates. The next term is $D_\mu \Phi_\alpha$ the covariant derivative of Higgs followed by the terms in the last line of the equation, Y_{ab} bilinear terms in the Higgs potential and Z_{abcd} quadrilinear terms in the Higgs potential.

To our decay rate calculation, not all the terms above make a contribution. Therefore we write down the interaction Hamiltonians for the ones that make a contribution.

In the three particle decay interaction Hamiltonian, we have fermion-anti fermion and a charge-less Higgs and a Yukawa coupling terms. And in the two particle decay interaction Hamiltonian, we have fermion-lepton-charged Higgs with the Yukawa coupling term, fermion-lepton-W boson with the mixing matrix fermion-lepton-photon and photon with the charged Higgs and W bosons.

1.2 Quantum Operators Acting on the Fields

In this section, we demonstrate how the creation and annihilation operators act on the photon field and fermionic fields. Below we see the description of the photon field.

$$A_\mu(\vec{x}) = \int \frac{d^3p}{(2\pi^3)} \frac{1}{\sqrt{2|\vec{p}|}} \sum_{\lambda=0}^{3} \epsilon_\mu^\lambda(\vec{p}) \Big[a_{\vec{p}}^\lambda \, e^{i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^{\lambda\dagger} e^{-i\vec{p}\cdot\vec{x}} \Big] \tag{3}$$

When the boson annihilation operator acts on the photon field, and after the commutation relations we get, we get

$$a_{\vec{k}_b}^{\lambda_b} A_\mu(\vec{x}) = \frac{1}{\sqrt{2|\vec{k}_b|}} \epsilon_\mu^{\lambda_b}(\vec{k}) e^{-ik_b\cdot\vec{x}} \tag{4}$$

And below, the fermionic fields are described.

$$\psi_a(x) = \sum_{r=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[b^r_{\vec{p}_a} u^r(\vec{p}) e^{-ip\cdot\vec{x}} + b^{r\dagger}_{\vec{p}_a} v^r(\vec{p}) e^{ip\cdot\vec{x}} \right]$$

$$\bar{\psi}_a(x) = \sum_{r=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[b^{r\dagger}_{\vec{p}_a} \bar{u}^r(\vec{p}) e^{ip\cdot\vec{x}} + b^r_{\vec{p}_a} \bar{v}^r(\vec{p}) e^{-ip\cdot\vec{x}} \right]. \tag{5}$$

When the annihilation operator acts on the fermion field and after the commutation relations, we get

$$b^{s_a}_{\vec{k}} \psi_j(x) = \delta_{aj} \frac{1}{\sqrt{2E_k}} v^s(\vec{k}) e^{ikx} \tag{6}$$

where the index a indicates the particle that the operator $b^{s(a)}$ *acts on by our choosing.* And for the field $\bar{\psi}_j(x)$, annihilation operator acts, and after applying the commutation relations we get

$$b^{s_a}_{\vec{k}} \bar{\psi}_j(x) = \delta_{aj} \frac{1}{\sqrt{2E_k}} \bar{u}^s(\vec{k}) e^{ik\cdot\vec{x}} \tag{7}$$

And for the $\bar{\psi}_j(x)$ field we get

$$\bar{\psi}_j(x) b^{s_i^\dagger}_{\vec{p}} = \delta_{aj} \frac{1}{\sqrt{2E_{p_{(i)}}}} \bar{v}^s(\vec{p}_{(i)}) e^{-i\vec{p}_{(i)}\cdot\vec{x}} \tag{8}$$

Finally for the field $b^{s\dagger}_{\vec{p}}$ acting on the field $\psi_j(x)$ and after the commutation relations, we get

$$\psi_j(x) b^{s_i^\dagger}_{\vec{p}} = \delta_{ij} \frac{1}{\sqrt{2E_{\vec{p}_{(i)}}}} u^s(\vec{p}_{(i)}) e^{-i\vec{p}_{(i)}\cdot\vec{x}} \tag{9}$$

1.3 Majorana Fermions Representation

In the further sections, when we get the spinors from the neutrino fields, we will encounter an anti-particle spinors. With the help of the Majorana representation we will be able to change the order of the spinors since the Majorana particles are their own anti-particles.

We can see this representation below for the $\bar{v}_2 F v_1$ spinors by writing the transpose of the the two spinors, including the terms between them.

$$\begin{aligned} \bar{v}_2 F v_1 &= (\gamma_0 C u_2^*)^\dagger \gamma_0 F \gamma_0 C u_1^* = (u_2^\top C^{-1} F \gamma_0 C u_1^*)^\top \\ &= u_1^\dagger C \gamma_0^\top F^\top C^{-1} u_2 = -\bar{u}_1 C F^\top C^{-1} u_2 \end{aligned} \tag{10}$$

in a similar fashion, we can show this relation for the $u - v$ *pair.*

$$\begin{aligned} \bar{u}_2 F v_1 &= (\gamma_0 C v_2^*)^\dagger \gamma_0 F \gamma_0 C u_1^* = (v_2^\top C^{-1} F \gamma_0 C u_1^*)^\top \\ &= u_1^\dagger C \gamma_0^\top F^\top C^{-1} v_2 = -\bar{u}_1 C F^\top C^{-1} v_2 \end{aligned} \tag{11}$$

where F is some numerical matrix between the two spinors and C is a matrix that has a unitarity property. And if we have a γ matrix between the spinors the relation changes as

$$\bar{u}_2\gamma^\mu v_1 = -\bar{u}_1 C(\gamma^\mu)^\top C^{-1} v_2 = -\bar{u}_1(-\gamma^\mu)v_2 = \bar{u}_1\gamma^\mu v_2 \tag{12}$$

2 Sterile Neutrino Decays into 3 particles

In the calculation of the amplitude for the sterile neutrino $\nu_s \to \nu_j + \nu_k + \nu_\ell$ we define our initial and final states as below.

$$\begin{aligned} |i\rangle &= \sqrt{2E_i}\, b^{s_i^\dagger}_{\vec{p}_i}|0\rangle \\ |f\rangle &= \sqrt{8E_aE_bE_c}\, b^{s_a^\dagger}_{\vec{k}_a} b^{s_b^\dagger}_{\vec{k}_b} b^{s_c^\dagger}_{\vec{k}_c}|0\rangle. \end{aligned} \tag{13}$$

where we can see the creation and annihilation operators that will act on the fields of the interaction Hamiltonian.

In our interaction Hamiltonian, we have two terms for the three body decay. The terms of the first part can represent the one sterile neutrino decays into another neutrino and three different Higgs which decays into another two neutrinos, and one sterile neutrino decays into a Goldstone boson that decays into two other neutrinos. And the second part of the interaction Hamiltonian represents the sterile neutrino decays into a neutrino and the Z boson that decays into two neutrinos, since one sterile neutrino does not decay into three neutrinos immediately.

2.1 The Interaction Hamiltonian for the 3 particle decay

Here below, we write down the interaction Hamiltonian that is going to be used for the three body decay.

$$\mathcal{H}_{\text{int}} = \Big(Y_{j_1\ell_1}\bar{\psi}_{j_1}\psi_{\ell_1}\phi_1 + V_{j_1\ell_1}(V^\dagger)_{\ell_1k_1}\bar{\psi}_{j_1}\gamma^\mu P_L\psi_{k_1}Z_{1\mu}\Big) \tag{14}$$

We use the second order $\mathcal{H}_{\text{int}}$, since we have four particles total in the decay as it is shown below.

$$\langle f|\exp\left[\int dx_1\mathcal{H}_{\text{int}}\right]|i\rangle = \sqrt{16E_iE_aE_bE_c}\,\langle 0|b^{s_a}_{\vec{k}_a}b^{s_b}_{\vec{k}_b}b^{s_c}_{\vec{k}_c}(\int dx_1\,\mathcal{H}_{\text{int}})^2\, b^{s_i^\dagger}_{\vec{p}_i}|0\rangle$$

The equation (15) gives the matrix element for the decay rate of the sterile neutrino.

$$\begin{aligned}(2\pi)^4\delta(p_i-k_a-k_b-k_c)\mathcal{M} &:= \sqrt{16E_iE_aE_bE_c}\langle 0|b^{s_a}_{\vec{k}_a}b^{s_b}_{\vec{k}_b}b^{s_c}_{\vec{k}_c} \\ &\Big(\int dx_1\, Y_{j_1\ell_1}\bar{\psi}_{j_1}\psi_{\ell_1}\phi_1 + V_{j_1\ell_1}(V^\dagger)_{\ell_1k_1}\bar{\psi}_{j_1}\gamma^\mu P_L\psi_{k_1}Z_{1\mu}\Big)^2\, b^{s_i^\dagger}_{\vec{p}_i}|0\rangle\end{aligned} \tag{15}$$

where the Y is the Yukawa coupling constant and the ψ and $\bar{\psi}$ terms represent the neutrinos and the ϕ terms represent the Higgs and the Goldstone bosons.

Please note that we do not need to write the product of these two terms when we write the square of the elements since their interaction does not have a meaningful value, since they do not couple, so, the vacuum expectation value of the product would be zero. Therefore, elements that is squared are only our two terms that are multiplied by themselves.

2.2 Operators acting on the interaction Hamiltonian

In this section, we make quantum operators act on the fields that yield spinors. We will use those spinors to write the amplitude of our particle interaction.

The annihilation operator $b^{s_c}_{\vec{k}_c}$ acts on the interaction Hamiltonian $\mathcal{H}_{\text{int}_G}$, where it represents the first diagram of the 3 body decay of neutrino that contains the Higgs and the Goldstone as it is shown below

$$\begin{aligned} b^{s_c}_{\vec{k}_c}\mathcal{H}_{\text{int}_G} = Y_{j_1\ell_1}Y_{j_2\ell_2}\Big[&\delta_{cj_1}e^{ik_cx_1}\bar{u}^{s_c}\psi_{\ell_1}\bar{\psi}_{j_2}\psi_{\ell_2} - \delta_{c\ell_1}e^{ik_cx_1}\bar{\psi}_{j_1}v^{s_c}\bar{\psi}_{j_2}\psi_{\ell_2} \\ &+ \delta_{cj_2}e^{ik_cx_2}\bar{\psi}_{j_1}\psi_{\ell_1}\bar{u}^{s_c}\psi_{\ell_2} - \delta_{c\ell_2}e^{ik_cx_2}\bar{\psi}_{j_1}\psi_{\ell_1}\bar{\psi}_{j_2}v^{s_c} + \bar{\psi}_{j_1}\psi_{\ell_1}\bar{\psi}_{j_2}\psi_{\ell_2}b_{k_c}\Big](\phi_1\phi_2) \end{aligned} \tag{16}$$

When the operator $b^{s_c}_{\vec{k}_c}$ acts on the neutrino fields, the remaining annihilation operator vanishes when acting on the vacuum, so the whole term vanishes, too. We can use the renaming of the summation indices to see, that there are only 2 independent term:

$$\begin{aligned} b^{s_c}_{\vec{k}_c}\mathcal{H}_{\text{int}_G} =& Y_{j_3\ell_3}Y_{j_4\ell_4}\Big[\delta_{cj_3}e^{ik_cx_3}\bar{u}^{s_c}\psi_{\ell_3}\bar{\psi}_{j_4}\psi_{\ell_4} - \delta_{c\ell_3}e^{ik_cx_3}\bar{\psi}_{j_3}v^{s_c}\bar{\psi}_{j_4}\psi_{\ell_4}\Big](\phi_3\phi_4) \\ &+ Y_{j_4\ell_4}Y_{j_3\ell_3}\Big[\delta_{cj_3}e^{ik_cx_3}\bar{\psi}_{j_4}\psi_{\ell_4}\bar{u}^{s_c}\psi_{\ell_3} - \delta_{c\ell_3}e^{ik_cx_3}\bar{\psi}_{j_4}\psi_{\ell_4}\bar{\psi}_{j_3}v^{s_c}\Big](\phi_4\phi_3) \\ =& Y_{j_3\ell_3}Y_{j_4\ell_4}\Big[\delta_{cj_3}e^{ik_cx_3}\bar{u}^{s_c}\psi_{\ell_3}\bar{\psi}_{j_4}\psi_{\ell_4} - \delta_{c\ell_3}e^{ik_cx_3}\bar{\psi}_{j_3}v^{s_c}\bar{\psi}_{j_4}\psi_{\ell_4}\Big](\phi_3\phi_4 + \phi_4\phi_3) \\ =& 2\Big[Y_{c\ell_1}Y_{j_2\ell_2}e^{ik_cx_1}\bar{u}^{s_c}\psi_{\ell_1}\bar{\psi}_{j_2}\psi_{\ell_2} - Y_{j_1c}Y_{j_2\ell_2}e^{ik_cx_1}\bar{\psi}_{j_1}v^{s_c}\bar{\psi}_{j_2}\psi_{\ell_2}\Big](\phi_1\phi_2) \end{aligned} \tag{17}$$

Now the $b^{s_b}_{\vec{k}_b}$ acts on the $b^{s_c}_{\vec{k}_c}\mathcal{H}_{\text{int}_G}$. And we get

$$\begin{aligned} b^{s_b}_{\vec{k}_b}b^{s_c}_{\vec{k}_c}\mathcal{H}_{\text{int}_G} = \; 2\Big[&Y_{cb}Y_{j_2\ell_2}e^{i(k_cx_1+k_bx_1)}\bar{u}^{s_c}v^{s_b}\bar{\psi}_{j_2}\psi_{\ell_2} - Y_{c\ell_1}Y_{b\ell_2}e^{i(k_cx_1+k_bx_2)}\bar{u}^{s_c}\psi_{\ell_1}\bar{u}^{s_b}\psi_{\ell_2} \\ &+ Y_{c\ell_1}Y_{j_2b}e^{i(k_cx_1+k_bx_2)}\bar{u}^{s_c}\psi_{\ell_1}\bar{\psi}_{j_2}v^{s_b} - Y_{bc}Y_{j_2\ell_2}e^{i(k_cx_1+k_bx_1)}\bar{u}^{s_b}v^{s_c}\bar{\psi}_{j_2}\psi_{\ell_2} \\ &+ Y_{j_1c}Y_{b\ell_2}e^{i(k_cx_1+k_bx_2)}\bar{\psi}_{j_1}v^{s_c}\bar{u}^{s_b}\psi_{\ell_2} - Y_{j_1c}Y_{j_2b}e^{i(k_cx_1+k_bx_2)}\bar{\psi}_{j_1}v^{s_c}\bar{\psi}_{j_2}v^{s_b} \\ &\Big](\phi_1\phi_2) \end{aligned} \tag{18}$$

Then the $b^{s_a}_{\vec{k}_a}$ acts on the $b^{s_b}_{\vec{k}_b} b^{s_c}_{\vec{k}_c}\mathcal{H}_{\text{int}_G}$

$$\begin{aligned}
b^{s_a}_{\vec{k}_a} b^{s_b}_{\vec{k}_b} b^{s_c}_{\vec{k}_c}\mathcal{H}_{\text{int}_G} = \; & 2\Big[Y_{cb}Y_{a\ell_2}e^{i(k_cx_1+k_bx_1+k_ax_2)}\bar{u}^{s_c}v^{s_b}\bar{u}^{s_a}\psi_{\ell_2} - Y_{cb}Y_{j_2a}e^{i(k_cx_1+k_bx_1+k_ax_2)}\bar{u}^{s_c}v^{s_b}\bar{\psi}_{j_2}v^{s_a} \\
& - Y_{ca}Y_{b\ell_2}e^{i(k_cx_1+k_bx_2+k_ax_1)}\bar{u}^{s_c}v^{s_a}\bar{u}^{s_b}\psi_{\ell_2} + Y_{c\ell_1}Y_{ba}e^{i(k_cx_1+k_bx_2+k_ax_2)}\bar{u}^{s_c}\psi_{\ell_1}\bar{u}^{s_b}v^{s_a} \\
& + Y_{ca}Y_{j_2b}e^{i(k_cx_1+k_bx_2+k_ax_1)}\bar{u}^{s_c}v^{s_a}\bar{\psi}_{j_2}v^{s_b} - Y_{c\ell_1}Y_{ab}e^{i(k_cx_1+k_bx_2+k_ax_2)}\bar{u}^{s_c}\psi_{\ell_1}\bar{u}^{s_a}v^{s_b} \\
& - Y_{bc}Y_{a\ell_2}e^{i(k_cx_1+k_bx_1+k_ax_2)}\bar{u}^{s_b}v^{s_c}\bar{u}^{s_a}\psi_{\ell_2} + Y_{bc}Y_{j_2a}e^{i(k_cx_1+k_bx_1+k_ax_2)}\bar{u}^{s_b}v^{s_c}\bar{\psi}_{j_2}v^{s_a} \\
& + Y_{ac}Y_{b\ell_2}e^{i(k_cx_1+k_bx_2k_ax_1)}\bar{u}^{s_a}v^{s_c}\bar{u}^{s_b}\psi_{\ell_2} - Y_{j_1c}Y_{ba}e^{i(k_cx_1+k_bx_2+k_ax_2)}\bar{\psi}_{j_1}v^{s_c}\bar{u}^{s_b}v^{s_a}) \\
& - Y_{ac}Y_{j_2b}e^{i(k_cx_1+k_bx_2+k_ax_1)}\bar{u}^{s_a}v^{s_c}\bar{\psi}_{j_2}v^{s_b} + Y_{j_1c}Y_{ab}e^{i(k_cx_1+k_bx_2+k_ax_2)}\bar{\psi}_{j_1}v^{s_c}\bar{u}^{s_a}v^{s_b}) \\
& \Big](\phi_1\phi_2) \qquad (19)
\end{aligned}$$

Now our last operator $b^{s_i^\dagger}_{p_i}$ acts on the $b^{s_a}_{\vec{k}_a} b^{s_b}_{\vec{k}_b} b^{s_c}_{\vec{k}_c}\mathcal{H}_{\text{int}_G}$. And we get

$$\begin{aligned}
b^{s_a}_{\vec{k}_a} b^{s_b}_{\vec{k}_b} b^{s_c}_{\vec{k}_c}\mathcal{H}_{\text{int}_G} b^{s_i^\dagger}_{p_i} = \; & 2\Big[e^{i(k_cx_1+k_bx_1+k_ax_2-p_ix_2)}(Y_{cb}\bar{u}^{s_c}v^{s_b} - Y_{bc}\bar{u}^{s_b}v^{s_c})(Y_{ai}\bar{u}^{s_a}u^{s_i} - Y_{ia}\bar{v}^{s_i}v^{s_a}) \\
& - e^{i(k_cx_1+k_bx_2+k_ax_1-p_ix_2)}(Y_{ca}\bar{u}^{s_c}v^{s_a} - Y_{ac}\bar{u}^{s_a}v^{s_c})(Y_{bi}\bar{u}^{s_b}u^{s_i} - Y_{ib}\bar{v}^{s_i}v^{s_b}) \\
& + e^{i(k_cx_1+k_bx_2+k_ax_2-p_ix_1)}(Y_{ci}\bar{u}^{s_c}u^{s_i} - Y_{ic}\bar{v}^{s_i}v^{s_c})(Y_{ba}\bar{u}^{s_b}v^{s_a} - Y_{ab}\bar{u}^{s_a}v^{s_b}) \\
& \Big](\phi_1\phi_2) \qquad (20)
\end{aligned}$$

Using the Majorana representation relations, we can write the interaction Hamiltonian as

$$\begin{aligned}
b^{s_a}_{\vec{k}_a} b^{s_b}_{\vec{k}_b} b^{s_c}_{\vec{k}_c}\mathcal{H}_{\text{int}_G} b^{s_i^\dagger}_{p_i} = 2\Big[& e^{i(k_cx_1+k_bx_1+k_ax_2-p_ix_2)}(Y_{cb}+Y_{bc})\bar{u}^{s_c}v^{s_b}(Y_{ai}+Y_{ia})\bar{u}^{s_a}u^{s_i} \\
& - e^{i(k_cx_1+k_bx_2+k_ax_1-p_ix_2)}(Y_{ca}+Y_{ac})\bar{u}^{s_c}v^{s_a}(Y_{bi}+Y_{ib})\bar{u}^{s_b}u^{s_i} \\
& + e^{i(k_cx_1+k_bx_2+k_ax_2-p_ix_1)}(Y_{ci}+Y_{ic})\bar{u}^{s_c}u^{s_i}(Y_{ba}+Y_{ab})\bar{u}^{s_b}v^{s_a} \\
& \Big](\phi_1\phi_2) \qquad (21)
\end{aligned}$$

Now, we can write the part of the Hamiltonian that contains Z boson in the diagram. Since, the operators acts on the fields as the same way with the Higgs and the Goldstone boson part, we will only show the part that looks different for the mixing matrix product. The mixing matrix is a unitary matrix and it refers to the mass eigenvalues of the coupling particles. Our reference equation for this is (20).

That part starts when we start summing the indices for the mixing matrices after all the quantum operators act on the fields where in the first part, we were summing the Yukawa coupling constants. Since the internal index is the same for the mixing matrices we can introduce a new parameter 4×4 U_{ij} matrix that is not a unitary matrix as a substitution of product of the mixing matrices.

$$U_{ij} = V_{i\ell}(V^\dagger)_{\ell j} \qquad (22)$$

where ℓ is an internal index of this product of the mixing matrices that represents the three neutrinos that can couple with Z boson.

From now on we can write our equation as

$$\begin{aligned} b^{s_a}_{\vec{k}_a} b^{s_b}_{\vec{k}_b} b^{s_c}_{\vec{k}_c} \mathcal{H}_{\text{int}_Z} b^{s_i^\dagger}_{p_i} = \ & 2\Big[e^{i(k_c x_1 + k_b x_1 + k_a x_2 - p_i x_2)} (U_{cb} \bar{u}^{s_c} \gamma^\mu v^{s_b} - U_{bc} \bar{u}^{s_b} \gamma^\nu v^{s_c})(U_{ai} \bar{u}^{s_a} \gamma^\mu u^{s_i} - U_{ia} \bar{v}^{s_i} \gamma^\nu v^{s_a}) \\ & - e^{i(k_c x_1 + k_b x_2 + k_a x_1 - p_i x_2)} (U_{ca} \bar{u}^{s_c} \gamma^\mu v^{s_a} - U_{ac} \bar{u}^{s_a} \gamma^\nu v^{s_c})(U_{bi} \bar{u}^{s_b} \gamma^\mu u^{s_i} - U_{ib} \bar{v}^{s_i} \gamma^\nu v^{s_b}) \\ & + e^{i(k_c x_1 + k_b x_2 + k_a x_2 - p_i x_1)} (U_{ci} \bar{u}^{s_c} \gamma^\mu u^{s_i} - U_{ic} \bar{v}^{s_i} \gamma^\nu v^{s_c})(U_{ba} \bar{u}^{s_b} \gamma^\mu v^{s_a} - U_{ab} \bar{u}^{s_a} \gamma^\nu v^{s_b}) \\ & \Big](Z_{1\mu} Z_{2\nu}) \end{aligned} \tag{23}$$

And using the Majorana representation relations we can write our equation as

$$\begin{aligned} b^{s_a}_{\vec{k}_a} b^{s_b}_{\vec{k}_b} b^{s_c}_{\vec{k}_c} \mathcal{H}_{\text{int}_Z} b^{s_i^\dagger}_{p_i} = \ & 2\Big[e^{i(k_c x_1 + k_b x_1 + k_a x_2 - p_i x_2)} (U_{cb} - U_{bc}) \bar{u}^{s_c} \gamma^\mu v^{s_b} (U_{ai} - U_{ia}) \bar{u}^{s_a} \gamma^\nu u^{s_i} \\ & - e^{i(k_c x_1 + k_b x_2 + k_a x_1 - p_i x_2)} (U_{ca} - U_{ac}) \bar{u}^{s_c} \gamma^\mu v^{s_a} (U_{bi} - U_{ib}) \bar{u}^{s_b} \gamma^\nu u^{s_i} \\ & + e^{i(k_c x_1 + k_b x_2 + k_a x_2 - p_i x_1)} (U_{ci} - U_{ic}) \bar{u}^{s_c} \gamma^\mu u^{s_i} (U_{ba} - U_{ab}) \bar{u}^{s_b} \gamma^\nu v^{s_a} \\ & \Big](Z_{1\mu} Z_{2\nu}) \end{aligned} \tag{24}$$

2.3 Integrals of the propagators with respect to momentum

Now, we write the propagator that is going to give a part of the matrix element that we will use in the phase space integral to find the decay rate. We get it from the $\phi_1\phi_2$ fields as

$$\begin{aligned} \langle 0| b^{s_a}_{\vec{k}_a} b^{s_b}_{\vec{k}_b} b^{s_c}_{\vec{k}_c} \mathcal{H}_{\text{int}_G} b^{s_i^\dagger}_{p_i} |0\rangle = & \int \frac{d^4k}{(2\pi)^4} d^4x_1 d^4x_2 \, \frac{e^{ik(x_1 - x_2)}}{k^2 - m^2 + i\epsilon} \\ & 2\Big[e^{i(k_c x_1 + k_b x_1 + k_a x_2 - p_i x_2)} (Y_{cb} + Y_{bc}) \bar{u}^{s_c} v^{s_b} (Y_{ai} + Y_{ia}) \bar{u}^{s_a} u^{s_i} \\ & - e^{i(k_c x_1 + k_b x_2 + k_a x_1 - p_i x_2)} (Y_{ca} + Y_{ac}) \bar{u}^{s_c} v^{s_a} (Y_{bi} + Y_{ib}) \bar{u}^{s_b} u^{s_i} \\ & + e^{i(k_c x_1 + k_b x_2 + k_a x_2 - p_i x_1)} (Y_{ci} + Y_{ic}) \bar{u}^{s_c} u^{s_i} (Y_{ba} + Y_{ab}) \bar{u}^{s_b} v^{s_a}\Big] \end{aligned} \tag{25}$$

The integral above is depending on a momentum and the difference of x_1 and x_2.

When we write our exponential terms with their corresponding x fields and integrate our terms first with respect to exponential term of the fields and then the momentum, we have our amplitude $\mathcal{M}$ as

$$\begin{aligned} (2\pi)^4 \delta(p_i - k_a - k_b - k_c)\mathcal{M} = -2i\Big[& (2\pi)^4 \delta(k_a - p_i + k_c + k_b) \frac{Y_{(cb)} Y_{(ai)}}{(k_a - p_i)^2 - m^2 + i\epsilon} \bar{u}^{s_c} v^{s_b} \bar{u}^{s_a} u^{s_i} \\ & - (2\pi)^4 \delta(k_b - p_i + k_c + k_a) \frac{Y_{(ba)} Y_{(ci)}}{(k_b - p_i)^2 - m^2 + i\epsilon} \bar{u}^{s_c} v^{s_a} \bar{u}^{s_b} u^{s_i} \\ & + (2\pi)^4 \delta(k_c - p_i + k_b + k_a) \frac{Y_{(ca)} Y_{(bi)}}{(k_c - p_i)^2 - m^2 + i\epsilon} \bar{u}^{s_c} u^{s_i} \bar{u}^{s_b} v^{s_a}\Big] \end{aligned} \tag{26}$$

2.4 Casimir's Trick

Now, we can get the square of the matrix element in (26) where its integral over momenta is part of our decay rate calculation. For that process, we use a method called Casimir's trick. What we do is getting the average of all spin configurations and sum them over all final spin configurations. The complete matrix element after averaging the spin configuration can be found in the appendix, equation (133) where below show the first two terms of it.

$$
\begin{aligned}
|\mathcal{M}|^2 =& 16\frac{\left|Y_{(cb)}Y_{(ai)}\right|^2}{\left[(k_a-p_i)^2-m_\phi^2\right]^2}\bar{u}^{s_c}v^{s_b}\bar{u}^{s_a}u^{s_i}\bar{v}^{s_b}u^{s_c}\bar{u}^{s_i}u^{s_a} \\
&+16\frac{Y_{(cb)}Y_{(ai)}Y^*_{(ba)}Y^*_{(ci)}}{\left[(k_a-p_i)^2-m_\phi^2\right]\left[(k_c-p_i)^2-m_\phi^2\right]}\bar{u}^{s_c}v^{s_b}\bar{u}^{s_a}u^{s_i}\bar{u}^{s_i}u^{s_c}\bar{v}^{s_a}u^{s_b} \\
&\dots
\end{aligned} \tag{27}
$$

2.5 Trace of the matrix elements

Now, we get the traces of the matrices from our averaged spinor terms (that were multiplied by their complex conjugates above). Then we evaluate those traces using the relations below. We will plug those into our averaged spinor terms and use it to integrate over the phase space to find the 3 body decay rate.

$$
\begin{aligned}
&Tr(\not{a}\not{b}) = 4a\cdot b \\
&Tr(\not{a}\not{b}\not{c}\not{d}) = 4(a\cdot b\, c\cdot d - a\cdot c\, b\cdot d + a\cdot d\, b\cdot c) \\
&Tr(\gamma^5\not{a}\not{b}\not{c}\not{d}) = 4i\epsilon^{\mu\nu\lambda\sigma}a_\mu b_\nu c_\lambda d_\sigma \\
&Tr = (\gamma^5\not{a}\not{b}) = 0
\end{aligned} \tag{28}
$$

where the odd number of γ matrices gives zero. Note that we may need to change the order of the spinors. For an easy read, we only write the first term of the spinors in each part of the matrix element.

The first step of this calculation is writing the order of the spinors, so we can write the traces. We should order them in a way that we do not change the order of the pairs and we close the order with the index of the spinor that we have in the first place that is the complex conjugate. And writing the trace starts with the spinor that is not a complex conjugate.

$$
\sum_{spins}\bar{u}^{s_c}v^{s_b}\bar{u}^{s_a}u^{s_i}\bar{v}^{s_b}u^{s_c}\bar{u}^{s_i}u^{s_a} = \sum_{spins}\bar{u}^{s_c}v^{s_b}\bar{v}^{s_b}u^{s_c}\bar{u}^{s_i}u^{s_a}\bar{u}^{s_a}u^{s_i} \tag{29}
$$

We can write the trace as follows:

$$
Tr\Big[(\not{k}_b - m_b)(\not{k}_c + m_c)\Big]\cdot Tr\Big[(\not{k}_a + m_a)(\not{p}_i + m_i)\Big] \tag{30}
$$

And once we evaluate the trace following the relations above,

$$4(b\cdot c-m_bm_c)\cdot 4(a\cdot p_i+m_am_i) \tag{31}$$

Please note that we wrote the indices of the momenta to to show the products for an easy read. For example; b is k_b in the equation (31) and in rest of the terms in the appendix. And we can write these products in terms of energy and mass as it is shown below.

$$\begin{aligned}4(a\cdot p_i+m_am_i) = &\Big[2\big(m_i^2-2m_iE_a+m_a^2-m_b^2-m_c^2\big)-4m_bm_c\Big]\\ &\cdot 4\Big[m_iE_a+m_am_i\Big]\end{aligned} \tag{32}$$

The rest of the trace operations and the evaluations can be found at the appendix from equation (134) to (141).

Now, we deal with the trace of the matrices of the spinors in part of our interaction Hamiltonian that has a coupling with the Z boson. Since we have gamma matrices and the projection operator between the spinors, the traces will look a bit different now.

$$\begin{aligned}&\sum_{spins}\bar{u}^{s_c}\gamma^{\mu}P_Lv^{s_b}\bar{u}^{s_a}\gamma^{\nu}P_Lu^{s_i}(-g_{\mu\nu})\bar{v}^{s_b}\gamma^{\lambda}P_Lu^{s_c}\bar{u}^{s_i}\gamma^{\rho}P_Lu^{s_a}(-g_{\lambda\rho})\\ &=\sum_{spins}\bar{u}^{s_c}\gamma^{\mu}P_Lv^{s_b}\bar{v}^{s_b}\gamma^{\lambda}P_Lu^{s_c}\;\;\bar{u}^{s_i}\gamma_{\mu}P_Lu^{s_a}\bar{u}^{s_a}\gamma_{\lambda}P_Lu^{s_i}\end{aligned} \tag{33}$$

Since we got the ordering, we can now write the trace as follows:

$$\begin{aligned}&Tr\Big[(\not{k}_b-m_b)\gamma^{\lambda}P_L(\not{k}_c+m_c)\gamma^{\mu}P_L\Big]\cdot Tr\Big[(\not{k}_a+m_a)\gamma_{\lambda}P_L(\not{p}_i+m_i)\gamma_{\mu}P_L\Big]\\ &=Tr\Big[\not{k}_b\gamma^{\lambda}\not{k}_c\gamma^{\mu}P_L\Big]\cdot Tr\Big[\not{k}_a\gamma_{\lambda}\not{p}_i\gamma_{\mu}P_L\Big]\\ &=\frac{1}{2}\Big(Tr\Big[\not{k}_b\gamma^{\lambda}\not{k}_c\gamma^{\mu}\cdot 1\Big]-Tr\Big[\not{k}_b\gamma^{\lambda}\not{k}_c\gamma^{\mu}\gamma^5\Big]\Big)\cdot\frac{1}{2}\Big(Tr\Big[\not{k}_a\gamma_{\lambda}\not{p}_i\gamma_{\mu}\cdot 1\Big]\Big)-Tr\Big[\not{k}_a\gamma_{\lambda}\not{p}_i\gamma_{\mu}\gamma^5\Big]\Big)\end{aligned} \tag{34}$$

We can write the explicit calculation as follows:

$$\begin{aligned}&\frac{1}{2}\Big(4\Big[k_b^{\lambda}k_c^{\mu}-(b\cdot c)g^{\lambda\mu}+k_b^{\mu}k_c^{\lambda}\Big]-\Big[4i\epsilon^{\beta\lambda\alpha\mu}k_{b\beta}k_{c\alpha}\Big]\Big)\\ &\quad\cdot\frac{1}{2}\Big(4\Big[k_{a\lambda}p_{i\mu}-(a\cdot i)g_{\lambda\mu}+k_{a\mu}p_{i\lambda}\Big]-\Big[4i\epsilon_{\beta'\lambda\alpha'\mu}k_a^{\beta'}p_i^{\alpha'}\Big]\Big)\\ &=\Big[4k_b^{\lambda}k_c^{\mu}k_{a\lambda}p_{i\mu}-4k_b^{\lambda}k_c^{\mu}(a\cdot i)g_{\lambda\mu}+4k_b^{\lambda}k_c^{\mu}k_{a\mu}p_{i\lambda}\\ &\quad-4(b\cdot c)g^{\lambda\mu}k_{a\lambda}p_{i\mu}-4(b\cdot c)g^{\lambda\mu}(a\cdot i)g_{\lambda\mu}-4(b\cdot c)g^{\lambda\mu}k_{a\mu}p_{i\lambda}\\ &\quad+4k_b^{\mu}k_c^{\lambda}k_{a\lambda}p_{i\mu}-4k_b^{\mu}k_c^{\lambda}(a\cdot i)g_{\lambda\mu}+4k_b^{\mu}k_c^{\lambda}k_{a\mu}p_{i\lambda}\Big]-\Big[(4\epsilon^{\beta\lambda\alpha\mu}k_{b\beta}k_{c\alpha})(\epsilon_{\beta'\lambda\alpha'\mu}k_a^{\beta'}p_i^{\alpha'})\Big]\end{aligned} \tag{35}$$

Please note that only multiplied the terms that has a product of two ϵ and the one that has not since the product of terms that has one ϵ gives zero.

$$\begin{aligned}
&4(b\cdot a)(c\cdot i) - 4(b\cdot c)(a\cdot i) + 4(b\cdot i)(c\cdot a)\\
&\quad - 4(b\cdot c)(a\cdot i) + 16(b\cdot c)(a\cdot i) - 4(b\cdot c)(a\cdot i)\\
&\quad + 4(b\cdot i)(a\cdot c) - 4(b\cdot c)(a\cdot i) + 4(b\cdot a)(c\cdot i)\\
&\quad - 4\Big[-2(\delta^{\beta}_{\beta'}\delta^{\alpha}_{\alpha'} - \delta^{\beta}_{\alpha'}\delta^{\alpha}_{\beta'})(k_{b\beta}k_{c\alpha})(k_a^{\beta'}p_i^{\alpha'})\Big] \to 8[(a\cdot b)(c\cdot i) - (i\cdot b)(c\cdot a)]\\
&= 16(b\cdot a)(c\cdot i)
\end{aligned} \tag{36}$$

We can once again write the relations in the terms of energy and mass.

$$16(b\cdot a)(c\cdot i) = 8(m_i^2 - 2E_c m_i + m_c^2 - m_b^2 - m_a^2)(m_i E_c) \tag{37}$$

The rest of the trace operations and the evaluations of the spinors for the Z boson can be found at the appendix from the equation (142) to (149).

2.6 Integral for the Decay rate to 3 particles

Now, we can integrate our phase space to find the decay rate of the sterile neutrino decays into three neutrinos. The equation for the decay rate is given below.

$$d\Gamma = \frac{S}{2\hbar m_1}\int |\mathcal{M}|^2 \left(\frac{d^3p_a}{(2\pi)^3 2E_a}\right)\left(\frac{d^3p_b}{(2\pi)^3 2E_b}\right)\left(\frac{d^3p_c}{(2\pi)^3 2E_c}\right) \times (2\pi)^4\delta^4(p_i - p_a - p_b - p_c) \tag{38}$$

2.6.1 Boundary conditions for the phase space and Integration

We start with peeling the δ function, we will first integrate over the E_a.

$$\begin{aligned}
d\Gamma &= \int |\mathcal{M}|^2 \left(\frac{d^3p_a}{(2\pi)^3 2E_a}\right)\left(\frac{d^3p_b}{(2\pi)^3 2E_b}\right)\left(\frac{d^3p_c}{(2\pi)^3 2E_c}\right) 2\pi\delta(m_i - E_a - E_b - E_c)\\
&\quad \cdot (2\pi)^3\delta^3(\vec{0} - p_a - p_b - p_c)\\
&= \frac{S}{2\hbar m_1}\int |\mathcal{M}|^2 \left(\frac{1}{E_a}\right)\left(\frac{d^3p_b}{(2\pi)^3 2E_b}\right)\left(\frac{d^3p_c}{(2\pi)^3 2E_c}\right)\pi\delta(m_i - E_a - E_b - E_c)
\end{aligned} \tag{39}$$

To solve the E_a integral, we introduce the u parameter which described as:

$$E_a^2 = m_a^2 + |p_b|^2 + |p_c|^2 + 2|p_b||p_c|\cos\theta \equiv u^2 \tag{40}$$

We can set the fixed polar axis on p_c by our choosing and solve the equation (40) for $d\cos\theta$, and We get

$$\begin{aligned}
2udu &= 2p_b p_c d\cos\theta\\
&= d\cos\theta = \frac{udu}{p_b p_c}
\end{aligned} \tag{41}$$

with this two equations (40) and (41), we can substitute E_a with u and write it in terms of p_b and p_c as we can see at the equation (46).

Now, by following the conservation of momentum and energy we can now decide the boundaries of the rest of the integrals. using E_a, we get

$$m_a \leq E_a \leq \frac{m_i^2 + m_a^2 - (m_b + m_c)^2}{2m_i}. \tag{42}$$

Then we get to the integral of (E_b). For the maximum value for our momentum p_b the $\cos\theta$ should be -1. When we solve this equality for the p_b we find

$$\begin{aligned} & m_a^2 + 2p_b p_c \cos\theta + p_b^2 + p_c^2 \leq \left(\frac{m_i^2 + m_a^2 - (m_b + m_c)^2}{2m_i}\right)^2 \\ & 2p_b p_c \cos\theta + p_b^2 + p_c^2 - \left(\frac{m_i^2 + m_a^2 - (m_b + m_c)^2}{2m_i}\right)^2 + m_a^2 \leq 0 \end{aligned} \tag{43}$$

And our solution for p_b is

$$p_b = -p_c \cos\theta \pm \sqrt{(p_c \cos\theta)^2 - p_c^2 - m_a^2 + \left(\frac{m_i^2 + m_a^2 - (m_b + m_c)^2}{2m_i}\right)^2} \tag{44}$$

To use this boundary condition in the integral dE_b below we have to write it in terms of energy instead of momentum.

$$\begin{aligned} E_{bmin} &= m_b \\ E_{bmax} &= \sqrt{m_b^2 + \left(-p_c + \sqrt{-m_a^2 + \left(\frac{m_i^2 + m_a^2 - (m_b + m_c)^2}{2m_i}\right)^2}\right)^2} \end{aligned} \tag{45}$$

And for the integrand dp_c, our boundary conditions are from 0 to p_c. And when we make substitutions for E_a with u in our equations we get

$$\begin{aligned} d\Gamma = & \int_0^{p_{c_{max}}} \int_{E_{b_{min}}}^{E_{b_{max}}} |M^2| \left(\frac{1}{u p_b p_c}\right) \left(\frac{p_b^2 dp_b}{(2\pi)^3 2E_b}\right) \left(\frac{d^3 p_c}{(2\pi)^3 2E_c}\right) u du d\phi \\ & \pi\delta(m_i - |u| - E_b - E_c) \end{aligned} \tag{46}$$

The integral of u is only gives 1 if the $m_i - E_b - E_c$ is between the minimum and the maximum value of u. Therefore, we make those changes in the $|\mathcal{M}|^2$ for E_a. We can see this form for all the terms in the $|\mathcal{M}|^2$ that has E_a. Below, we show it for one term

$$\begin{aligned} = & \int_0^{p_{c_{max}}} \int_{E_{b_{min}}}^{E_{b_{max}}} \frac{64}{p_b p_c} \frac{\left|Y_{(cb)} Y_{(ai)}\right|^2}{\left[m_a^2 - 2(m_i - E_b - E_c)) + m_i^2 - m_\phi^2\right]^2} \\ & \left[\frac{1}{2}\left(m_i^2 - 2(m_i - E_b - E_c) + m_a^2 - m_b^2 - m_c^2\right) - m_b m_c\right] \\ & \cdot 4\left[(m_i - E_b - E_c) + m_a m_i\right] \ldots \left(\frac{p_b^2 dp_b}{(2\pi)^3 2E_b}\right) \left(\frac{d^3 p_c}{(2\pi)^3 2E_c}\right) \end{aligned} \tag{47}$$

where the complete form of $|\mathcal{M}|^2$ can be found in the appendix, equation (150).

Using the following relation $E_b^2 = m_b^2 + p_b^2$, and derive the it as $2E_b dE_b = 2p_b dp_b$ to write $p_b dp_b$ in terms of dE_b.

$$= \int_0^{p_{c_{max}}} \int_{E_{b_{min}}}^{E_{b_{max}}} \frac{|M^2|}{p_c} \left(\frac{2\pi p_b dp_b}{(2\pi)^2 4E_b} \right) \left(\frac{d^3 p_c}{(2\pi)^3 2E_c} \right) \tag{48}$$

$$= \int_0^{p_{c_{max}}} \int_{E_{b_{min}}}^{E_{b_{max}}} \frac{|M^2|}{p_c} \left(\frac{p_b dp_b}{(2\pi) 4E_b} \right) \left(\frac{d^3 p_c}{(2\pi)^3 2E_c} \right) \tag{49}$$

So, we get

$$d\Gamma = \int_0^{p_{c_{max}}} \int_{E_{b_{min}}}^{E_{b_{max}}} \frac{|M^2|}{p_c} \left(\frac{dE_b}{4(2\pi)} \right) \left(\frac{d^3 p_c}{(2\pi)^3 2E_c} \right) \tag{50}$$

When we write the spherical coordinates for the p_c, we can see that nothing else depends on ϕ or θ so we simply get 4π.

$$d\Gamma = \int_0^{p_{c_{max}}} \int_{E_{b_{min}}}^{E_{b_{max}}} \frac{|M^2|}{p_c} \left(\frac{dE_b}{2} \right) \left(\frac{p_c^2 dp_c}{(2\pi)^3 2E_c} \right) \tag{51}$$

After this equation (51), we have all our boundary conditions and we do not have any other integrals left that are not dependent on E_b and p_c.

$$\Gamma = \int_0^{p_{c_{max}}} \int_{E_{b_{min}}}^{E_{b_{max}}} \frac{64}{p_c} \frac{\left| Y_{(cb)} Y_{(ai)} \right|^2}{\left[m_a^2 - 2(m_i - E_b - E_c)) + m_i^2 - m_\phi^2 \right]^2}$$
$$\left[\frac{1}{2} (m_i^2 - 2(m_i - E_b - E_c) + m_a^2 - m_b^2 - m_c^2) - m_b m_c \right]$$
$$\cdot 4 \left[(m_i - E_b - E_c) + m_a m_i \right] \dots \left(\frac{dE_b}{2} \right) \left(\frac{p_c^2 dp_c}{(2\pi)^3 2E_c} \right) \tag{52}$$

We can see that the form of the integral is product of energies with the sums of the masses that are constants in the numerator. And in the denominator, we can see a similar quadratic form of energy products. So, we can integrate over E_b and then plug in the boundary conditions. In the step of integrating over p_c, we have two scenarios.

First one is that we ignore the mass of the outgoing particles by them being an infinitesimal, and since the m_ϕ has much larger value, we will have an integral where we can calculate the decay rate of the sterile neutrino goes into three neutrinos.

Other scenario is that we do not ignore the mass of the outgoing particles, meaning that our integral is going to have terms such that after validating the integral, we are going to make an expansion of the mass terms in the result there are going to be a lot of terms that have much longer values since we will have integrands in the denominator in the form of forth power polynomial according to our approach.

3 Sterile Neutrino Decays into 2 particles

In the calculation of the amplitude for the sterile neutrino decays into a neutrino and a photon, we define our initial and final states as below.

$$\begin{aligned}
|i\rangle &= \sqrt{2E_i}\, b^{s_i^\dagger}_{\vec{p}_i}|0\rangle \\
|f\rangle &= \sqrt{4E_aE_b}\, b^{s_a^\dagger}_{\vec{k}_a} a^{b^\dagger}_{\vec{k}_b}|0\rangle.
\end{aligned} \tag{53}$$

where we can see the creation and annihilation operators that will act on the fields of the interaction Hamiltonian.

In our interaction Hamiltonian we have four terms that represent a Feynman diagram of a sterile neutrino decays into a photon and a neutrino with four different loops. The first loop of the Feynman diagram has charged Higgs particles where in the second loop of the Feynman diagram there is $W^\pm$ bosons. The third loop of the Feynman diagram has a charged lepton and a charged Higgs. And finally, our last loop has a charged lepton and a $W^\pm$ boson.

3.1 The Interaction Hamiltonian for 2 particle decay

That being said, we now show the interaction Hamiltonian for the 2 particle decay below.

$$\begin{aligned}
\mathcal{H}_{\text{int}_1} =& \Big(Y_{j_1\ell_1}\bar{\psi}_{j_1}\psi'_{\ell_1}\phi_1^+ + (Y^\dagger)_{j_1\ell_1}\bar{\psi}'_{j_1}\psi_{\ell_1}\phi_1^- + \bar{\psi}'_{j_1}\gamma^\mu\psi'_{j_1}A_{1\mu} \\
&+ V_{j_1\ell_1}\bar{\psi}_{j_1}\gamma^\mu\psi'_{\ell_1}W^+_{1\mu} + (V^\dagger)_{\ell_1 j_1}\bar{\psi}'_{\ell_1}\gamma^\mu\psi_{j_1}W^-_{1\mu} \\
&+ \big[4i\eta g A_\alpha W^+_\beta(\partial^\alpha W^{-\beta} - \partial^\beta W^{-\alpha}) + 4i\eta g A_\alpha W^-_\beta(-\partial^\alpha W^{+\beta} + \partial^\beta W^{+\alpha}) \\
&+ 4i\eta g\{W^+_\alpha W^-_\beta(\partial^\alpha A^\beta - \partial^\beta A^\alpha)\} + \big[\partial_\mu\phi^- i\eta_e e A^\mu\phi^+ - \partial^\mu\phi^+ i\eta_e e A_\mu\phi^-\big]\Big)
\end{aligned} \tag{54}$$

To get the amplitude of the two particle decay, we need the third order of the (54) as it is shown below.

$$\langle f|e^{\mathcal{H}_{\text{int}}}|i\rangle = \sqrt{8E_iE_aE_b}\,\langle 0|b^{s_a}_{\vec{k}_a}a^{s_b}_{\vec{k}_b}(\int dx_1\,\mathcal{H}_{\text{int}_1})^3\, b^{s_i^\dagger}_{\vec{p}_i}|0\rangle \tag{55}$$

3.2 The spinors and the polarization vectors of fields

When the annihilation operator $b^{s_a}_{\vec{k}_a}$ acts on the neutrino fields, the interaction Hamiltonian becomes

$$\begin{aligned}
b^{s_a}_{\vec{k}_a}\mathcal{H}_{\text{int}} =& e^{ik_ax_1}\Big[Y_{j_1\ell_1}\delta_{aj_1}\bar{u}^{s_a}\psi'_{\ell_1}\phi_1^+ + (Y^\dagger)_{j_1\ell_1}\delta_{a\ell_1}\bar{\psi}'_{j_1}v^{s_a}\phi_1^- \\
&+ V_{aj_1}\bar{u}^{s_a}\gamma^\mu\psi'_{j_1}W^+_{1\mu} + (V^\dagger)_{j_1a}\bar{\psi}'_{j_1}\gamma^\mu v^{s_a}W^-_{1\mu}\Big].
\end{aligned} \tag{56}$$

Notice that there are no more neutrino fields for the annihilation operator to act and the rest of the terms in the interaction Hamiltonian yield zero for this operator.

Then, the next annihilation operator $b^{s_a}_{\vec{k}_a}$ acts on the photon fields where the terms in interaction Hamiltonian has the index 2. Therefore, we have

$$\begin{aligned}a^{\lambda_b}_{\vec{k}_b}\mathcal{H}_{\text{int}} =& e^{-ik_b x_2}\Big[\bar{\psi}'_{j_2}\gamma^{\mu}\psi'_{j_2}\delta^{\lambda\lambda_b}\epsilon^{\lambda}_{\mu} + \big[4i\eta g\delta^{\lambda\lambda_b}\epsilon^{\lambda}_{\alpha}W^{+}_{2\beta}(\partial^{\alpha}W_2^{-\beta} - \partial^{\beta}W_2^{-\alpha})\\ &+ 4i\eta g\delta_{\lambda\lambda_b}\epsilon^{\lambda}_{\alpha}W^{-}_{2\beta}(-\partial^{\alpha}W_2^{+\beta} + \partial^{\beta}W_2^{+\alpha})\\ &+ 4i\eta g(W^{+}_{\alpha}W^{-}_{\beta})\big[-ik^{b\alpha}\delta^{\lambda\lambda_b}\epsilon^{\lambda}_{\beta} + ik^{b\beta}\delta^{\lambda\lambda_b}\epsilon^{\lambda}_{\alpha}\big]\big]\\ &+ \big[\partial_{\mu}\phi^{-}_{2}i\eta_e e\delta_{\lambda\lambda_b}\epsilon^{\lambda}_{\mu}\phi^{+}_{2} + \partial_{\mu}\phi^{+}_{2}i\eta_e e\delta^{\lambda\lambda_b}\epsilon^{\lambda}_{\mu}\phi^{-}_{2}\big]\Big]\end{aligned} \tag{57}$$

Once again, there are no more photon fields that the annihilation operator can act on and the rest of the terms yield zero. Finally, the creation operator $b^{s_i^\dagger}_{\vec{p}_i}$ acts on the neutrino fields where the terms in the interaction Hamiltonian have the index 3 and we get

$$\begin{aligned}\mathcal{H}_{\text{int}}b^{s_i^\dagger}_{\vec{p}_i} =& e^{-ik_i x_3}\Big[Y_{j_3\ell_3}\delta_{ij_3}\bar{v}^{s_i}\psi'_{\ell_3}\phi^{+}_{3} + (Y^{\dagger})_{j_3\ell_3}\delta_{i\ell_3}\bar{\psi}'_{j_3}u^{s_i}\phi^{-}_{3}\\ &+ V_{ij_3}\bar{v}^{s_i}\gamma^{\mu}\psi'_{j_3}W^{+}_{3\mu} + (V^{\dagger})_{j_3 i}\bar{\psi}'_{j_3}\gamma^{\mu}u^{s_i}W^{-}_{3\mu}\Big].\end{aligned} \tag{58}$$

Now, we used all the quantum operators on the fields and we can get to the next step which is writing the whole interaction Hamiltonian with the particles that can couple and give the terms for the propagators. Below, we can see the the interaction Hamiltonian formulation that the quantum operators acted on;

$$\begin{aligned}\langle 0|b^{s_a}_{\vec{k}_a}a^{s_b}_{\vec{k}_b}(\int dx_j\,\mathcal{H}_{\text{int}_j})^3 b^{s_i^\dagger}_{\vec{p}_i}|0\rangle =& \langle 0|(b^{s_a}_{\vec{k}_a}\int dx_1\,\mathcal{H}_{\text{int}_1})(a^{s_b}_{\vec{k}_b}\int dx_2\,\mathcal{H}_{\text{int}_2})(\int dx_3\,\mathcal{H}_{\text{int}_3}b^{s_i^\dagger}_{\vec{p}_i})|0\rangle\\ =& \langle 0|\int dx_1\int dx_2\int dx_3\Big[Y_{j_1\ell_1}\delta_{aj_1}\bar{u}^{s_a}e^{ik_a x_1}\psi'_{\ell_1}\phi^{+}_{1}\Big]\\ &\Big[\partial_{\mu}\phi^{-}_{2}i\eta_e e\delta_{\lambda b}\epsilon^{\lambda}_{\mu}e^{-ik_b x_2}\phi^{+}_{2} + \partial_{\mu}\phi^{+}_{2}i\eta_e e\delta^{\lambda b}\epsilon^{\lambda}_{\mu}e^{-ik_b x_2}\phi^{-}_{2}\Big]\\ &\Big[(Y^{\dagger})_{j_3\ell_3}\delta_{i\ell_3}\bar{\psi}'_{j_3}u^{s_i}e^{-ik_i x_3}\phi^{-}_{3}\Big]\dots\end{aligned} \tag{59}$$

where the full version of it is in the appendix equation (151).

3.3 Integrals of the propagators with respect to momentum (grouped terms)

We can now group the terms we have depending on the type and the number of their coupling particles in the propagators such as 2 fermions 1 Higgs or, 1 fermion and 2 Higgs and write the propagators afterwards from the vacuum expectation value that will help us writing the amplitude $\mathcal{M}$ for our particle interactions.

3.3.1 1 Fermion 2 Higgs propagators

We can start with the terms that have 1 fermion and 2 Higgs boson. What do we do below is writing the terms that will give the first propagator that have different spinors before them wherein the first term we have $\bar{u}^{s_a}$ and in the second term $\bar{v}^{s_i}$. To write these two terms together we take the transpose of it as per the Majorana representation relation and write their propagators from the vacuum expectation value, where we include the propagators for the Higgs boson in our terms in the equation (64):

$$T_{1^{st}\psi.\phi.\phi} = \int dx_1 \int dx_2 \int dx_3 e^{i(k_a x_1 - k_b x_2 - k_i x_3)} \Big(\delta_{aj_1} \delta_{\lambda\lambda_b} \delta_{i\ell_3} i\eta_e e Y_{j_1\ell_1} (Y^\dagger)_{j_3\ell_3}$$
$$\left(\bar{u}^{s_a} \sum_j \int \frac{d^4k_1}{(2\pi)^4} \frac{e^{ik_1(x_1-x_3)}(k_1^\mu \gamma_\mu - m_j)}{(k_1^2 - m_j^2 + i\epsilon)} \delta_{j\ell_1} \delta_{jj_3} u^{s_i} \right) \langle 0| \Big[\phi_1^+ \partial_\mu \phi_2^- \epsilon_\mu^\lambda \phi_2^+ \phi_3^- \Big] 0| \rangle$$
$$- \delta_{a\ell_1} \delta^{\lambda\lambda_b} \delta_{ij_3} i\eta_e e (Y^\dagger)_{j_1\ell_1} Y_{j_3\ell_3}$$
$$\left(\bar{v}^{s_i} \sum_j \int \frac{d^4k_1}{(2\pi)^4} \frac{e^{ik_1(x_3-x_1)}(k_1^\mu \gamma_\mu - m_j)}{k_1^2 - m_j^2 + i\epsilon} \delta_{j\ell_3} \delta_{jj_1} v^{s_a} \right)^\top \langle 0| \Big[\phi_1^- \partial_\mu \phi_2^+ \epsilon_\mu^\lambda \phi_2^- \phi_3^+ \Big] |0\rangle \Big) \tag{60}$$

where the $T_{1^{st}\psi.\phi.\phi}$ represents the first pair of terms for the fermion-Higgs-Higgs propagator. We will use the similar formulation for labeling the equations.

The next step is summing the indices of the δ functions with the Yukawa coupling terms, and get the transpose of the second term and write them together using the Majorana representation relations by writing the $\bar{v}^{s_i} v^{s_a}$ as $\bar{u}^{s_a} u^{s_i}$.

$$T_{1^{st}\psi.\phi.\phi} = i\eta_e e e^{i(k_a x_1 - k_b x_2 - k_i x_3)} \sum_j$$
$$\Big(Y_{aj} (Y^\dagger)_{ji} \int \frac{d^4k_1}{(2\pi)^4} \frac{e^{ik_1(x_1-x_3)} \bar{u}^{s_a} (k_1^\mu \gamma_\mu - m_j) u^{s_i}}{(k_1^2 - m_j^2 + i\epsilon)} \langle 0| \Big[\phi_1^+ \partial_\mu \phi_2^- \epsilon_\mu^{\lambda_b} \phi_2^+ \phi_3^- \Big] |0\rangle$$
$$- (Y^\dagger)_{ja} Y_{ij} \int \frac{d^4k_1}{(2\pi)^4} \frac{e^{ik_1(x_3-x_1)} \bar{u}^{s_a} (k_1^\mu - m_j) u^{s_i}}{k_1^2 - m_j^2 + i\epsilon} \langle 0| \Big[\phi_1^- \partial_\mu \phi_2^+ \epsilon_\mu^{\lambda_b} \phi_2^- \phi_3^+ \Big] |0\rangle \Big) \tag{61}$$

Then we can change the positions x_1 and x_3 by changing the integral d^4k_1 with a minus sign

$$
\begin{aligned}
T_{1^{st}\psi.\phi.\phi} =& i\eta_e e e^{i(k_a x_1 - k_b x_2 - k_i x_3)} \sum_j \\
&\Big(Y_{aj}(Y^\dagger)_{ji} \int \frac{d^4k_1}{(2\pi)^4} \frac{e^{ik_1(x_1-x_3)}\bar{u}^{s_a}(k_1^\mu - m_j)u^{s_i}}{(k_1^2 - m_j^2 + i\epsilon)} \langle 0| \Big[\phi_1^+ \partial_\mu \phi_2^- \epsilon_\mu^{\lambda_b} \phi_2^+ \phi_3^- \Big] |0\rangle \\
&- (Y^\dagger)_{ja} Y_{ij} \int \frac{d^4(-k_1)}{(2\pi)^4} \frac{e^{-ik_1(x_3-x_1)}\bar{u}^{s_a}(k_1^\mu - m_j)u^{s_i}}{(k_1^2 - m_j^2 + i\epsilon)} \langle 0| \Big[\phi_1^- \partial_\mu \phi_2^+ \epsilon_\mu^{\lambda_b} \phi_2^- \phi_3^+ \Big] |0\rangle \Big) \\
=& i\eta_e e e^{i(k_a x_1 - k_b x_2 - k_i x_3)} \sum_j \int \frac{d^4k_1}{(2\pi)^4} \frac{e^{ik_1(x_1-x_3)}}{k_1^2 - m_j^2 + i\epsilon} \\
&\Big(Y_{aj}(Y^\dagger)_{ji} \bar{u}^{s_a}(k_1^\mu \gamma_\mu - m_j)u^{s_i} \langle 0| \Big[\phi_1^+ \partial_\mu \phi_2^- \epsilon_\mu^{\lambda_b} \phi_2^+ \phi_3^- \Big] |0\rangle \\
&+ (Y^\dagger)_{ja} Y_{ij} \bar{u}^{s_a}(k_1^\mu \gamma_\mu - m_j)u^{s_i} \langle 0| \Big[\phi_1^- \partial_\mu \phi_2^+ \epsilon_\mu^{\lambda_b} \phi_2^- \phi_3^+ \Big] |0\rangle \Big)
\end{aligned} \tag{62}
$$

So, we can write the terms with the spinors together as

$$
\begin{aligned}
T_{1^{st}\psi.\phi.\phi} =& i\eta_e e e^{i(k_a x_1 - k_b x_2 - k_i x_3)} \sum_j \int \frac{d^4k_1}{(2\pi)^4} \frac{e^{ik_1(x_1-x_3)}}{k_1^2 - m_j^2 + i\epsilon} \bar{u}^{s_a}(k_1^\mu - m_j)u^{s_i} \\
&\Big(Y_{aj}(Y^\dagger)_{ji} \langle 0| \Big[\phi_1^+ \partial_\mu \phi_2^- \epsilon_\mu^{\lambda_b} \phi_2^+ \phi_3^- \Big] |0\rangle \\
&+ (Y^\dagger)_{ja} Y_{ij} \langle 0| \Big[\phi_1^- \partial_\mu \phi_2^+ \epsilon_\mu^{\lambda_b} \phi_2^- \phi_3^+ \Big] |0\rangle \Big)
\end{aligned} \tag{63}
$$

And when we write the propagators from the ϕ fields, we get the equation below.

$$
\begin{aligned}
T_{1^{st}\psi.\phi.\phi} =& \int dx_1 \int dx_2 \int dx_3 i\eta_e e e^{i(k_a x_1 - k_b x_2 - k_i x_3)} \\
&\sum_j \int \frac{d^4k_1}{(2\pi)^4} \frac{e^{ik_1(x_1-x_3)}}{k_1^2 - m_j^2 + i\epsilon} \bar{u}^{s_a}(k_1^\mu \gamma_\mu - m_j)u^{s_i} \\
&\Big(Y_{aj}(Y^\dagger)_{ji} \Big[\int \frac{d^4k_2}{(2\pi)^4} \frac{-ik_{2\mu} e^{ik_2(x_1-x_2)}}{k_2^2 - m_\phi^2 + i\epsilon} \epsilon_\mu^\lambda \int \frac{d^4k_3}{(2\pi)^4} \frac{e^{ik_3(x_2-x_3)}}{k_3^2 - m_\phi^2 + i\epsilon} \Big] \\
&+ (Y^\dagger)_{ja} Y_{ij} \Big[\int \frac{d^4k_2}{(2\pi)^4} \frac{-ik_{2\mu} e^{ik_2(x_1-x_2)}}{k_2^2 - m_\phi^2 + i\epsilon} \epsilon_\mu^\lambda \int \frac{d^4k_3}{(2\pi)^4} \frac{e^{ik_3(x_2-x_3)}}{k_3^2 - m_\phi^2 + i\epsilon} \Big] \Big)
\end{aligned} \tag{64}
$$

Then the Yukawa coupling terms can be summed up and the propagators term can be written once since they have the same components.

$$\begin{aligned}
T_{1^{st}\psi.\phi.\phi} = &\int dx_1 \int dx_2 \int dx_3 i\eta_e e e^{i(k_a x_1 - k_b x_2 - k_i x_3)} \\
&\sum_j \int \frac{d^4k_1}{(2\pi)^4} \frac{e^{ik_1(x_1-x_3)}}{k_1^2 - m_j^2 + i\epsilon} \bar{u}^{s_a}(k_1^\mu - m_j) u^{s_i} \\
&\Big(Y_{aj}(Y^\dagger)_{ji} + (Y^\dagger)_{ja} Y_{ij} \Big) \Big[\int \frac{d^4k_2}{(2\pi)^4} \frac{-ik_{2\nu} e^{ik_2(x_1-x_2)}}{k_2^2 - m_\phi^2 + i\epsilon} \epsilon_\nu^\lambda \int \frac{d^4k_3}{(2\pi)^4} \frac{e^{ik_3(x_2-x_3)}}{k_3^2 - m_\phi^2 + i\epsilon} \Big]
\end{aligned} \tag{65}$$

Once we integrate over the x_i fields and the momentum we get

$$\begin{aligned}
T_{1^{st}\psi.\phi.\phi} = &(2\pi)^4 \delta(k_a - k_b - k_i) \Big(Y_{aj}(Y^\dagger)_{ji} + (Y^\dagger)_{ja} Y_{ij} \Big) i\eta_e e \\
&\int \frac{d^4k_3}{(2\pi)^4} \Big[\frac{\bar{u}^{s_a}(-\not{k}_3 + \not{k}_b - \not{k}_a - m_j) u^{s_i}}{((k_3 - k_b + k_a)^2 - m_j^2 + i\epsilon)} \frac{-i(k_3 - k_b)_\mu}{(k_3 - k_b)^2 - m_\phi^2 + i\epsilon} \epsilon_\mu^\lambda \frac{1}{k_3^2 - m_\phi^2 + i\epsilon} \Big]
\end{aligned} \tag{66}$$

In the same fashion, we can also write the pair of term that has the partial derivative $\partial\phi$ on the 3^{rd} index wherein the previous pair, it was on the 2^{nd} ϕ field.

$$\begin{aligned}
T_{2^{nd}\psi.\phi.\phi} = &\int dx_1 \int dx_2 \int dx_3 e^{i(k_a x_1 - k_b x_2 - k_i x_3)} \Big(\delta_{aj_1} \delta^{\lambda\lambda_b} \delta_{i\ell_3} Y_{j_1\ell_1} (Y^\dagger)_{j_3\ell_3} i\eta_e e \\
&\Big(\bar{u}^{s_a} \sum_j \int \frac{d^4k_1}{(2\pi)^4} \frac{e^{ik_1(x_1-x_3)}(\not{k}_1 - m_j)}{(k_1^2 - m_j^2 + i\epsilon)} \delta_{j\ell_1} \delta_{jj_3} u^{s_i} \Big) \langle 0 | \Big[\phi_1^+ \phi_2^- \epsilon_\nu^\lambda \partial_\nu \phi_2^+ \phi_3^- \Big] | 0 \rangle \\
&- \delta_{a\ell_1} \delta_{\lambda\lambda_b} \delta_{ij_3} (Y^\dagger)_{j_1\ell_1} Y_{j_3\ell_3} i\eta_e e \\
&\Big(\bar{v}^{s_i} \sum_j \int \frac{d^4k_1}{(2\pi)^4} \frac{e^{ik_1(x_3-x_1)}(\not{k}_1 - m_j)}{(k_1^2 - m_j^2 + i\epsilon)} \delta_{j\ell_3} \delta_{jj_1} v^{s_a} \Big)^\top \langle 0 | \Big[\phi_1^- \phi_2^+ \epsilon_\mu^\lambda \partial_\mu \phi_2^- \phi_3^+ \Big] | 0 \rangle \Big)
\end{aligned} \tag{67}$$

Following the same steps above we can write the equation as

$$\begin{aligned}
T_{2^{nd}\psi.\phi.\phi} = &\int dx_1 \int dx_2 \int dx_3 i\eta_e e e^{i(k_a x_1 - k_b x_2 - k_i x_3)} \\
&\sum_j \int \frac{d^4k_1}{(2\pi)^4} \frac{e^{ik_1(x_1-x_3)}}{k_1^2 - m_j^2 + i\epsilon} \bar{u}^{s_a}(k_1^\mu \gamma_\mu - m_j) u^{s_i} \\
&\Big(Y_{aj}(Y^\dagger)_{ji} + (Y^\dagger)_{ja} Y_{ij} \Big) \Big[\int \frac{d^4k_2}{(2\pi)^4} \frac{e^{ik_2(x_1-x_2)}}{k_2^2 - m_\phi^2 + i\epsilon} \epsilon_\nu^\lambda \int \frac{d^4k_3}{(2\pi)^4} \frac{ik_{3\nu} e^{ik_3(x_2-x_3)}}{k_3^2 - m_\phi^2 + i\epsilon} \Big]
\end{aligned} \tag{68}$$

Once we integrate over the x fields in the exponential term and the momentum we have

$$T_{2^{nd}\psi.\phi.\phi} = (2\pi)^4\delta(k_a - k_b - k_i)\Big(Y_{aj}(Y^\dagger)_{ji} + (Y^\dagger)_{ja}Y_{ij}\Big)i\eta_e e$$
$$\int \frac{d^4k_3}{(2\pi)^4}\Big[\frac{\bar{u}^{s_a}(-\not{k}_3 + \not{k}_b - \not{k}_a - m_j)u^{s_i}}{((k_3 - k_b + k_a)^2 - m_j^2 + i\epsilon)}\frac{1}{(k_3 - k_b)^2 - m_\phi^2 + i\epsilon}\epsilon_\nu^\lambda\frac{ik_{3\nu}}{k_3^2 - m_\phi^2 + i\epsilon}\Big] \tag{69}$$

3.3.2 2 Fermions 1 Higgs Propagators

The following equation shows terms that have the coupling of 2 fermions and 1 Higgs.

$$T_{\psi.\psi.\phi} = \int dx_1 \int dx_2 \int dx_3 e^{i(k_a x_1 - ik_b x_2 - ik_i x_3)}\Bigg(\delta_{aj_1}\delta^{\lambda\lambda_b}\delta_{i\ell_3}Y_{j_1\ell_1}(Y^\dagger)_{j_3\ell_3}$$
$$\Bigg(\bar{u}^{s_a}\sum_j \int \frac{d^4k_1}{(2\pi)^4}\frac{e^{ik_1(x_1-x_2)}(\not{k}_1 - m_j)}{(k_1^2 - m_j^2 + i\epsilon)}\delta_{j\ell_1}\delta_{jj_2}\gamma^\mu$$
$$\times \sum_{\bar{j}} \int \frac{d^4k_2}{(2\pi)^4}\frac{e^{ik_2(x_2-x_3)}(\not{k}_2 - m_{\bar{j}})}{(k_2^2 - m_{\bar{j}}^2 + i\epsilon)}\delta_{\bar{j}j_2}\delta_{\bar{j}j_3}u^{s_i}\Bigg)\langle 0|\Big[\epsilon_\mu^\lambda\phi_1^+\phi_3^-\Big]|0\rangle$$
$$-\delta_{a\ell_1}\delta^{\lambda\lambda_b}\delta_{ij_3}(Y^\dagger)_{j_1\ell_1}Y_{j_3\ell_3}$$
$$\Bigg(\bar{v}^{s_i}\sum_j \int \frac{d^4k_1}{(2\pi)^4}\frac{e^{ik_1(x_3-x_2)}(\not{k}_1 - m_j)}{(k_1^2 - m_j^2 + i\epsilon)}\delta_{j\ell_3}\delta_{jj_2}\gamma^\mu$$
$$\times \sum_{\bar{j}} \int \frac{d^4k_2}{(2\pi)^4}\frac{e^{ik_2(x_2-x_1)}(\not{k}_2 - m_{\bar{j}})}{(k_2^2 - m_{\bar{j}}^2 + i\epsilon)}\delta_{\bar{j}j_2}\delta_{\bar{j}j_1}v^{s_a}\Bigg)^\top\langle 0|\Big[\epsilon_\mu^\lambda\phi_1^-\phi_3^+\Big]|0\rangle\Bigg) \tag{70}$$

Now, we have a bit different form since there are two fermions. The steps are almost the similar for grouping these two terms. But we have an additional index to sum up which is $\bar{j}$. We sum j_2 with $\bar{j}$ and then $\bar{j}$ with j after taking the transpose of the second term once again. And changing the boundaries of the last two integral, so we can change the order of the $x_3 - x_2$ and $x_2 - x_1$ in the exponential functions, we can write them in the same brackets as:

$$T_{\psi.\psi.\phi} = \int dx_1 \int dx_2 \int dx_3 e^{i(k_a x_1 - ik_b x_2 - ik_i x_3)}\Big(Y_{aj}(Y^\dagger)_{ji} + (Y^\dagger)_{ja}Y_{ij}\Big)$$
$$\Bigg(\bar{u}^{s_a}\sum_j \int \frac{d^4k_1}{(2\pi)^4}\frac{e^{ik_1(x_1-x_2)}(k_1^\mu\gamma_\mu - m_j)}{(k_1^2 - m_j^2 + i\epsilon)}\gamma^\mu$$
$$\int \frac{d^4k_2}{(2\pi)^4}\frac{e^{ik_2(x_2-x_3)}(k_2^\nu\gamma_\nu - m_j)}{(k_2^2 - m_j^2 + i\epsilon)}u^{s_i}\epsilon_\mu^\lambda \int \frac{d^4k_3}{(2\pi)^4}\frac{e^{ik_3(x_1-x_3)}}{k_3^2 - m_\phi^2 + i\epsilon}\Bigg) \tag{71}$$

We have now all our fields grouped with their relevant denominators according to the coupling of our particles and the quantum operators acted on every field and we wrote the propagators as well.

Once we integrate over the x fields and get the delta functions and integrate over the momentum the equation becomes

$$
\begin{aligned}
T_{\psi.\psi.\phi} =&(2\pi)^4\delta(k_a - k_b - k_i)\Big(Y_{aj}(Y^\dagger)_{ji} + (Y^\dagger)_{ja}Y_{ij}\Big)i\eta_e e \\
&\int \frac{d^4k_3}{(2\pi)^4}\Big[\frac{\bar{u}^{s_a}(-\not{k}_a - \not{k}_3 - m_j)}{(k_a + k_3)^2 - m_j^2 + i\epsilon)}\gamma^\mu \frac{(-\not{k}_3 - \not{k}_a + \not{k}_b - m_j)u^{s_i}}{(-k_3 - k_a + k_b)^2 - m_j^2 + i\epsilon}\epsilon^\lambda_\mu \frac{1}{k_3^2 - m_\phi^2 + i\epsilon}\Big]
\end{aligned} \tag{72}
$$

But during the integration over the momenta, we always have one leftover momentum at the last integral. For that, we are going to have to use the Feynman parametrization and the Wick rotation since that is an integral of a loop and we need to evaluate it with these steps. The description of these methods are shown below.

3.3.3 1 Fermion 2 $W^\pm$ boson Propagators

Now, we get to the terms with 1 fermion and 2 W bosons. Starting with the one that shows the W^+

$$
\begin{aligned}
T_{\psi.W^+.W^-} =& \int dx_1 \int dx_2 \int dx_3 4i\eta g V_{aj_1}\delta^{\lambda\lambda_b}V_{ij_3} \\
& e^{ik_ax_1}e^{-ik_bx_2}e^{-ik_ix_3}\langle 0|\Big[\bar{u}^{s_a}\gamma^\mu\psi'_{j_1}\bar{\psi}'_{j_3}\gamma^\nu u^{s_i}\Big] \\
& \Big[\epsilon^\lambda_\alpha W^+_{1\mu}\partial^\alpha W_2^{-\beta}W^+_{2\beta}W^-_{3\nu} + \epsilon^\lambda_\alpha W^+_{1\mu}(-\partial^\beta W_2^{-\alpha})W^+_{2\beta}W^-_{3\nu} \\
& + \epsilon^\lambda_\alpha W^+_{1\mu}W^-_{2\beta}(-\partial^\alpha W_2^{+\beta})W^-_{3\nu} + \epsilon^\lambda_\alpha W^+_{1\mu}W^-_{2\beta}(\partial^\beta W_2^{+\alpha})W^-_{3\nu} \\
& W^+_{1\mu}W^-_{2\upsilon}\big(-ik^{b\theta}\epsilon^{\upsilon\lambda} + ik^{b\upsilon}\epsilon^{\theta\lambda}\big)W^+_{2\theta}W^-_{3\nu}\Big]|0\rangle
\end{aligned} \tag{73}
$$

where the following terms, showing the W^- boson

$$
\begin{aligned}
T_{\psi.W^-W^+} =& -\int dx_1 \int dx_2 \int dx_3 4i\eta g V_{aj_1}\delta^{\lambda\lambda_b}V_{ij_3} \\
& e^{ik_ax_1}e^{-ik_bx_2}e^{-ik_ix_3}\langle 0|\Big[\bar{v}^{s_i}\gamma^\nu\psi'_{j_3}\bar{\psi}'_{j_1}\gamma^\mu v^{s_a}\Big] \\
& \Big[\epsilon^\lambda_\alpha W^-_{1\mu}W^+_{2\beta}(\partial^\alpha W_2^{-\beta})W^+_{3\nu} + \epsilon^\lambda_\alpha W^-_{1\mu}W^+_{2\beta}(-\partial^\beta W_2^{-\alpha})W^+_{3\nu} \\
& + \epsilon^\lambda_\alpha W^-_{1\mu}(-\partial^\alpha W_2^{+\beta})W^-_{2\beta}W^+_{3\nu} + \epsilon^\lambda_\alpha W^-_{1\mu}(\partial^\beta W_2^{+\alpha})W^-_{2\beta}W^+_{3\nu} \\
& + W^-_{1\mu}W^+_{2\upsilon}(-ik^{b\upsilon}\epsilon^{\theta\lambda} + ik^{b\theta}\epsilon^{\upsilon\lambda})W^-_{2\theta}W^+_{3\nu}\Big]|0\rangle
\end{aligned} \tag{74}
$$

We attempt to write them with the same constants that they have in common, then we have

$$
\begin{aligned}
T_{\psi.W^{\pm}} = & \int dx_1 \int dx_2 \int dx_3 4i\eta g V_{aj_1} \delta^{\lambda\lambda_b} V_{ij_3} e^{ik_a x_1} e^{-ik_b x_2} e^{-ik_i x_3} \\
& \langle 0| \Big[\bar{u}^{s_a} \gamma^{\mu} \psi'_{j_1} \bar{\psi}'_{j_3} \gamma^{\nu} u^{s_i} - \bar{v}^{s_i} \gamma^{\nu} \psi'_{j_3} \bar{\psi}'_{j_1} \gamma^{\mu} v^{s_a} \Big] \\
& \Big[\epsilon^{\lambda}_{\alpha} W^{+}_{1\mu} \partial^{\alpha} W^{-\beta}_{2} W^{+}_{2\beta} W^{-}_{3\nu} + \epsilon^{\lambda}_{\alpha} W^{+}_{1\mu} (-\partial^{\beta} W^{-\alpha}_{2}) W^{+}_{2\beta} W^{-}_{3\nu} \\
& + \epsilon^{\lambda}_{\alpha} W^{+}_{1\mu} W^{-}_{2\beta} (-\partial^{\alpha} W^{+\beta}_{2}) W^{-}_{3\nu} + \epsilon^{\lambda}_{\alpha} W^{+}_{1\mu} W^{-}_{2\beta} (\partial^{\beta} W^{+\alpha}_{2}) W^{-}_{3\nu} \Big] |0\rangle \\
+ & \int dx_1 \int dx_2 \int dx_3 4i\eta g V_{aj_1} \delta^{\lambda\lambda_b} V_{ij_3} e^{ik_a x_1} e^{-ik_b x_2} e^{-ik_i x_3} \\
& \langle 0| \Big[\bar{u}^{s_a} \gamma^{\mu} \psi'_{j_1} \bar{\psi}'_{j_3} \gamma^{\nu} u^{s_i} W^{+}_{1\mu} W^{-}_{2\upsilon} \big(-ik^{b\theta} \epsilon^{\upsilon\lambda} + ik^{b\upsilon} \epsilon^{\theta\lambda} \big) W^{+}_{2\theta} W^{-}_{3\nu} \\
& - (\bar{v}^{s_i} \gamma^{\nu} \psi'_{j_3} \bar{\psi}'_{j_1} \gamma^{\mu} v^{s_a}) W^{-}_{1\mu} W^{+}_{2\upsilon} (-ik^{b\upsilon} \epsilon^{\theta\lambda} + ik^{b\theta} \epsilon^{\upsilon\lambda}) W^{-}_{2\theta} W^{+}_{3\nu} \Big] |0\rangle
\end{aligned} \tag{75}
$$

Using the Majorana representation relation, we can write the term with $\bar{v}^{s_i}$ as $\bar{u}^{s_a}$ and change the order of the $x_3 - x_1$ as we did before as

$$
\begin{aligned}
& \langle 0| \bar{u}^{s_a} \gamma^{\mu} \psi'_{j_1} \bar{\psi}'_{j_3} \gamma^{\nu} u^{s_i} - \bar{v}^{s_i} \gamma^{\nu} \psi'_{j_3} \bar{\psi}'_{j_1} \gamma^{\mu} v^{s_a} |0\rangle \\
= & \int dx_1 \int dx_2 \int dx_3 V_{aj_1} \delta^{\lambda\lambda_b} V_{ij_3} \\
& \int \frac{d^4k_1}{(2\pi)^4} \frac{e^{ik_1(x_1-x_3)} \bar{u}^{s_a} \gamma^{\mu} (\not{k}_1 - m_j) \gamma^{\nu} u^{s_i}}{k_1^2 - m_j^2 + i\epsilon} \delta_{jj_1} \delta_{jj_3} \\
& - \int dx_1 \int dx_2 \int dx_3 V_{aj_1} \delta^{\lambda\lambda_b} V_{ij_3} \\
& \int \frac{d^4k_1}{(2\pi)^4} \frac{e^{ik_1(x_3-x_1)} (\bar{v}^{s_i} \gamma^{\nu} (\not{k}_1 - m_j) \gamma^{\mu} v^{s_a})^{\top}}{k_1^2 - m_j^2 + i\epsilon} \delta_{jj_1} \delta_{jj_3} \\
= & \int dx_1 \int dx_2 \int dx_3 V_{aj} \delta^{\lambda\lambda_b} V_{ij} \\
& \int \frac{d^4k_1}{(2\pi)^4} \frac{e^{ik_1(x_1-x_3)} \bar{u}^{s_a} \gamma^{\mu} (\not{k}_1 - m_j) \gamma^{\nu} u^{s_i}}{k_1^2 - m_j^2 + i\epsilon} \\
& - \int dx_1 \int dx_2 \int dx_3 V_{aj} \delta^{\lambda\lambda_b} V_{ij} \\
& \int \frac{d^4(-k_1)}{(2\pi)^4} \frac{e^{-ik_1(x_3-x_1)} \bar{u}^{s_a} \gamma^{\mu} (\not{k}_1 - m_j) \gamma^{\nu} u^{s_i}}{k_1^2 - m_j^2 + i\epsilon} \\
= & \int dx_1 \int dx_2 \int dx_3 V_{aj} \delta^{\lambda\lambda_b} V_{ij} \\
& \int \frac{d^4k_1}{(2\pi)^4} \frac{e^{ik_1(x_1-x_3)} \bar{u}^{s_a} \gamma^{\mu} (\not{k}_1 - m_j) \gamma^{\nu} u^{s_i}}{k_1^2 - m_j^2 + i\epsilon}
\end{aligned} \tag{76}
$$

And then write the propagators from the vacuum expectation value as

$$
\begin{aligned}
T_{\psi.W^{\pm}} = & \int dx_1 \int dx_2 \int dx_3 4i\eta g V_{aj} \delta^{\lambda\lambda_b} V_{ij} e^{ik_a x_1} e^{-ik_b x_2} e^{-ik_i x_3} \\
& \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int \frac{d^4k_3}{(2\pi)^4} \Big[\bar{u}^{s_a} \gamma^{\mu} e^{ik_1(x_1-x_3)} (\not{k}_1 - m_j) \gamma^{\nu} u^{s_i} \\
& + \bar{u}^{s_a} \gamma^{\mu} e^{ik_1(x_1-x_3)} (\not{k}_1 - m_j) \gamma^{\nu} u^{s_i} \\
& \Big(\epsilon^{\lambda}_{\alpha} (g^{\beta}_{\mu} i k^{\alpha}_2 e^{ik_2(x_1-x_2)}) (-g_{\beta\nu} e^{ik_3(x_2-x_3)}) \\
& + \epsilon^{\lambda}_{\alpha} (g^{\alpha}_{\mu} i k^{\beta}_2 e^{ik_2(x_1-x_2)}) (-g_{\beta\nu} e^{ik_3(x_2-x_3)}) \\
& + \epsilon^{\lambda}_{\alpha} (-g_{\mu\beta} e^{ik_2(x_1-x_2)}) (-g^{\beta}_{\nu} i k^{\alpha}_2 e^{ik_3(x_2-x_3)}) \\
& + \epsilon^{\lambda}_{\alpha} (-g_{\mu\beta} e^{ik_2(x_1-x_2)}) (-g^{\alpha}_{\nu} i k^{\beta}_2 e^{ik_3(x_2-x_3)}) \Big) \\
& \Big((k_1^2 - m_j^2 + i\epsilon)(k_2^2 - m_W^2 + i\epsilon)(k_3^2 - m_W^2 + i\epsilon) \Big)^{-1} \Big] \\
& + \int dx_1 \int dx_2 \int dx_3 4i\eta g V_{aj} \delta^{\lambda\lambda_b} V_{ij} e^{ik_a x_1} e^{-ik_b x_2} e^{-ik_i x_3} \\
& \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int \frac{d^4k_3}{(2\pi)^4} \Big[(\bar{u}^{s_a} \gamma^{\mu} e^{ik_1(x_1-x_3)} (\not{k}_1 - m_j) \gamma^{\nu} u^{s_i}) \\
& (-g_{\mu\upsilon} e^{ik_2(x_1-x_2)}) \Big(-ik^{b\theta} \epsilon^{\upsilon\lambda} + ik^{b\upsilon} \epsilon^{\theta\lambda} \Big) (-g_{\theta\nu} e^{ik_3(x_2-x_3)}) \\
& + (\bar{u}^{s_a} \gamma^{\mu} e^{ik_1(x_1-x_3)} (\not{k}_1 - m_j) \gamma^{\nu} u^{s_i}) \\
& (-g_{\mu\upsilon} e^{ik_2(x_1-x_2)}) (-ik^{b\upsilon} \epsilon^{\theta\lambda} + ik^{b\theta} \epsilon^{\upsilon\lambda}) (-g_{\theta\nu} e^{ik_3(x_2-x_3)}) \\
& \Big((k_1^2 - m_j^2 + i\epsilon)(k_2^2 - m_W^2 + i\epsilon)(k_3^2 - m_W^2 + i\epsilon) \Big)^{-1} \Big]
\end{aligned} \tag{77}
$$

We can carry the exponentials out for they are the same for each term

$$
\begin{aligned}
T_{\psi.W^{\pm}} =& \int dx_1 \int dx_2 \int dx_3 4i\eta g V_{aj}\delta^{\lambda\lambda_b}V_{ij}e^{ik_a x_1}e^{-ik_b x_2}e^{-ik_i x_3} \\
& \int \frac{d^4k_1}{(2\pi)^4}\int \frac{d^4k_2}{(2\pi)^4}\int \frac{d^4k_3}{(2\pi)^4}e^{ik_1(x_1-x_3)}e^{ik_2(x_1-x_2)}e^{ik_3(x_2-x_3)} \\
& \Big[\bar{u}^{s_a}\gamma^{\mu}(\not{k}_1 - m_j)\gamma^{\nu}u^{s_i} + \bar{u}^{s_a}\gamma^{\nu}(\not{k}_1 - m_j)\gamma^{\mu}u^{s_i} \\
& \Big(- \epsilon^{\lambda}_{\alpha}(g^{\beta}_{\mu}ik^{\alpha}_2)(g_{\beta\nu}) - \epsilon^{\lambda}_{\alpha}(g^{\alpha}_{\mu}ik^{\beta}_2)(g_{\beta\nu}) \\
& + \epsilon^{\lambda}_{\alpha}(g_{\mu\beta})(g^{\beta}_{\nu}ik^{\alpha}_2) + \epsilon^{\lambda}_{\alpha}(g_{\mu\beta})(g^{\alpha}_{\nu}ik^{\beta}_2)\Big) \\
& \Big((k_1^2 - m_j^2 + i\epsilon)(k_2^2 - m_W^2 + i\epsilon)(k_3^2 - m_W^2 + i\epsilon)\Big)^{-1}\Big] \\
+& \int dx_1 \int dx_2 \int dx_3 4i\eta g V_{aj}\delta^{\lambda\lambda_b}V_{ij}e^{ik_a x_1}e^{-ik_b x_2}e^{-ik_i x_3} \\
& \int \frac{d^4k_1}{(2\pi)^4}\int \frac{d^4k_2}{(2\pi)^4}\int \frac{d^4k_3}{(2\pi)^4}e^{ik_1(x_1-x_3)}e^{ik_2(x_1-x_2)}e^{ik_3(x_2-x_3)} \\
& \Big[(\bar{u}^{s_a}\gamma^{\mu}(\not{k}_1 - m_j)\gamma^{\nu}u^{s_i}) \\
& (-g_{\mu\upsilon})\big(- ik^{b\theta}\epsilon^{\upsilon\lambda} + ik^{b\upsilon}\epsilon^{\theta\lambda}\big)(-g_{\theta\nu}) \\
& - e^{ik_1(x_1-x_3)}e^{ik_2(x_1-x_2)}e^{ik_3(x_2-x_3)} \\
& (\bar{v}^{s_i}\gamma^{\nu}(\not{k}_1 - m_j)\gamma^{\mu}v^{s_a}) \\
& (-g_{\mu\upsilon})(-ik^{b\upsilon}\epsilon^{\theta\lambda} + ik^{b\theta}\epsilon^{\upsilon\lambda})(-g_{\theta\nu}) \\
& \Big((k_1^2 - m_j^2 + i\epsilon)(k_2^2 - m_W^2 + i\epsilon)(k_3^2 - m_W^2 + i\epsilon)\Big)^{-1}\Big]
\end{aligned}
\tag{78}
$$

Now, we can integrate over the position to get the δ functions for the momenta integral

$$
\begin{aligned}
T_{\psi.W^{\pm}} =& (2\pi)^4\delta(k_a - k_b - k_i)8i\eta g V_{aj}\delta^{\lambda\lambda_b}V_{ij} \\
& \int \frac{d^4k_3}{(2\pi)^4}\Big[\Big(\bar{u}^{s_a}\gamma^{\mu}(-\not{k}_3 + \not{k}_b - \not{k}_a - m_j)\gamma^{\nu}u^{s_i}\Big) \\
& \big(- g^{\beta}_{\mu}\epsilon^{\lambda}_{\alpha}i(-k^{\alpha}_3 + k^{\alpha}_b)g_{\beta\nu} - g^{\alpha}_{\mu}\epsilon^{\lambda}_{\alpha}i(-k^{\beta}_3 + k^{\beta}_b)g_{\beta\nu} \\
& + g_{\mu\beta}\epsilon^{\lambda}_{\alpha}i(-k^{\alpha}_3 + k^{\alpha}_b)g^{\beta}_{\nu} + g_{\mu\beta}\epsilon^{\lambda}_{\alpha}i(-k^{\beta}_3 + k^{\beta}_b)g^{\alpha}_{\nu}\big) \\
& + (g_{\mu\upsilon})\big(- ik^{b\theta}\epsilon^{\upsilon\lambda} + ik^{b\upsilon}\epsilon^{\theta\lambda}\big)(g_{\theta\nu}) \\
& + (g_{\mu\upsilon})(-ik^{b\upsilon}\epsilon^{\theta\lambda} + ik^{b\theta}\epsilon^{\upsilon\lambda})(g_{\theta\nu}) \\
& \Big(((k_3 - k_b + k_a)^2 - m_j^2 + i\epsilon)((k_3 - k_b)^2 - m_W^2 + i\epsilon)(k_3^2 - m_W^2 + i\epsilon)\Big)^{-1}\Big]
\end{aligned}
\tag{79}
$$

After we sum the indices we get

$$
\begin{aligned}
T_{\psi.W^{\pm}} =&(2\pi)^4\delta(k_a - k_b - k_i)8i\eta g V_{aj}V_{ij} \\
&\int \frac{d^4k_3}{(2\pi)^4}\Big[\big(\bar{u}^{s_a}\gamma^{\mu}(-\not{k}_3 + \not{k}_b - \not{k}_a - m_j)\gamma^{\nu}u^{s_i}\big)\big(\epsilon_{\mu}^{\lambda_b}i(k_{3\nu}) - \epsilon_{\nu}^{\lambda_b}i(k_{3\mu})\big) \\
&\Big(((k_3 - k_b + k_a)^2 - m_j^2 + i\epsilon)((k_3 - k_b)^2 - m_W^2 + i\epsilon)(k_3^2 - m_W^2 + i\epsilon)\Big)^{-1}\Big]
\end{aligned}
\tag{80}
$$

3.3.4 2 Fermions 1 $W^{\pm}$ Boson Propagators

And finally for the 2 fermions 1 W Boson terms, we write the equation as

$$
\begin{aligned}
T_{\psi.\psi.W^{\pm}} =& \int dx_1 \int dx_2 \int dx_3 V_{aj_1}\delta^{\lambda\lambda_b}Vij_3 e^{i(k_a x_1 - k_b x_2 - k_i x_3)} \\
&\Big(- \bar{v}^{s_i}\gamma^{\rho}\int \frac{d^4k_1}{(2\pi)^4}\frac{e^{ik_1(x_3-x_2)}(\not{k}_1 - m_j)}{(k_1^2 - m_j^2 + i\epsilon)}\gamma^{\nu}\int \frac{d^4k_2}{(2\pi)^4}\frac{e^{ik_2(x_2-x_1)}(\not{k}_2 - m_j)}{(k_2^2 - m_j^2 + i\epsilon)}\gamma^{\mu}v^{s_a}\epsilon_{\nu}^{\lambda} \\
&+ \bar{u}^{s_a}\gamma^{\mu}\int \frac{d^4k_1}{(2\pi)^4}\frac{e^{ik_1(x_1-x_2)}(\not{k}_1 - m_j)}{(k_1^2 - m_j^2 + i\epsilon)}\gamma^{\nu}\int \frac{d^4k_2}{(2\pi)^4}\frac{e^{ik_2(x_2-x_3)}(\not{k}_2 - m_j)}{(k_2^2 - m_j^2 + i\epsilon)}\gamma^{\rho}u^{s_i}\epsilon_{\nu}^{\lambda}\Big) \\
&\int \frac{d^4k_3}{(2\pi)^4}\frac{-g_{\mu\rho}e^{ik_3(x_1-x_3)}}{(k_3^2 - m_W^2 + i\epsilon)}
\end{aligned}
\tag{81}
$$

having the first term's transpose, we can write them together as we did in the previous pairs similarly. So, we have

$$
\begin{aligned}
T_{\psi.\psi.W^{\pm}} =& \int dx_1 \int dx_2 \int dx_3 V_{aj_1}\delta^{\lambda\lambda_b}V_{ij_3} e^{i(k_a x_1 - k_b x_2 - k_i x_3)} \\
&\Big(- \bar{u}^{s_a}\gamma^{\mu}\int \frac{d^4k_1}{(2\pi)^4}\frac{e^{ik_1(x_1-x_2)}(\not{k}_2 + m_j)}{(k_1^2 - m_j^2 + i\epsilon)}\gamma^{\nu}\int \frac{d^4k_2}{(2\pi)^4}\frac{e^{ik_2(x_2-x_3)}(\not{k}_1 + m_j)}{(k_2^2 - m_j^2 + i\epsilon)}\gamma^{\rho}u^{s_i}\epsilon_{\nu}^{\lambda} \\
&+ \bar{u}^{s_a}\gamma^{\mu}\int \frac{d^4k_1}{(2\pi)^4}\frac{e^{ik_1(x_1-x_2)}(\not{k}_1 - m_j)}{(k_1^2 - m_j^2 + i\epsilon)}\gamma^{\nu}\int \frac{d^4k_2}{(2\pi)^4}\frac{e^{ik_2(x_2-x_3)}(\not{k}_2 - m_j)}{(k_2^2 - m_j^2 + i\epsilon)}\gamma^{\rho}u^{s_i}\epsilon_{\nu}^{\lambda}\Big) \\
&\int \frac{d^4k_3}{(2\pi)^4}\frac{-g_{\mu\rho}e^{ik_3(x_1-x_3)}}{(k_3^2 - m_W^2 + i\epsilon)}
\end{aligned}
\tag{82}
$$

So after integrating over the position and the momentum, we can write our terms as

$$
\begin{aligned}
T_{\psi.\psi.W^{\pm}} =&(2\pi)^4\delta(k_a - k_b - k_i)V_{aj}V_{ij} \\
&\Big[\Big(\int \frac{d^4k_3}{(2\pi)^4}\bar{u}^{s_a}\left[8(\not{k}_3 + \not{k}_a)m_j - 8\not{k}_b m_j \not{\epsilon}^{\lambda}u^{s_i}\right] \\
&\Big(((k_a + k_3)^2 - m_j^2 + i\epsilon)((-k_3 - k_a + k_b)^2 - m_j^2 + i\epsilon)(k_3^2 - m_W^2 + i\epsilon)\Big)^{-1}\Big]
\end{aligned}
\tag{83}
$$

where the explicit calculation can be found at the appendix equation (154) and (155).

3.4 The Feynman parametrization for the denominators

For the evaluation of the loop integrals, we now introduce the Feynman parametrization. And then we will apply it to the four different denominators. We can see the formulation as follows

$$\frac{1}{AB^n} = \int_0^1 dx\,dy\,\delta(x+y-1)\frac{ny^{n-1}}{\left[xA+yB\right]^{n+1}} \tag{84}$$

$$\frac{1}{A_1\cdots A_n} = (n-1)!\int_0^1 \cdots dx_1 \int_0^1 dx_n \frac{\delta(1-\sum_{k=1}^n x_k)}{(\sum_{k=1}^n x_k A_k)^n} \tag{85}$$

in the integral after the Feynman parametrization we will have dx_1, dx_2 and dx_3 for the terms wherein the previous pages we had the same notation for the positions. But from now on it is for the integral of the parameterized values.

3.4.1 Wick rotation for the momentum

After using the Feynman parametrization we are left with one integral over the momentum k^μ that is in the Minkowski space. Therefore, we use the Wick rotation to write momentum in the Euclidean space.

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-\Delta+i\epsilon)^n} \tag{86}$$

where the Δ is the terms in the denominator that does not contain momentum k. This integral has poles at $k_0 = \sqrt{\vec{k}^2+\Delta+i\epsilon}$ and $k_0 = -\sqrt{\vec{k}^2+\Delta+i\epsilon}$. We substitute $k_0 \to ik_0$, so that $k^2 \to -k_0^2 - \vec{k}^2 = -k_E^2$ where $k_E^2 = k_0^2 + \vec{k}^2$ is now the Euclidean momentum. This process is called the Wick rotation. From now on, we will not need the $i\epsilon$ term anymore. So, we set it to be zero. Using the formula for the surface area of the Euclidean 4-sphere $\int \Omega_4 = 2\pi^2$ we can evaluate the rest of the integrals.

3.4.2 4 Denominators (that represents 4 different diagrams)

To be able to calculate the loop integrals, first we must use the Feynman parameters to write the denominators as in the steps below. Wick rotation which is going to help us write the integral in polar coordinates where at the end we will have not d^4k but a simpler integral.

$$\begin{aligned}
\text{Denom}_{\psi.\phi.\phi} =& x_2((k_3 - k_b + k_a)^2 - m_j^2 + i\epsilon) \\
&+ (x_1 - x_2)((k_3 - k_b)^2 - m_\phi^2 + i\epsilon) + (1 - x_1)(k_3^2 - m_\phi^2 + i\epsilon) \\
&= x_2(k_3^2 + k_b^2 + k_a^2 - 2k_3.(k_b - k_a) - 2k_a.k_b - m_j^2 + i\epsilon) \\
&+ (x_1 - x_2)(k_3^2 + k_b^2 - 2k_3.k_b - m_\phi^2 + i\epsilon) + (1 - x_1)(k_3^2 - m_\phi^2 + i\epsilon) \\
&= k_3^2(x_2 + x_1 - x_2 + 1 - x_1) - 2k_3.(x_2 k_b - x_2 k_a + (x_1 - x_2)k_b) \\
&+ x_2(k_b^2 + k_a^2 - 2k_a.k_b - m_j^2) + (x_1 - x_2)k_b^2 \\
&- (x_1 - x_2)m_\phi^2 + (1 - x_1)(-m_\phi^2) + (x_2 + x_1 - x_2 + 1 - x_1)i\epsilon \\
&= k_3^2 - 2k_3.(x_1 k_b - x_2 k_a) + x_2(k_i^2 - m_j^2) + (x_1 - x_2)k_b^2 - (1 - x_2)m_\phi^2 + i\epsilon \\
&= (k_3 - x_1 k_b + x_2 k_a)^2 \\
&- (x_1 k_b - x_2 k_a)^2 + x_2(k_i^2) + (x_1 - x_2)k_b^2 \\
&- x_2 m_j^2 - m_\phi^2 + x_2 m_\phi^2 + i\epsilon
\end{aligned} \tag{87}$$

When we write the second line of the last equation in a more expanded way we get

$$\begin{aligned}
&- (x_1 k_b - x_2 k_a)^2 + x_2(k_i^2) + (x_1 - x_2)k_b^2 \\
&= -x_1^2 k_b^2 + x_1 x_2\, 2k_b.k_a - x_2^2 k_a^2 + x_2 k_i^2 + x_1 k_b^2 - x_2 k_b^2
\end{aligned} \tag{88}$$

And when we write the k_b and k_a momentum in terms of k_i, and since they are also the momentum of outer going particles, we can write them in terms of their masses.

$$2k_b.k_a = (k_a - k_b)^2 - k_a^2 - k_b^2 = k_i^2 - k_a^2 - k_b^2 = m_i^2 - m_a^2 - m_b^2 \tag{89}$$

for the integration $k_3 \to k$ with $k = k_3 - x_1 k_b + x_2 k_a$ or $k_3 = k + x_1 k_b - x_2 k_a$.

We can write the integral of the neutrino-Higgs-Higgs term now as

$$\begin{aligned}
T_{1^{st}\psi.\phi.\phi} =& (2\pi)^4 \delta(k_a - k_b - k_i)\Big(Y_{aj}(Y^\dagger)_{ji} + (Y^\dagger)_{ja} Y_{ij}\Big) i\eta_e e \\
& \int_0^1 dx_1 \int_0^{x_1} dx_2 \int \frac{d^4k}{(2\pi)^4} \bar{u}^{s_a}(-\not{k} - x_1(\not{k}_a - \not{k}_i) + x_2 \not{k}_a - \not{k}_i - m_j)u^{s_i} \\
& i(-k - x_1 k_b + x_2 k_a + k_b).\epsilon^\lambda \\
& \Big[k^2 - x_1^2 m_b^2 + x_1 x_2(m_i^2 - m_a^2 - m_b^2) - x_2^2 m_a^2 + x_2 m_i^2 + x_1 m_b^2 - x_2 m_b^2 \\
& - x_2 m_j^2 - m_\phi^2 + x_2 m_\phi^2 + i\epsilon\Big]^{-3}
\end{aligned} \tag{90}$$

and we can introduce the term Δ where

$$\begin{aligned}
\Delta &= -x_1^2 m_b^2 + x_1 x_2(m_i^2 - m_a^2 - m_b^2) - x_2^2 m_a^2 + x_2 m_i^2 + x_1 m_b^2 - x_2 m_b^2 \\
&\quad - x_2 m_j^2 - m_\phi^2 + x_2 m_\phi^2 \\
&= x_1 x_2(m_i^2 - m_a^2) - x_2^2 m_a^2 + x_2 m_i^2 - x_2 m_j^2 - m_\phi^2 + x_2 m_\phi^2
\end{aligned} \tag{91}$$

when we use, that $m_b = m_\gamma$ and we know the photon is mass less.

Then the integral becomes

$$
\begin{aligned}
T_{1^{st}\psi.\phi.\phi} &= (2\pi)^4\delta(k_a - k_b - k_i)\Big(Y_{aj}(Y^\dagger)_{ji} + (Y^\dagger)_{ja}Y_{ij}\Big)i\eta_e e \\
&\int_0^1 dx_1 \int_0^{x_1} dx_2 \int \frac{d^4k}{(2\pi)^4}\bar{u}^{s_a}(-\not{k} - x_1(m_a - m_i) + x_2 m_a - m_i - m_j)u^{s_i} \\
&\qquad i(-k - x_1 k_b + x_2 k_a + k_b).\epsilon^\lambda\Big[k^2 + \Delta + i\epsilon\Big]^{-3} \\
&= (2\pi)^4\delta(k_a - k_b - k_i)Y_{j_1\ell_1}(Y^\dagger)_{j_3\ell_3}\delta_{i\ell_3}\delta_{aj_1}\delta_{\lambda\lambda_b}i\eta_e e \\
&\int_0^1 dx_1 \int_0^{x_1} dx_2 \int \frac{d^4k}{(2\pi)^4}\Big\{\bar{u}^{s_a}\gamma_\mu u^{s_i} i\epsilon^\lambda_\nu \frac{k^\mu k^\nu}{[k^2 + \Delta + i\epsilon]^3} \\
&\quad + (-x_1(m_a - m_i) + x_2 m_a - m_i - m_j)\bar{u}^{s_a}u^{s_i} i x_2(k_a.\epsilon^\lambda)\Big[k^2 + \Delta + i\epsilon\Big]^{-3} \\
&\quad + (-x_1(m_a - m_i) + x_2 m_a - m_i - m_j)\bar{u}^{s_a}u^{s_i} i(1 - x_1)(k_b.\epsilon^\lambda)\Big[k^2 + \Delta + i\epsilon\Big]^{-3}\Big\}
\end{aligned}
\tag{92}
$$

When we shift $k_0 \to ik_0$, so that $k^2 \to -k_0^2 - \vec{k}^2 = -k_E^2$, we get our Euclidean momentum. When we look at the poles, we could see that it is at $k_0 = \pm\sqrt{\vec{k}^2 + \Delta} \mp i\epsilon$. After shifting the terms and using $k_b.\epsilon^\lambda = 0$, our integral becomes

$$
\begin{aligned}
T_{1^{st}\psi.\phi.\phi} =& (2\pi)^4\delta(k_a - k_b - k_i)\Big(Y_{aj}(Y^\dagger)_{ji} + (Y^\dagger)_{ja}Y_{ij}\Big)i\eta_e e \\
&\frac{-i}{(2\pi)^4}\int_0^1 dx_1 \int_0^{x_1} dx_2 \int d\Omega_4 \int_0^\infty k_E^3 dk_E \Big\{\bar{u}^{s_a}\gamma_\mu u^{s_i} i\epsilon^\lambda_\nu \frac{g^{\mu\nu}(-k_E^2)}{[-k_E^2 + \Delta + i\epsilon]^3} \\
&\quad + (-x_1(m_a - m_i) + x_2 m_a - m_i - m_j)\bar{u}^{s_a}u^{s_i} i x_2(k_a.\epsilon^\lambda)\Big[-k_E^2 + \Delta + i\epsilon\Big]^{-3}\Big\}
\end{aligned}
\tag{93}
$$

Since the $\int d\Omega_4$ yields $2\pi^2$ the integral becomes

$$
\begin{aligned}
T_{1^{st}\psi.\phi.\phi} =& (2\pi)^4\delta(k_a - k_b - k_i)\Big(Y_{aj}(Y^\dagger)_{ji} + (Y^\dagger)_{ja}Y_{ij}\Big)i\eta_e e \\
&\frac{-i}{8\pi^2}\int_0^1 dx_1 \int_0^{x_1} dx_2 \int d\Omega_4 \int_0^\infty k_E^3 dk_E \Big\{\bar{u}^{s_a}\gamma_\mu u^{s_i} i\epsilon^\lambda_\nu \frac{g^{\mu\nu}(-k_E^2)}{[-k_E^2 + \Delta]^3} \\
&\quad + (-x_1(m_a - m_i) + x_2 m_a - m_i - m_j)\bar{u}^{s_a}u^{s_i} i x_2(k_a.\epsilon^\lambda)\Big[-k_E^2 + \Delta\Big]^{-3}\Big\}
\end{aligned}
\tag{94}
$$

We can write the Feynman parametrization for the integral of the other neutrino-Higgs-Higgs pair, so, the integral becomes

$$\begin{aligned}
T_{2^{nd}\psi.\phi.\phi} = &= (2\pi)^4\delta(k_a - k_b - k_i)\Big(Y_{aj}(Y^\dagger)_{ji} + (Y^\dagger)_{ja}Y_{ij}\Big)i\eta_e e \\
&\int_0^1 dx_1 \int_0^{x_1} dx_2 \int \frac{d^4k}{(2\pi)^4}\bar{u}^{s_a}(-\not{k} - x_1(\not{k}_a - \not{k}_i) + x_2\not{k}_a - \not{k}_i - m_j)u^{s_i} \\
&i(k + x_1k_b - x_2k_a).\epsilon^\lambda \\
&\Big[k^2 - x_1^2m_b^2 + x_1x_2(m_i^2 - m_a^2 - m_b^2) - x_2^2m_a^2 + x_2m_i^2 + x_1m_b^2 - x_2m_b^2 \\
&- x_2m_j^2 - m_\phi^2 + x_2m_\phi^2 + i\epsilon\Big]^{-3} \\
&= (2\pi)^4\delta(k_a - k_b - k_i)\Big(Y_{aj}(Y^\dagger)_{ji} + (Y^\dagger)_{ja}Y_{ij}\Big)i\eta_e e \\
&\int_0^1 dx_1 \int_0^{x_1} dx_2 \int \frac{d^4k}{(2\pi)^4}\bar{u}^{s_a}(-\not{k} - x_1(m_a - m_i) + x_2m_a - m_i - m_j)u^{s_i} \\
&i(k + x_1k_b - x_2k_a).\epsilon^\lambda\Big[k^2 + \Delta + i\epsilon\Big]^{-3} \\
= &(2\pi)^4\delta(k_a - k_b - k_i)\Big(Y_{aj}(Y^\dagger)_{ji} + (Y^\dagger)_{ja}Y_{ij}\Big)i\eta_e e \\
&\int_0^1 dx_1 \int_0^{x_1} dx_2 \int \frac{d^4k}{(2\pi)^4}\Big\{\bar{u}^{s_a}\gamma_\mu u^{s_i}i\epsilon_\nu^\lambda\frac{-k^\mu k^\nu}{[k^2 + \Delta + i\epsilon]^3} \\
&+ (-x_1(m_a - m_i) + x_2m_a - m_i - m_j)\bar{u}^{s_a}u^{s_i}ix_2(-k_a.\epsilon^\lambda)\Big[k^2 + \Delta + i\epsilon\Big]^{-3} \\
&+ (-x_1(m_a - m_i) + x_2m_a - m_i - m_j)\bar{u}^{s_a}u^{s_i}i(x_1)(k_b.\epsilon^\lambda)\Big[k^2 + \Delta + i\epsilon\Big]^{-3}\Big\}
\end{aligned} \tag{95}$$

And after the Wick rotations, we get

$$\begin{aligned}
T_{2^{nd}\psi.\phi.\phi} =&(2\pi)^4\delta(k_a - k_b - k_i)\Big(Y_{aj}(Y^\dagger)_{ji} + (Y^\dagger)_{ja}Y_{ij}\Big)i\eta_e e \\
&\frac{-i}{8\pi^2}\int_0^1 dx_1 \int_0^{x_1} dx_2 \int d\Omega_4 \int_0^\infty k_E^3dk_E\Big\{\bar{u}^{s_a}\gamma_\mu u^{s_i}i\epsilon_\nu^\lambda\frac{g^{\mu\nu}(k_E^2)}{[-k_E^2 + \Delta]^3} \\
&+ (-x_1(m_a - m_i) + x_2m_a - m_i - m_j)\bar{u}^{s_a}u^{s_i}ix_2(-k_a.\epsilon^\lambda)\Big[-k_E^2 + \Delta\Big]^{-3}\Big\}
\end{aligned} \tag{96}$$

The other denominator that represents the diagram that has one charged two fermions and a charged Higgs is shown below

$$\begin{aligned}
\mathrm{Denom}_{\psi.\psi.\phi} =& x_2\big((k_a+k_3)^2-m_j^2+i\epsilon\big)+(x_1-x_2)((-k_b+k_a+k_3)^2-m_j^2+i\epsilon) \\
& +(1-x_1)(k_3^2-m_\phi^2+i\epsilon) \\
=& x_2(k_a^2+k_3^2+2k_ak_3-m_j^2+i\epsilon) \\
& +(x_1-x_2)(k_b^2+k_a^2+k_3^2-2k_b(k_3-k_a)+2k_ak_3-m_j^2+i\epsilon) \\
& (1-x_1)(k_3^2-m_\phi^2+i\epsilon) \\
=& k_3^2(x_2+x_1-x_2+1-x_1)+(x_1-x_2)(k_b^2+k_a^2-2k_bk_a-2k_bk_3-m_j^2) \\
& -(1-x_1)m_\phi^2+i\epsilon(x_1-x_2+1-x_1+x_2) \\
=& k_3^2-(x_1-x_2)m_j^2-x_2m_j^2-(1-x_1)m_\phi^2+i\epsilon \\
& +(x_1-x_2)(k_b^2+k_a^2-2k_bk_a-2k_bk_3+2k_ak_3) \\
=& k_3^2-(x_1-x_2)m_j^2-x_2m_j^2-(1-x_1)m_\phi^2+i\epsilon \\
& +(x_1-x_2)(k_i^2)+k_3(x_1-x_2)(-2k_b+2k_a) \\
=& \big(k_3-(x_1-x_2)(k_a-k_b)\big)^2 \\
& -\big((x_1-x_2)(k_a-k_b)\big)^2-(x_1-x_2)m_j^2-x_2m_j^2 \\
& +(x_1-x_2)k_i^2-m_\phi^2+x_1m_\phi^2+i\epsilon
\end{aligned} \tag{97}$$

When we expand the terms that do not have a mass term we get

$$\begin{aligned}
& -\big((x_1-x_2)(k_a-k_b)\big)^2+(x_1-x_2)k_i^2 \\
& =-x_1k_a^2+x_2k_a^2-x_1k_b^2+x_2k_b^2+2k_ak_bx_1-2k_ak_bx_2-x_1k_i^2+x_2k_i^2
\end{aligned} \tag{98}$$

for the integration of the fermion-fermion-Higgs term $k_3 \to k$ with $k = k_3-(x_1-x_2)(k_a-k_b)$ or $k_3 = k+(x_1-x_2)(k_a-k_b)$. And we get

$$\begin{aligned}
T_{\psi.\psi.\phi} =& (2\pi)^4\delta(k_a-k_b-k_i)\Big(Y_{aj}(Y^\dagger)_{ji}+(Y^\dagger)_{ja}Y_{ij}\Big)i\eta_e e \\
& \int_0^1 dx_1\int_0^1 dx_2\int\frac{d^4k_3}{(2\pi)^4}\bar{u}^{s_a}(-\not{k}_a-\not{k}-(x_1-x_2)(\not{k}_a-\not{k}_b)-m_j) \\
& \gamma^\mu(-\not{k}-(x_1-x_2)(\not{k}_a-\not{k}_b)-\not{k}_a+\not{k}_b-m_j)u^{s_i}\epsilon_\mu^\lambda \\
& \Big[\big(k+(x_1-x_2)(k_a-k_b)-(x_1-x_2)(k_a-k_b)\big)^2 \\
& -\big((x_1-x_2)(k_a-k_b)\big)^2-(x_1-x_2)m_j^2-x_2m_j^2+(x_1-x_2)k_i^2-m_\phi^2+x_1m_\phi^2+i\epsilon\Big]^{-3}
\end{aligned} \tag{99}$$

once we use the relation $m_b = m_\gamma$, and use the equation (98), we can write the integral as

$$\begin{aligned}
T_{\psi.\psi.\phi} =& (2\pi)^4\delta(k_a-k_b-k_i)\Big(Y_{aj}(Y^\dagger)_{ji}+(Y^\dagger)_{ja}Y_{ij}\Big)i\eta_e e \\
& \int_0^1 dx_1\int_0^1 dx_2\int\frac{d^4k_3}{(2\pi)^4}\bar{u}^{s_a}(-m_a-\not{k}-(x_1-x_2)(m_a)-m_j) \\
& \gamma^\mu(-\not{k}-(x_1-x_2)(m_a)-m_a-m_j)u^{s_i}\epsilon_\mu^\lambda\Big[k^2+\Delta+i\epsilon\Big]^{-3}
\end{aligned} \tag{100}$$

where

$$\begin{aligned}\Delta &= -x_1k_a^2 + x_2k_a^2 - x_1k_b^2 + x_2k_b^2 + 2k_ak_bx_1 - 2k_ak_bx_2 - x_1k_i^2 + x_2k_i^2 \\ &- (x_1 - x_2)m_j^2 - x_2m_j^2 - m_\phi^2 + x_1m_\phi^2 \\ &= -2x_1m_a^2 + 2x_2m_a^2 - x_1m_j + x_2m_j^2 - x_2m_j^2 - m_\phi^2 + x_1m_\phi^2\end{aligned} \tag{101}$$

So, our integral becomes

$$\begin{aligned}T_{\psi.\psi.\phi} =&(2\pi)^4\delta(k_a - k_b - k_i)\Big(Y_{aj}(Y^\dagger)_{ji} + (Y^\dagger)_{ja}Y_{ij}\Big)i\eta_e e \\ &\int_0^1 dx_1 \int_0^1 dx_2 \int \frac{d^4k_3}{(2\pi)^4}\Big\{\bar{u}^{s_a}\gamma^\mu u^{s_i}\epsilon_\mu^\lambda \frac{k^\mu k^\nu}{\Big[k^2 + \Delta + i\epsilon\Big]^{-3}} \\ &+ \big(-m_a - (x_1 - x_2)m_a - m_j\big)^2 \bar{u}^{s_a}\gamma^\mu u^{s_i}\epsilon_\mu^\lambda\Big[k^2 + \Delta + i\epsilon\Big]^{-3}\Big\}\end{aligned} \tag{102}$$

After the Wick rotations we get

$$\begin{aligned}T_{\psi.\psi.\phi} =&(2\pi)^4\delta(k_a - k_b - k_i)\Big(Y_{aj}(Y^\dagger)_{ji} + (Y^\dagger)_{ja}Y_{ij}\Big)i\eta_e e \\ &\frac{-i}{8\pi^2}\int_0^1 dx_1 \int_0^1 dx_2 \int_0^\infty k_E^3 dk_E \Big\{\bar{u}^{s_a}\gamma^\mu u^{s_i}\epsilon_\mu^\lambda \frac{g^{\mu\nu}(-k_E^2)}{\Big[-k_E^2 + \Delta\Big]^{-3}} \\ &+ \big(-m_a - (x_1 - x_2)m_a - m_j\big)^2 \bar{u}^{s_a}\gamma^\mu u^{s_i}\epsilon_\mu^\lambda\Big[-k_E^2 + \Delta\Big]^{-3}\Big\}\end{aligned} \tag{103}$$

Since the rest of the two denominators have the same form of these two, fermion-fermion-w boson or fermion-w boson-w boson, we can immediately write them. The only different term would be the non-fermion particle terms. We can write them below and make the calculations of Feynman parametrization and the Wick rotation for the loop integrals.

$$\begin{aligned}\text{Denom}_{\psi.W^\pm.W^\pm} =&(k_3 - x_1k_b + x_2k_a)^2 \\ &- (x_1k_b - x_2k_a)^2 + x_2(k_i^2) + (x_1 - x_2)k_b^2 \\ &- x_2m_j^2 - m_W^2 + x_2m_W^2 + i\epsilon\end{aligned} \tag{104}$$

We can write our term as

$$\begin{aligned}T_{\psi.W^\pm.W^\pm} =&(2\pi)^4\delta(k_a - k_b - k_i)8i\eta g V_{aj}V_{ij} \\ &\int_0^1 dx_1 \int_0^1 dx_2 \int \frac{d^4k_3}{(2\pi)^4} \\ &\big(\bar{u}^{s_a}\gamma^\mu(-\not{k} - x_1(\not{k}_a - \not{k}_i) + x_2\not{k}_a - \not{k}_i - m_j)\gamma^\nu u^{s_i}\big) \\ &\big(\epsilon_\mu^{\lambda_b} i(k + x_1k_b - x_2k_a)_\nu - \epsilon_\nu^{\lambda_b} i(-k - x_1k_b + x_2k_a)_\mu\big) \\ &\Big[k^2 + \Delta + i\epsilon\Big]^{-3}\end{aligned} \tag{105}$$

We can use the commutation relation of γ matrix. So, the equation (105) becomes

$$\begin{aligned}T_{\psi.W^{\pm}.W^{\pm}} =&(2\pi)^4\delta(k_a-k_b-k_i)8i\eta g V_{aj}V_{ij}\\&\int_0^1 dx_1\int_0^1 dx_2\int\frac{d^4k_3}{(2\pi)^4}\\&-i\bar{u}^{s_a}\big(2k_3^2\not{\epsilon}^\lambda+2\not{\epsilon}^\lambda(k_ik_3)+2\not{k}_3(k_i\epsilon^\lambda)\\&+2m_j(\epsilon^\lambda k_3)-2(\epsilon^\lambda k_3)\not{k}_i\big)u^{s_i}\\&\Big[k^2+\Delta+i\epsilon\Big]^{-3}\end{aligned}\tag{106}$$

So, after the Wick rotation, the equation looks like

$$\begin{aligned}T_{\psi.W^{\pm}.W^{\pm}} =&(2\pi)^4\delta(k_a-k_b-k_i)8i\eta g V_{aj}V_{ij}\\&\int_0^1 dx_1\int_0^1 dx_2\int_0^\infty k_E^3dk_E\\&-i\bar{u}^{s_a}\big(2(k-x_2k_a)^2\not{\epsilon}^\lambda+2\not{\epsilon}^\lambda(k_i(k-x_2k_a))+2(-x_2\not{k}_a)(k_i\epsilon^\lambda)\\&+2m_j(\epsilon^\lambda(k-x_2k_a))-2(\epsilon^\lambda(k-x_2k_a))\not{k}_i\big)u^{s_i}\\&\Big[k_E^2+\Delta\Big]^{-3}\end{aligned}\tag{107}$$

And the last denominator can be written as

$$\begin{aligned}\text{Denom}_{\psi.\psi.W^{\pm}} =&\big(k_3-(x_1-x_2)(k_a-k_b)\big)^2\\&-\big((x_1-x_2)(k_a-k_b)\big)^2-(x_1-x_2)m_j^2-x_2m_j^2\\&+(x_1-x_2)k_i^2-m_W^2+x_1m_W^2+i\epsilon\end{aligned}\tag{108}$$

where its numerator looks like

$$\begin{aligned}T_{\psi.\psi.W^{\pm}} =&(2\pi)^4\delta(k_a-k_b-k_i)V_{aj}V_{ij}\\&\Big[\Big(\int\frac{d^4k_3}{(2\pi)^4}\bar{u}^{s_a}\big[8(\not{k}_3+\not{k}_a)m_j-8\not{k}_bm_j\not{\epsilon}^\lambda u^{s_i}\big]\\&\Big(k^2+\Delta+i\epsilon\Big)^{-1}\Big]\end{aligned}\tag{109}$$

and after the Wick rotation where we substitute $k_3 = k+(x_1-x_2)(k_a-k_b)$, the equation looks like

$$\begin{aligned}T_{\psi.\psi.W^{\pm}} =&(2\pi)^4\delta(k_a-k_b-k_i)V_{aj}V_{ij}\\&\Big[\Big(\frac{-i}{8\pi^2}\int_0^1 dx_1\int_0^1 dx_2\int_0^\infty k_E^3dk_E\\&\bar{u}^{s_a}\big[8(\not{k}+(x_1-x_2)\not{k}_a+\not{k}_a)\big]m_j\not{\epsilon}^\lambda u^{s_i}\big]\\&\Big(k_E^2+\Delta\Big)^{-1}\Big]\end{aligned}\tag{110}$$

3.4.3 Loop integrals for the left-over momentum

We now have written all the terms with their propagators that have one left-over momenta. We used the Feynman parametrization to solve the last integral after using the Wick rotation. Now, we can integrate the equation to find the amplitude of our particle interaction. We can seperate our terms by the ones with the Yukawa coupling constants using the notation for the equation T_{1-2-Y}, and terms with the product of mixing matrices using the notation T_{1-2-V}. Starting with the former as

$$\begin{aligned}
T_{1-2-Y} =&(2\pi)^4\delta(k_a-k_b-k_i)\Big(Y_{aj}(Y^\dagger)_{ji}+(Y^\dagger)_{ja}Y_{ij}\Big)i\eta_e e \\
&\frac{-i}{8\pi^2}\int_0^1 dx_1\int_0^{x_1}dx_2\int_0^\infty k_E^3 dk_E\Big\{\bar{u}^{s_a}\gamma_\mu u^{s_i} i\epsilon_\nu^\lambda\frac{g^{\mu\nu}(-k_E^2)}{[-k_E^2+\Delta]^3} \\
&\quad+(-x_1(m_a-m_i)+x_2m_a-m_i-m_j)\bar{u}^{s_a}u^{s_i}ix_2(k_a.\epsilon^\lambda)\Big[-k_E^2+\Delta\Big]^{-3}\Big\} \\
&+(2\pi)^4\delta(k_a-k_b-k_i)\Big(Y_{aj}(Y^\dagger)_{ji}+(Y^\dagger)_{ja}Y_{ij}\Big)i\eta_e e \\
&\frac{-i}{8\pi^2}\int_0^1 dx_1\int_0^{x_1}dx_2\int_0^\infty k_E^3 dk_E\Big\{\bar{u}^{s_a}\gamma_\mu u^{s_i} i\epsilon_\nu^\lambda\frac{g^{\mu\nu}(k_E^2)}{[-k_E^2+\Delta]^3} \\
&\quad+(-x_1(m_a-m_i)+x_2m_a-m_i-m_j)\bar{u}^{s_a}u^{s_i}ix_2(-k_a.\epsilon^\lambda)\Big[-k_E^2+\Delta\Big]^{-3}\Big\} \\
&+(2\pi)^4\delta(k_a-k_b-k_i)\Big(Y_{aj}(Y^\dagger)_{ji}+(Y^\dagger)_{ja}Y_{ij}\Big)i\eta_e e \\
&\frac{-i}{8\pi^2}\int_0^1 dx_1\int_0^1 dx_2\int_0^\infty k_E^3 dk_E\Big\{\bar{u}^{s_a}\gamma^\mu u^{s_i}\epsilon_\mu^\lambda\frac{-k_E^2}{\Big[-k_E^2+\Delta\Big]^{-3}} \\
&+\big(-m_a-(x_1-x_2)m_a-m_j\big)^2\bar{u}^{s_a}\gamma^\mu u^{s_i}\epsilon_\mu^\lambda\Big[-k_E^2+\Delta\Big]^{-3}\Big\}
\end{aligned} \tag{111}$$

We can use the following relation to integrate with the boundaries from $0\to\infty$

$$\begin{aligned}
(z-\Delta)^2 &= z^2-2z\Delta+\Delta^2 \\
z^2 &= (z-\Delta)^2+2z\Delta+\Delta^2
\end{aligned} \tag{112}$$

and for the $d(k_E^2)$ we can make the substitution as

$$\begin{aligned}
k_E^2 &= z \\
&= d(k_E^2) = dz \\
&= 2k_E dk_E = dz \\
dk_E &= \frac{dz}{2k_E}
\end{aligned} \tag{113}$$

So, we can re-write our equation as

$$
\begin{aligned}
T_{1-2-Y} =&(2\pi)^4\delta(k_a-k_b-k_i)\Big(Y_{aj}(Y^\dagger)_{ji}+(Y^\dagger)_{ja}Y_{ij}\Big)i\eta_e e\\
&\frac{-i}{8\pi^2}\int_0^1 dx_1\int_0^{x_1}dx_2\int_0^{\text{cut off}}\frac{1}{2}\Big\{\bar{u}^{s_a}\gamma_\mu u^{s_i}i\epsilon_\nu^\lambda g^{\mu\nu}\Big[\frac{dz}{[z-\Delta]}+\frac{2\Delta dz}{[z-\Delta]^2}+\frac{3\Delta^2 dz}{[z-\Delta]^3}\Big]\\
&\quad-(-x_1(m_a-m_i)+x_2m_a-m_i-m_j)\bar{u}^{s_a}u^{s_i}ix_2(k_a.\epsilon^\lambda)dz\Big[z-\Delta\Big]^{-3}\Big\}\\
&+(2\pi)^4\delta(k_a-k_b-k_i)\Big(Y_{aj}(Y^\dagger)_{ji}+(Y^\dagger)_{ja}Y_{ij}\Big)i\eta_e e\\
&\frac{-i}{8\pi^2}\int_0^1 dx_1\int_0^{x_1}dx_2\int_0^{\text{cut off}}\frac{1}{2}\Big\{\bar{u}^{s_a}\gamma_\mu u^{s_i}i\epsilon_\nu^\lambda g^{\mu\nu}\Big[-\frac{dz}{[z-\Delta]}-\frac{2\Delta dz}{[z-\Delta]^2}-\frac{3\Delta^2 dz}{[z-\Delta]^3}\Big]\\
&\quad+(-x_1(m_a-m_i)+x_2m_a-m_i-m_j)\bar{u}^{s_a}u^{s_i}ix_2(k_a.\epsilon^\lambda)dz\Big[z^2-\Delta\Big]^{-3}\Big\}\\
&+(2\pi)^4\delta(k_a-k_b-k_i)\Big(Y_{aj}(Y^\dagger)_{ji}+(Y^\dagger)_{ja}Y_{ij}\Big)i\eta_e e\\
&\frac{-i}{8\pi^2}\int_0^1 dx_1\int_0^1 dx_2\int_0^{\text{cut off}}\frac{1}{2}\Big\{\bar{u}^{s_a}\gamma_\mu u^{s_i}\epsilon_\nu^\lambda g^{\mu\nu}\Big[\frac{dz}{[z-\Delta]}+\frac{2\Delta dz}{[z-\Delta]^2}+\frac{3\Delta^2 dz}{[\Delta-z]^3}\Big]\\
&-\Big(-m_a-(x_1-x_2)m_a-m_j\Big)^2\bar{u}^{s_a}\gamma_\mu u^{s_i}\epsilon_\nu^\lambda dz\Big[z-\Delta\Big]^{-3}\Big\}
\end{aligned}
\tag{114}
$$

We can see that most of the terms cancels due to the opposite signs, so we only have

$$
\begin{aligned}
T_{1-2-Y} =&(2\pi)^4\delta(k_a-k_b-k_i)\Big(Y_{aj}(Y^\dagger)_{ji}+(Y^\dagger)_{ja}Y_{ij}\Big)i\eta_e e\\
&\frac{-i}{8\pi^2}\int_0^1 dx_1\int_0^1 dx_2\int_0^{\text{cut off}}\frac{1}{2}\Big\{\bar{u}^{s_a}\gamma_\mu u^{s_i}\epsilon_\nu^\lambda\Big[\frac{dz}{[z-\Delta]}+\frac{2\Delta dz}{[z-\Delta]^2}+\frac{3\Delta^2 dz}{[z-\Delta]^3}\Big]\\
&-\Big(-m_a-(x_1-x_2)m_a-m_j\Big)^2\bar{u}^{s_a}\gamma_\mu u^{s_i}\epsilon_\nu^\lambda dz\Big[z-\Delta\Big]^{-3}\Big\}
\end{aligned}
\tag{115}
$$

where we used the $g^{\mu\nu}$ on ϵ_ν^λ and lowered the index and sum with the index of γ^μ we get $\not\epsilon^\lambda$. When we integrate over z we can see the eqaution

$$
\begin{aligned}
T_{1-2-Y} =&\delta(k_a-k_b-k_i)\Big(Y_{aj}(Y^\dagger)_{ji}+(Y^\dagger)_{ja}Y_{ij}\Big)i\eta_e e\\
&-i\pi^2\int_0^1 dx_1\int_0^1 dx_2\Big\{\bar{u}^{s_a}\not\epsilon^\lambda u^{s_i}\Big[\ln(z-\Delta)+\frac{2\Delta}{[\Delta-z]}+\frac{3\Delta^2}{2[\Delta-z]^2}\Big]\\
&-2\Big(-m_a-(x_1-x_2)m_a-m_j\Big)^2\bar{u}^{s_a}\not\epsilon^\lambda u^{s_i}\Big[\Delta-z\Big]^{-2}\Big\}\Big|_0^{\text{cut off}}
\end{aligned}
\tag{116}
$$

Once we put off the cut off values to the terms that have z, in the denominator goes to infinity, so the whole term goes to zero.

$$
\begin{aligned}
T_{1-2-Y} =&\delta(k_a-k_b-k_i)\Big(Y_{aj}(Y^\dagger)_{ji}+(Y^\dagger)_{ja}Y_{ij}\Big)i\eta_e e\\
&(-i\pi^2)\int_0^1 dx_1\int_0^1 dx_2\Big\{\bar{u}^{s_a}\not\epsilon^\lambda u^{s_i}\Big[\ln\frac{(\text{cut off}-\Delta)}{\Delta}-2-\frac{3}{2}\Big]\\
&+2\Big(-m_a-(x_1-x_2)m_a-m_j\Big)^2\bar{u}^{s_a}\not\epsilon^\lambda u^{s_i}\Delta^{-2}\Big\}
\end{aligned}
\tag{117}
$$

And below, we write the terms with the product of mixing matrices. After the substitution for k_E^2 to z, and summation of the indices and the loop integrals, this is how the equation looks like

$$
\begin{aligned}
T_{1-2-V} =& \delta(k_a - k_b - k_i) 8 i\eta g V_{aj} V_{ij} \\
& \frac{-i\pi^2}{2} \int_0^1 dx_1 \int_0^1 dx_2 \\
& - i\bar{u}^{s_a} \big(2(-x_2 k_a)^2 \not\epsilon^\lambda + 2\not\epsilon^\lambda (k_i(-x_2 k_a)) + 2(-x_2 \not k_a)(k_i \epsilon^\lambda) \\
& + 2m_j(\epsilon^\lambda(-x_2 k_a)) - 2(\epsilon^\lambda(-x_2 k_a))\not k_i \big) u^{s_i} \big[\Delta\big]^{-3} \\
& + 4\delta(k_a - k_b - k_i) V_{aj} V_{ij} \\
& \frac{-i\pi^2}{2} \int_0^1 dx_1 \int_0^1 dx_2 \bar{u}^{s_a} \big[8((x_1 - x_2)\not k_a + \not k_a) \big] m_j \not\epsilon^\lambda u^{s_i} [\Delta]^{-2}
\end{aligned} \tag{118}
$$

Once we solved the loop integrals, we have our amplitude with δ functions and their coefficients. As we can see from the terms; to get the square of the matrix element we have one type of spinor order which is $\bar{u}^{s_a} \not\epsilon^\lambda u^{s_i}$ with different coefficients. We can go the next step now, writing the $|\mathcal{M}|^2$ by summing over the average of the spinors by their complex conjugates as we can see in the next section.

3.5 Square of the matrix elements

We can now write the absolute square of it $|\mathcal{M}|^2$ by once again using the Casimir's trick and get the average over the sum of the indices. In this case in total, we have four spinors in the numerator not eight as we had in the previous chapter. We also have gamma matrices and polarization vectors too in them. So, the trace of the matrix elements will look different once again.

$$
\sum_{\text{spins}} = \bar{u}^{s_a}((c_1 \not\epsilon^\lambda) + c_2(\not k_a \not\epsilon^\lambda) u^{s_i}) \bar{u}^{s_i} (c_1^* \not\epsilon^{\lambda *} + c_2^* \not\epsilon^{\lambda *}) u^{s_a} \tag{119}
$$

Remember, this is one of the three terms that is the part of $|\mathcal{M}|^2$.

3.6 Trace of the matrix elements

Once we wrote the square of the matrix element, we can get to the step of taking the traces of the matrices. The difference of the traces between the first chapter and this one is the number of the averaged spinor. We have 4 spinors in total, not 8 with different terms in between such as the product of γ matrices and momenta. Therefore, for the terms that have γ matrix products in between the spinors, we can write the 4×4 γ matrices with a basis of 16 matrices to parameterize them and find their coefficients.

We can calculate the trace of the element of

$$
\bar{u}^{s_a} \big[8((x_1 - x_2)\not k_a + \not k_a) \big] m_j \not\epsilon^\lambda u^{s_i} \tag{120}
$$

After the sum trace of the element would look like this

$$Tr[(c_1 \not{k}_a \not{\epsilon}^\lambda + c_2 \not{k}_a \not{\epsilon}^\lambda)(k_i + m_i)(c_1^* \not{k}_a \not{\epsilon}^{\lambda^*} + c_2^* \not{k}_a \not{\epsilon}^{\lambda^*})(k_a + m_a)] \tag{121}$$

We multiply everything out and see

$$\begin{aligned} &c_1 \not{k}_a \not{\epsilon}^\lambda k_i + c_1 \not{k}_a \not{\epsilon}^\lambda m_i + c_1 \not{k}_a \not{\epsilon}^\lambda c_1^* \not{k}_a \not{\epsilon}^{\lambda^*} + c_1 \not{k}_a \not{\epsilon}^\lambda c_2^* \not{k}_a \not{\epsilon}^{\lambda^*} + c_1 \not{k}_a \not{\epsilon}^\lambda k_a + c_1 \not{k}_a \not{\epsilon}^\lambda m_a \\ &+ c_2 \not{k}_a \not{\epsilon}^\lambda k_i + c_2 \not{k}_a \not{\epsilon}^\lambda m_i + c_2 \not{k}_a \not{\epsilon}^\lambda c_1^* \not{k}_a \not{\epsilon}^{\lambda^*} + c_2 \not{k}_a \not{\epsilon}^\lambda c_2^* \not{k}_a \not{\epsilon}^{\lambda^*} + c_2 \not{k}_a \not{\epsilon}^\lambda k_a + c_2 \not{k}_a \not{\epsilon}^\lambda m_a \end{aligned} \tag{122}$$

And these terms yield

$$\begin{aligned} &-4c_1 \not{k}_a \not{k}_a c_1^* + -4c_2 \not{k}_a \not{k}_a c_2^* + \\ &= -4c_1^2(p_a^2) - 4c_2^2(p_a^2) \end{aligned} \tag{123}$$

Using this term we could have an idea of how the whole matrix element square looks like. We could write it as sum of two more terms that are similar to one that we show below.

$$|\mathcal{M}| = 32 V_{aj} V_{ij} \frac{-4c_1^2(p_a^2) - 4c_2^2(p_a^2)}{\Delta} \cdots \tag{124}$$

where c_1 and c_2 are constants with several mass terms.

3.7 Integral for the Decay rate to 2 particles

Since we have our matrix element square in terms of energy and momentum, we can now start integration of the phase space to find the decay rate of the sterile neutrino two a neutrino and a photon. The Golden Rule formula for this calculation is shown as

$$\Gamma = \frac{S}{32\pi^2 \hbar m_i} \int \Big(|\mathcal{M}|^2 \delta^4(p_i - p_a - p_b) \frac{d^3 p_a}{(2\pi)^3 2E_a} \frac{d^3 p_b}{(2\pi)^3 2E_b} \Big) \tag{125}$$

We can peel the $\delta^4(p_1 - p_2 - p_3)$ function as we did in the previous chapter to two delta functions $\delta(p_1^0 - p_2^0 - p_3^0)\delta^3(p_1 - p_2 - p_3)$ and substitute the terms in the δ function (energies) as

$$\frac{S}{32\pi^2 \hbar m_i} \int \Big(|\mathcal{M}|^2 \frac{\delta\big(m_i - \sqrt{p_a^2 + m_a^2} - p_b\big)}{(2\pi)^6 4 p_b \sqrt{p_a^2 + m_a^2}} \delta^{(3)}(\emptyset - p_a - p_b) d^3 p_a d^3 p_b \tag{126}$$

when we integrate over p_b that leaves integral $-p_a$. And the equation becomes

$$\frac{S}{32\pi^2 \hbar m_i} \int \Big(|\mathcal{M}|^2 \frac{\delta\big(m_i - \sqrt{p_a^2 + m_a^2} - p_a\big)}{(2\pi)^6 4 p_a \sqrt{p_a^2 + m_a^2}} d^3 p_a \tag{127}$$

For the d^3p_a integral, we can go to the spherical coordinates where we introduce a component $r = |p_a|$ and the integral goes as $d^3p_a \to r^2 \sin\theta dr d\theta d\phi$. And we have

$$\frac{S}{32\pi^2\hbar m_i}\int \Big(|\mathcal{M}|^2 \frac{\delta\big(m_i - \sqrt{r^2+m_a^2} - r\big)}{(2\pi)^6 4r\sqrt{r^2+m_a^2}} r^2 \sin\theta dr d\theta d\phi \tag{128}$$

Since there are no dependence on the integrals of ϕ and θ they yield the numerical value 4π. Then we can simplify the delta function introducing u parameter as

$$u \equiv \sqrt{r^2+m_a^2} + r \tag{129}$$

where

$$\begin{aligned} \frac{du}{dr} &= \frac{r}{\sqrt{r^2+m_a^2}} + 1 \\ &= \frac{r+\sqrt{r^2+m_a^2}}{\sqrt{r^2+m_a^2}} \\ &= \frac{u}{\sqrt{r^2+m_a^2}} \end{aligned} \tag{130}$$

So, we can write the integral as

$$\Gamma = \frac{S}{32\pi^2\hbar m_i}\int |\mathcal{M}|^2 \delta(m_i - u)\frac{r}{u}du = \frac{|\mathcal{M}|^2 S|p|}{8\pi\hbar m_i^2 c} \tag{131}$$

where the $|p|$ is the momenta of outgoing particles in case of the incoming particle m_i is larger than the mass of the outgoing particles since otherwise the δ function causes the Γ to be zero.

Having that one term out of three elements of the $|\mathcal{M}|^2 S|p|$, we at least could write one part of the total decay rate, since the other terms are independent summations with this term.

$$\begin{aligned} \Gamma &= \left(\frac{-4p_a^3c_2^2 - 4p_a^3c_1^2}{3} + \cdots\right)\frac{S\sqrt{m_i^4}}{8\pi\hbar m_i^2} \\ &= \left(\frac{-4p_a^3c_2^2 - 4p_a^3c_1^2}{3} + \cdots\right)\frac{S}{8\pi\hbar} \end{aligned} \tag{132}$$

So, this is the sterile neutrino decay rate to two particles for one of our $W^\pm$ boson propagator. The rest of the two terms could be found in the same way and summed together to find the total decay rate where that part is missing from our calculation.

4 Conclusion

In this paper, we have calculated the sterile neutrino decays in two chapters by using the interaction Hamiltonian of the Grimus Neufeld Model. In the first chapter of this paper, we investigated the sterile neutrino decay into three neutrinos. As a mediating particle we had Higgs boson and a Z boson. Once we have written the propagators, we integrated over position and the momentum to find the amplitude of the interacting particles. Then averaging over sum of the indices we have found the matrix element that was the part of the phase space integration. And when we integrated over the phase space, ignoring the mass of the outgoing particles except the Higgs and the Z boson, we can found the decay rate of the sterile neutrino to three neutrinos. We can see the complete version of the result in the appendix.

In the second chapter, sterile neutrino decay into a neutrino and a photon Once we have written the propagators and integrated over the position and the momentum, we had one leftover momentum and to solve it we had to introduce the Feynman parametrization and the Wick rotation to calculate the loop integrals. We have completed this part for one term in summation of three terms and gave one of the coefficient for the matrix element square. Taking the trace of the matrix element was still possible since after the simplifications in the terms we have found out that we had the same spinor order and the type for all the terms. We have calculated the matrix element square for the same term and used it in the integration of the phase space. And by summing the two decay rates in these two chapters, we could find the total decay rate of the sterile neutrino that we tried to calculate.

A Appendix

A.1 Square of the matrix element with spinor sums

We can see below the square of the matrix element for the 3 body decay with showing the averaged spinors.

$$
\begin{aligned}
|\mathcal{M}|^2 =& 16\frac{\left|Y_{(cb)}Y_{(ai)}\right|^2}{\left[(k_a-p_i)^2-m_\phi^2\right]^2}\bar{u}^{s_c}v^{s_b}\bar{u}^{s_a}u^{s_i}\bar{v}^{s_b}u^{s_c}\bar{u}^{s_i}u^{s_a} \\
&+16\frac{Y_{(cb)}Y_{(ai)}Y^*_{(ba)}Y^*_{(ci)}}{\left[(k_a-p_i)^2-m_\phi^2\right]\left[(k_c-p_i)^2-m_\phi^2\right]}\bar{u}^{s_c}v^{s_b}\bar{u}^{s_a}u^{s_i}\bar{u}^{s_i}u^{s_c}\bar{v}^{s_a}u^{s_b} \\
&-16\frac{Y_{(cb)}Y_{(ai)}Y^*_{(ca)}Y^*_{(bi)}}{\left[(k_a-p_i)^2-m_\phi^2\right]\left[(k_b-p_i)^2-m_\phi^2\right]}\bar{u}^{s_c}v^{s_b}\bar{u}^{s_a}u^{s_i}\bar{v}^{s_a}u^{s_c}\bar{u}^{s_i}u^{s_b} \\
&+16\frac{\left|Y_{(ba)}Y_{(ci)}\right|^2}{\left[(k_c-p_i)^2-m_\phi^2\right]^2}\bar{u}^{s_c}u^{s_i}\bar{u}^{s_b}v^{s_a}\bar{u}^{s_i}u^{s_c}\bar{v}^{s_a}u^{s_b} \\
&+16\frac{Y_{(ba)}Y_{(ci)}Y^*_{(cb)}Y^*_{(ai)}}{\left[(k_c-p_i)^2-m_\phi^2\right]\left[(k_a-p_i)^2-m_\phi^2\right]}\bar{u}^{s_c}u^{s_i}\bar{u}^{s_b}v^{s_a}\bar{v}^{s_b}u^{s_c}\bar{u}^{s_i}u^{s_a} \\
&-16\frac{Y_{(ba)}Y_{(ci)}Y^*_{(ca)}Y^*_{(bi)}}{\left[(k_c-p_i)^2-m_\phi^2\right]\left[(k_b-p_i)^2-m_\phi^2\right]}\bar{u}^{s_c}u^{s_i}\bar{u}^{s_b}v^{s_a}\bar{v}^{s_a}u^{s_c}\bar{u}^{s_i}u^{s_b} \\
&+16\frac{\left|Y_{(ca)}Y_{(bi)}\right|^2}{\left[(k_b-p_i)^2-m_\phi^2\right]^2}\bar{u}^{s_c}v^{s_a}\bar{u}^{s_b}u^{s_i}\bar{v}^{s_a}u^{s_c}\bar{u}^{s_i}u^{s_b} \\
&-16\frac{Y_{(ca)}Y_{(bi)}Y^*_{(cb)}Y^*_{(ai)}}{\left[(k_b-p_i)^2-m_\phi^2\right]\left[(k_a-p_i)^2-m_\phi^2\right]}\bar{u}^{s_c}v^{s_a}\bar{u}^{s_b}u^{s_i}\bar{v}^{s_b}u^{s_c}\bar{u}^{s_i}u^{s_a} \\
&-16\frac{Y_{(ca)}Y_{(bi)}Y^*_{(ba)}Y^*_{(ci)}}{\left[(k_b-p_i)^2-m_\phi^2\right]\left[(k_c-p_i)^2-m_\phi^2\right]}\bar{u}^{s_c}v^{s_a}\bar{u}^{s_b}u^{s_i}\bar{u}^{s_i}u^{s_c}\bar{v}^{s_a}u^{s_b} \\
&+16\frac{\left|U_{[cb]}U_{[ai]}\right|^2}{\left[(k_a-p_i)^2-m_z^2\right]^2}\bar{u}^{s_c}v^{s_b}\bar{u}^{s_a}u^{s_i}\bar{v}^{s_b}u^{s_c}\bar{u}^{s_i}u^{s_a} \\
&+16\frac{U_{[cb]}U_{[ai]}U^*_{[ba]}U^*_{[ci]}}{\left[(k_a-p_i)^2-m_z^2\right]\left[(k_c-p_i)^2-m_z^2\right]}\bar{u}^{s_c}v^{s_b}\bar{u}^{s_a}u^{s_i}\bar{u}^{s_i}u^{s_c}\bar{v}^{s_a}u^{s_b}
\end{aligned}
$$

$$
\begin{aligned}
&- 16\frac{U_{[cb]}U_{[ai]}U^*_{[ca]}U^*_{[bi]}}{\Big[(k_a-p_i)^2-m_z^2\Big]\Big[(k_b-p_i)^2-m_z^2\Big]}\bar{u}^{s_c}v^{s_b}\bar{u}^{s_a}u^{s_i}\bar{v}^{s_a}u^{s_c}\bar{u}^{s_i}u^{s_b}\\
&+ 16\frac{\Big|U_{[ba]}U_{[ci]}\Big|^2}{\Big[(k_c-p_i)^2-m_z^2\Big]^2}\bar{u}^{s_c}u^{s_i}\bar{u}^{s_b}v^{s_a}\bar{u}^{s_i}u^{s_c}\bar{v}^{s_a}u^{s_b}\\
&+ 16\frac{U_{[ba]}U_{[ci]}U^*_{[cb]}U^*_{[ai]}}{\Big[(k_c-p_i)^2-m_z^2\Big]\Big[(k_a-p_i)^2-m_z^2\Big]}\bar{u}^{s_c}u^{s_i}\bar{u}^{s_b}v^{s_a}\bar{v}^{s_b}u^{s_c}\bar{u}^{s_i}u^{s_a}\\
&- 16\frac{U_{[ba]}U_{[ci]}U^*_{[ca]}U^*_{[bi]}}{\Big[(k_c-p_i)^2-m_z^2\Big]\Big[(k_b-p_i)^2-m_z^2\Big]}\bar{u}^{s_c}u^{s_i}\bar{u}^{s_b}v^{s_a}\bar{v}^{s_a}u^{s_c}\bar{u}^{s_i}u^{s_b}\\
&+ 16\frac{\Big|U_{[ca]}U_{[bi]}\Big|^2}{\Big[(k_b-p_i)^2-m_z^2\Big]^2}\bar{u}^{s_c}v^{s_a}\bar{u}^{s_b}u^{s_i}\bar{v}^{s_a}u^{s_c}\bar{u}^{s_i}u^{s_b}\\
&- 16\frac{U_{[ca]}U_{[bi]}U^*_{[cb]}U^*_{[ai]}}{\Big[(k_b-p_i)^2-m_z^2\Big]\Big[(k_a-p_i)^2-m_z^2\Big]}\bar{u}^{s_c}v^{s_a}\bar{u}^{s_b}u^{s_i}\bar{v}^{s_b}u^{s_c}\bar{u}^{s_i}u^{s_a}\\
&- 16\frac{U_{[ca]}U_{[bi]}U^*_{[ba]}U^*_{[ci]}}{\Big[(k_b-p_i)^2-m_z^2\Big]\Big[(k_c-p_i)^2-m_z^2\Big]}\bar{u}^{s_c}v^{s_a}\bar{u}^{s_b}u^{s_i}\bar{u}^{s_i}u^{s_c}\bar{v}^{s_a}u^{s_b}
\end{aligned}
\tag{133}
$$

where the $(2\pi)^4$ and the δ functions are excluded in the equation since they are not the part of the matrix element.

A.2 Traces of the Matrices for the Higgs and the Goldstone bosons

Below we have written explicit calculation of the traces for the matrix elements of the Higgs and the Goldstone boson spinors from the terms of the second fraction.

2^{nd} term is

$$\sum_{spins} \bar{u}^{s_c} v^{s_b} \bar{u}^{s_a} u^{s_i} \bar{u}^{s_i} u^{s_c} \bar{v}^{s_a} u^{s_b} = \sum_{spins} \bar{u}^{s_a} u^{s_i} \bar{u}^{s_i} u^{s_c} \bar{u}^{s_c} v^{s_b} (-\bar{v}^{s_b} u^{s_a})$$
$$= -Tr\Big[(\not{p}_i + m_i)(\not{k}_c + m_c)(\not{k}_b - m_b)(\not{k}_a + m_a)\Big]$$
$$= -4\Big((p_i \cdot c)(b \cdot a) - (b \cdot p_i)(c \cdot a) + (p_i \cdot a)(b \cdot c)\Big)$$
$$- 4m_i m_c (b \cdot a) + 4m_i m_b (c \cdot a) - 4m_i m_a (c \cdot b) + 4m_i m_c m_b m_a$$
$$+ 4m_c m_b (p_i \cdot a) - 4m_c m_a (p_i \cdot b) + 4m_b m_a (p_i \cdot c)$$
$$= -4\Big[(m_i(E_c + m_c))\frac{1}{2}\big(m_i^2 - 2m_i E_c + m_c^2 - m_b^2 - m_a^2\big)$$
$$- (m_i(E_b + m_b))\frac{1}{2}\big(m_i^2 - 2m_i E_b + m_b^2 - m_c^2 - m_a^2\big)$$
$$+ (m_i(E_a + m_a))\frac{1}{2}\big(m_i^2 - 2m_i E_a + m_a^2 - m_b^2 - m_c^2\big)$$
$$- m_i m_c m_b m_a - m_c m_b (m_i E_a) + m_c m_a (m_i E_b)$$
$$+ m_b m_a (m_i E_c)\Big] \tag{134}$$

3^{rd} term is

$$\sum_{spins} \bar{u}^{s_c} v^{s_b} \bar{u}^{s_a} u^{s_i} \bar{v}^{s_a} u^{s_c} \bar{u}^{s_i} u^{s_b} = \sum_{spins} (-\bar{u}^{s_b} v^{s_c}) \bar{u}^{s_a} u^{s_i} \bar{v}^{s_c} u^{s_a} \bar{u}^{s_i} u^{s_b}$$
$$= -Tr[(\not{k}_a + m_a)(\not{p}_i + m_i)(\not{k}_b + m_b)(\not{k}_c - m_c)]$$
$$= -4\big[(a \cdot p_i)(b \cdot c) - (b \cdot a)(p_i \cdot c) + (a \cdot c)(b \cdot p_i)\big]$$
$$- 4m_a m_i (b \cdot c) - 4m_a m_b (p_i \cdot c)$$
$$+ 4m_a m_c (p_i \cdot b) + 4m_a m_i m_b m_c - 4m_i m_b (a \cdot c)$$
$$+ 4m_i m_c (a \cdot b) + 4m_b m_c (a \cdot p_i)$$
$$= -4\Big[(m_i(E_a + m_a))\frac{1}{2}\big(m_i^2 - 2m_i E_a + m_a^2 - m_b^2 - m_c^2\big)$$
$$- (m_i(E_c + m_c))\frac{1}{2}\big(m_i^2 - 2m_i E_c + m_c^2 - m_b^2 - m_a^2\big)$$
$$+ (m_i(E_b + m_b))\frac{1}{2}\big(m_i^2 - 2m_i E_b + m_b^2 - m_a^2 - m_c^2\big)$$
$$+ m_a m_b (m_i E_c) - m_a m_c (m_i E_b) - m_a m_i m_b m_c$$
$$- m_b m_c (m_i E_a)\Big] \tag{135}$$

4^{th} term is

$$\begin{aligned}\sum_{spins}\bar{u}^{s_c}u^{s_i}\bar{u}^{s_b}v^{s_a}\bar{u}^{s_i}u^{s_c}\bar{v}^{s_a}u^{s_b} &= \sum_{spins}\bar{u}^{s_i}u^{s_c}\bar{u}^{s_c}u^{s_i} \quad \bar{v}^{s_a}u^{s_b}\bar{u}^{s_b}v^{s_a} \\ &= Tr[(\not{p}_i + m_i)(\not{k}_c + m_c)]\cdot Tr[(\not{k}_a - m_a)(\not{k}_b + m_b)] \\ &= 4(p_i\cdot c + m_i m_c)\cdot 4(a\cdot b - m_a m_b) \\ &= 4\Big[(m_i E_c) + m_i m_c\Big]\cdot 4\Big[\frac{1}{2}\big(m_i^2 - 2m_i E_b \\ &\quad + m_b^2 - m_c^2 - m_a^2\big) - m_a m_b\Big] \end{aligned} \tag{136}$$

5^{th} term is

$$\begin{aligned}\sum_{spins}\bar{u}^{s_c}u^{s_i}\bar{u}^{s_b}v^{s_a}\bar{v}^{s_b}u^{s_c}\bar{u}^{s_i}u^{s_a} &= \sum_{spins}\bar{u}^{s_c}u^{s_i}\ \ \bar{u}^{s_i}u^{s_a}\ \ \bar{u}^{s_b}v^{s_a}\ \ \bar{v}^{s_b}u^{s_c} \\ &= \sum_{spins}\bar{u}^{s_c}u^{s_i}\ \ \bar{u}^{s_i}u^{s_a}\ \ (-\bar{u}^{s_a}v^{s_b})\ \ \bar{v}^{s_b}u^{s_c} \\ &= -Tr[(\not{p}_i + m_i)(\not{k}_a + m_a)(\not{k}_b - m_b)(\not{k}_c + m_c)] \\ &= -4\Big[(p_i\cdot a)(b\cdot c) - (b\cdot p_i)(a\cdot c) + (p_i\cdot c)(b\cdot a)\Big] \\ &\quad + 4m_i m_a(b\cdot c) - 4m_i m_b(a\cdot c) \\ &\quad + 4m_i m_c(a\cdot b) - 4m_i m_a m_b m_c \\ &\quad - 4m_a m_b(p_i\cdot c) + 4m_a m_c(p_i\cdot b) - 4m_b m_c(p_i\cdot a) \\ &= -4\Big[(m_i(E_a - m_a))\frac{1}{2}\big(m_i^2 - 2m_i E_a + m_a^2 - m_b^2 - m_c^2\big) \\ &\quad - (m_i(E_b - m_b))\frac{1}{2}\big(m_i^2 - 2m_i E_b + m_b^2 - m_a^2 - m_c^2\big) \\ &\quad + (m_i(E_c - m_c))\frac{1}{2}\big(m_i^2 - 2m_i E_c + m_c^2 - m_b^2 - m_a^2\big) \\ &\quad - m_i m_a m_b m_c - m_a m_b(m_i E_c) - m_a m_c(m_i E_b) \\ &\quad - m_b m_c(m_i E_a)\Big] \end{aligned} \tag{137}$$

6^{th} term is

$$\begin{aligned}
\sum_{spins} \bar{u}^{s_c} u^{s_i} \bar{u}^{s_b} v^{s_a} \bar{v}^{s_a} u^{s_c} \bar{u}^{s_i} u^{s_b} &= \sum_{spins} \bar{u}^{s_b} v^{s_a} \bar{v}^{s_a} u^{s_c} \bar{u}^{s_c} u^{s_i} \bar{u}^{s_i} u^{s_b} \\
&= Tr[(\not{k}_a - m_a)(\not{k}_c + m_c)(\not{p}_i + m_i)(\not{k}_b + m_b)] \\
&= 4\Big[(a \cdot c)(p_i \cdot b) - (p_i \cdot a)(c \cdot b) + (a \cdot b)(p_i \cdot c)\Big] - 4m_a m_c (p_i \cdot b) \\
&\quad - 4m_a m_i (c \cdot b) - 4m_a m_b (c \cdot p_i) - 4m_a m_c m_i m_b + 4m_c m_i (a \cdot b) \\
&\quad + 4m_c m_b (a \cdot p_i) + 4m_i m_b (a \cdot c) \\
&= 4\Big[(m_i E_b + m_i m_b)\frac{1}{2}\big(m_i^2 - 2m_i E_b + m_b^2 - m_a^2 - m_c^2\big) \\
&\quad - (m_i E_a + m_a m_i)\frac{1}{2}\big(m_i^2 - 2m_i E_a + m_a^2 - m_c^2 - m_b^2\big) \\
&\quad + (m_i E_c + 4m_c m_i)\frac{1}{2}\big(m_i^2 - 2m_i E_c + m_c^2 - m_a^2 - m_b^2\big) \\
&\quad - m_a m_c (m_i E_b) - m_a m_b (m_i E_c) - m_a m_c m_i m_b \\
&\quad + m_c m_b (m_i E_a)\Big] \qquad (138)
\end{aligned}$$

7^{th} term is

$$\begin{aligned}
\sum_{spins} \bar{u}^{s_c} v^{s_a} \bar{u}^{s_b} u^{s_i} \bar{v}^{s_a} u^{s_c} \bar{u}^{s_i} u^{s_b} &= \sum_{spins} \bar{u}^{s_c} v^{s_a} \bar{v}^{s_a} u^{s_c} \;\; \bar{u}^{s_i} u^{s_b} \bar{u}^{s_b} u^{s_i} \\
&= Tr[(\not{k}_a - m_a)(\not{k}_c + m_c)] \cdot Tr[(\not{k}_b + m_b)(\not{p}_i + m_i)] \\
&= 4(a \cdot c - m_a m_c) \cdot 4(b \cdot p_i + m_b m_i) \\
&= 4\Big[\frac{1}{2}\big(m_i^2 - 2m_i E_b + m_b^2 - m_a^2 - m_c^2\big) - m_a m_c\Big] \\
&\quad \cdot 4\Big[m_i E_b - m_b m_i\Big] \qquad (139)
\end{aligned}$$

8^{th} term is

$$\begin{aligned}
\sum_{spins} \bar{u}^{s_c} v^{s_a} \bar{u}^{s_b} u^{s_i} \bar{v}^{s_b} u^{s_c} \bar{u}^{s_i} u^{s_a} &= \sum_{spins} \bar{u}^{s_a} v^{s_c} \bar{u}^{s_b} u^{s_i} \bar{v}^{s_c} u^{s_b} \bar{u}^{s_i} u^{s_a} \\
&= Tr[(\not{k}_c - m_c)(\not{k}_b + m_b)(\not{p}_i + m_i)(\not{k}_a + m_a)] \\
&= 4\Big[\frac{1}{2}(m_i(E_a + m_a))\big(m_i^2 - 2m_i E_a + m_a^2 - m_c^2 - m_b^2\big) \\
&\quad - (m_i(E_c + m_c))\frac{1}{2}\big(m_i^2 - 2m_i E_c - m_c^2 - m_b^2 - m_a^2\big) \\
&\quad + (m_i(E_b + m_b))\frac{1}{2}\big(m_i^2 - 2m_i E_b + m_b^2 - m_c^2 - m_a^2\big) \\
&\quad - m_c m_b (m_i E_a) - m_c m_a (m_i E_b) - m_c m_b m_i m_a \\
&\quad + m_b m_a (m_i E_c)\Big] \qquad (140)
\end{aligned}$$

9^{th} term is

$$\sum_{spins} \bar{u}^{s_c} v^{s_a} \bar{u}^{s_b} u^{s_i} \bar{u}^{s_i} u^{s_c} \bar{v}^{s_a} u^{s_b} = \sum_{spins} \bar{u}^{s_i} u^{s_c} \bar{u}^{s_c} v^{s_a} \bar{v}^{s_a} u^{s_b} \bar{u}^{s_b} u^{s_i}$$
$$= Tr[(\not{p}_i + m_i)(\not{k}_c + m_c)(\not{k}_a - m_a)(\not{k}_b + m_b)]$$
$$= 4\Big[(m_i(E_c + m_c))\frac{1}{2}\big(m_i^2 - 2m_iE_c + m_c^2 - m_a^2 - m_b^2\big)$$
$$- (m_i(E_a + m_a))\frac{1}{2}\big(m_i^2 - 2m_iE_a + m_a^2 - m_c^2 - m_b^2\big)$$
$$+ (m_i(E_b + m_b))\frac{1}{2}\big(m_i^2 - 2m_iE_b + m_b^2 - m_a^2 - m_c^2\big)$$
$$- m_im_cm_am_b - m_cm_a(m_iE_b) + m_cm_b(m_iE_a)$$
$$- m_am_b(m_iE_c)\Big] \tag{141}$$

A.3 Traces of the Matrices for the Z boson Part

Below we have written explicit calculation of the traces for the matrix elements of the Z boson spinors from the terms of the second fraction.

2^{nd} term is

$$\sum_{spins} \bar{u}^{s_c} \gamma^\mu P_L v^{s_b} \bar{u}^{s_a} \gamma^\nu P_L u^{s_i} (-g_{\mu\nu}) \bar{u}^{s_i} \gamma^\lambda P_L u^{s_c} \bar{v}^{s_a} \gamma^\rho u^{s_b} (-g_{\lambda\rho})$$
$$\sum_{spins} \bar{u}^{s_a} \gamma^\mu P_L u^{s_i} \;\; \bar{u}^{s_i} \gamma^\lambda P_L u^{s_c} \;\; \bar{u}^{s_c} \gamma^\mu P_L v^{s_b} \;\; (-\bar{v}^{s_b} \gamma^\lambda P_L u^{s_a})$$
$$= -Tr\Big[(\not{p}_i + m_i)\gamma^\lambda P_L(\not{k}_c + m_c)\gamma^\mu P_L(\not{k}_b - m_b)\gamma_\lambda P_L(\not{k}_a + m_a)\gamma_\mu P_L\Big]$$
$$= -Tr\Big[\not{p}_i \gamma^\lambda P_L \not{k}_c \gamma^\mu P_L \not{k}_b \gamma_\lambda P_L \not{k}_a \gamma_\mu P_L\Big]$$
$$= -Tr[\not{p}_i(-2\not{k}_b \gamma^\mu \not{k}_c)\not{k}_a \gamma^\mu P_L]$$
$$= 2Tr[\not{p}_i \not{k}_b 4(c \cdot a)P_L]$$
$$= 8(a \cdot c) \cdot \frac{1}{2} \cdot 4(p_i \cdot k_b)$$
$$= 16(a \cdot c)(b \cdot i)$$
$$= 8(m_i^2 - 2m_im_b + m_b^2 - m_a^2 - m_c^2)(m_iE_b) \tag{142}$$

3^{rd} term is

$$
\begin{aligned}
&\sum_{spins} \bar{u}^{s_c}\gamma^{\mu}P_L v^{s_b}\bar{u}^{s_a}\gamma^{\nu}P_L u^{s_i}(-g_{\mu\nu})\bar{v}^{s_a}\gamma^{\lambda}P_L u^{s_c}\bar{u}^{s_i}\gamma^{\rho}P_L u^{s_b}(-g_{\lambda\rho}) \\
&\quad = \sum_{spins} \bar{v}^{s_c}\gamma_{\lambda}P_L u^{s_a} \;\; \bar{u}^{s_a}\gamma^{\mu}P_L u^{s_i} \;\; \bar{u}^{s_i}\gamma^{\lambda}P_L u^{s_b} \;\; (-\bar{u}^{s_b}\gamma_{\mu}P_L v^{s_c}) \\
&\quad = -Tr[\not{k}_a\gamma^{\mu}P_L\not{p}_i\gamma^{\lambda}P_L\not{k}_b\gamma_{\mu}P_L\not{k}_c\gamma_{\lambda}P_L] \\
&\quad = 2Tr[\not{k}_a(\not{p}_i\gamma^{\lambda}\not{k}_b)\not{k}_c\gamma_{\lambda}P_L] \\
&\quad = 8Tr[\not{k}_a\not{p}_i(b\cdot c)P_L] \\
&\quad = 8(a\cdot i)\frac{1}{2}4(b\cdot c) = 16(a\cdot i)(b\cdot c) \\
&\quad = 8(m_i^2 - 2E_a m_i + m_a^2 - m_b^2 - m_c^2)(m_i E_a) \qquad (143)
\end{aligned}
$$

4^{th} term is

$$
\begin{aligned}
&\sum_{spins} \bar{u}^{s_c}\gamma^{\mu}P_L u^{s_i}\bar{u}^{s_b}\gamma^{\nu}P_L v^{s_a}(-g_{\mu\nu})\bar{u}^{s_i}\gamma^{\lambda}P_L u^{s_c}\bar{v}^{s_a}\gamma^{\rho}P_L u^{s_b}(-g_{\lambda\rho}) \\
&= \sum_{spins} \bar{u}^{s_i}\gamma^{\lambda}P_L u^{s_c}\bar{u}^{s_c}\gamma^{\mu}P_L u^{s_i} \quad \bar{v}^{s_a}\gamma_{\lambda}P_L u^{s_b}\bar{u}^{s_b}\gamma_{\mu}P_L v^{s_a} \\
&= Tr[(\not{p}_i + m_i)\gamma^{\mu}P_L(\not{k}_c + m_c)\gamma^{\lambda}P_L]\cdot Tr[(\not{k}_a - m_a)\gamma_{\mu}P_L(\not{k}_b + m_b)\gamma_{\lambda}P_L] \\
&= Tr[\not{p}_i\gamma^{\mu}P_L\not{k}_c\gamma^{\lambda}P_L]\cdot Tr[\not{k}_a\gamma_{\mu}P_L\not{k}_b\gamma_{\lambda}P_L] \\
&= \frac{1}{2}\Big(Tr[\not{p}_i\gamma^{\mu}\not{k}_c\gamma^{\lambda}\cdot 1] - Tr[\not{p}_i\gamma^{\mu}\not{k}_c\gamma^{\lambda}\gamma^5]\Big) \\
&\quad \cdot\frac{1}{2}\Big(Tr[\not{k}_a\gamma_{\mu}\not{k}_b\gamma_{\lambda}\cdot 1] - Tr[\not{k}_a\gamma_{\mu}\not{k}_b\gamma_{\lambda}\gamma^5]\Big) \\
&= \frac{1}{2}\Big(4\big[p_i^{\mu}k_c^{\lambda} - (c\cdot i)g^{\mu\lambda} + p_i^{\lambda}k_c^{\mu}\big] - \big[4i\epsilon^{\beta\lambda\alpha\mu}p_{i\beta}k_{c\alpha}\big]\Big) \\
&\quad \cdot\frac{1}{2}\Big(4\big[k_{a\mu}k_{b\lambda} - (a\cdot b)g_{\mu\lambda} + k_{a\lambda}k_{b\mu}\big] - \big[4i\epsilon_{\beta'\lambda\alpha'\mu}k_a^{\beta'}k_b^{\alpha'}\big]\Big) \\
&= \big[4p_i^{\mu}k_c^{\lambda}k_{a\mu}k_{b\lambda} - 4p_i^{\mu}k_c^{\lambda}(a\cdot b)g_{\mu\lambda} + 4p_i^{\mu}k_c^{\lambda}k_{a\lambda}k_{b\mu} \\
&\quad - 4(c\cdot i)g^{\mu\lambda}k_{a\mu}k_{b\lambda} + 4(c\cdot i)g^{\mu\lambda}(a\cdot b)g_{\mu\lambda} - 4(c\cdot i)g^{\mu\lambda}k_{a\lambda}k_{b\mu} \\
&\quad + 4p_i^{\lambda}k_c^{\mu}k_{a\mu}k_{b\lambda} - 4p_i^{\lambda}k_c^{\mu}(a\cdot b)g_{\mu\lambda} + 4p_i^{\lambda}k_c^{\mu}k_{a\lambda}k_{b\mu}\big] - [(4\epsilon^{\beta\lambda\alpha\mu}p_{i\beta}k_{c\alpha})(\epsilon_{\beta'\lambda\alpha'\mu}k_a^{\beta'}k_b^{\alpha'})] \\
&= 4(i\cdot a)(b\cdot c) - 4(i\cdot c)(a\cdot b)g_{\mu\lambda} + 4(i\cdot b)(a\cdot c) \\
&\quad - 4(c\cdot i)(a\cdot b)g^{\mu\lambda} + 4(c\cdot i)g^{\mu\lambda}(a\cdot b)g_{\mu\lambda} - 4(c\cdot i)g^{\mu\lambda}(a\cdot b) \\
&\quad + 4(i\cdot b)(a\cdot c) - 4(i\cdot c)g_{\mu\lambda}(a\cdot b) + 4(i\cdot a)(b\cdot c) \\
&= -4\big[-2\big(\delta^{\beta}_{\beta'}\delta^{\alpha}_{\alpha'} - \delta^{\beta}_{\alpha'}\delta^{\alpha'}_{\beta}\big)p_{i\beta}k_{c\alpha}k_a^{\beta'}k_b^{\alpha'}\big] \to 8[(a\cdot i)(c\cdot b) - (i\cdot b)(a\cdot c)] \\
&= 16(i\cdot a)(b\cdot c) = 16(m_i E_a)(m_b m_c) \\
&= 8(m_i^2 - 2E_a m_i + m_a^2 - m_b^2 - m_c^2)(m_i E_a) \qquad (144)
\end{aligned}
$$

5^{th} term is

$$\sum_{spins} \bar{u}^{s_c}\gamma^\mu P_L u^{s_i}\bar{u}^{s_b}\gamma^\nu P_L v^{s_a}(-g_{\mu\nu})\bar{v}^{s_b}\gamma^\lambda P_L u^{s_c}\bar{u}^{s_i}\gamma^\rho P_L u^{s_a}(-g_{\lambda\rho})$$
$$\begin{aligned}
&= \sum_{spins} \bar{u}^{s_c}\gamma^\mu P_L u^{s_i} \;\; \bar{u}^{s_i}\gamma^\lambda P_L u^{s_a} \;\; (-\bar{u}^{s_a}\gamma_\mu P_L v^{s_b}) \;\; \bar{v}^{s_b}\gamma_\lambda P_L u^{s_c} \\
&= -Tr[\not{p}_i \gamma^\lambda P_L \not{k}_a \gamma_\mu P_L \not{k}_b \gamma_\lambda P_L \not{k}_c \gamma^\mu P_L] \\
&= -Tr[\not{p}_i \gamma_\lambda \not{k}_a (\not{k}_b \gamma^\lambda \not{k}_c)] \\
&= 2Tr[\not{p}_i \gamma_\lambda \not{k}_a (\not{k}_b \gamma^\lambda \not{k}_c)] \\
&= 8(a\cdot b)Tr[\not{p}_i \not{k}_c P_L] \\
&= 16(a\cdot b)(c\cdot i) \\
&= 8(m_i^2 - 2E_c m_i + m_c^2 - m_b^2 - m_a^2)(m_i E_c)
\end{aligned} \tag{145}$$

6^{th} term is

$$\sum_{spins} \bar{u}^{s_c}\gamma^\mu P_L u^{s_i}\bar{u}^{s_b}\gamma^\nu P_L v^{s_a}(-g_{\mu\nu})\bar{v}^{s_a}\gamma^\lambda P_L u^{s_c}\bar{u}^{s_i}\gamma^\rho P_L u^{s_b}(-g_{\lambda\rho})$$
$$\begin{aligned}
&= \sum_{spins} \bar{u}^{s_b}\gamma^\mu P_L v^{s_a}\bar{v}^{s_a}\gamma^\lambda P_L u^{s_c} \;\; \bar{u}^{s_c}\gamma_\mu P_L u^{s_i}\bar{u}^{s_i}\gamma_\lambda P_L u^{s_b} \\
&= Tr[(\not{k}_a - m_a)\gamma^\lambda P_L(\not{k}_c + m_c)\gamma_\mu P_L(\not{p}_i + m_i)\gamma_\lambda P_L(\not{k}_b + m_b)\gamma^\mu P_L] \\
&= Tr[\not{k}_a \gamma^\lambda P_L \not{k}_c \gamma_\mu P_L \not{p}_i \gamma_\lambda P_L \not{k}_b \gamma^\mu P_L] \\
&= Tr[\not{k}_a \gamma^\lambda \not{k}_c (-2\not{p}_i \gamma_\lambda \not{k}_b) P_L] \\
&= -2Tr[\not{k}_a \not{k}_b 4(c\cdot i) P_L] \\
&= -16(a\cdot b)(c\cdot i) \\
&= -8(m_i^2 - 2E_c m_i + m_c^2 - m_b^2 - m_a^2)(m_i E_c)
\end{aligned} \tag{146}$$

7^{th} term is

$$
\begin{aligned}
&\sum_{spins} \bar{u}^{s_c}\gamma^{\mu}P_L v^{s_a}\bar{u}^{s_b}\gamma^{\nu}P_L u^{s_i}\bar{v}^{s_a}\gamma^{\lambda}P_L u^{s_c}\bar{u}^{s_i}\gamma^{\rho}P_L u^{s_b} \\
&\quad = \sum_{spins} \bar{u}^{s_c}\gamma^{\mu}P_L v^{s_a}\bar{v}^{s_a}\gamma^{\lambda}P_L u^{s_c} \;\; \bar{u}^{s_i}\gamma_{\mu}P_L u^{s_b}\bar{u}^{s_b}\gamma_{\lambda}P_L u^{s_i} \\
&\quad = Tr[\not{k}_a\gamma^{\mu}P_L\not{k}_c\gamma^{\lambda}P_L]\cdot Tr[\not{k}_b\gamma_{\mu}P_L\not{p}_i\gamma_{\lambda}P_L] \\
&\quad = \frac{1}{2}\big(Tr[\not{k}_a\gamma^{\mu}\not{k}_c\gamma^{\lambda}\cdot 1] - Tr[\not{k}_a\gamma^{\mu}\not{k}_c\gamma^{\lambda}\gamma^5]\big)\cdot\frac{1}{2}Tr[\not{k}_b\gamma_{\mu}\not{p}_i\gamma_{\lambda}\cdot 1] - Tr[\not{k}_b\gamma_{\mu}\not{p}_i\gamma_{\lambda}\gamma^5] \\
&\quad = \frac{1}{2}\Big(4\big[k_a^{\mu}k_c^{\lambda} - (a\cdot c)g^{\mu\lambda} + k_a^{\lambda}k_c^{\mu}\big] - 4[i\epsilon^{\beta\lambda\alpha\mu}k_{a\beta}k_{c\alpha}]\Big) \\
&\qquad \cdot\frac{1}{2}\Big(4\big[k_{b\mu}p_{i\lambda} - (b\cdot i)g_{\mu\lambda} + k_{b\lambda}p_{i\mu}\big] - 4[i\epsilon_{\beta'\lambda\alpha'\mu}k_b^{\beta'}p_i^{\alpha'}]\Big) \\
&\quad = 4k_a^{\mu}k_c^{\lambda}k_{b\mu}p_{i\lambda} - 4k_a^{\mu}k_c^{\lambda}(b\cdot i)g_{\mu\lambda} + 4k_a^{\mu}k_c^{\lambda}k_{b\lambda}p_{i\mu} \\
&\qquad - 4(a\cdot c)g^{\mu\lambda}k_{b\mu}p_{i\lambda} + 4(a\cdot c)g^{\mu\lambda}(b\cdot i)g_{\mu\lambda} - 4(a\cdot c)g^{\mu\lambda}k_{b\lambda}p_{i\mu} \\
&\qquad + 4k_a^{\lambda}k_c^{\mu}k_{b\mu}p_{i\lambda} - 4k_a^{\lambda}k_c^{\mu}(b\cdot i)g_{\mu\lambda} + 4k_a^{\lambda}k_c^{\mu}k_{b\lambda}p_{i\mu} - 4[(\epsilon^{\beta\lambda\alpha\mu}k_{a\beta}k_{c\alpha})(\epsilon_{\beta'\lambda\alpha'\mu}k_b^{\beta'}p_i^{\alpha'})] \\
&\quad = 4(a\cdot b)(c\cdot i) - 4(a\cdot c)(b\cdot i) + 4(a\cdot i)(b\cdot c) \\
&\qquad - 4(a\cdot c)(b\cdot i) + 16(a\cdot c)(b\cdot i) - 4(a\cdot c)(b\cdot i) \\
&\qquad + 4(a\cdot i)(b\cdot c) - 4(a\cdot c)(b\cdot i) + 4(a\cdot b)(c\cdot i) \\
&\qquad - 4\big[-2(\delta^{\beta}_{\beta'}\delta^{\alpha}_{\alpha'} - \delta^{\beta}_{\alpha'}\delta^{\alpha}_{\beta'})k_{a\beta}k_{c\alpha}k_b^{\beta'}p_i^{\alpha'}\big] \to 8\big[(a\cdot i)(c\cdot b) - (a\cdot b)(c\cdot i)\big] \\
&\quad = 16(a\cdot i)(b\cdot c) \\
&\quad = 8(m_i^2 - 2E_a m_i + m_a^2 - m_b^2 - m_c^2)(m_i E_a)
\end{aligned}
\tag{147}
$$

8^{th} term is

$$
\begin{aligned}
&\sum_{spins} \bar{u}^{s_c}\gamma^{\mu}P_L v^{s_a}\bar{u}^{s_b}\gamma^{\nu}P_L u^{s_i}\bar{v}^{s_b}\gamma^{\lambda}P_L u^{s_c}\bar{u}^{s_i}\gamma^{\rho}P_L u^{s_a} \\
&\quad = \sum_{spins} \bar{u}^{s_a}\gamma^{\mu}P_L v^{s_c}\bar{v}^{s_c}\gamma^{\lambda}P_L u^{s_b}\bar{u}^{s_b}\gamma_{\mu}P_L u^{s_i}\bar{u}^{s_i}\gamma_{\lambda}P_L u^{s_a} \\
&\quad = Tr[\not{k}_c\gamma^{\lambda}P_L\not{k}_b\gamma_{\mu}P_L\not{p}_i\gamma_{\lambda}P_L\not{k}_a\gamma^{\mu}P_L] \\
&\quad = Tr[\not{k}_c(-2\not{k}_b\gamma_{\mu}\not{p}_i)\not{k}_a\gamma^{\mu}P_L] \\
&\quad = -8Tr[\not{k}_c\not{k}_b(a\cdot i)P_L] \\
&\quad = -16(c\cdot b)(a\cdot i) \\
&\quad = -8(m_i^2 - 2E_a m_i + m_a^2 - m_b^2 - m_c^2)(m_i E_a)
\end{aligned}
\tag{148}
$$

9^{th} term is

$$
\begin{aligned}
&\sum_{spins} \bar{u}^{s_c}\gamma^{\mu}P_L v^{s_a}\bar{u}^{s_b}\gamma^{\nu}P_L u^{s_i}\bar{u}^{s_i}\gamma^{\lambda}P_L u^{s_c}\bar{v}^{s_a}\gamma^{\rho}P_L u^{s_b} \\
&\quad = \sum_{spins} \bar{u}^{s_i}\gamma^{\lambda}P_L u^{s_c}\bar{u}^{s_c}\gamma^{\mu}P_L v^{s_a}\bar{v}^{s_a}\gamma_{\lambda}P_L u^{s_b}\bar{u}^{s_b}\gamma_{\mu}P_L u^{s_i} \\
&\quad = Tr[\not{p}_i\gamma^{\mu}P_L\not{k}_c\gamma_{\lambda}P_L\not{k}_a\gamma_{\mu}P_L\not{k}_b\gamma^{\lambda}P_L] \\
&\quad = Tr[\not{p}_i(-2\not{k}_c\gamma_{\mu}\not{k}_a)\not{k}_b\gamma^{\mu}P_L] \\
&\quad = -8Tr[\not{p}_i\not{k}_c(a\cdot b)P_L] \\
&\quad = -16(c\cdot i)(a\cdot b) \\
&\quad = -8(m_i^2 - 2E_c m_i + m_c^2 - m_b^2 - m_a^2)(m_i E_c) \qquad (149)
\end{aligned}
$$

A.4 The complete square of matrix element of 3 body decay

The square of the matrix elements that we will use in the phase space integration for the 3 body decay is shown below.

$$
\begin{aligned}
|\mathcal{M}|^2 =& 64\frac{\left|Y_{(cb)}Y_{(ai)}\right|^2}{\left[m_a^2 - 2m_iE_a + m_i^2 - m_\phi^2\right]^2}\left[\frac{1}{2}\left(m_i^2 - 2m_iE_a + m_a^2 - m_b^2 - m_c^2\right) - m_bm_c\right] \\
& \cdot 4\left[m_iE_a + m_am_i\right] \\
& - 64\frac{Y_{(cb)}Y_{(ai)}Y^*_{(ba)}Y^*_{(ci)}}{\left[m_a^2 - 2m_iE_a + m_i^2 - m_\phi^2\right]\left[m_c^2 - 2m_iE_c + m_i^2 - m_\phi^2\right]} \\
& \Big[(m_i(E_c + m_c))\frac{1}{2}\left(m_i^2 - 2m_iE_c + m_c^2 - m_b^2 - m_a^2\right) \\
& - (m_i(E_b + m_b))\frac{1}{2}\left(m_i^2 - 2m_iE_b + m_b^2 - m_c^2 - m_a^2\right) \\
& + (m_i(E_a + m_a))\frac{1}{2}\left(m_i^2 - 2m_iE_a + m_a^2 - m_b^2 - m_c^2\right) \\
& - m_im_cm_bm_a - m_cm_b(m_iE_a) + m_cm_a(m_iE_b) \\
& + m_bm_a(m_iE_c)\Big] \\
& + 64\frac{Y_{(cb)}Y_{(ai)}Y^*_{(ca)}Y^*_{(bi)}}{\left[m_a^2 - 2m_iE_a + m_i^2 - m_\phi^2\right]\left[m_b^2 - 2m_iE_b + m_i^2 - m_\phi^2\right]} \\
& \Big[(m_i(E_a + m_a))\frac{1}{2}\left(m_i^2 - 2m_iE_a + m_a^2 - m_b^2 - m_c^2\right) \\
& - (m_i(E_c + m_c))\frac{1}{2}\left(m_i^2 - 2m_iE_c + m_c^2 - m_b^2 - m_a^2\right) \\
& + (m_i(E_b + m_b))\frac{1}{2}\left(m_i^2 - 2m_iE_b + m_b^2 - m_a^2 - m_c^2\right) \\
& + m_am_b(m_iE_c) - m_am_c(m_iE_b) - m_am_im_bm_c \\
& - m_bm_c(m_iE_a)\Big] \\
& + 64\frac{\left|Y_{(ba)}Y_{(ci)}\right|^2}{\left[m_c^2 - 2m_iE_c + m_i^2 - m_\phi^2\right]^2}\left[(m_iE_c) + m_im_c\right] \cdot 4\Big[\frac{1}{2}\Big(m_i^2 - 2m_iE_b \\
& + m_b^2 - m_c^2 - m_a^2\Big) - m_am_b\Big]
\end{aligned}
$$

$$
\begin{aligned}
&- 64\frac{Y_{(ba)}Y_{(ci)}Y^*_{(cb)}Y^*_{(ai)}}{\Big[m_c^2 - 2m_iE_c + m_i^2 - m_\phi^2\Big]\Big[m_a^2 - 2m_iE_a + m_i^2 - m_\phi^2\Big]} \\
&\Big[(m_i(E_a - m_a))\frac{1}{2}\big(m_i^2 - 2m_iE_a + m_a^2 - m_b^2 - m_c^2\big) \\
&- (m_i(E_b - m_b))\frac{1}{2}\big(m_i^2 - 2m_iE_b + m_b^2 - m_a^2 - m_c^2\big) \\
&+ (m_i(E_c - m_c))\frac{1}{2}\big(m_i^2 - 2m_iE_c + m_c^2 - m_b^2 - m_a^2\big) \\
&- m_im_am_bm_c - m_am_b(m_iE_c) - m_am_c(m_iE_b) \\
&- m_bm_c(m_iE_a)\Big] \\
&- 64\frac{Y_{(ba)}Y_{(ci)}Y^*_{(ca)}Y^*_{(bi)}}{\Big[m_c^2 - 2m_iE_c + m_i^2 - m_\phi^2\Big]\Big[m_b^2 - 2m_iE_b + m_i^2 - m_\phi^2\Big]} \\
&\Big[(m_iE_b + m_im_b)\frac{1}{2}\big(m_i^2 - 2m_iE_b + m_b^2 - m_a^2 - m_c^2\big) \\
&- (m_iE_a + m_am_i)\frac{1}{2}\big(m_i^2 - 2m_iE_a + m_a^2 - m_c^2 - m_b^2\big) \\
&+ (m_iE_c + 4m_cm_i)\frac{1}{2}\big(m_i^2 - 2m_iE_c + m_c^2 - m_a^2 - m_b^2\big) \\
&- m_am_c(m_iE_b) - m_am_b(m_iE_c) - m_am_cm_im_b \\
&+ m_cm_b(m_iE_a)\Big] \\
&+ 64\frac{\Big|Y_{(ca)}Y_{(bi)}\Big|^2}{\Big[m_b^2 - 2m_iE_b + m_i^2 - m_\phi^2\Big]^2}\Big[\frac{1}{2}\big(m_i^2 - 2m_iE_b + m_b^2 - m_a^2 - m_c^2\big) - m_am_c\Big] \\
&\cdot 4\Big[m_iE_b - m_bm_i\Big] \\
&- 64\frac{Y_{(ca)}Y_{(bi)}Y^*_{(cb)}Y^*_{(ai)}}{\Big[m_b^2 - 2m_iE_b + m_i^2 - m_\phi^2\Big]\Big[m_a^2 - 2m_iE_a + m_i^2 - m_\phi^2\Big]} \\
&\Big[\frac{1}{2}(m_i(E_a + m_a))\big(m_i^2 - 2m_iE_a + m_a^2 - m_c^2 - m_b^2\big) \\
&- (m_i(E_c + m_c))\frac{1}{2}\big(m_i^2 - 2m_iE_c - m_c^2 - m_b^2 - m_a^2\big) \\
&+ (m_i(E_b + m_b))\frac{1}{2}\big(m_i^2 - 2m_iE_b + m_b^2 - m_c^2 - m_a^2\big) \\
&- m_cm_b(m_iE_a) - m_cm_a(m_iE_b) - m_cm_bm_im_a \\
&+ m_bm_a(m_iE_c)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&- 64\frac{Y_{(ca)}Y_{(bi)}Y^*_{(ba)}Y^*_{(ci)}}{\Big[m_b^2 - 2m_iE_b + m_i^2 - m_\phi^2\Big]\Big[m_c^2 - 2m_iE_c + m_i^2 - m_\phi^2\Big]} \\
&\quad \Big[(m_i(E_c + m_c))\frac{1}{2}\big(m_i^2 - 2m_iE_c + m_c^2 - m_a^2 - m_b^2\big) \\
&\quad - (m_i(E_a + m_a))\frac{1}{2}\big(m_i^2 - 2m_iE_a + m_a^2 - m_c^2 - m_b^2\big) \\
&\quad + (m_i(E_b + m_b))\frac{1}{2}\big(m_i^2 - 2m_iE_b + m_b^2 - m_a^2 - m_c^2\big) \\
&\quad - m_im_cm_am_b - m_cm_a(m_iE_b) + m_cm_b(m_iE_a) \\
&\quad - m_am_b(m_iE_c)\Big] \\
&+ 128\frac{\Big|U_{[cb]}U_{[ai]}\Big|^2}{\Big[m_a^2 - 2m_iE_a + m_i^2 - m_z^2\Big]^2}(m_i^2 - 2E_cm_i + m_c^2 - m_b^2 - m_a^2)(m_iE_c) \\
&+ 128\frac{U_{[cb]}U_{[ai]}U^*_{[ba]}U^*_{[ci]}}{\Big[m_a^2 - 2m_iE_a + m_i^2 - m_z^2\Big]\Big[m_c^2 - 2m_iE_c + m_i^2 - m_z^2\Big]}(m_i^2 - 2E_bm_i + m_b^2 - m_a^2 - m_c^2)(m_iE_b) \\
&- 128\frac{U_{[cb]}U_{[ai]}U^*_{[ca]}U^*_{[bi]}}{\Big[m_a^2 - 2m_iE_a + m_i^2 - m_z^2\Big]\Big[m_b^2 - 2m_iE_b + m_i^2 - m_z^2\Big]}(m_i^2 - 2E_am_i + m_a^2 - m_b^2 - m_c^2)(m_iE_a) \\
&+ 128\frac{\Big|U_{[ba]}U_{[ci]}\Big|^2}{\Big[m_c^2 - 2m_iE_c + m_i^2 - m_z^2\Big]^2}(m_i^2 - 2E_am_i + m_a^2 - m_b^2 - m_c^2)(m_iE_a) \\
&+ 128\frac{U_{[ba]}U_{[ci]}U^*_{[cb]}U^*_{[ai]}}{\Big[m_c^2 - 2m_iE_c + m_i^2 - m_z^2\Big]\Big[m_a^2 - 2m_iE_a + m_i^2 - m_z^2\Big]}(m_i^2 - 2E_cm_i + m_c^2 - m_b^2 - m_a^2)(m_iE_c) \\
&+ 128\frac{U_{[ba]}U_{[ci]}U^*_{[ca]}U^*_{[bi]}}{\Big[m_c^2 - 2m_iE_c + m_i^2 - m_z^2\Big]\Big[m_b^2 - 2m_iE_b + m_i^2 - m_z^2\Big]}(m_i^2 - 2E_cm_i + m_c^2 - m_b^2 - m_a^2)(m_iE_c) \\
&+ 128\frac{\Big|U_{[ca]}U_{[bi]}\Big|^2}{\Big[m_b^2 - 2m_iE_b + m_i^2 - m_z^2\Big]^2}(m_i^2 - 2E_am_i + m_a^2 - m_b^2 - m_c^2)(m_iE_a) \\
&+ 128\frac{U_{[ca]}U_{[bi]}U^*_{[cb]}U^*_{[ai]}}{\Big[m_b^2 - 2m_iE_b + m_i^2 - m_z^2\Big]\Big[m_a^2 - 2m_iE_a + m_i^2 - m_z^2\Big]}(m_i^2 - 2E_am_i + m_a^2 - m_b^2 - m_c^2)(m_iE_a) \\
&+ 128\frac{U_{[ca]}U_{[bi]}U^*_{[ba]}U^*_{[ci]}}{\Big[m_c^2 - 2m_iE_c + m_i^2 - m_z^2\Big]\Big[m_b^2 - 2m_iE_b + m_i^2 - m_z^2\Big]}(m_i^2 - 2E_cm_i + m_c^2 - m_b^2 - m_a^2)(m_iE_c)
\end{aligned}
\tag{150}
$$

A.5 Integrating over the phase space

We show below the decay rate of the sterile neutrino to three neutrinos; where n is the m_i, b is E_b maximum value p is the mass of Higgs m_z is the mass of the Z boson, c is the maximum value for E_c and Y and the U are the Yukawa coupling constants and the product of the mixing matrices written in shorthand.

$$-\frac{2M\left(\frac{n}{2}-c\right)^{3/2}}{3}=\text{integrate}(\text{integrate}(\frac{128\,YY\,(n+c)\left(n^2-2\sqrt{\frac{n}{2}-c}\,n\right)}{(-p^2-2nc+n^2)^2}-\frac{32\,YYYY\,c\,n\left(\frac{n\left(n-\sqrt{\frac{n}{2}-c}-c\right)\left(n^2-2n\left(n-\sqrt{\frac{n}{2}-c}-c\right)\right)}{2}-\frac{\sqrt{\frac{n}{2}-c}\,n\left(n^2-2\sqrt{\frac{n}{2}-c}\,n\right)}{2}+n^2-n\left(n-\sqrt{\frac{n}{2}-c}-c\right)+\sqrt{\frac{n}{2}-c}\,n+cn-n-2c\right)}{(-p^2+n^2-2c)\left(-p^2+n^2-2n\left(n-\sqrt{\frac{n}{2}-c}-c\right)\right)}+$$

$$\frac{64\,YYYY\left(\frac{n\left(n-\sqrt{\frac{n}{2}-c}-c\right)\left(n^2-2n\left(n-\sqrt{\frac{n}{2}-c}-c\right)\right)}{2}+\frac{n\left(\sqrt{\frac{n}{2}-c}\right)\left(n^2-2\sqrt{\frac{n}{2}-c}\,n\right)}{2}-\frac{cn(n^2-2c)}{2}-n\left(n-\sqrt{\frac{n}{2}-c}-c\right)-\sqrt{\frac{n}{2}-c}\,n-n+c\right)\left(-p^2+n^2-2\sqrt{\frac{n}{2}-c}\,n\right)}{-p^2+n^2-2n\left(n-\sqrt{\frac{n}{2}-c}-c\right)},\sqrt{\frac{n}{2}-c})-64\,YYYY\,($$

$$\frac{\sqrt{\frac{n}{2}-c}\left(6p^4+\left(6n^2+(-6m_b-18c)n\right)p^2+3n^4+(-3m_b-12c)n^3+\left(6cm_b+18c^2+6\right)n^2+(-6c^2-6c+6)n-6c\right)+\left(\frac{n}{2}-c\right)\left(-6np^2-6n^3+(3m_b+12c)n^2\right)+4\left(\frac{n}{2}-c\right)^{3/2}n^2}{6}-$$

$$\frac{\left(p^6+(-m_b-3c)np^4+\left(\left(cm_b+4c^2+2\right)n^2+(-2c^2-2c+2)n-2c\right)p^2+(-2c^3-2c)n^3+(2c^3+2c^2-2c)n^2+2c^2n\right)\log\left(p^2-n^2+2\sqrt{\frac{n}{2}-c}\,n\right)}{2n})-64\,YYYY\,(-p^2+n^2-2c)$$

$$\left(\frac{\sqrt{\frac{n}{2}-c}\,(n^2-2cn+2)}{2}-\frac{\left((n^2-2cn+2)p^2-cn^3+(n(c)+2c^2)n^2+(-2c-2)n-2cn(c)-2c\right)\log\left(p^2-n^2+2\sqrt{\frac{n}{2}-c}\,n\right)}{4n}\right)+128\,YY\left(\frac{(2p^2-n^2+2n)\log\left(p^2-n^2+2\sqrt{\frac{n}{2}-c}\,n\right)}{4n}+\frac{p^4+(2n-n^2)p^2}{4np^2-4n^3+8\sqrt{\frac{n}{2}-c}\,n^2}-\frac{\sqrt{\frac{n}{2}-c}}{2}\right)+128\,YY$$

$$\left(-\frac{(2p^2-n^2-2n)\log\left(-p^2-n^2+2\sqrt{\frac{n}{2}-c}\,n+2cn\right)}{4n}+\frac{p^4+(-n^2-2n)p^2}{-4np^2-4n^3+8\sqrt{\frac{n}{2}-c}\,n^2+8cn^2}-\frac{\sqrt{\frac{n}{2}-c}}{2}\right)+$$

$$\left(32\,YYYY\,(\left(\frac{n}{2}-c\right)\left((-3n\,nb-3n^3+(6c-3)n^2-6n)p^2+(3n^3-6n^2)nb+6n^5+(21-15c)n^4+(6c^2-3c+12)n^3+(-30c-6)n^2+(-6c^2-6c)n\right)+\sqrt{\frac{n}{2}-c}\left(\left(-6n\,nb+3n^4+(18-9c)n^3+(6c^2-3c+6)n^2+(-30c-6)n-6c^2-6c\right)p^2+6n^3nb-\right.\right.$$

$$/\left(3(-p^2+n^2-2c)\right)-$$

$$\left(32\,YYYY\,(\sqrt{\frac{n}{2}-c}\left(\left(3n^4-9cn^3+\left(-3n(c-m_c)+6c^2+6\right)n^2+6cn(c-m_c)\right)p^2+3n^6-15cn^5+\left(-3n(c-m_c)+24c^2+6\right)n^4+\left(6cn(c-m_c)-12c^3-12c\right)n^3+6cn(c-m_c)n^2-12c^2n(c-m_c)n\right)+\left(\frac{n}{2}-c\right)\left((6cn^2-3n^3)p^2-6n^5+21cn^4+\left(3n(c-m_c)-18c^2\right.\right.\right.$$

$$/\left(3(-p^2+n^2-2c)\right),c)+128\,UUUU\,n\,(-$$

$$\left(\left(\left(\frac{\left(-3n^4+\left(6m_z^2+4\right)n^3-10m_z^2n^2+4m_z^4n\right)\log\left(\frac{-2\sqrt{n\left(n-2m_z^2\right)}+4\sqrt{\frac{n}{2}-c}\,n-2n}{2\sqrt{n\left(n-2m_z^2\right)}+4\sqrt{\frac{n}{2}-c}\,n-2n}\right)}{2\sqrt{n\left(n-2m_z^2\right)}}-\frac{\left(6n^4+\left(-6m_z^2-8\right)n^3+\left(12m_z^2-3m_z^4\right)n^2+2m_z^6\right)\log\left(2\left(\frac{n}{2}-c\right)n-2\sqrt{\frac{n}{2}-c}\,n+m_z^2\right)}{4n}+\frac{\left(\frac{n}{2}-c\right)^{3/2}(40n^2-30n^3)+\left(\frac{n}{2}-c\right)\left(-45n^3+\left(60-45m_z^2\right)n^2-30m_z^2n+30m_z^4\right)+\cdots}{\cdots}\right.\right.\right.$$

$$/(2n^2)-\frac{n\left(\frac{cn}{2}-\frac{c^2}{2}\right)-\frac{2\left(\left(\frac{n}{2}-c\right)^{3/2}\left(5m_z^2-5n^2\right)+6\left(\frac{n}{2}-c\right)^{5/2}n\right)}{15}}{2n})\left(-2nc+n^2-m_z^2\right)-128\,UUUU\,n\,(-$$

$$\frac{\left(m_z^2n^2-m_z^4\right)\left(\frac{\left(\frac{n}{2}-c\right)\log\left(-n^2+2\sqrt{\frac{n}{2}-c}\,n+2cn-m_z^2\right)}{2}-\frac{\frac{\left(n-2m_z^2\right)\log\left(\frac{-2\sqrt{n\left(n-2m_z^2\right)}+4\sqrt{\frac{n}{2}-c}\,n-2n}{2\sqrt{n\left(n-2m_z^2\right)}+4\sqrt{\frac{n}{2}-c}\,n-2n}\right)}{2\sqrt{n\left(n-2m_z^2\right)}}+\frac{\left(n-m_z^2\right)\log\left(2\left(\frac{n}{2}-c\right)n-2\sqrt{\frac{n}{2}-c}\,n+m_z^2\right)}{2n}+\frac{2\left(\frac{n}{2}-c\right)+2\sqrt{\frac{n}{2}-c}}{2}}{2}\right)}{2n^2}-\frac{\frac{2\left(\left(\frac{n}{2}-c\right)^{3/2}\left(5n^2-5m_z^2\right)+6\left(\frac{n}{2}-c\right)^{5/2}n\right)}{15}+n\left(\frac{cn}{2}-\frac{c^2}{2}\right)}{2n})$$

$$\left(-2nb+n^2-m_z^2\right)+$$

$$128\,UU\,\Bigg(-\frac{\left(2n^3+\left(-3m_z^2+m_c^2-4\right)n^2+\left(10m_z^2-2m_c^2\right)n-5m_z^4+3m_c^2m_z^2\right)\log\left(\frac{-2\sqrt{n\left(n-2m_z^2\right)}+4\sqrt{\frac{n}{2}-c}\,n-2n}{2\sqrt{n\left(n-2m_z^2\right)}+4\sqrt{\frac{n}{2}-c}\,n-2n}\right)}{8n\sqrt{n\left(n-2m_z^2\right)}}-\frac{\left(2n^3+\left(-m_z^2+m_c^2-4\right)n^2+\left(6m_z^2-2m_c^2\right)n-m_z^4+m_c^2m_z^2\right)\log\left(2\left(\frac{n}{2}-c\right)n-2\sqrt{\frac{n}{2}-c}\,n+m_z^2\right)}{8n^2}+\frac{\left(\frac{n}{2}-c\right)\left(-3n^2+6n-3m_z^2+3m_c^2\right)+\sqrt{\frac{n}{2}\cdots}}{\cdots}$$

$$\cdots n-256\,UUUU\,\Bigg(\frac{\left(n^{10}-3n^9+\left(3-4m_z^2\right)n^8+\left(8m_z^2-1\right)n^7+\left(6m_z^4-4m_z^2\right)n^6-6m_z^4n^5+\left(-4m_z^6-m_z^4\right)n^4+m_z^4n^3+\left(m_z^8+2m_z^6\right)n^2+m_z^8n\right)\operatorname{atan}\left(\frac{\sqrt{2}\sqrt{\frac{n}{2}-c}}{\sqrt{n^2-n-m_z^2}}\right)}{2^{5/2}\sqrt{n^2-n-m_z^2}}+$$

$$\frac{\left(n^8-2n^7+\left(1-3m_z^2\right)n^6+4m_z^2n^5+\left(3m_z^4-m_z^2\right)n^4-2m_z^4n^3-m_z^6n^2\right)\log\left(n^2-n+2\left(\frac{n}{2}-c\right)-m_z^2\right)}{16}+$$

$$\sqrt{\frac{n}{2}-c}\left(-210n^8+420n^7+\left(630m_z^2-210\right)n^6-630m_z^2n^5-630m_z^4n^4+\left(210m_z^6+210m_z^4\right)n^2+210m_z^6n\right)+\left(\frac{n}{2}-c\right)^{3/2}\left(140n^6-140n^5-280m_z^2n^4+140m_z^4n^2+140m_z^4n\right)+\left(\frac{n}{2}-c\right)\left(-105n^6+105n^5+210m_z^2n^4-105m_z^2n^3-105m_z^4n^2\right)+\left(\frac{n}{2}-c\cdots\right.$$

$$\cdots 840)-512\,UUUU\,\Bigg(\frac{\left(n^9-2n^8+\left(1-4m_z^2\right)n^7+6m_z^2n^6+\left(6m_z^4-2m_z^2\right)n^5-6m_z^4n^4+\left(m_z^4-4m_z^6\right)n^3+2m_z^6n^2+m_z^8n\right)\operatorname{atan}\left(\frac{\sqrt{2}\sqrt{\frac{n}{2}-c}}{\sqrt{n^2-n-m_z^2}}\right)}{2^{5/2}\sqrt{n^2-n-m_z^2}}-\frac{\left(n^8-2n^7+\left(1-3m_z^2\right)n^6+4m_z^2n^5+\left(3m_z^4-m_z^2\right)n^4-2m_z^4n^3-m_z^6n^2\right)\log\left(n^2-n+2\left(\frac{n}{2}-c\right)-m_z^2\right)}{16}+$$

$$\sqrt{\frac{n}{2}-c}\left(-30n^7+30n^6+90m_z^2n^5-60m_z^2n^4-90m_z^4n^3+30m_z^4n^2+30m_z^6n\right)+\left(\frac{n}{2}-c\right)\left(15n^6-15n^5-30m_z^2n^4+15m_z^2n^3+15m_z^4n^2\right)+\left(\frac{n}{2}-c\right)^{3/2}\left(20n^5-40m_z^2n^3+20m_z^4n\right)+\left(\frac{n}{2}-c\right)^2\left(15m_z^2n^2-15n^4\right)+\left(\frac{n}{2}-c\right)^{5/2}\left(24m_z^2n-24n^3\right)+20\left(\frac{n}{2}\cdots\right.$$

$$\cdots 120)+$$

$$128\,UU\,n\,\Bigg(\frac{\left(15n^5-39n^4+\left(18-30m_z^2\right)n^3+\left(39m_z^2+6\right)n^2+\left(15m_z^4+6m_z^2\right)n\right)\operatorname{atan}\left(\frac{\sqrt{2}\sqrt{\frac{n}{2}-c}}{\sqrt{n^2-n-m_z^2}}\right)}{2^{5/2}\sqrt{n^2-n-m_z^2}}+\frac{\left(12n^3-15n^2-12m_z^2n\right)\log\left(n^2-n+2\left(\frac{n}{2}-c\right)-m_z^2\right)}{8}-\frac{\sqrt{\frac{n}{2}-c}\left(6n^5-18n^4+\left(8-12m_z^2\right)n^3+\left(18m_z^2+4\right)n^2+\left(6m_z^4+4m_z^2\right)n\right)-6n^5+15n^4+\left(12m_z^2-9\right)n^3-15n\cdots}{8n^2-8n+16\left(\frac{n}{2}-c\right)-8m^2\cdots}$$

$$128\,UU\,\Bigg(-\frac{\left(2n^3+\left(-3m_z^2+m_c^2-4\right)n^2+\left(10m_z^2-2m_c^2\right)n-5m_z^4+3m_c^2m_z^2\right)\log\left(\frac{-2\sqrt{n\left(n-2m_z^2\right)}+4\sqrt{\frac{n}{2}-c}\,n-2n}{2\sqrt{n\left(n-2m_z^2\right)}+4\sqrt{\frac{n}{2}-c}\,n-2n}\right)}{8n\sqrt{n\left(n-2m_z^2\right)}}-\frac{\left(2n^3+\left(-m_z^2+m_c^2-4\right)n^2+\left(6m_z^2-2m_c^2\right)n-m_z^4+m_c^2m_z^2\right)\log\left(2\left(\frac{n}{2}-c\right)n-2\sqrt{\frac{n}{2}-c}\,n+m_z^2\right)}{8n^2}+\frac{\left(\frac{n}{2}-c\right)\left(-3n^2+6n-3m_z^2+3m_c^2\right)+\sqrt{\frac{n}{2}\cdots}}{\cdots}$$

$$\cdots n-256\,UUUU\,\Bigg(\frac{\left(n^{10}-3n^9+\left(3-4m_z^2\right)n^8+\left(8m_z^2-1\right)n^7+\left(6m_z^4-4m_z^2\right)n^6-6m_z^4n^5+\left(-4m_z^6-m_z^4\right)n^4+m_z^4n^3+\left(m_z^8+2m_z^6\right)n^2+m_z^8n\right)\operatorname{atan}\left(\frac{\sqrt{2}\sqrt{\frac{n}{2}-c}}{\sqrt{n^2-n-m_z^2}}\right)}{2^{5/2}\sqrt{n^2-n-m_z^2}}+$$

$$\frac{\left(n^8-2n^7+\left(1-3m_z^2\right)n^6+4m_z^2n^5+\left(3m_z^4-m_z^2\right)n^4-2m_z^4n^3-m_z^6n^2\right)\log\left(n^2-n+2\left(\frac{n}{2}-c\right)-m_z^2\right)}{16}+$$

$$\sqrt{\frac{n}{2}-c}\left(-210n^8+420n^7+\left(630m_z^2-210\right)n^6-630m_z^2n^5-630m_z^4n^4+\left(210m_z^6+210m_z^4\right)n^2+210m_z^6n\right)+\left(\frac{n}{2}-c\right)^{3/2}\left(140n^6-140n^5-280m_z^2n^4+140m_z^4n^2+140m_z^4n\right)+\left(\frac{n}{2}-c\right)\left(-105n^6+105n^5+210m_z^2n^4-105m_z^2n^3-105m_z^4n^2\right)+\left(\frac{n}{2}-c\cdots\right.$$

$$\cdots 840)-512\,UUUU\,\Bigg(\frac{\left(n^9-2n^8+\left(1-4m_z^2\right)n^7+6m_z^2n^6+\left(6m_z^4-2m_z^2\right)n^5-6m_z^4n^4+\left(m_z^4-4m_z^6\right)n^3+2m_z^6n^2+m_z^8n\right)\operatorname{atan}\left(\frac{\sqrt{2}\sqrt{\frac{n}{2}-c}}{\sqrt{n^2-n-m_z^2}}\right)}{2^{5/2}\sqrt{n^2-n-m_z^2}}-\frac{\left(n^8-2n^7+\left(1-3m_z^2\right)n^6+4m_z^2n^5+\left(3m_z^4-m_z^2\right)n^4-2m_z^4n^3-m_z^6n^2\right)\log\left(n^2-n+2\left(\frac{n}{2}-c\right)-m_z^2\right)}{16}+$$

$$\sqrt{\frac{n}{2}-c}\left(-30n^7+30n^6+90m_z^2n^5-60m_z^2n^4-90m_z^4n^3+30m_z^4n^2+30m_z^6n\right)+\left(\frac{n}{2}-c\right)\left(15n^6-15n^5-30m_z^2n^4+15m_z^2n^3+15m_z^4n^2\right)+\left(\frac{n}{2}-c\right)^{3/2}\left(20n^5-40m_z^2n^3+20m_z^4n\right)+\left(\frac{n}{2}-c\right)^2\left(15m_z^2n^2-15n^4\right)+\left(\frac{n}{2}-c\right)^{5/2}\left(24m_z^2n-24n^3\right)+20\left(\frac{n}{2}\cdots\right.$$

$$\cdots 120)+$$

$$128\,UU\,n\,\Bigg(\frac{\left(15n^5-39n^4+\left(18-30m_z^2\right)n^3+\left(39m_z^2+6\right)n^2+\left(15m_z^4+6m_z^2\right)n\right)\operatorname{atan}\left(\frac{\sqrt{2}\sqrt{\frac{n}{2}-c}}{\sqrt{n^2-n-m_z^2}}\right)}{2^{5/2}\sqrt{n^2-n-m_z^2}}+\frac{\left(12n^3-15n^2-12m_z^2n\right)\log\left(n^2-n+2\left(\frac{n}{2}-c\right)-m_z^2\right)}{8}-\frac{\sqrt{\frac{n}{2}-c}\left(6n^5-18n^4+\left(8-12m_z^2\right)n^3+\left(18m_z^2+4\right)n^2+\left(6m_z^4+4m_z^2\right)n\right)-6n^5+15n^4+\left(12m_z^2-9\right)n^3-15n\cdots}{8n^2-8n+16\left(\frac{n}{2}-c\right)-8m^2\cdots}$$

A.6 Complete interaction Hamiltonian for the 2 particle decay

We introduce the parameter D to fit the whole equation to the page.

$$
\begin{aligned}
D &= \langle 0| b^{s_a}_{\vec{k}_a} a^{s_b}_{\vec{k}_b} \Big(\int dx_j\, \mathcal{H}_{\mathrm{int}_j}\Big)^3 b^{s_i^\dagger}_{\vec{p}_i} |0\rangle \\
D &= \langle 0| \int dx_1 \int dx_2 \int dx_3 \Big[Y_{j_1\ell_1}\delta_{aj_1}\bar{u}^{s_a} e^{ik_a x_1} \psi'_{\ell_1}\phi_1^+ \Big] \\
&\quad \Big[\partial_\mu \phi_2^- i\eta_e e \delta_{\lambda b}\epsilon^\lambda_\mu e^{-ik_b x_2}\phi_2^+ + \partial_\mu \phi_2^+ i\eta_e e \delta^{\lambda b}\epsilon^\lambda_\mu e^{-ik_b x_2}\phi_2^- \Big] \\
&\quad \Big[(Y^\dagger)_{j_3\ell_3}\delta_{i\ell_3}\bar{\psi}'_{j_3} u^{s_i} e^{-ik_i x_3}\phi_3^- \Big] \\
&+ \Big[V_{aj_1}\bar{u}^{s_a} e^{ik_a x_1}\gamma^\mu \psi'_{j_1} W^+_{1\mu} \Big] \\
&\quad \Big[4i\eta g \delta^{\lambda b}\epsilon^\lambda_\alpha e^{-ik_b x_2} W^+_{2\beta}(\partial^\alpha W_2^{-\beta} - \partial^\beta W_2^{-\alpha}) \\
&\quad + 4i\eta g \delta_{\lambda b}\epsilon^\lambda_\alpha e^{-ik_b x_2} W^-_{2\beta}(-\partial^\alpha W_2^{+\beta} + \partial^\beta W_2^{+\alpha}) \\
&\quad + 4i\eta g \big[W^+_\alpha W^-_\beta\big]\big[-ik^{b\alpha}\delta^{\lambda b}\epsilon^\lambda_\beta e^{-ik_b x_2} + ik^{b\beta}\delta^{\lambda b}\epsilon^\lambda_\alpha e^{-ik_b x_2}\big]\Big] \\
&\quad \Big[V_{ij_3} e^{-ik_i x_3}\bar{\psi}'_{j_3}\gamma^\mu u^{s_i} W^-_{3\mu} \Big] \\
&+ \Big[V_{aj_1} e^{ik_a x_1}\bar{\psi}'_{j_1}\gamma^\mu v^{s_a} W^-_{1\mu} \Big] \\
&\quad \Big[4i\eta g \delta^{\lambda b}\epsilon^\lambda_\alpha e^{-ik_b x_2} W^+_{2\beta}(\partial^\alpha W_2^{-\beta} - \partial^\beta W_2^{-\alpha}) \\
&\quad + 4i\eta g \delta_{\lambda b}\epsilon^\lambda_\alpha e^{-ik_b x_2} W^-_{2\beta}(-\partial^\alpha W_2^{+\beta} + \partial^\beta W_2^{+\alpha}) \\
&\quad + 4i\eta g [W^+_\alpha W^-_\beta]\big[-ik^{b\alpha}\delta^{\lambda b}\epsilon^\lambda_\beta e^{-ik_b x_2} + ik^{b\beta}\delta^{\lambda b}\epsilon^\lambda_\alpha e^{-ik_b x_2}\big]\Big] \\
&\quad \Big[V_{ij_3}\bar{v}^{s_i} e^{-ik_i x_3}\gamma^\mu \psi'_{j_3} W^+_{3\mu} \Big] \\
&+ \Big[(Y^\dagger)_{j_1\ell_1}\delta_{a\ell_1}\bar{\psi}'_{j_1} v^{s_a} e^{ik_a x_1}\phi_1^- \Big] \\
&\quad \Big[\partial_\mu \phi_2^- i\eta_e e \delta_{\lambda b}\epsilon^\lambda_\mu e^{-ik_b x_2}\phi_2^+ + \partial_\mu \phi_2^+ i\eta_e e \delta^{\lambda b}\epsilon^\lambda_\mu e^{-ik_b x_2}\phi_2^- \Big] \\
&\quad \Big[Y_{j_3\ell_3}\delta_{ij_3}\bar{v}^{s_i} e^{-ik_i x_3}\psi'_{\ell_3}\phi_3^+ \Big] \\
&+ \Big[Y_{j_1\ell_1}\delta_{aj_1}\bar{u}^{s_a} e^{ik_a x_1}\psi'_{\ell_1}\phi_1^+ \Big]\Big[\bar{\psi}'_{j_2}\gamma^\mu \psi'_{j_2}\delta^{\lambda b}\epsilon^\lambda_\mu e^{-ik_b x_2}\Big]\Big[(Y^\dagger)_{j_3\ell_3}\delta_{i\ell_3}\bar{\psi}'_{j_3} u^{s_i} e^{-ik_i x_3}\phi_3^- \Big] \\
&+ \Big[(Y^\dagger)_{j_1\ell_1}\delta_{a\ell_1}\bar{\psi}'_{j_1} v^{s_a} e^{ik_a x_1}\phi_1^- \Big]\Big[\bar{\psi}'_{j_2}\gamma^\mu \psi'_{j_2}\delta^{\lambda b}\epsilon^\lambda_\mu e^{-ik_b x_2}\Big]\Big[Y_{j_3\ell_3}\delta_{ij_3}\bar{v}^{s_i} e^{-ik_i x_3}\psi'_{\ell_3}\phi_3^+ \Big] \\
&+ \Big[V_{aj_1} e^{ik_a x_1}\bar{\psi}'_{j_1}\gamma^\mu v^{s_a} W^-_{1\mu} \Big]\Big[\bar{\psi}'_{j_2}\gamma^\mu \psi'_{j_2}\delta^{\lambda b}\epsilon^\lambda_\mu e^{-ik_b x_2}\Big]\Big[V_{ij_3}\bar{v}^{s_i} e^{-ik_i x_3}\gamma^\mu \psi'_{j_3} W^+_{3\mu} \Big] \\
&+ \Big[V_{aj_1}\bar{u}^{s_a} e^{ik_a x_1}\gamma^\mu \psi'_{j_1} W^+_{1\mu} \Big]\Big[\bar{\psi}'_{j_2}\gamma^\mu \psi'_{j_2}\delta^{\lambda b}\epsilon^\lambda_\mu e^{-ik_b x_2}\Big]\Big[V_{ij_3} e^{-ik_i x_3}\bar{\psi}'_{j_3}\gamma^\mu u^{s_i} W^-_{3\mu} \Big]|0\rangle
\end{aligned}
\tag{151}
$$

A.7 Majorana Representation Explicitly

This is an example of the explicit Majorana representation relation that we used for the term in equation (82).

$$\begin{aligned}&\bar{v}^{s_i}\gamma^\rho(\not{k}_1 - m_j)\gamma^\nu(\not{k}_2 - m_j)\gamma^\mu v^{s_a}\\ =&(\gamma_0 C u^{s_i *})^\dagger \gamma_0 \gamma^\rho(\not{k}_1 - m_j)\gamma^\nu(\not{k}_2 - m_j)\gamma^\mu \gamma_0 C u^{s_a *}\end{aligned} \tag{152}$$

And once we get the transpose, we have

$$\begin{aligned}&((u^{s_i *\dagger} C^\dagger \gamma_0^\dagger)\gamma_0\gamma^\rho(\not{k}_2 - m_j)\gamma^\nu(\not{k}_1 - m_j)\gamma^\mu\gamma_0 C u^{s_a *})^\top\\ =&u^{s_a\dagger}(-C)(-C^{-1}\gamma_0 C)(-C^{-1}\gamma^\mu C)[(-C^{-1}\not{k}C) - m_j](-C^{-1}\gamma^\nu C)\\ &[(-C^{-1}\not{k}C) - m_j](-C^{-1}\gamma^\rho C)(-C^{-1}\gamma_0 C)(-C^{-1}\gamma_0 C)(-C^{-1})u^{s_i}\\ &= \bar{u}^{s_a}\gamma^\mu(\not{k}_2 + m_j)\gamma^\nu(\not{k}_1 + m_j)\gamma^\rho u^{s_i}\end{aligned} \tag{153}$$

A.8 $\psi\psi W^\pm$ Propagator numerator simplification

We can simplify the terms of the equation by separating it into two equation first. Starting with the first one; We can simplify the terms of the equation by separating it into two equation first. Starting with the first one;

$$\begin{aligned}&\gamma^\mu(\not{k}_2 + m_j)\gamma^\nu(\not{k}_2 - \not{k}_b + m_j)\gamma^\rho\\ &= \gamma^\mu\not{k}_2\gamma^\nu\not{k}_2\gamma^\rho\\ &+ \gamma^\mu m_j\gamma^\nu\not{k}_2\gamma^\rho\\ &- \gamma^\mu m_j\gamma^\nu\not{k}_b\gamma^\rho\\ &+ \gamma^\mu m_j\gamma^\nu m_j\gamma^\rho\\ &- \gamma^\mu\not{k}_2\gamma^\nu\not{k}_b\gamma^\rho\\ &+ \gamma^\mu\not{k}_2\gamma^\nu m_j\gamma^\rho\end{aligned} \tag{154}$$

And the other term is shown as

$$\begin{aligned}&- \gamma^\mu(\not{k}_2 - \not{k}_b - m_j)\gamma^\nu(\not{k}_2 - m_j)\gamma^\rho\\ &= -\gamma^\mu\not{k}_2\gamma^\nu\not{k}_2\gamma^\rho\\ &+ \gamma^\mu m_j\gamma^\nu\not{k}_2\gamma^\rho\\ &- \gamma^\mu m_j\gamma^\nu m_j\gamma^\rho\\ &+ \gamma^\mu\not{k}_2\gamma^\nu m_j\gamma^\rho\\ &+ \gamma^\mu\not{k}_b\gamma^\nu\not{k}_2\gamma^\rho\\ &- \gamma^\mu\not{k}_b\gamma^\nu m_j\gamma^\rho\end{aligned} \tag{155}$$

After the simplifications and summing up the indices, we have

$$u^{s_a}\left(8\not{k}_2 m_j - 8\not{k}_b m_j\right)\not{\epsilon}^\lambda u^{s_i} \tag{156}$$

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