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VALUE DISTRIBUTION OF LERCH AND SELBERG ZETA-FUNCTIONS

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Notation

$\mathbb{N}$	the set of all positive integers
$\mathbb{Z}$	the set of all integers
$\mathbb{Q}$	the set of all rational numbers
$\mathbb{R}$	the set of all real numbers
$\mathbb{C}$	the set of all complex numbers
s	the complex variable, $s = \sigma + it$
$\overline{s}$	the conjugate complex variable, $s = \sigma - it$
$\arg(s)$	the argument of complex number s
$\zeta(s)$	the Riemann zeta-function
$\zeta(s, \alpha)$	the Hurwitz zeta-function
$\mathrm{Li}_s(q)$	the polylogarithm function
$L(\lambda, \alpha, s)$	the Lerch zeta-function
$\Phi(q, s, \alpha)$	the Lerch transcendent function
$Z_C(s)$	the Selberg zeta-function defined on the compact Riemann surface
$Z_\Gamma(s)$	the Selberg zeta-function defined for the modular group $\Gamma = SL(2, \mathbb{Z})$

Introduction

This thesis deals with problems I studied as a PHD student at Vilnius University. In particular, I have investigated the limit theorems for the Lerch zeta-function, zero distribution of the Lerch transcendent function and the monotonicity properties of Selberg zeta-functions.

Actuality. Zeta-functions play a major role in the analytic number theory. The Riemann zeta-function and the distribution of prime numbers are related by Riemann hypothesis (RH) or, in other words, the divisibility properties of integer numbers are described by the value distribution of the Riemann zeta-function, especially by the distribution of zeros. The Riemann hypothesis, stated by German mathematician Riemann [39] in 1859 is still an open problem and attracts a lot of attention in Mathematics.

The precise definition of the Riemann zeta function, its relation to the prime number distribution and the formulation of the Riemann hypothesis are given further in section **Summary and main results**.

To understand the behavior of the Riemann zeta-function better, other zeta-functions are also investigated, for instance, polylogarithm function, Lerch, Hurwitz, Selberg zeta-functions, and Lerch transcendental function.

Problems. The aim of this thesis is to obtain new theorems for Lerch and related zeta-functions. More precisely:

- In the forties of the last century Selberg proved that on the critical line $\sigma = 1/2$ suitably normalized logarithm of the Riemann zeta-function is asymptotically normally distributed. The Riemann zeta-function is a special case $\zeta(s) = L(1,1,s)$ of the more general Lerch zeta-function

$$L(\lambda, \alpha, s) = \sum_{n=0}^{\infty} \frac{e^{2\pi i \lambda n}}{(n+\alpha)^s},$$

where $0 < \lambda \leq 1$ and $0 < \alpha \leqslant 1$.

Let $f(n)$ be an arithmetic function. The series $\sum f(n)$ has an Euler product if it can be expressed as an absolutely convergent infinite product,

$$\sum_{n=1}^{\infty} f(n) = \prod_{p} \left(1 + f(p) + f(p^2) + \ldots\right),$$

where the product is extended over all prime numbers.

In general, the Lerch zeta-function has no Euler product. The proof of the limit theorem for the Riemann zeta-function relies on the Euler product. The challenging problem is to prove limit theorem for the Lerch zeta-function.

- Let

$$Li_s(q) = \sum_{n=1}^{\infty} q^n n^{-s}$$

be the polylogarithm function. In 1957, Wiener and Wintner [50] pointed to a possible relationship between the behavior of the zeros in the right-half-plane $\sigma > 1$ of the polylogarithm function and the Riemann Hypothesis, which asserts that all non-trivial zeros of the Riemann zeta-function $\zeta(s)$ are located on the line $\mathrm{Re}(s) = 1/2$. They proved that the Riemann Hypothesis is true if there exists a number $0 < \varepsilon < 1$ such that

$$\sum_{n=1}^{\infty} q^n n^{-s} \neq 0,$$

for $\sigma > 1$ and $1 - \varepsilon < q < 1$.

However, in 1983, Montgomery [34] pointed that the polylogarithm function $Li_s\left(e^{-1/N}\right)$ has zeros in the region $\sigma > 1$ for all sufficiently large integers N, making Wiener and Winter theorem vacuous.

We investigate the Lerch transendent function

$$\Phi(q, s, \alpha) = \sum_{n=0}^{\infty} \frac{q^n}{(n+\alpha)^s},$$

as a generalization of function $Li_s(q)$. The main goal is to compare the distribution of zeros of functions $\Phi(q, s, \alpha)$ and $Li_s(q)$ in the half-plane $\mathrm{Re}(s) > 1$ expecting that Wieners and Winters criteria (probably in a different way) is still somehow related to the Riemann hypothesis.

Let $0 < q < 1$ and $1/2 < \alpha \leq 1$. We expect that the Lerch transcendent function $\Phi(q, s, \alpha)$ has zeros in $\mathrm{Re}(s) > 1$, if q is sufficiently near to 1. For $\alpha = 1$, this is due to Montgomery [34], but Montgomery showed only the existence of such zero (zeros).

- In the paper [40] Saidak and Zvengrowski proved the following fact for the modulus of the Riemann zeta-function: for $0 \leqslant \Delta \leqslant 1/2$ and $t \geqslant 2\pi + 1$ holds the inequality

$$|\zeta(1/2 - \Delta + it)| \geqslant |\zeta(1/2 + \Delta + it)|.$$

The authors also pointed that if the inequality could be strengthened to show that, for $0 < \Delta \leqslant 1/2$, one has

$$|\zeta(1/2 - \Delta + it)| > |\zeta(1/2 + \Delta + it)|,$$

then the Riemann Hypothesis would be true. In the paper of Matiyasevich, Saidak and Zvengrowski [33] there was proved the following relation between functions $\zeta(s)$ and $\xi(s)$. The functions ζ and ξ satisfy the inequality

$$\mathrm{Re}\left(\frac{\zeta'(s)}{\zeta(s)}\right) < \mathrm{Re}\left(\frac{\xi'(s)}{\xi(s)}\right),$$

for $|t| \geqslant 8$, $\sigma < 1/2$. Sondow and Dumitrescu [43] proved that the following three statements are equivalent.

I. If t is any fixed real number, then $|\xi(\sigma + it)|$ is increasing for $1/2 < \sigma < \infty$.

II. If t is any fixed real number, then $|\xi(\sigma + it)|$ is decreasing for $-\infty < \sigma < 1/2$.

III. The Riemann hypothesis is true.

We are interested in determining whether the Selberg zeta-functions have a similar properties. Note that for the Selberg zeta-functions the analog of Riemann hypothesis is usually valid.

Main Methods. The main methods used in this thesis are theory of functions of complex variable, ideas of analytic number theory, also the probabilistic methods of measure convergence, and the computer calculations with program MATHEMATICA.

Conferences. The main results of the thesis were presented at the following international conferences:

- *50th Conference of Lihuanian Mathematical Society*, Vilnius, Lithuania, 2009.
- *Number Theory and its Applications*, An International Conference Dedicated to Kálmán Győry, Attila Pethő, János Pintz and András Sárközy, Debrecen, Hungary, 2010.
- *16th International Conference on Mathematical Modelling and Analysis*, Sigulda, Latvia, 2011.
- *27th Journées Arithmétiques*, Vilnius, Lithuania, 2011.
- *International Conference in Honour of Jonas Kubilius*, Palanga, Lithuania, 2011.
- *Arctic Number Theory School*, Helsinki, Finland, 2011.

Moreover, the results of the thesis were presented at the seminar at the Department of Probability Theory and Number Theory of the Faculty of Mathematics and Informatics of Vilnius University.

Publications. The main results of the thesis were published in the following papers:

- R. Garunkštis, A. Grigutis, A. Laurinčikas, *Selberg's central limit theorem on the critical line and the Lerch zeta-function*, New Directions in Value Distribution Theory of zeta and L-Functions: proceedings of Würzburg Conference, October 6-10, 2008, Shaker Verlag, 57–64, 2009.

- R. Garunkštis, A. Grigutis, *Zeros of the Lerch Transcendent Function*, Math. Model. Anal., Volume 17, Number 2, Taylor&Francis and VGTU, 245–250, 2012.
- A. Grigutis, *Selberg's Central Limit Theorem on the Critical Line and the Lerch Zeta-Function II*, Šiauliai Mathematical Seminar, Šiauliai, 31–40, 2012.

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Summary and main results. We start with notations and definitions of Dirichlet series. We use notations of a big O and $\ll$ interchangeably to describe the limiting behavior of a function when its variable tends towards infinity. We write

$$f(t) = O(g(t)) \text{ or } f(t) \ll g(t)$$

if and only if there exists a positive real number C and a real number t_0 such that

$$|f(t)| \leqslant C|g(t)|$$

for all $t > t_0$.

In this thesis T always tends to infinity and ε is a positive number as small as we want.

The set of complex numbers is denoted by $\mathbb{C}$. Let $s = \sigma + it$ be a complex variable. By *Dirichlet series* we mean a series of the form

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

where the coefficients a_n are any given numbers. The more general series

$$\sum_n a_n e^{-\lambda_n s}$$

is called a *general Dirichlet series* or sometimes also just a *Dirichlet series* if it is not confusing. The special type $\sum_{n\geqslant 1} a_n n^{-s}$ may be obtained by putting $\lambda_n = \log n$ in to a general Dirichlet series. The best known Dirichlet series, with coefficients $a_n \equiv 1$, is the *Riemann zeta-function*, which is defined as a series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

for $\sigma > 1$ and can be analytically continued to the whole complex plane, except a point $s = 1$, where it has a simple pole with residue 1. The famous Riemann hypothesis, formulated in Riemann [39], asserts that $\zeta(s) \neq 0$, for $\sigma > 1/2$. The relation between this conjecture and prime number distribution is well illustrated using a dependence given by Von Koch [30]:

$$\pi(x) = \int_2^x \frac{du}{\log u} + O\left(x^{\alpha+\varepsilon}\right) \quad \Leftrightarrow \quad \zeta(s) \neq 0, \ \textit{for } \sigma > \alpha,$$

where $1/2 \leqslant \alpha < 1$ and $\pi(x)$ is a function counting a prime numbers up to a given number x, i.e.

$$\pi(x) = \sum_{p\leqslant x} 1.$$

Zeta-functions are generated by Dirichlet series with respect to the coefficients a_n and λ_n. For example, the *Hurwitz zeta-function*

$$\sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^s}, \ \sigma > 1, \ 0 < \alpha \leqslant 1$$

is a general Dirichlet series with $a_n = 1$ and $\lambda_n = \log(n+\alpha)$ or the *Lerch zeta-function*

$$L(\lambda, \alpha, s) = \sum_{n=0}^{\infty} \frac{e^{2\pi i \lambda n}}{(n+\alpha)^s}, \ \sigma > 1, \ 0 < \lambda, \alpha \leqslant 1$$

is also a general Dirichlet series with $a_n = e^{2\pi i \lambda n}$ and $\lambda_n = \log(n+\alpha)$.

In Mathematics, a branch of number theory that uses methods from mathematical analysis to investigate the properties of the integer numbers is called an *Analytic Number Theory*. Books of Apostol [1], Davenport [5], Montgomery [35], Iwaniec and Kowalski [25], Newmann [36] are known as a classic of analytic number theory. Monographs of Titchmarsh [45] and [44] are well known as a classic of the theory of the Riemann zeta-function and the theory of functions.

Limit theorems for the Lerch zeta-function

In the first half of the last century, Selberg proved that on the critical line $\sigma = 1/2$ the suitably normalized logarithm of the Riemann zeta-function is asymptotically normally distributed. Here, we formulate his result separately for real and imaginary parts. Denote by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du$$

the standard normal distribution function. We have

$$\lim_{T\to\infty} \frac{1}{T} \text{meas} \left\{ t \in [0,T] : \frac{\log|\zeta(1/2+it)|}{\sqrt{2^{-1}\log\log T}} < x \right\} = \Phi(x)$$

and the same equality is true if we replace $\log|\zeta(1/2+it)|$ by $\arg\zeta(1/2+it)$. The limit law remains the same if σ is "near" to the critical line. Let $\beta_T = \log T$ for $1/2 \leqslant \sigma \leqslant 1/2 + 1/\log T$ and $\beta_T = \log(1/(\sigma - 1/2))$ for $1/2 + 1/\log T < \sigma < 1/2 + o(1)$ as $T \to \infty$. Then

$$\lim_{T\to\infty} \frac{1}{T} \text{meas} \left\{ t \in [0,T] : \frac{\log|\zeta(\sigma+it)|}{\sqrt{2^{-1}\log\beta_T}} < x \right\} = \Phi(x).$$

Similar results, when σ is "near" to the critical line, are known for $\arg\zeta(\sigma+it)$. For these results and for their proofs see Bombieri and Hejhal [3], Joyner [26], Laurinčikas [28], Selberg [42], Tsang [46].

The Riemann zeta-function is a special case of the more general Lerch zeta-function

$$L(\lambda, \alpha, s) = \sum_{n=0}^{\infty} \frac{e^{2\pi i\lambda n}}{(n+\alpha)^s},$$

where $0 < \lambda \leq 1$ and $0 < \alpha \leqslant 1$ (see Laurinčikas and Garunkštis [29] for more information). In general, the Lerch zeta-function has no Euler product. The proof of the limit theorem for the Riemann zeta-function relies on the Euler product, and it is not clear whether limit theorems can be obtained for the Lerch zeta-function.

Bellow we formulate several cases of limit theorems for the Lerch zeta-functions.

Theorem 1.1. *Let*

$$1 - T^{-1-\varepsilon} < \lambda < 1 - e^{-T/\log T} \quad \textit{or} \quad \lambda = 1 \quad \textit{or} \quad |\lambda - 1/2| < T^{-1-\varepsilon}$$

and let

$$1 - T^{-1-\varepsilon} < \alpha \leq 1 \quad \textit{or} \quad |\alpha - 1/2| < T^{-1-\varepsilon}.$$

Then

$$\lim_{T\to\infty} \nu_T \left(\frac{\log |L(\lambda, \alpha, 1/2 + it)|}{\sqrt{2^{-1} \log\log T}} < x \right) = \Phi(x).$$

Let

$$E(\lambda, \alpha, t) = -(2\pi/t)^{it} \exp\left\{ it + \frac{3\pi i}{4} - 2\pi i \lambda\alpha + 2\pi i \alpha \right\} \frac{e^{-\pi t}}{(1 - \{\lambda\})^{1/2-it}}.$$

To obtain a limit law when λ tends to 1 too "quickly", we need a certain modification.

Theorem 1.2. *Let* $1 - e^{-T/\log T} \leq \lambda < 1$ *and let*

$$1 - T^{-1-\varepsilon} < \alpha \leq 1 \quad \textit{or} \quad |\alpha - 1/2| < T^{-1-\varepsilon}.$$

Then

$$\lim_{T\to\infty} \nu_T \left(\frac{\log |L(\lambda, \alpha, 1/2 + it) - E(\lambda, \alpha, t)|}{\sqrt{2^{-1} \log\log T}} < x \right) = \Phi(x).$$

Below we formulate the limit theorems for $\arg L(\lambda, \alpha, s)$.

Theorem 1.3. *Let*

$$1 - T^{-1-\varepsilon} < \lambda < 1 - e^{-T/\log T} \quad \textit{or} \quad \lambda = 1 \quad \textit{or} \quad |\lambda - 1/2| < T^{-1-\varepsilon}$$

and let $1 - T^{-1-\varepsilon} < \alpha \leq 1$. *Then*

$$\lim_{T\to\infty} \nu_T \left(\frac{\arg L(\lambda, \alpha, 1/2 + it)}{\sqrt{2^{-1} \log\log T}} < x \right) = \Phi(x).$$

THEOREM 1.4. *Let*

$$1-T^{-1-\varepsilon}<\lambda<1-e^{-T/\log T} \quad or \quad \lambda=1 \quad or \quad |\lambda-1/2|<T^{-1-\varepsilon}$$

and let $|\alpha-1/2|<T^{-1-\varepsilon}$. *Then*

$$\lim_{T\to\infty}\nu_T\left(\frac{\arg L(\lambda,\alpha,1/2+it)-t\log 2/2\pi}{\sqrt{2^{-1}\log\log T}}<x\right)=\Phi(x).$$

These theorems are proved in Section 1.

In Theorems 1.1 – 1.4, we investigate the limit laws when (λ,α) is near to $(1,1)$, $(1,1/2)$, $(1/2,1)$ and $(1/2,1/2)$. However one may ask, if the limit law remains similar when (λ,α) is near to $(0,0)$, $(0,1/2)$, $(0,1)$, $(1/2,0)$ and $(1,0)$. All these cases may be presented in a following table:

(0,0)	(1/2,0)	(1,0)
(0,1/2)	(1/2,1/2)	(1,1/2)
(0,1)	(1/2,1)	(1,1)

Let

$$\Lambda(\lambda,\alpha,t)=\left(\frac{2\pi}{t}\right)^{it}\exp\left\{it+\frac{\pi i}{4}-2\pi i\lambda\alpha\right\}\frac{1}{\lambda^{1/2-it}}.$$

Consider the cases:

a) $$0<\lambda\leqslant\frac{1}{T(\log T)^{1+\varepsilon}} \quad \text{and} \quad 1-\frac{1}{T(\log T)^{1+\varepsilon}}\leqslant\alpha\leqslant 1,$$

b) $$0<\lambda\leqslant\frac{1}{T(\log T)^{1+\varepsilon}} \quad \text{and} \quad |\alpha-1/2|\leqslant\frac{1}{T(\log T)^{1+\varepsilon}},$$

c) $$0<\lambda\leqslant\frac{1}{T(\log T)^{1+\varepsilon}} \quad \text{and} \quad 0<\alpha\leqslant\frac{1}{T(\log T)^{1+\varepsilon}},$$

d) $$|\lambda-1/2|\leqslant\frac{1}{T(\log T)^{1+\varepsilon}} \quad \text{and} \quad 0<\alpha\leqslant\frac{1}{T(\log T)^{1+\varepsilon}},$$

e) $$1-\frac{1}{T(\log T)^{1+\varepsilon}}\leqslant\lambda<1-e^{-T/\log T} \quad \text{and} \quad 0<\alpha\leqslant\frac{1}{T(\log T)^{1+\varepsilon}},$$

f) $$1-e^{-T/\log T}\leqslant\lambda<1 \quad \text{and} \quad 0<\alpha\leqslant\frac{1}{T(\log T)^{1+\varepsilon}}.$$

THEOREM 1.5. *If we ssume a) or b) then*

$$\lim_{T\to\infty} \nu_T \left(\frac{\log |L(\lambda, \alpha, 1/2 + it) - \Lambda(\lambda, \alpha, t)|}{\sqrt{2^{-1} \log \log T}} < x \right) = \Phi(x).$$

If we assume c) then

$$\lim_{T\to\infty} \nu_T \left(\frac{\log |L(\lambda, \alpha, 1/2 + it) - \Lambda(\lambda, \alpha, t) - 1/\alpha^{1/2+it}|}{\sqrt{2^{-1} \log \log T}} < x \right) = \Phi(x).$$

If we assume d) or e) then

$$\lim_{T\to\infty} \nu_T \left(\frac{\log |L(\lambda, \alpha, 1/2 + it) - 1/\alpha^{1/2+it}|}{\sqrt{2^{-1} \log \log T}} < x \right) = \Phi(x).$$

If we assume f) then

$$\lim_{T\to\infty} \nu_T \left(\frac{\log |L(\lambda, \alpha, 1/2 + it) - E(\lambda, \alpha, t) - 1/\alpha^{1/2+it}|}{\sqrt{2^{-1} \log \log T}} < x \right) = \Phi(x),$$

where $E(\lambda, \alpha, t)$ is a function of the form

$$E(\lambda, \alpha, t) = -(2\pi/t)^{it} \exp\left\{ it + \frac{3\pi i}{4} - 2\pi i \lambda \alpha + 2\pi i \alpha \right\} \frac{e^{-\pi t}}{(1 - \{\lambda\})^{1/2-it}}.$$

For cases from a) to f), similar theorems can be formulated for $\arg L(\lambda, \alpha, 1/2 + it)$. We formulate only one example below.

THEOREM 1.6. *If we assume a), then*

$$\lim_{T\to\infty} \nu_T \left(\frac{\arg(L(\lambda, \alpha, 1/2 + it) - \Lambda(\lambda, \alpha, t))}{\sqrt{2^{-1} \log \log T}} < x \right) = \Phi(x).$$

As an interesting fact, we note that $L(0, \alpha, 1/2 + it) = L(1, \alpha, 1/2 + it)$, but the limit behavior of $L(\lambda, \alpha, s)$ differs when λ tends to 0 and 1.

Theorem 1.5 is also proved in Section 1.

ZERO DISTRIBUTION OF THE LERCH TRANSCENDENT FUNCTION

The Lerch transcendent function is the analytic continuation of the series

$$\Phi(q, s, \alpha) = \sum_{n=0}^{\infty} \frac{q^n}{(n + \alpha)^s},$$

which converges for any real number $\alpha > 0$ if q and s are complex numbers with either $|q| < 1$, or $|q| = 1$, and $\sigma > 1$. Here we consider $\Phi(q, s, \alpha)$ as a function of s with the parameters $q \in \mathbb{C}$, $0 < |q| \leq 1$, and $0 < \alpha \leq 1$.

The Riemann zeta-function has no zeros in the right-half-plane $\sigma \geq 1$. In the left-half-plane $\sigma \leq 0$, it has only trivial zeros at even negative integers. The famous Riemann hypothesis (RH) asserts that the remaining, nontrivial, zeros lie on the critical line $\sigma = 1/2$.

The Hurwitz zeta-function $\zeta(s, \alpha)$ has infinitely many zeros in $1 < \sigma < 1 + \alpha$ if α is transcendental or rational $\neq 1/2$, 1 (Davenport and Heilbronn [6]). This result was extended by Cassels [4] for α algebraic irrational number. Let $1/2 < \sigma_1 < \sigma_2 < 1$. Then, Voronin [48] (for rational $\alpha \neq 1/2, 1$) and Gonek [20] (for transcendental α) proved that the number of zeros of $\zeta(s, \alpha)$ in the rectangle $\sigma_1 < \sigma < \sigma_2$, $0 < t < T$ is of order T for sufficiently large T. Gonek [21] also showed that for certain values of α the proportion of zeros of $\zeta(s, \alpha)$ on $\sigma = 1/2$ is definitely less than 1. In the complex s-plane, trajectories of zeros $\rho = \rho(\alpha)$ of the Hurwitz zeta function were considered in Garunkštis and Steuding [16], [17]. Based on these trajectories, the classification of nontrivial zeros of the Riemann zeta function were introduced. For the zero distribution of the Lerch zeta-function see [14], [11], [12], [18], [29].

Fornberg and Kölbig [8] investigated trajectories of zeros $\rho = \rho(q)$ of the polylogarithm function $Li_s(q)$ for real q with $|q| < 1$. They found that some trajectories tend towards the zeros of $\zeta(s)$ as $q \to -1$ and approach these zeros closely as $q \to 1 - \delta$ for small but finite $\delta > 0$. However, the later trajectories appear to descend to the point $s = 1$ as $\delta \to 0$. Both, for $q \to -1$ and $q \to 1$, there are trajectories which do not tend towards zeros of $\zeta(s)$.

We consider the zeros of $\Phi(q, s, \alpha)$ for $0 < \alpha < 1$ and $q \in \mathbb{C}, 0 < |q| < 1$. Let $N_\Phi(\sigma_1, \sigma_2, T) = N_\Phi(\sigma_1, \sigma_2, T, q, \alpha)$ denote the number of zeros of $\Phi(q, s, \alpha)$ in the region $\{s : \sigma_1 < \mathrm{Re}(s) < \sigma_2, 0 < \mathrm{Im}\ \mathrm{s} \leqslant T\}$. Let $\sigma_0 = \sigma_0(q, \alpha)$ be a real number defined by the equality

$$\sum_{n=1}^{\infty} \frac{|q|^n}{\left(\frac{n}{\alpha} + 1\right)^{\sigma_0}} = 1.$$

It is easy to see that $\sigma_0 \leq c = 1.73\ldots$, where $\zeta(c) = \sum_{n=1}^{\infty} n^{-c} = 2$, and that σ_0 can take any value between $-\infty$ and c.

THEOREM 2.1. *Let $q \in \mathbb{C}$, $0 < |q| < 1$. Let $0 < \alpha < 1$ be a transcendental number. Then, for any fixed strip $\sigma_1 < \sigma < \sigma_2 \le \sigma_0$, we have that*

$$T \ll N_\Phi(\sigma_1, \sigma_2, T) \ll T$$

and $\Phi(q, s, \alpha)$ has no zeros for $\sigma > \sigma_0$.

Theorem is proved in Section 2.

Wiener and Wintner [50, Section 4] pointed to a possible relationship between the behavior of the zeros in the right-half-plane $\sigma > 1$ of the polylogarithm function and the Riemann Hypothesis. It was shown that the Riemann Hypothesis is true if there exists a number $0 < \varepsilon < 1$, such that

$$Li_s(q) = \sum_{n=1}^{\infty} q^n n^{-s} \neq 0,$$

for $\sigma > 1$ and $1 - \varepsilon < q < 1$. However, Montgomery [34] stated that the polylogarithm function $Li_s\left(e^{-1/N}\right)$ has zeros in the region $\sigma > 1$ for all sufficiently large integers N, making Wiener and Winter theorem vacuous. Theorem 2.1 shows that the Lerch transcendent function $\Phi(q, s, \alpha)$ also has zeros in the region $\sigma > 1$ for $0.92 < q < 1$ and transcendental α, $1/2 < \alpha < 1$; however, computer calculations indicate the different behavior of zeros of $\Phi(q, s, \alpha)$ in $\mathrm{Re}(s) > 1$ dependently on $\alpha = 1$ or $\alpha \neq 1$.

Tables of computer calculations are presented in Subsection 2.1.

On the behavior of the Selberg zeta-functions in the critical strip

Recall that the Riemann zeta-function $\zeta(s)$ can be analytically continued to the whole complex plane except for a simple pole at s = 1 with residue 1, and it satisfies the important functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s)\zeta(1-s).$$

or

$$\xi(s) = \xi(1-s),$$

where

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s).$$

This functional equation plays a central role in the theory of zeta function, however in [23] p. 137 Grosswald says: "... the functional equation - probably our most powerful tool, so far - does not give as much information on what happens for $0 \leqslant \sigma \leqslant 1$, in the so called critical strip."

In the paper [40] Saidak and Zvengrowski found that $|\zeta(\sigma + it)|$ behaves very nicely with respect to σ, as contrast to incredibly complicated behavior with respect to t. Contrary to Grosswald's claim, it was also shown that one can apply certain sharp estimates of the Euler-MacLaurin type (together with the functional equation $\xi(s) = \xi(1-s)$) in order to obtain proofs of some interesting new results concerning the horizontal behavior of $|\zeta|$ in the critical strip. Saidak and Zvengrowski proved the following fact for the modulus of the Riemann zeta-function.

Theorem A. *For* $0 \leqslant \Delta \leqslant 1/2$ *and* $t \geqslant 2\pi + 1$ *we have*

$$|\zeta(1/2 - \Delta + it)| \geqslant |\zeta(1/2 + \Delta + it)|.$$

The authors also pointed that if the inequality could be strengthened to show that, for $0 < \Delta \leqslant 1/2$, one has $|\zeta(1/2 - \Delta + it)| > |\zeta(1/2 + \Delta + it)|$, then the Riemann Hypothesis would follow. The authors also give accurate but simple asymptotic estimates for the quotient

$$\alpha(\Delta, t) := \frac{|\zeta(1/2 - \Delta + it)|}{|\zeta(1/2 + \Delta + it)|}.$$

They proved: If Δ and t are in the range $0 \leqslant \Delta \leqslant 1/2$, then

$$\alpha(\Delta, t) \sim \left(\frac{|s|}{2\pi}\right)^{\Delta} \sim \left(\frac{t}{2\pi}\right)^{\Delta}.$$

In the paper of Matiyasevich, Saidak, and Zvengrowski [33] there was proved the following relation between functions $\zeta(s)$ and $\xi(s)$.

Theorem B. *The functions* ζ *and* ξ *satisfy the inequality*

$$\operatorname{Re}\left(\frac{\zeta'(s)}{\zeta(s)}\right) < \operatorname{Re}\left(\frac{\xi'(s)}{\xi(s)}\right),$$

for $|t| \geqslant 8$, $\sigma < 1/2$.

Sondow and Dumitrescu [43] proved the following theorem for the function ξ.

THEOREM C. *The xi function is increasing in modulus along every horizontal half-line lying in any open right half-plane that contains no xi zeros. Similarly, the modulus decreases on each horizontal half-line in any zero-free, open left half-plane.*

In the same paper, there was given the following reformulation for the Riemann hypothesis.

THEOREM D. *The following statements are equivalent.*
I. *If* t *is any fixed real number, then* $|\xi(\sigma+it)|$ *is increasing for* $1/2 < \sigma < \infty$.
II. *If* t *is any fixed real number, then* $|\xi(\sigma+it)|$ *is decreasing for* $-\infty < \sigma < 1/2$.
III. *The Riemann hypothesis is true.*

We ask whether Selberg zeta-functions have a similar properties as the Riemann-zeta function has in theorems A-D. Note that for Selberg zeta-functions the analog of RH is usually valid.

For the modular group $\Gamma = SL(2,\mathbb{Z})$, the Selberg zeta-function is defined by the Euler product (cf. Fischer [9], Hejhal [24])

$$Z_\Gamma(s) = \prod_{\{P\}} \prod_{k=0}^{\infty} (1 - N(P)^{-s-k})\,(\sigma > 1)$$

where $\{P\}$ runs trough all the primitive hyperbolic conjugacy classes of Γ and $N(P) = \alpha^2$ if the eigenvalues of P are α and α^{-1} ($|\alpha| > 1$).

Similarly as the Riemann zeta-function, the Selberg zeta-function $Z_\Gamma(s)$ for the modular group $\Gamma = SL(2,\mathbb{Z})$ also has a meromorfic continuation to the whole complex plane, and it satisfies the symmetric functional equation (see Kurokawa [27])

$$\Xi(s) = \Xi(1-s),$$

where

$$\Xi(s) = Z_\Gamma(s) Z_{id}(s) Z_{ell}(s) Z_{par}(s)$$

and

$$Z_{id}(s) = \left(\frac{(2\pi)^s}{\Gamma(s)}\right)^{1/6} (\Gamma_2(s))^{1/3},$$

$$Z_{ell}(s) = \Gamma\left(\frac{s}{2}\right)^{-1/2} \Gamma\left(\frac{s+1}{2}\right)^{1/2} \Gamma\left(\frac{s}{3}\right)^{-2/3} \Gamma\left(\frac{s+2}{3}\right)^{2/3},$$

$$Z_{par}(s) = \frac{\pi^s}{\Gamma(s)\zeta(2s)\Gamma(s+1/2)2^s},$$

where $\Gamma(s)$ denotes the Euler gamma function and $\zeta(s)$ is the Riemann zeta function. The function $\Gamma_2(s)$ is called the double gamma function of Barnes. It is defined by the canonical product

$$\frac{1}{\Gamma_2(s+1)} =$$
$$(2\pi)^{\frac{s}{2}} \exp\left\{-\frac{s}{2} - \frac{(\gamma+1)s^2}{2}\right\} \prod_{k=1}^{\infty} \left\{\left(1+\frac{s}{k}\right)^k \exp\left(-s+\frac{s^2}{2k}\right)\right\},$$

where γ denotes the Euler's constant. The function $\Gamma_2(s)$ has the following properties

$$\Gamma_2(1) = 1,\ \Gamma_2(s+1) = \frac{\Gamma_2(s)}{\Gamma(s)},\ \Gamma_2(n+1) = \frac{1^2 \cdot 2^2 \cdots n^2}{(n!)^n},$$

see Sarnak [41] and Vignéras [47].

We obtain the following theorem.

Theorem 3.1. *There exists a positive number C, such that*

$$\mathrm{Re}\left(\frac{\Xi'(s)}{\Xi(s)}\right) < 0$$

for $t > C$ and $0 < \sigma < 1/2$.

Further more, if we assume the Riemann hypothesis for $\zeta(s)$, then there exists a positive number C_1, such that

$$\mathrm{Re}\left(\frac{Z'_\Gamma(s)}{Z_\Gamma(s)}\right) < \mathrm{Re}\left(\frac{\Xi'(s)}{\Xi(s)}\right)$$

for $t > C_1$ and $0 < \sigma < 1/4$.

Let F denote a compact Riemann surface of genus $g \geqslant 2$. F can be represented as a quotient space $\Gamma\backslash H$, where $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ is a strictly hyperbolic Fuchsian group, and H is the upper half-plane. The Γ conjugacy class determined by

$P \in \Gamma$ will be denoted by $\{P\}$ and its norm by $N\{P\}$. By P_0 will be denoted the primitive element of Γ. The Selberg zeta-function for the compact Riemann surface F is given by (Hejhal [24])

$$Z_C(s) = \prod_{\{P_0\}} \prod_{k=0}^{\infty} (1 - N(P_0)^{-s-k}), (\sigma > 1).$$

It is an entire function of order 2 with a functional equation

$$Z_C(s) = f(s) Z_C(1-s),$$

where

$$f(s) = \exp\left(4\pi(g-1)\int_0^{s-1/2} v\tan(\pi v)\,dv\right).$$

The above functional equation is equivalent to $M(s) = M(1-s)$, where

$$M(s) = Z_C(s)\exp\left(2\pi(g-1)\int_0^{1/2-s} v\tan\pi v\,dv\right).$$

THEOREM 3.4. *There exists a number $B > 0$ such that the functions $Z_C(s)$ and $M(s)$, for $t > B$, $0 < \sigma < 1/2$ satisfies the inequality*

$$\operatorname{Re}\left(\frac{Z_C'(s)}{Z_C(s)}\right) < \operatorname{Re}\left(\frac{M'(s)}{M(s)}\right) < 0.$$

For the modulus of the function $Z_C(s)$ in the critical strip holds the following theorem.

THEOREM 3.5. *For $0 < \Delta \leqslant 1/2$ and $t > t_0$*

$$|Z_C(1/2-\Delta+it)| > |Z_C(1/2+\Delta+it)|,$$

where

$$t_0 = \frac{1}{\pi}\log\frac{2}{\sqrt{5}-1} = 0.15\ldots$$

Theorems 3.1, 3.4, and 3.5 are proved in Section 3.

1. Limit theorems for the Lerch zeta-function

In the 1940s, Selberg proved that on the critical line $\sigma = 1/2$ suitably normalized logarithm of the Riemann zeta-function is asymptotically normally distributed. We formulate his result separately for real and imaginary parts of the Riemann zeta-function. Denote by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du$$

the standard normal distribution function. We have

$$\lim_{T\to\infty} \frac{1}{T} \text{meas} \left\{ t \in [0,T] : \frac{\log|\zeta(1/2+it)|}{\sqrt{2^{-1}\log\log T}} < x \right\} = \Phi(x)$$

and the same equality is true if we replace $\log|\zeta(1/2+it)|$ by $\arg\zeta(1/2+it)$. The limit law remains the same if σ is "near" to the critical line. Let $\beta_T = \log T$ for $1/2 \leqslant \sigma \leqslant 1/2 + 1/\log T$ and $\beta_T = \log(1/(\sigma - 1/2))$ for $1/2 + 1/\log T < \sigma < 1/2 + o(1)$ as $T \to \infty$. Then

$$\lim_{T\to\infty} \frac{1}{T} \text{meas} \left\{ t \in [0,T] : \frac{\log|\zeta(\sigma+it)|}{\sqrt{2^{-1}\log\beta_T}} < x \right\} = \Phi(x).$$

Similar results, when σ is "near" to the critical line, are known for $\arg\zeta(\sigma+it)$. For these results and their proofs see Bombieri and Hejhal [3], Joyner [26], Laurinčikas [28], Selberg [42], Tsang [46].

The Riemann zeta-function is a special case of the more general Lerch zeta-function

$$L(\lambda, \alpha, s) = \sum_{n=0}^{\infty} \frac{e^{2\pi i \lambda n}}{(n+\alpha)^s},$$

where $0 < \lambda \leq 1$ and $0 < \alpha \leqslant 1$. In general, the Lerch zeta-function has no Euler product, and the proof of the limit theorem for the Riemann zeta-function relies on the Euler product, and it is not clear whether limit theorems can be obtained for all Lerch zeta-functions. Similarly as Selberg's limit theorem remains true for $\zeta(\sigma+it)$ when σ is near to 1/2, one expects that the same is true for $L(\lambda, \alpha, 1/2+it)$ when (λ, α) is near to $(1,1)$, $(1/2,1)$, $(1,1/2)$ or $(1/2,1/2)$. For these four cases, we have

$$L(1,1,s)=\zeta(s),\quad L(1,1/2,s)=(2^s-1)\zeta(s), \tag{1.1}$$
$$L(1/2,1,s)=(1-2^{1-s})\zeta(s),\quad \text{and}\quad L(1/2,1/2,s)=2^s L(s,\chi),$$

where χ is a Dirichlet character $\mod 4$ with $\chi(3)=-1$.

First, we consider limit theorems for $\log|L(\lambda,\alpha,1/2+it)|$. We use notation

$$\nu_T(\dots)=T^{-1}\text{meas}\{t\in[0,T]:\dots\},$$

where in place of dots a condition satisfied by t is written. By ε we always denote a positive fixed number which can be as small as we want.

Theorem 1.1. *Let*

$$1-T^{-1-\varepsilon}<\lambda<1-e^{-T/\log T}\quad or\quad \lambda=1\quad or\quad |\lambda-1/2|<T^{-1-\varepsilon}$$

and let

$$1-T^{-1-\varepsilon}<\alpha\le 1\quad or\quad |\alpha-1/2|<T^{-1-\varepsilon}.$$

Then

$$\lim_{T\to\infty}\nu_T\left(\frac{\log|L(\lambda,\alpha,1/2+it)|}{\sqrt{2^{-1}\log\log T}}<x\right)=\Phi(x). \tag{1.2}$$

If the parameter λ is "very" close to 1, then the Lerch zeta-function becomes "very" large. From the approximation by a finite sum (1.4) (see Section 1) we have that

$$|L(\lambda,\alpha,1/2+it)|=\frac{e^{-\pi t}}{(1-\{\lambda\})^{1/2}}+O\left(t^{1/4}+\frac{1}{\sqrt{\lambda}}+\frac{1}{\sqrt{\alpha}}\right)$$

uniformly for $0<\lambda,\alpha\le 1$, as $t\to\infty$. We can remove that large term and again obtain central limit theorem when λ is "very" close to one. Let

$$E(\lambda,\alpha,t)=-(2\pi/t)^{it}\exp\left\{it+\frac{3\pi i}{4}-2\pi i\lambda\alpha+2\pi i\alpha\right\}\frac{e^{-\pi t}}{(1-\{\lambda\})^{1/2-it}}.$$

Theorem 1.2. *Let* $1-e^{-T/\log T}\le\lambda<1$ *and let*

$$1-T^{-1-\varepsilon}<\alpha\le 1\quad or\quad |\alpha-1/2|<T^{-1-\varepsilon}.$$

Then
$$\lim_{T\to\infty} \nu_T \left(\frac{\log|L(\lambda,\alpha,1/2+it)-E(\lambda,\alpha,t)|}{\sqrt{2^{-1}\log\log T}} < x \right) = \Phi(x).$$

Now we turn to limit theorems for $\arg L(\lambda,\alpha,s)$. The Lerch zeta-function $L(\lambda,\alpha,s)$ does not vanish for $\Re s > 1+\alpha$ (see [29]). We follow tradition and define $\arg L(\lambda,\alpha,s)$ by continuous displacement from the point $s=2$ (choosing $-\pi < \arg L(\lambda,\alpha,2) \le \pi$) along the straight lines connecting the points $s = 2, 2+it$, and $\sigma+it$. If on this way there is a zero, then $\arg L(\lambda,\alpha,s)$ remains undefined. We note that countably many zeros do not affect our measures.

We start with cases when (λ,α) tends to $(1,1)$ and $(1/2,1)$.

Theorem 1.3. *Let*
$$1-T^{-1-\varepsilon} < \lambda < 1-e^{-T/\log T} \quad \textit{or} \quad \lambda = 1 \quad \textit{or} \quad |\lambda - 1/2| < T^{-1-\varepsilon}$$
and let $1-T^{-1-\varepsilon} < \alpha \le 1$. *Then*
$$\lim_{T\to\infty} \nu_T \left(\frac{\arg L(\lambda,\alpha,1/2+it)}{\sqrt{2^{-1}\log\log T}} < x \right) = \Phi(x).$$

In the cases when (λ,α) tends to $(1,1/2)$ and $(1/2,1/2)$, we consider $L(1,1/2,s) = (2^s-1)\zeta(s)$ and $L(1/2,1/2,s) = 2^s L(s,\chi)$. We have that $\arg\left(2^{1/2+it}-1\right)$ and $\arg\left(2^{1/2+it}\right)$ are equal to $\frac{\log 2}{2\pi}t + O(1)$. Selberg's limit theorem is also valid for argument of Dirichlet L-functions (Bombieri and Hejhal [3]). Thus, we obtain the following theorem.

Theorem 1.4. *Let*
$$1-T^{-1-\varepsilon} < \lambda < 1-e^{-T/\log T} \quad \textit{or} \quad \lambda = 1 \quad \textit{or} \quad |\lambda - 1/2| < T^{-1-\varepsilon}$$
and let $|\alpha - 1/2| < T^{-1-\varepsilon}$. *Then*
$$\lim_{T\to\infty} \nu_T \left(\frac{\arg L(\lambda,\alpha,1/2+it) - t\log 2/2\pi}{\sqrt{2^{-1}\log\log T}} < x \right) = \Phi(x).$$

Similarly as before, Theorems 1.3 and 1.4 remain true for $1-e^{-T/\log T} \le \lambda < 1$ if we replace $\arg L(\lambda,\alpha,1/2+it)$ and $\arg L(\lambda,\alpha,1/2+it) - t\log 2/2\pi$ by $\arg\left(L(\lambda,\alpha,1/2+it)-E(\lambda,\alpha,t)\right)$ and $\arg\left(L(\lambda,\alpha,1/2+it)-E(\lambda,\alpha,t)\right) - t\log 2/2\pi$, accordingly.

Now, we turn our attention to the limit theorems for the Lerch zeta-function $L(\lambda, \alpha, 1/2+it)$ when the pair of parameters (λ, α) is near to $(0,0)$, $(0,1/2)$, $(0,1)$, $(1/2,0)$ and $(1,0)$. For a clarity, we set up a table of 9 different pairs of (λ, α):

(0,0)	(1/2,0)	(1,0)
(0,1/2)	(1/2,1/2)	(1,1/2)
(0,1)	(1/2,1)	(1,1)

In theorems 1.1 − 1.4 we investigated the limit laws when (λ, α) are near to $(1,1)$, $(1,1/2)$, $(1/2,1)$ and $(1/2,1/2)$. In the following theorems, we deal with 5 remaining cases from the table above.

Recall that

$$\Lambda(\lambda, \alpha, t) = \left(\frac{2\pi}{t}\right)^{it} \exp\left\{it + \frac{\pi i}{4} - 2\pi i\lambda\alpha\right\} \frac{1}{\lambda^{1/2-it}}.$$

and

$$E(\lambda, \alpha, t) = -\left(\frac{2\pi}{t}\right)^{it} \exp\left\{it + \frac{3\pi i}{4} - 2\pi i\lambda\alpha + 2\pi i\alpha\right\} \frac{e^{-\pi t}}{(1-\{\lambda\})^{1/2-it}}.$$

Consider the cases:

a)
$$0 < \lambda \leqslant \frac{1}{T(\log T)^{1+\varepsilon}} \quad \text{and} \quad 1 - \frac{1}{T(\log T)^{1+\varepsilon}} \leqslant \alpha \leqslant 1,$$

b)
$$0 < \lambda \leqslant \frac{1}{T(\log T)^{1+\varepsilon}} \quad \text{and} \quad |\alpha - 1/2| \leqslant \frac{1}{T(\log T)^{1+\varepsilon}},$$

c)
$$0 < \lambda \leqslant \frac{1}{T(\log T)^{1+\varepsilon}} \quad \text{and} \quad 0 < \alpha \leqslant \frac{1}{T(\log T)^{1+\varepsilon}},$$

d)
$$|\lambda - 1/2| \leqslant \frac{1}{T(\log T)^{1+\varepsilon}} \quad \text{and} \quad 0 < \alpha \leqslant \frac{1}{T(\log T)^{1+\varepsilon}},$$

e)
$$1 - \frac{1}{T(\log T)^{1+\varepsilon}} \leqslant \lambda < 1 - e^{-T/\log T} \quad \text{and} \quad 0 < \alpha \leqslant \frac{1}{T(\log T)^{1+\varepsilon}},$$

f)
$$1 - e^{-T/\log T} \leqslant \lambda < 1 \quad \text{and} \quad 0 < \alpha \leqslant \frac{1}{T(\log T)^{1+\varepsilon}}.$$

Theorem 1.5. *If we assume a) or b) then*

$$\lim_{T\to\infty} \nu_T \left(\frac{\log |L(\lambda,\alpha,1/2+it) - \Lambda(\lambda,\alpha,t)|}{\sqrt{2^{-1}\log\log T}} < x \right) = \Phi(x). \tag{1.3}$$

If we assume c) then

$$\lim_{T\to\infty} \nu_T \left(\frac{\log |L(\lambda,\alpha,1/2+it) - \Lambda(\lambda,\alpha,t) - 1/\alpha^{1/2+it}|}{\sqrt{2^{-1}\log\log T}} < x \right) = \Phi(x).$$

If we assume d) or e) then

$$\lim_{T\to\infty} \nu_T \left(\frac{\log |L(\lambda,\alpha,1/2+it) - 1/\alpha^{1/2+it}|}{\sqrt{2^{-1}\log\log T}} < x \right) = \Phi(x).$$

If we assume f) then

$$\lim_{T\to\infty} \nu_T \left(\frac{\log |L(\lambda,\alpha,1/2+it) - E(\lambda,\alpha,t) - 1/\alpha^{1/2+it}|}{\sqrt{2^{-1}\log\log T}} < x \right) = \Phi(x).$$

For cases from a) to f), similar theorems can be formulated for $\arg L(\lambda,\alpha,1/2+it)$. We formulate only one example below.

Theorem 1.6. *If we assume a), then*

$$\lim_{T\to\infty} \nu_T \left(\frac{\arg(L(\lambda,\alpha,1/2+it) - \Lambda(\lambda,\alpha,t))}{\sqrt{2^{-1}\log\log T}} < x \right) = \Phi(x).$$

As an interesting fact, we note that $L(0,\alpha,1/2+it) = L(1,\alpha,1/2+it)$, but the limit behavior of $L(\lambda,\alpha,s)$ differs when λ tends to 0 and 1, compare equalities (1.2) and (1.3).

We note that Theorems 1.1 − 1.4 remain true if in the conditions of them we replace $T^{-1-\varepsilon}$ by $1/T(\log T)^{1+\varepsilon}$.

Theorems are proved in the next section.

1.1. **Proofs.**

First we prove several lemmas and later derive theorems.

Lemma 1.7. *For (λ,α) equal to $(1,1)$, $(1/2,1)$, $(1,1/2)$ or $(1/2,1/2)$, we have that*

$$\lim_{T\to\infty} \nu_T \left(\frac{\log |L(\lambda,\alpha,1/2+it)|}{\sqrt{2^{-1}\log\log T}} < x \right) = \Phi(x).$$

Proof. One proof of Selberg's central limit theorem can be found in Bombieri and Hejhal [3]. Their formulation

$$\lim_{T\to\infty} \nu_T \left(\frac{\log|\zeta(1/2+it)|}{\sqrt{2^{-1}\log\log t}} < x \right) = \Phi(x)$$

is equivalent to

$$\lim_{T\to\infty} \nu_T \left(\frac{\log|\zeta(1/2+it)|}{\sqrt{2^{-1}\log\log T}} < x \right) = \Phi(x)$$

because of the bound (see Lemma 5 in Bombieri and Hejhal [3])

$$\int_T^{2T} |\log\zeta(1/2+it)|dt = O\left(T\sqrt{\log\log T}\right),$$

where $T \to \infty$. Bombieri and Hejhal considered general class of zeta functions, which includes Dirichlet L-functions. Thus, Lemma 1.7 is true for $(\lambda, \alpha) = (1, 1)$ and $(1/2, 1/2)$.

If $(\lambda, \alpha) = (1/2, 1)$, then the lemma follows by (1.1) and

$$\begin{aligned} \frac{\log|L(1/2,1,1/2+it)|}{\sqrt{2^{-1}\log\log T}} &= \frac{\log|\zeta(1/2+it)|}{\sqrt{2^{-1}\log\log T}} + \frac{\log|1-2^{1/2-it}|}{\sqrt{2^{-1}\log\log T}} \\ &= \frac{\log|\zeta(1/2+it)|}{\sqrt{2^{-1}\log\log T}} + o(1). \end{aligned}$$

The remaining case is analogous. □

Remark. Lemma 1.7 remains true if (λ, α) equal to $(0, 1)$, $(0, 1/2)$. This is because of the obvious fact that $L(0, \alpha, s) = L(1, \alpha, s)$.

Using a limit theorem for the argument of Dirichlet L-functions (see Bombieri and Hejhal [3]), Lemma 1.7 can be extended to $\arg L(\lambda, \alpha, 1/2 + it)$.

The following lemma will be needed for Lemma 1.9.

Lemma 1.8. *If a sequence of distribution functions $F_n(x)$ converges weakly to continuous distribution function $F(x)$, then this convergence is uniform in x, $-\infty < x < \infty$.*

Proof. The lemma can be found in Petrov [37]. □

If two functions are near to each other, then, of course, we expect they are distributed similarly.

Lemma 1.9. *Let $l(T) \to +\infty$ and $h(T) \to +\infty$ for $T \to \infty$. Let f and f_T be two measurable complex functions defined on real numbers. Let f and f_T have only countably many zeros. Assume that*

$$\lim_{T\to\infty} \nu_T \left(\frac{\log|f(t)|}{\sqrt{2^{-1}\log\log T}} < x \right) = \Phi(x)$$

and, for $t \in [T/h(T), T]$, assume that $|f_T(t) - f(t)| < \exp\left(-l(T)\sqrt{\log\log T}\right)$. Then

$$\lim_{T\to\infty} \nu_T \left(\frac{\log|f_T(t)|}{\sqrt{2^{-1}\log\log T}} < x \right) = \Phi(x).$$

Proof. If $|f(t)| \neq 0$ then

$$\log|f_T(t)| = \log|f(t)| + \log\left(1 + \frac{|f_T(t)| - |f(t)|}{|f(t)|}\right).$$

From the last equality we see that $\log|f_T(t)|$ is "near" to $\log|f(t)|$ if $|f(t)|$ is not very "small". We expect that there are "not many" t for which $|f(t)|$ is very "small". Accordingly, we divide the interval $[T/h(T), T]$ in to two subsets

$$J_T = \left\{ t \in [T/h(T), T] : \frac{\log|f(t)|}{\sqrt{2^{-1}\log\log T}} < -l(T) \right\}$$

and

$$I_T = \{t \in [T/h(T), T] : t \notin J_T\}.$$

By Lemma 1.8 and conditions of the lemma (recall that $\Phi(x)$ denotes the standard normal distribution function) we see that

$$\begin{aligned} \frac{1}{T}\operatorname{meas}\{J_T\} =& \nu_T \left(\frac{\log|f(t)|}{\sqrt{2^{-1}\log\log T}} < -l(T) \right) + o(1) \\ =& \Phi\left(-l(T)\right) + o(1) = o(1), \end{aligned}$$

as $T \to \infty$. For $t \in I_T$, we have

$$\begin{aligned} \log|f_T(t)| =& \log|f(t)| + \log\left(1 + O\left(\frac{|f_T(t) - f(t)|}{|f(t)|}\right)\right) \\ =& \log|f(t)| + o(1). \end{aligned}$$

Now, we can finish the proof. As $T \to \infty$, we obtain

$$\begin{aligned}
&\nu_T\left(\frac{\log|f_T(t)|}{\sqrt{2^{-1}\log\log T}} < x\right)\\
&= \frac{1}{T}\text{meas}\{t \in I_T : \frac{\log|f_T(t)|}{\sqrt{2^{-1}\log\log T}} \leq x\}\\
&+ \frac{1}{T}\text{meas}\{t \in J_T : \frac{\log|f_T(t)|}{\sqrt{2^{-1}\log\log T}} \leq x\} + o(1)\\
&= \frac{1}{T}\text{meas}\{t \in I_T : \frac{\log|f(t)|}{\sqrt{2^{-1}\log\log T}} + o(1) \leq x\} + O\left(\frac{1}{T}\text{meas}\{t \in J_T\}\right) + o(1)\\
&= \nu_T\left(\frac{\log|f(t)|}{\sqrt{2^{-1}\log\log T}} \leq x + o(1)\right) + o(1) = \Phi(x) + o(1).
\end{aligned}$$

□

We consider how close two Lerch zeta-functions are if their parameters are also close. In the previous section, we have introduced the functions

$$E(\lambda, \alpha, t) = -\left(\frac{2\pi}{t}\right)^{it} \exp\left\{it + \frac{3\pi i}{4} - 2\pi i\lambda\alpha + 2\pi i\alpha\right\}\frac{e^{-\pi t}}{(1 - \{\lambda\})^{1/2 - it}}$$

and

$$\Lambda(\lambda, \alpha, t) = \left(\frac{2\pi}{t}\right)^{it} \exp\left\{it + \frac{\pi i}{4} - 2\pi i\lambda\alpha\right\}\frac{1}{\lambda^{1/2 - it}}.$$

Lemma 1.10. *Let* $0 < \lambda_1, \lambda_2, \alpha_1, \alpha_2 \leqslant 1$. *Let* $T \to \infty$ *and* $\max\{|\lambda_1 - \lambda_2|, |\alpha_1 - \alpha_2|\} \ll T^{-3/4}$. *Then*

$$\begin{aligned}
&L(\lambda_1, \alpha_1, 1/2 + iT) - E(\lambda_1, \alpha_1, T) - \Lambda(\lambda_1, \alpha_1, T) - \alpha_1^{-1/2 - iT}\\
&- L(\lambda_2, \alpha_2, 1/2 + iT) + E(\lambda_2, \alpha_2, T) + \Lambda(\lambda_2, \alpha_2, T) + \alpha_2^{-1/2 - iT}\\
&\ll T\left(|\lambda_1 - \lambda_2| + |\alpha_1 - \alpha_2|\right) + T^{-1/4}.
\end{aligned}$$

Proof. The Lerch zeta-function can be approximated by a finite sum. We have (see Garunkštis [19]) that

$$L(\lambda,\alpha,1/2+it)=\sum_{0\le n\le\sqrt{t/2\pi}}\frac{e^{2\pi i\lambda n}}{(n+\alpha)^{1/2+it}} \tag{1.4}$$
$$+\left(\frac{2\pi}{t}\right)^{it}\exp\left\{it+\frac{\pi i}{4}-2\pi i\lambda\alpha\right\}\Bigg(\sum_{0\le n\le\sqrt{t/2\pi}}\frac{e^{-2\pi i\alpha n}}{(n+\lambda)^{1/2-it}}-\frac{e^{-\pi t+\pi i/2+2\pi i\alpha}}{(1-\{\lambda\})^{1/2-it}}\Bigg)$$
$$+O(t^{-1/4})$$

uniformly in λ and α, $0<\lambda,\alpha\leqslant 1$. Thus

$$L(\lambda_1,\alpha_1,1/2+iT)-E(\lambda_1,\alpha_1,T)-\Lambda(\lambda_1,\alpha_1,T)-\alpha_1^{-1/2-iT}$$
$$-L(\lambda_2,\alpha_2,1/2+iT)+E(\lambda_2,\alpha_2,T)+\Lambda(\lambda_2,\alpha_2,T)+\alpha_2^{-1/2-iT}$$
$$\ll\left|\sum_{1\leqslant n\leqslant\sqrt{T}}\frac{e^{2\pi i\lambda_1 n}(n+\alpha_2)^{1/2+iT}-e^{2\pi i\lambda_2 n}(n+\alpha_1)^{1/2+iT}}{((n+\alpha_1)(n+\alpha_2))^{1/2+iT}}\right| \tag{1.5}$$
$$+\left|\sum_{1\leqslant n\leqslant\sqrt{T}}\frac{e^{-2\pi i(\alpha_1 n+\alpha_1\lambda_1)}(n+\lambda_2)^{1/2-iT}-e^{-2\pi i(\alpha_2 n+\alpha_2\lambda_2)}(n+\lambda_1)^{1/2-iT}}{((n+\lambda_1)(n+\lambda_2))^{1/2-iT}}\right|$$
$$+O(T^{-1/4}):=A+B+O(T^{-1/4}).$$

We consider the first sum in (1.5).

$$A=\sum_{1\leqslant n\leqslant\sqrt{T}}(n+\alpha_1)^{-1/2-iT}e^{2\pi i\lambda_1 n}\left(1-e^{2\pi in(\lambda_2-\lambda_1)}\left(1+\frac{\alpha_1-\alpha_2}{n+\alpha_2}\right)^{1/2+iT}\right)$$
$$\ll\sum_{1\leqslant n\leqslant\sqrt{T}}(n+\alpha_1)^{-1/2}\times$$
$$\times\left(1-\exp\{2\pi in(\lambda_2-\lambda_1)\}\exp\left\{\left(\frac{1}{2}+iT\right)\log\left(1+\frac{\alpha_1-\alpha_2}{n+\alpha_2}\right)\right\}\right).$$

By Taylor expansion of functions e^x and $\log x$, we obtain

$$A\ll\sum_{1\leqslant n\leqslant\sqrt{T}}(n+\alpha_1)^{-1/2}\left(n|\lambda_2-\lambda_1|+\frac{T|\alpha_2-\alpha_1|}{n+\alpha_2}+T|\lambda_2-\lambda_1||\alpha_2-\alpha_1|\right).$$

The bounds

$$\sum_{1\leqslant n\leqslant\sqrt{T}}n^{1/2}\ll T^{3/4},\qquad\sum_{1\leqslant n\leqslant\sqrt{T}}n^{-3/2}<\infty\quad\text{and}\quad\sum_{1\leqslant n\leqslant\sqrt{T}}n^{-1/2}\ll T^{1/4}$$

leads to

$$A \ll T^{3/4}|\lambda_2-\lambda_1| + T|\alpha_2-\alpha_1| + T^{5/4}|\lambda_2-\lambda_1||\alpha_2-\alpha_1|.$$

Similarly, we derive that the second sum in (1.5) is

$$\begin{aligned} B \ll & T^{3/4}|\alpha_2-\alpha_1| + T|\lambda_2-\lambda_1| + T^{1/4}|\alpha_2\lambda_2-\alpha_1\lambda_1| \\ & + T^{5/4}|\lambda_2-\lambda_1||\alpha_2-\alpha_1| + T|\alpha_2\lambda_2-\alpha_1\lambda_1||\lambda_2-\lambda_1|. \end{aligned}$$

The lemma is proved. □

Proof of Theorems 1.1 and 1.2 follows by Lemmas 1.7, 1.9, and 1.10.

□

Proof of Theorems 1.3 and 1.4 is similar to the proof of Theorem 1.1. Note that Lemmas 1.7 and 1.9 can be easily rewritten for $\arg L(\lambda, \alpha, 1/2+it)$ (see the note after the proof of Lemma 1.7).

□

Proof of Theorem 1.5

We proof only the case a), where (λ, α), depending on T, is close to $(0,1)$. Remaining cases are analogous. Recall that $L(0,1,1/2+it) = \zeta(1/2+it)$.

If $|\zeta(1/2+it)| \neq 0$, then

$$\begin{aligned} & \log|L(\lambda,\alpha,1/2+it) - \Lambda(\lambda,\alpha,t)| \\ & = \log|\zeta(1/2+it)| + \log\left(1 + \frac{|L(\lambda,\alpha,1/2+it)-\Lambda(\lambda,\alpha,t)| - |\zeta(1/2+it)|}{|\zeta(1/2+it)|}\right). \end{aligned}$$

From the last equality, we see that $\log|L(\lambda,\alpha,1/2+it) - \Lambda(\lambda,\alpha,t)|$ is “near” to $\log|\zeta(1/2+it)|$ if $|\zeta(1/2+it)|$ is not very “small”. We expect that there are “not many” t for which $|\zeta(1/2+it)|$ is very “small”. For this reason, we choose some monotone function $K(T)$, which satisfies the following conditions:

$$K(T) \to +\infty \quad \text{as} \quad T \to +\infty \quad \text{and} \quad K(T) \ll \sqrt{\log\log T}.$$

Accordingly, we divide the interval $[0, T]$ into two intervals: $[0, T/K(T))$ and $[T/K(T), T]$. The second interval we divide in to two subsets

$$J_T = \left\{ t \in [T/K(T), T] : \frac{\log|\zeta(1/2+it)|}{\sqrt{2^{-1}\log\log T}} < -K(T) \right\}$$

and

$$I_T = \{ t \in [T/K(T), T] : t \notin J_T \}.$$

By Lemma 1.8 (recall that $\Phi(x)$ denotes the standard normal distribution function) we see that

$$\begin{aligned} \frac{1}{T}\text{meas}\{J_T\} =& \nu_T \left(\frac{\log|\zeta(1/2+it)|}{\sqrt{2^{-1}\log\log T}} < -K(T) \right) + o(1) \\ =& \Phi(-K(T)) + o(1) = o(1), \end{aligned}$$

as $T \to \infty$.

For $t \in I_T$, we have

$$\begin{aligned} &\log|L(\lambda, \alpha, 1/2+it) - \Lambda(\lambda, \alpha, t)| \\ &= \log|\zeta(1/2+it)| + \log\left(1 + O\left(\frac{|L(\lambda,\alpha,1/2+it) - \Lambda(\lambda,\alpha,t) - \zeta(1/2+it)|}{|\zeta(1/2+it)|}\right)\right) \end{aligned}$$

Since $t \in I_T$, by Lemma 1.10 we see that

$$\begin{aligned} &\frac{|L(\lambda,\alpha,1/2+it) - \Lambda(\lambda,\alpha,t) - \zeta(1/2+it)|}{|\zeta(1/2+it)|} \\ &\ll \exp\left(K(T)\sqrt{\log\log T}\right) |L(\lambda,\alpha,1/2+it) - \Lambda(\lambda,\alpha,t) - \zeta(1/2+it)| \\ &\ll \exp\left(K(T)\sqrt{\log\log T}\right) \times \\ &\times \left(\exp\left(-\frac{\pi T}{K(T)}\right)(1 - \{\lambda\})^{-1/2} + T\lambda + T|\alpha - 1| + \left(\frac{K(T)}{T}\right)^{1/4} \right) \\ &\ll \log T \left(\exp\left(-T/\log\log T\right) + (\log T)^{-1-\varepsilon} + \left(\frac{\log\log T}{T}\right)^{1/4} \right) \ll o(1). \end{aligned}$$

Finally, for $t \in I_T$, we have

$$\log|L(\lambda, \alpha, 1/2+it) - \Lambda(\lambda, \alpha, t)| = \log|\zeta(1/2+it)| + o(1).$$

Now we can finish the proof. As $T \to \infty$, we obtain

$$\nu_T\left(\frac{\log|L(\lambda,\alpha,1/2+it)-\Lambda(\lambda,\alpha,t)|}{\sqrt{2^{-1}\log\log T}} < x\right)$$
$$= \frac{1}{T}\text{meas}\{t \in [0, T/K(T)) : \frac{\log|L(\lambda,\alpha,1/2+it)-\Lambda(\lambda,\alpha,t)|}{\sqrt{2^{-1}\log\log T}} \leqslant x\}$$
$$+ \frac{1}{T}\text{meas}\{t \in I_T : \frac{\log|L(\lambda,\alpha,1/2+it)-\Lambda(\lambda,\alpha,t)|}{\sqrt{2^{-1}\log\log T}} \le x\}$$
$$+ \frac{1}{T}\text{meas}\{t \in J_T : \frac{\log|L(\lambda,\alpha,1/2+it)-\Lambda(\lambda,\alpha,t)|}{\sqrt{2^{-1}\log\log T}} \le x\} + o(1)$$
$$= \frac{1}{T}\text{meas}\{t \in I_T : \frac{\log|\zeta(1/2+it)|}{\sqrt{2^{-1}\log\log T}} + o(1) \le x\} + O\left(\frac{1}{T}\text{meas}\{t \in J_T\}\right) + o(1)$$
$$= \nu_T\left(\frac{\log|\zeta(1/2+it)|}{\sqrt{2^{-1}\log\log T}} \le x + o(1)\right) + o(1) = \Phi(x) + o(1).$$

□

Proof of Theorem 1.6 is similar to the proof of Theorem 1.5. Note that Lemma 1.7 and proof of Theorem 1.5 can be rewritten for $\arg L(\lambda,\alpha,1/2+it)$ (see the note after the proof of Lemma 1.7).

□

1.2. **The limit law illustration.**

In previous chapter, we proved theorems on asymptotical distribution of the values of the Lerch zeta-function. We found conditions when the values of the Lerch zeta-function have an asymptotical standard normal distribution. The standard normal distribution function

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-x^2/2}\,dx$$

may be easily plotted with many of computational softwares. The goal of this chapter is to check computationally whether the distribution of values of Lerch zeta-function is "visible" when T is "small".

Using the mathematical program MATHEMATICA, we fix a number T ($T = 200$) and calculate values of the following distribution functions numerically:

$$D_{1,1}(x) := \frac{1}{T}\text{meas}\left\{t \in [0,T] : \frac{\log|\zeta(1/2+it)|}{\sqrt{2^{-1}\log\log T}} < x\right\}$$

and

$$D_{\lambda,\alpha}(x) := \frac{1}{T}\text{meas}\left\{t \in [0,T] : \frac{\log|L(\lambda,\alpha,1/2+it)|}{\sqrt{2^{-1}\log\log T}} < x\right\}.$$

We compare values of $D_{1,1}(x)$ and $D_{\lambda,\alpha}(x)$ with the values of the standard normal distribution function.

In the tables and graphics below, we present results of calculation when $T = 200$, x takes values from the set

$$\{-3, -2, -1.75, -1.5, -1.25, \ldots, 1.25, 1.5, 1.75, 2, 3\},$$

and parameters (λ, α) are taken from the set

$$\{(1, 0.995), (0.8, 0.8), (1, 0.5), (0.5, 1), (0.5, 0.5)\}.$$

We remind that $\zeta(1/2+it) = L(1, 1, 1/2+it)$.

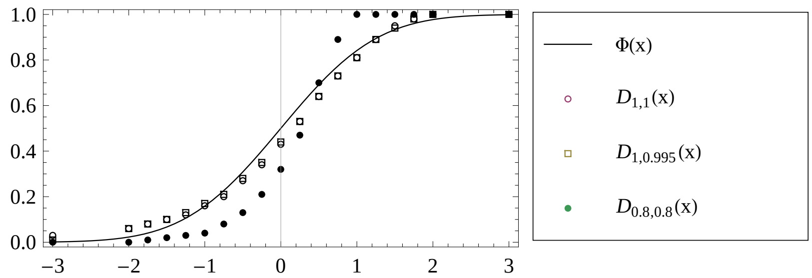

FIGURE 1. Graph of the functions $\Phi(x)$, $D_{1,1}(x)$ and $D_{\lambda,\alpha}(x)$ when $T = 200$ and $-3 \leqslant x \leqslant 3$.

x	$\Phi(x)$	$D_{1,1}(x)$	$D_{1,0.995}(x)$	$D_{0.8,0.8}(x)$
-3	0.00	0.03	0.01	0.00
-2	0.02	0.06	0.06	0.00
-1.75	0.04	0.08	0.08	0.01
-1.5	0.07	0.10	0.10	0.02
-1.25	0.11	0.12	0.13	0.03
-1	0.16	0.16	0.17	0.04
-0.75	0.23	0.20	0.21	0.08
-0.5	0.31	0.27	0.28	0.13
-0.25	0.4	0.34	0.35	0.21
0	0.50	0.43	0.44	0.32
0.25	0.60	0.53	0.53	0.47
0.5	0.69	0.64	0.64	0.70
0.75	0.77	0.73	0.73	0.89
1	0.84	0.81	0.81	1.00
1.25	0.89	0.89	0.89	1.00
1.5	0.93	0.95	0.94	1.00
1.75	0.96	0.98	0.98	1.00
2	0.98	1.00	1.00	1.00
3	1.00	1.00	1.00	1.00

TABLE 1. Values of the functions $\Phi(x)$, $D_{1,1}(x)$ and $D_{\lambda,\alpha}(x)$ when $T = 200$.

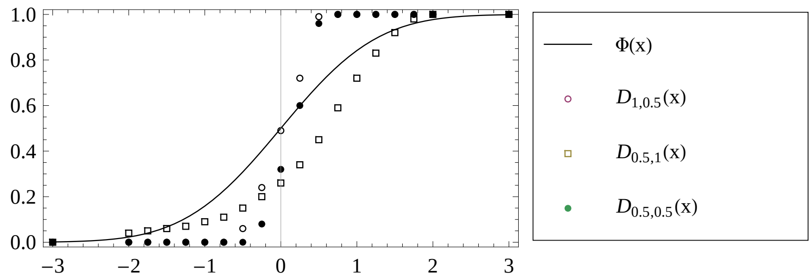

Figure 2. Graph of the functions $\Phi(x)$ and $D_{\lambda,\alpha}(x)$ when $T = 200$ and $-3 \leqslant x \leqslant 3$.

x	$\Phi(x)$	$D_{1,0.5}(x)$	$D_{0.5,1}(x)$	$D_{0.5,0.5}(x)$
-3	0.00	0.00	0.00	0.00
-2	0.02	0.00	0.04	0.00
-1.75	0.04	0.00	0.05	0.00
-1.5	0.07	0.00	0.06	0.00
-1.25	0.11	0.00	0.07	0.00
-1	0.16	0.00	0.09	0.00
-0.75	0.23	0.00	0.11	0.00
-0.5	0.31	0.06	0.15	0.00
-0.25	0.4	0.24	0.20	0.00
0	0.50	0.49	0.26	0.08
0.25	0.60	0.72	0.34	0.32
0.5	0.69	0.99	0.45	0.60
0.75	0.77	1.00	0.59	0.96
1	0.84	1.00	0.72	1.00
1.25	0.89	1.00	0.83	1.00
1.5	0.93	1.00	0.92	1.00
1.75	0.96	1.00	0.98	1.00
2	0.98	1.00	1.00	1.00
3	1.00	1.00	1.00	1.00

Table 2. Values of the functions $\Phi(x)$ and $D_{\lambda,\alpha}(x)$ when $T = 200$.

2. Zero distribution of the Lerch transcendent function

The Lerch transcendent function is the analytic continuation of the series

$$\Phi(q, s, \alpha) = \sum_{n=0}^{\infty} q^n (n+\alpha)^{-s},$$

which converges for any real number $\alpha > 0$ if q and s are complex numbers with either $|q| < 1$, or $|q| = 1$ and $\sigma > 1$. Here, we consider $\Phi(q, s, \alpha)$ as a function of s with the parameters $q \in \mathbb{C}$, $0 < |q| \leq 1$, and $0 < \alpha \leq 1$. Special cases include the Riemann zeta-function $\zeta(s) = \Phi(1, s, 1)$, the Hurwitz zeta-function $\zeta(s, \alpha) = \Phi(1, s, \alpha)$, the polylogarithm function $Li_s(q) = q\Phi(q, s, 1)$, and the Lerch zeta-function $L(\lambda, \alpha, s) = \Phi(\exp(2\pi i\lambda), s, \alpha)$.

The Riemann zeta-function has no zeros in the right-half-plane $\sigma \geq 1$. In the left-half-plane $\sigma \leq 0$, it has only trivial zeros at even negative integers. The famous Riemann hypothesis (RH) asserts that the remaining, nontrivial, zeros lie on the critical line $\sigma = 1/2$.

The Hurwitz zeta-function $\zeta(s, \alpha)$ has infinitely many zeros in $1 < \sigma < 1+\alpha$ if α is transcendental or rational $\neq 1/2$, 1 (Davenport and Heilbronn [6]). This result was extended by Cassels [4] for α algebraic irrational. Let $1/2 < \sigma_1 < \sigma_2 < 1$. Then, Voronin [48] (for rational $\alpha \neq 1/2, 1$) and Gonek [20] (for transcendental α) proved that the number of zeros of $\zeta(s, \alpha)$ in the rectangle $\sigma_1 < \sigma < \sigma_2$, $0 < t < T$ is approximately equal to T for sufficiently large T. Gonek [21] also showed that for $\alpha = \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{6}$ or $\frac{5}{6}$ the proportion of zeros of $\zeta(s, \alpha)$ on $\sigma = 1/2$ is definitely less than 1. In the complex s-plane, trajectories of zeros $\rho = \rho(\alpha)$ of the Hurwitz zeta function were considered in [16] and [17]. Based on these trajectories, the classification of nontrivial zeros of the Riemann zeta function were introduced. For the zero distribution of the Lerch zeta-function see [14], [11], [12], [18], [29].

Fornberg and Kölbig [8] investigated trajectories of zeros $\rho = \rho(q)$ of the polylogarithm function $Li_s(q)$ for real q with $|q| < 1$. They found that some trajectories tend towards the zeros of $\zeta(s)$ as $q \to -1$, and approach these zeros closely as

$q \to 1 - \delta$ for small but finite $\delta > 0$. However, the later trajectories appear to descend to the point $s = 1$ as $\delta \to 0$. Both, for $q \to -1$ and $q \to 1$, there are trajectories which do not tend towards zeros of $\zeta(s)$.

Next, we consider the zeros of $\Phi(q, s, \alpha)$ for $0 < \alpha < 1$ and $q \in \mathbb{C}, 0 < |q| < 1$. Let $N_\Phi(\sigma_1, \sigma_2, T) = N_\Phi(\sigma_1, \sigma_2, T, q, \alpha)$ denote the number of zeros of $\Phi(q, s, \alpha)$ in the region $\{s : \sigma_1 < \mathrm{Re}(s) < \sigma_2, 0 < \mathrm{Im\ s} \leqslant T\}$. Let $\sigma_0 = \sigma_0(q, \alpha)$ be a real number defined by the equality

$$\sum_{n=1}^{\infty} \frac{|q|^n}{\left(\frac{n}{\alpha} + 1\right)^{\sigma_0}} = 1.$$

It is easy to see that $\sigma_0 \leq c = 1.73\ldots$, where $\zeta(c) = \sum_{n=1}^{\infty} n^{-c} = 2$, and that σ_0 can take any value between $-\infty$ and c.

Theorem 2.1. *Let $q \in \mathbb{C}, 0 < |q| < 1$. Let $0 < \alpha < 1$ be a transcendental number. Then we have that, for any fixed strip $\sigma_1 < \sigma < \sigma_2 \leq \sigma_0$,*

$$T \ll N_\Phi(\sigma_1, \sigma_2, T) \ll T$$

and $\Phi(q, s, \alpha)$ has no zeros for $\sigma > \sigma_0$.

Theorem 2.1 is proved in Section 2.2.

As already mentioned, Wiener and Wintner [50, Section 4] pointed to a possible relationship between the behavior of the zeros in the right-half-plane $\sigma > 1$ of the polylogarithm function and the Riemann Hypothesis. The authors showed that the Riemann Hypothesis is true if there exists a number $0 < \varepsilon < 1$ such that $\sum_{n=1}^{\infty} q^n n^{-s} \neq 0$, for $\sigma > 1$ and $1 - \varepsilon < q < 1$. However, Montgomery [34] pointed that the polylogarithm function $Li_s\left(e^{-1/N}\right)$ has zeros in the region $\sigma > 1$ for all sufficiently large integers N, making the mentioned criteria vacuous. Theorem 2.1 shows that the Lerch transcendent function $\Phi(q, s, \alpha)$ also has zeros in the region $\sigma > 1$ for $0.92 < q < 1$ and transcendental α, $1/2 < \alpha < 1$. In the section 2.1, we try to find explicit zeros in $\sigma > 1$. We see that it is relatively easy to find zeros if $\alpha \neq 1$. In the case $\alpha = 1$ the zeros in the right half-lane, $\sigma > 1$ currently are out of reach.

2.1. **Calculations.**

The calculations of this section were made with the program MATHEMATICA. To calculate the number N of zeros of $\Phi(q,s,\alpha)$ inside the contour Γ, we used the well known formula

$$N=\frac{1}{2\pi i}\int_\Gamma \frac{(\Phi(q,s,\alpha))'_s}{\Phi(q,s,\alpha)}ds.$$

If the interior of the contour Γ contains one zero ρ, then we find this zero using the following expression

$$\rho=\frac{1}{2\pi i}\int_\Gamma s\frac{(\Phi(q,s,\alpha))'_s}{\Phi(q,s,\alpha)}ds.$$

The zero ρ can be adjusted by MATHEMATICA command *FindRoot*.

Let

$$R=\{s:\operatorname{Re}(s)>1, 0<\operatorname{Im}(s)\leqslant 1000\}.$$

In Table 3, we present the number of zeros of function $\Phi(q,s,\alpha)$ for chosen q and α in the region R.

$\alpha \setminus$ q	**0.9**	**0.95**	**0.99**	**1**
0.9	2	8	34	40
0.95	4	10	37	46
0.99	14	27	41	45
1	0	0	0	0

TABLE 3. Number of zeros of the function $\Phi(q,s,\alpha)$ in the region R.

For example, we see that $\Phi(0.99,s,0.9)$ has 34 zeros in R. In Table 3, the last column describes zeros of the Hurwitz zeta-function, and the last raw describes zeros of the polylogarithm function. In view of Montgomery's result [34], we expect that $\Phi(q,s,1)$ has zeros in $\sigma>1$ for $q\geq 0.9$. If so, then Table 3 possibly indicates the different behavior of zeros of $\Phi(q,s,\alpha)$ in $\sigma>1$ dependently on $\alpha=1$ or $\alpha\neq 1$.

In Table 4, we present zeros of functions $\Phi(0.9, s, 0.9)$, $\Phi(0.9, s, 0.95)$, $\Phi(0.9, s, 0.99)$.

	$\Phi(0.9, s, 0.9)$	$\Phi(0.9, s, 0.95)$	$\Phi(0.9, s, 0.99)$
1	$1.02 + 550.55i$	$1.07 + 108.39i$	$1.05 + 480.29i$
2	$1.02 + 609.75i$	$1.01 + 135.21i$	$1.11 + 525.79i$
3	-	$1.09 + 169.68i$	$1.08 + 588.57i$
4	-	$1.07 + 196.67i$	$1.06 + 616.03i$
5	-	-	$1.11 + 651.27i$
6	-	-	$1.11 + 696.71i$
7	-	-	$1.13 + 724.38i$
8	-	-	$1.05 + 759.64i$
9	-	-	$1.15 + 787.05i$
10	-	-	$1.02 + 805.00i$
11	-	-	$1.12 + 849.96i$
12	-	-	$1.17 + 895.31i$
13	-	-	$1.09 + 958.10i$
14	-	-	$1.00 + 985.50i$

TABLE 4. Coordinates of zeros of the function $\Phi(q, s, \alpha)$ in the region R.

In this table, numbers were rounded up to two decimal places.

2.2. **Proof of Theorem 2.1.**

First, we formulate the theorems of Kronecker and Rouché (see Tichmarsh [45, Section 8.3] and Tichmarsh [44, Section 3.42]).

LEMMA 2.2. *(Kronecker's theorem) Let $a_1, a_2, \ldots, a_N$ be linearly independent real numbers, i.e. numbers such that relation $\lambda_1 a_1 + \cdots + \lambda_N a_N = 0$ is possible only if $\lambda_1 = \cdots = \lambda_N = 0$. Let $b_1, \ldots, b_N$ be any real numbers, and ε a given positive number. Then we can find a number t and integers $x_1, \ldots, x_N$ such that $|ta_n - b_n - x_n| < \varepsilon$, $n = 1, \ldots, N$.*

Lemma 2.3. *(Rouché's theorem) Suppose that $f(s)$ and $g(s)$ are analytic functions inside and on a regular closed curve γ, and that $|f(s)| > |g(s)|$ for all $s \in \gamma$. Then $f(s) + g(s)$ and $f(s)$ have the same number of zeros inside γ.*

The next lemma will be useful in the proof of Theorem 2.1.

Lemma 2.4. *Let $q \in \mathbb{C}, 0 < |q| < 1$, and $0 < \alpha < 1$ be a transcendental number. Let σ' be a real number. Let $a(n)$ be a sequence of complex numbers such that $|a(n)| = 1$. Let $\Phi_a(q, s, \alpha) = \sum_{n=0}^{\infty} a(n) q^n (n+\alpha)^{-s}$.*
Then for any $\varepsilon > 0$ there exist $\tau \in \mathbb{R}$ such that

$$|\Phi(q, s+i\tau, \alpha) - \Phi_a(q, s, \alpha)| < \varepsilon$$

for $Re(s) \geq \sigma'$.

Proof. The Dirichlet series of the Lerch transcendent function converges absolutely for any s if $|q| < 1$. Therefore, for given σ' there is a positive integer N such that, for any real number u and $\sigma \geq \sigma'$,

$$\left|\sum_{n=N+1}^{\infty} \frac{q^n}{(n+\alpha)^{s+iu}} - \sum_{n=N+1}^{\infty} \frac{q^n a(n)}{(n+\alpha)^s}\right| \leq 2 \sum_{n=N+1}^{\infty} \frac{|q|^n}{(n+\alpha)^{\sigma}} < \frac{\varepsilon}{2}. \tag{2.1}$$

Let

$$A = \sum_{n=0}^{N} \frac{|q|^n}{(n+\alpha)^{\sigma'}}.$$

There is a sequence of real numbers $b(n)$ such that $e^{-2\pi i b(n)} = a(n)$. The numbers $\log(n+\alpha)$ are linearly independent over $\mathbb{Q}$, since α is the transcendental number. By Kronecker's theorem (Lemma 2.2), there exist a real number τ and integers x_n such that

$$\left|\frac{\tau \log(n+\alpha)}{2\pi} - b(n) - x_n\right| < \frac{\varepsilon}{8\pi A}.$$

In view of the inequality $|e^z - 1| \leq 2|z|$, where $|z| < 1$, we obtain

$$|(n+\alpha)^{-i\tau} - a(n)| = \left|e^{-2\pi i\left(\frac{\tau \log(n+\alpha)}{2\pi} - b(n) - x_n\right)} - 1\right| < \frac{\varepsilon}{2A}.$$

By above, we see that there is τ such that, for $\mathrm{Re}(s) \geq \sigma'$,

$$\left|\sum_{n=0}^{N} \frac{q^n}{(n+\alpha)^{s+i\tau}} - \sum_{n=0}^{N} \frac{q^n a(n)}{(n+\alpha)^s}\right| \leq \sum_{n=0}^{N} \frac{|q|^n}{(n+\alpha)^{\sigma'}} |(n+\alpha)^{-i\tau} - a(n)| < \frac{\varepsilon}{2}.$$

This and inequality (2.1) in view of triangle inequality prove Lemma 2.4. $\square$

Proof of Theorem 2.1. For fixed q and α, the function $\Phi(q,s,\alpha)$ is bounded in any right half-plane of complex numbers. This, together with Theorem 9.62 of Titchmarsh [44], give the bound

$$N_\Phi(\sigma_1,\sigma_2,T) \ll T.$$

Further, if the strip $\sigma_1<\sigma<\sigma_2$ contains a zero of $\Phi(q,s,\alpha)$, then, arguing as in Lemma 1 of [11], we get the bound

$$N_\Phi(\sigma_1,\sigma_2,T) \gg T.$$

Next, we will show that the function $\Phi(q,s,\alpha)$ has a zero in the strip $\sigma_1<\sigma<\sigma_2$. We consider an auxiliary function $\Phi_a(q,\sigma,\alpha)=\sum_{n=0}^{\infty} a(n)q^n(n+\alpha)^{-\sigma}$. For fixed σ, q and α, let V be the set of values taken by $\Phi_a(q,\sigma,\alpha)$ for independent $a(0)$, $a(1)$, $\ldots$, where $a(n)\in\mathbb{C}$ and $|a(n)|=1$. If $\sigma<\sigma_0$, then by Tichmarsh [45, Section 11.5, p. 297] we see that

$$V=\left\{z : |z| \le \sum_{n=0}^{\infty} |q|^n(n+\alpha)^{-\sigma}\right\}.$$

Thus for $\sigma_1<\sigma'<\sigma_2$, q, and α there is a sequence $a(1)$, $a(2)$, $\ldots$, such that $\Phi_a(q,\sigma',\alpha)=0$.

Let $0<\varepsilon'<\min(\sigma'-\sigma_1,\sigma_2-\sigma')$ be such that $\Phi_a(q,s,\alpha)\neq 0$ for $|s-\sigma'|=\varepsilon'$. Let

$$\varepsilon=\min_{|s-\sigma'|=\varepsilon'} |\Phi_a(q,s,\alpha)|.$$

By Lemma 2.4, there is a real shift τ such that

$$|\Phi(q,s+i\tau,\alpha)-\Phi_a(q,s,\alpha)|<\varepsilon$$

for $\mathrm{Re}(s)\ge\sigma_1$. Hence, Rouché's theorem gives that $\Phi(q,s,\alpha)$ has a zero in the disk $|s-\sigma'-i\tau|<\varepsilon'$, which is contained in the strip $\sigma_1<\sigma<\sigma_2$. By this Theorem 2.1 is proved. $\square$

3. On the behavior of the Selberg zeta-functions in the critical strip

In the paper [40], Saidak and Zvengrowski proved the following fact for the modulus of the Riemann zeta-function.

Theorem A. *Let $s = 1/2 + \Delta + it$. For $0 \leqslant \Delta \leqslant 1/2$ and $t \geqslant 2\pi + 1$ we have*

$$|\zeta(1/2 - \Delta + it)| \geqslant |\zeta(1/2 + \Delta + it)|.$$

The authors also pointed that if the inequality could be strengthened to show that for $0 < \Delta \leqslant 1/2$, one has $|\zeta(1/2 - \Delta + it)| > |\zeta(1/2 + \Delta + it)|$, then the Riemann Hypothesis would follow.

In the paper Matiyasevich, Saidak, and Zvengrowski [33], there was proved the following relation between functions $\zeta(s)$ and $\xi(s)$.

Theorem B. *The functions ζ and ξ satisfy the inequality*

$$\mathrm{Re}\left(\frac{\zeta'(s)}{\zeta(s)}\right) < \mathrm{Re}\left(\frac{\xi'(s)}{\xi(s)}\right),$$

for $|t| \geqslant 8$, $\sigma < 1/2$.

Sondow and Dumitrescu [43] proved the following theorem for the function ξ.

Theorem C. *The xi function is increasing in modulus along every horizontal half-line lying in any open right half-plane that contains no xi zeros. Similarly, the modulus decreases on each horizontal half-line in any zero-free, open left half-plane.*

In the same paper, there was given the following reformulation for the Riemann hypothesis.

Theorem D. *The following statements are equivalent.*

I. *If t is any fixed real number, then $|\xi(\sigma + it)|$ is increasing for $1/2 < \sigma < \infty$.*

II. *If t is any fixed real number, then $|\xi(\sigma + it)|$ is decreasing for $-\infty < \sigma < 1/2$.*

III. *The Riemann hypothesis is true.*

Here, we reprove Theorem D in a slightly different way. If $|\xi(s)|$ is increasing along a half-line L (or decreasing on L), then $\xi(s)$ cannot have a zero on L. In view of functional equation $\xi(s) = \xi(1-s)$, we see that I implies III. Similarly II implies III. Conversely, if III holds, then $\xi(s) \neq 0$ on the right and left open half-planes of the critical line. Using the Hadamards formula, the logarithmic derivative of $\xi(s)$ may be expressed as

$$\frac{\xi'(s)}{\xi(s)} = \sum_{\rho} \frac{1}{s-\rho},$$

where the summation is over all zeros ρ of ξ taken in conjugate pairs and in order of increasing imaginary parts (see for instance Edwards [7]). If $\rho = \beta + i\gamma$, then we have that

$$\operatorname{Re}\left(\frac{1}{s-\rho}\right) = \frac{\sigma-\beta}{(\sigma-\beta)^2+(t-\gamma)^2}. \tag{3.1}$$

The statement III implies I and II in view of equation (3.1) and lemma 3.6, which is formulated in Subsection 3.1 below.

Here, we ask whether Selberg zeta-functions have similar properties as the Riemann-zeta function has in Theorems A-D.

For the modular group $\Gamma = SL(2,\mathbb{Z})$, the Selberg zeta-function is defined by the Euler product (cf. Fischer [9], Hejhal [24])

$$Z_\Gamma(s) = \prod_{\{P\}} \prod_{k=0}^{\infty} (1 - N(P)^{-s-k})$$

where $\{P\}$ runs trough all the primitive hyperbolic conjugacy classes of Γ and $N(P) = \alpha^2$ if the eigenvalues of P are α and α^{-1} ($|\alpha| > 1$).

Similarly as the Riemann zeta-function, the Selberg zeta-function $Z_\Gamma(s)$ for the modular group $\Gamma = SL(2,\mathbb{Z})$ has a meromorfic continuation to the whole complex plane, and it satisfies the symmetric functional equation (see Kurokawa [27])

$$\Xi(s) = \Xi(1-s), \tag{3.2}$$

where

$$\Xi(s) = Z_\Gamma(s) Z_{id}(s) Z_{ell}(s) Z_{par}(s)$$

and

$$
\begin{aligned}
Z_{id}(s) &= \left(\frac{(2\pi)^s}{\Gamma(s)}\right)^{1/6} (\Gamma_2(s))^{1/3}, \\
Z_{ell}(s) &= \Gamma\left(\frac{s}{2}\right)^{-1/2} \Gamma\left(\frac{s+1}{2}\right)^{1/2} \Gamma\left(\frac{s}{3}\right)^{-2/3} \Gamma\left(\frac{s+2}{3}\right)^{2/3}, \qquad (3.3) \\
Z_{par}(s) &= \frac{\pi^s}{\Gamma(s)\zeta(2s)\Gamma(s+1/2)2^s},
\end{aligned}
$$

where $\Gamma(s)$ denotes the Euler gamma function and $\zeta(s)$ is the Riemann zeta function. The function $\Gamma_2(s)$ is called the double gamma function of Barnes. It is defined by the canonic product

$$
\begin{aligned}
&\frac{1}{\Gamma_2(s+1)} = \qquad (3.4) \\
&(2\pi)^{\frac{s}{2}} \exp\left\{-\frac{s}{2} - \frac{(\gamma+1)s^2}{2}\right\} \prod_{k=1}^{\infty} \left\{\left(1+\frac{s}{k}\right)^k \exp\left(-s+\frac{s^2}{2k}\right)\right\},
\end{aligned}
$$

where γ denotes the Euler's constant. The function $\Gamma_2(s)$ has the following properties

$$
\Gamma_2(1) = 1,\ \Gamma_2(s+1) = \frac{\Gamma_2(s)}{\Gamma(s)},\ \Gamma_2(n+1) = \frac{1^2 \cdot 2^2 \cdots n^2}{(n!)^n},
$$

see for instance Sarnak [41], Vignéras [47] or Barnes [2].

The function $\Xi(s)$ is an entire function of order 2 with zeros on the critical line $\mathrm{Re}(s) = 1/2$ only, see Momotani [32]. In the same paper, we find that the function $Z_\Gamma(s)$ has poles and zeros at the following points:

Poles of $Z_\Gamma(s)$:

(1) $s = 0$; order 1,

(2) $s = 1/2 - k$, $(k \geqslant 0)$; order 1.

Zeros of $Z_\Gamma(s)$:

(1) $s = 1$; order 1,

(2) $s = -6k - j$ $(k \geqslant 0, j = 1, 2, 3, 4, 6)$; order $2k+1$,

(3) $s = -6k - 5$ $(k \geqslant 0)$; order $2k+3$,

(4) $s = \rho/2$ (ρ - non-trivial zeros of $\zeta(s)$),

(5) $s = 1/2 \pm ir_n$ $(n \geqslant 1)$.

We prove the following theorem.

Theorem 3.1. *There exists a positive number C, such that*

$$\mathrm{Re}\left(\frac{\Xi'(s)}{\Xi(s)}\right) < 0$$

for $t > C$ and $0 < \sigma < 1/2$.

Further more, if we assume the Riemann hypothesis for $\zeta(s)$, then there exists a positive number C_1, such that

$$\mathrm{Re}\left(\frac{Z_\Gamma'(s)}{Z_\Gamma(s)}\right) < \mathrm{Re}\left(\frac{\Xi'(s)}{\Xi(s)}\right)$$

for $t > C_1$ and $0 < \sigma < 1/4$.

Theorem 3.1 is proved in the next section. Below we formulate couple of corollaries of Theorem 3.1.

Corollary 3.2. *For fixed and sufficiently large t, the function $|\Xi(s)|$ is decreasing for $0 < \sigma < 1/2$, and it is increasing for $1/2 < \sigma < 1$ as a function of σ.*

Corollary 3.3. *Under the Riemann hypotesis for $\zeta(s)$ and for fixed sufficiently large t, the function $|Z_\Gamma(s)|$ is decreasing for $0 < \sigma < 1/4$ as a function of σ.*

Proofs of corollaries are obvious in view of Lemma 3.6.

We turn to Selberg zeta-functions attached to compact Riemann surfaces. Let F denote a compact Riemann surface of genus $g \geqslant 2$. F can be represented as a quotient space $\Gamma\backslash H$, where $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ is a strictly hyperbolic Fuchsian group, and H is the upper half-plane. The Γ conjugacy class determined by $P \in \Gamma$ will be denoted by $\{P\}$, and its norm by $N\{P\}$. By P_0 will be denoted the primitive element of Γ. The Selberg zeta-function for $\sigma > 1$ is given by (Hejhal [24])

$$Z_C(s) = \prod_{\{P_0\}} \prod_{k=0}^{\infty} (1 - N(P_0)^{-s-k}).$$

It is an entire function of order 2 with a functional equation

$$Z_C(s) = f(s) Z_C(1-s), \tag{3.5}$$

where

$$f(s) = \exp\left(4\pi(g-1)\int_0^{s-1/2} v\tan(\pi v)\,dv\right). \tag{3.6}$$

The above functional equation is equivalent to $M(s) = M(1-s)$, where

$$M(s) = Z_C(s)\exp\left(2\pi(g-1)\int_0^{1/2-s} v\tan\pi v\,dv\right) \tag{3.7}$$

and it is known that all zeros of $Z_C(s)$ lie on the critical line $\mathrm{Re}(s) = 1/2$, see Luo [31].

So, the analogue of the Riemann hypothesis holds for $Z_C(s)$.

THEOREM 3.4. *There exists a number $B > 0$ such that the functions $Z_C(s)$ and $M(s)$, for $t > B$, $0 < \sigma < 1/2$, satisfy the inequality*

$$\mathrm{Re}\left(\frac{Z_C'(s)}{Z_C(s)}\right) < \mathrm{Re}\left(\frac{M'(s)}{M(s)}\right) < 0.$$

A part of this theorem is proved in Luo [31]. In the mentioned paper is shown that

$$\mathrm{Re}\left(\frac{Z_C'(s)}{Z_C(s)}\right) < 0$$

for $-c \leqslant \mathrm{Re}(s) \leqslant 1/2$ and $\mathrm{Im}(s) \geqslant t_0 > 0$, where $c > 0$ is arbitrary and t_0 is a constant depending on c.

Just like for $|\zeta(s)|$ in the critical strip holds Theorem A, for $|Z_C(s)|$ in the critical strip holds the following theorem.

THEOREM 3.5. *For $0 < \Delta \leqslant 1/2$ and $t > t_0$,*

$$|Z_C(1/2 - \Delta + it)| > |Z_C(1/2 + \Delta + it)|,$$

where

$$t_0 = \frac{1}{\pi}\log\frac{2}{\sqrt{5}-1} = 0.15\ldots$$

Theorems 3.4 and 3.5 are proved in the Section 3.2.

3.1. **Proof of Theorem 3.1.**

First, we formulate several lemmas and then prove the theorem.

Lemma 3.6. (*a*) *Let f be holomorphic in an open domain D and not identically zero. Let us also suppose Re$(f'(s)/f(s)) < 0$ for all $s \in D$ such that $f(s) \neq 0$. Then $|f(s)|$ is strictly decreasing with respect to σ in D, i.e. for each $s_0 \in D$ there exists a $\delta > 0$ such that $|f(s)|$ is strictly monotonically decreasing with respect to σ on the horizontal interval from $s_0 - \delta$ to $s_0 + \delta$.*
(*b*) *Conversely, if $|f(s)|$ is decreasing with respect to σ in D, then Re$(f'(s)/f(s)) \leqslant 0$ for all $s \in D$ such that $f(s) \neq 0$.*

Proof. See Matiyasevich, Saidak, and Zvengrowski [33] for the proof. □

Note: Of course, the analogous results hold for monotone increasing $|f(s)|$ and $\mathrm{Re}\,(f'(s)/f(s)) > 0$.

Lemma 3.7. *Let $N(T)$ be the number of zeros of $\zeta(s)$ in the rectangle $0 < \sigma < 1$, $0 < t < T$. Then*

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + O(\log T). \tag{3.8}$$

Proof. See Titchmarsh [45] for the proof. □

Note: If we assume the Riemann hypothesis, then the error term in formula (3.8) is $O\left(\frac{\log T}{\log\log T}\right)$. This result also can be found in Titchmarsh [45].

Lemma 3.8. *Let $\Psi(s) = \Gamma'(s)/\Gamma(s)$ and $0 < \varepsilon < \pi$. In the sector of the complex plane $\varepsilon - \pi \leqslant \arg(s) \leqslant \pi - \varepsilon$, one has*

$$\Psi(s) = \log(s) - \frac{1}{2s} + R_0(s),$$

where $|R_0(s)| \leqslant \sec^2(\theta/2)\frac{B_2}{2|s|^2}$, with $B_2 = 1/6$ being the second Bernoulli number.

Proof. See Matiyasevich, Saidak, and Zvengrowski [33] for the proof. □

In the proof of Lemma 3.8 in Matiyasevich, Saidak, and Zvengrowski [33], the authors also give the following expression: in the sector $-\pi/2 < \arg(s) < \pi/2$ it holds

$$\operatorname{Re}(\Psi(s)) = \log|s| - \frac{\sigma}{2|s|^2} + \operatorname{Re}(R_0(s)),$$

where $|\operatorname{Re}(R_0(s)| < 2\sqrt{2}/(6|s|^2)$.

LEMMA 3.9. *Let γ denote an imaginary part of a non-trivial zero of the Riemann zeta-function $\zeta(s)$. We have that*

$$\sum_{\gamma>0} \frac{1}{1/4+(\gamma-t)^2} = \log\frac{t}{2\pi} + O\left(\frac{1}{t}\right),$$

where t is a fixed positive number.

PROOF. By Lemma 3.7 and by the partial summation we get

$$\begin{aligned}
&\sum_{\gamma>0} \frac{1}{1/4+(\gamma-t)^2} \qquad (3.9)\\
&= \int_0^\infty \frac{1}{1/4+(u-t)^2}\, d\left(\frac{u}{2\pi}\log\frac{u}{2\pi} - \frac{u}{2\pi} + \frac{7}{8} + O(\log u)\right)\\
&= \frac{1}{2\pi}\int_0^\infty \frac{\log(u/2\pi)\,du}{1/4+(u-t)^2} + O\left(\int_0^\infty \log u\, d\left(\frac{1}{1/4+(u-t)^2}\right)\right)\\
&= \frac{1}{2\pi}\int_0^\infty \frac{\log u\,du}{1/4+(u-t)^2} - \frac{\log 2\pi}{2\pi}\left(2\arctan(2t)+\pi\right)\\
&+ O\left(\int_0^\infty \log u\, d\left(\frac{1}{1/4+(u-t)^2}\right)\right)\\
&= \log\frac{t}{2\pi} + O\left(\frac{1}{t}\right).
\end{aligned}$$

The lemma is proved. □

Recall that $\Xi(s) = Z_\Gamma(s)Z_{id}(s)Z_{ell}(s)Z_{par}(s)$. Now we prove Theorem 3.1.

Proof of Theorem 3.1. First, we prove that

$$\operatorname{Re}\left(\frac{Z'_\Gamma(s)}{Z_\Gamma(s)}\right) < \operatorname{Re}\left(\frac{\Xi'(s)}{\Xi(s)}\right).$$

From logarithmic derivative of $\Xi(s)$ we obtain

$$\frac{\Xi'(s)}{\Xi(s)} = \frac{Z'_\Gamma(s)}{Z_\Gamma(s)} + \frac{Z'_{id}(s)}{Z_{id}(s)} + \frac{Z'_{ell}(s)}{Z_{ell}(s)} + \frac{Z'_{par}(s)}{Z_{par}(s)} =: \frac{Z'_\Gamma(s)}{Z_\Gamma(s)} + U(s). \qquad (3.10)$$

Hence, to complete the proof it is sufficient to show that

$$\operatorname{Re}(U(s)) > 0,\ t > C_1 > 0,\ 0 < \sigma < 1/4.$$

By (3.3) we easily obtain

$$\begin{aligned}U(s) =&a_0+\frac{1}{4}\left(\Psi\left(\frac{s}{2}+\frac{1}{2}\right)-\Psi\left(\frac{s}{2}\right)\right)+\frac{2}{9}\left(\Psi\left(\frac{s}{3}+\frac{2}{3}\right)-\Psi\left(\frac{s}{3}\right)\right)\\&+\frac{1}{3}\Psi_2(s)-\frac{7}{6}\Psi(s)-\Psi\left(s+\frac{1}{2}\right)-2\frac{\zeta'}{\zeta}(2s),\end{aligned}$$

where $a_0=\frac{1}{6}\log 2\pi+\log\frac{\pi}{2}=0.757\ldots$, $\Psi(s)=\Gamma'(s)/\Gamma(s)$,
$\Psi_2(s)=\Gamma_2'(s)/\Gamma_2(s)$ - logarithmic derivative of the double gamma function of Barnes, see equation (3.4).

To prove $\mathrm{Re}\,(U(s))>0$, we need to investigate the behavior of functions $\Psi(s)$, $\Psi_2(s)$ and $\zeta'(2s)/\zeta(2s)$ in the region $0<\sigma<1/4$ and $t>C_1>0$.

From Lemma 3.8 we see that, for $0<\sigma<1/4$ and $t>C_1>0$,

$$\mathrm{Re}(\Psi(s))=\log|s|+O\left(|s|^{-2}\right).$$

Recall the Weierstrass canonical product form for the double Gamma function of Barnes Γ_2:

$$\begin{aligned}&\frac{1}{\Gamma_2(s+1)}\\&=(2\pi)^{\frac{s}{2}}\exp\left\{-\frac{s}{2}-\frac{(\gamma+1)s^2}{2}\right\}\prod_{k=1}^{\infty}\left\{\left(1+\frac{s}{k}\right)^k\exp\left(-s+\frac{s^2}{2k}\right)\right\}.\end{aligned}$$

We deduce (see also Chapter 3 of Fischer [9])

$$\begin{aligned}&\frac{\Gamma_2'(s+1)}{\Gamma_2(s+1)}=\Psi_2(s+1)=\frac{1-\log 2\pi}{2}+(\gamma+1)s-\sum_{k=1}^{\infty}\left(\frac{k}{k+s}-1+\frac{s}{k}\right)\\&=-\frac{1+\log 2\pi}{2}+s-s\Psi(s),\ -s\notin\mathbb{N}.\end{aligned}$$

Here we used the equality

$$\sum_{k=1}^{\infty}\frac{s^2}{k\,(k+s)}=s\gamma+s\Psi\,(s)+1,$$

which can be easily derived from the Weierstrass product of the gamma function

$$\Gamma(s)=\frac{e^{-\gamma s}}{s}\prod_{k=1}^{\infty}\left\{\left(1+\frac{s}{k}\right)^{-1}e^{s/k}\right\},$$

$$s\in\mathbb{C}\setminus\mathbb{Z}_0^-;\ \mathbb{Z}_0^-:=\{0,-1,-2,\ldots\}.$$

Finally,

$$\begin{aligned}\mathrm{Re}(\Psi_2(s)) &= -\frac{3+2\log 2\pi}{2} + \sigma + (1-\sigma)\mathrm{Re}(\Psi(s-1) + t\mathrm{Im}(\Psi(s-1) \\ &= A + (1-\sigma)\log|s| + t\arctan\left(\frac{t}{\sigma}\right) + O(|s|^{-1}),\end{aligned}$$

for some absolute constant A.

The logarithmic derivative of the Riemann zeta-function can be expressed as (see Titchmarsh [45])

$$\frac{\zeta'}{\zeta}(s) = \sum_\rho \frac{1}{s-\rho} - \frac{1}{s-1} - \frac{1}{2}\Psi\left(\frac{s}{2}+1\right) + \frac{1}{2}\log\pi,$$

where the summation is over all nontrivial zeros of the Riemann zeta-function taken in conjugate pairs and in order of increasing imaginary parts.

Let $\rho = \beta + i\gamma$. The real part of $\zeta'/\zeta(s)$ is evaluated to

$$\begin{aligned}\mathrm{Re}\left(\frac{\zeta'}{\zeta}(s)\right) = &\sum_\rho \frac{\sigma-\beta}{(\sigma-\beta)^2+(t-\gamma)^2} - \frac{\sigma-1}{(\sigma-1)^2+t^2} \\ &- \frac{1}{2}\mathrm{Re}\left(\Psi\left(\frac{s}{2}+1\right)\right) + \frac{1}{2}\log\pi.\end{aligned} \tag{3.11}$$

We assume the Riemann hypothesis (take β=1/2) and bound the sum in (3.11). In view of Lemma 3.9, we obtain

$$\begin{aligned}&\sum_\gamma \frac{1}{(\sigma-1/2)^2+(t-\gamma)^2} \\ &= \sum_\gamma \frac{1}{1/4+(t-\gamma)^2} + \sum_\gamma \left(\frac{1}{(\sigma-1/2)^2+(t-\gamma)^2} - \frac{1}{1/4+(t-\gamma)^2}\right) \\ &= \sum_\gamma \frac{1}{1/4+(t-\gamma)^2} \\ &\quad + \sum_\gamma \left(\frac{\sigma(1-\sigma)}{(\sigma-1/2)^2/4+(t-\gamma)^2(1/4+(\sigma-1/2)^2)+(t-\gamma)^4}\right) \\ &= \sum_{\gamma>0} \left(\frac{1}{1/4+(t-\gamma)^2} + \frac{1}{1/4+(t+\gamma)^2}\right) + O(\log t) \\ &= \sum_{\gamma>0} \left(\frac{1}{1/4+(t-\gamma)^2}\right) + O(\log t) = O(\log t).\end{aligned}$$

Here we used also that

$$\sum_{\gamma}\left(\frac{\sigma(1-\sigma)}{(\sigma-1/2)^2/4+(t-\gamma)^2(1/4+(\sigma-1/2)^2)+(t-\gamma)^4}\right)$$
$$\ll \sum_{\gamma>0}\left(\frac{1}{1/4+(t-\gamma)^2}\right) = O(\log t).$$

From equation (3.11) and Lemma 3.9, we get that

$$\begin{aligned}\mathrm{Re}\left(\frac{\zeta'}{\zeta}(s)\right) &= (\sigma-1/2)\log t - \frac{1}{2}\log|s| + O(\log t)\\ &= (\sigma-1/2)\log t + O(\log t) = O(\log t).\end{aligned}$$

In view of above, we obtain

$$\begin{aligned}&\mathrm{Re}(U(s)) \qquad (3.12)\\ &= B_1 - \frac{13}{6}\log|s| + \frac{1}{3}\mathrm{Re}(\Psi_2(s)) - 2\mathrm{Re}\left(\frac{\zeta'}{\zeta}(2s)\right) + O(|s|^{-1})\\ &= B_1 - \frac{2\sigma+11}{6}\log|s| + \frac{t}{3}\arctan\left(\frac{t}{\sigma}\right) - 2\mathrm{Re}\left(\frac{\zeta'}{\zeta}(2s)\right) + O(|s|^{-1})\\ &= -\frac{5+26\sigma}{6}\log t + \frac{t}{3}\arctan\left(\frac{t}{\sigma}\right) + O(\log t) = \frac{t}{3}\arctan\left(\frac{t}{\sigma}\right) + O(\log t).\end{aligned}$$

It is easy to see that there exists a $C_1 > 0$ such that the last expression is positive for $t > C_1 > 0$ and $0 < \sigma < 1/4$. This proves that

$$\mathrm{Re}\left(\frac{Z_\Gamma'(s)}{Z_\Gamma(s)}\right) < \mathrm{Re}\left(\frac{\Xi'(s)}{\Xi(s)}\right).$$

We note that this inequality also holds for $0 < \sigma < 1/2$, with some restrictions of t though. Restrictions of t are due to the zeros of the function $\zeta(2s)$.

Now we prove that

$$\mathrm{Re}\left(\frac{\Xi'(s)}{\Xi(s)}\right) < 0,$$

for $t > C$ and $0 < \sigma < 1/2$.

The function $\Xi(s)$ is an entire function of order two, see Randol [38]. It has a canonical product expansion

$$\Xi(s) = e^{as^2+bs+c}s^n \prod_{\hat\rho}\left(1-\frac{s}{\hat\rho}\right)e^{s/\hat\rho+(1/2)(s/\hat\rho)^2}, \qquad (3.13)$$

where $\hat{\rho}$ runs over the nonzero roots of $\Xi(s)$, and a, b, c, and n are constants. Taking its logarithmic derivative, we obtain

$$\frac{\Xi'(s)}{\Xi(s)} = 2as + b + \frac{n}{s} + \sum_{\hat{\rho}} \frac{s^2}{\hat{\rho}^2(s-\hat{\rho})}.$$

If $\hat{\rho} = 1/2 + ir_n \, (n \geqslant 0)$, (concerning r_n see Momotani [32]) then

$$\operatorname{Re}\left(\frac{\Xi'(s)}{\Xi(s)}\right) = 2a\sigma + b + \frac{n\sigma}{\sigma^2+t^2} + \sum_{n\geqslant 0} \frac{\sigma(1/4 - r_n^2) + tr_n}{(1/4-r_n^2)^2 + r_n^2} \tag{3.14}$$
$$+ \sum_{n\geqslant 0} \frac{1/2}{1/4 + r_n^2} + \sum_{n\geqslant 0} \frac{\sigma - 1/2}{(\sigma-1/2)^2 + (t-r_n)^2}.$$

We see that the sum

$$\sum_{n\geqslant 0} \frac{\sigma(1/4 - r_n^2) + tr_n}{(1/4-r_n^2)^2 + r_n^2} = \frac{\sigma(1/4 - r_0^2) + tr_0}{(1/4-r_0^2)^2 + r_0^2} + \sum_{n\geqslant 1} \frac{\sigma(1/4 - r_n^2) + tr_n}{(1/4-r_n^2)^2 + r_n^2}$$

is positive and unbounded as t grows.

Then, from equation (3.14), we see that there exists a number C, such that

$$\operatorname{Re}\left(\frac{\Xi'(s)}{\Xi(s)}\right) > 0$$

for $t > C > 0$ and $1/2 < \sigma < 1$. By note after the Lemma 3.6, for fixed $t > C > 0$, the function $|\Xi(\sigma+it)|$ is monotone increasing as a function of σ, $1/2 < \sigma < 1$. In view of the functional equation $\Xi(s) = \Xi(1-s)$, the function $|\Xi(\sigma+it)|$ is monotone decreasing for $t > C > 0$ as a function of σ, $0 < \sigma < 1/2$. So, the real part of its logarithmic derivative is negative, and the theorem is proved. □

Moreover, following the proof of Theorem 3.1 we can bound the function $\operatorname{Re}\left(\frac{Z_\Gamma'(s)}{Z_\Gamma(s)}\right)$ in $3/4 < \sigma < 1$, $t > C_0 = \max(C, C_1) > 0$.

Corollary 3.10. *For $3/4 < \sigma < 1$ and $t > C_0 > 0$ holds*

$$Re\left(\frac{Z_\Gamma'(s)}{Z_\Gamma(s)}\right) > -t \cdot \frac{\pi}{6}\left(1 + O\left(\frac{\log t}{t}\right)\right).$$

Proof. In view of Theorem 3.1, the function $|Z_\Gamma(\sigma+it)|$ is also monotone decreasing for $t > C_0 > 0$ as a function of σ, $0 < \sigma < 1/4$.

From functional equation $\Xi(s) = \Xi(1-s)$ and

$$\operatorname{Re}\left(\frac{\partial}{\partial s}\log \Xi(s)\right) < 0, \ \textit{for } 0 < \sigma < 1/2, \ t > C_0 > 0,$$

we see that

$$\operatorname{Re}\left(\frac{Z_\Gamma'(1-s)}{Z_\Gamma(1-s)}\right) > -\operatorname{Re}\left(U(1-s)\right),\, 0<\sigma<1/4,\, t>C_0>0,$$

where the function $U(s)$ is defined by (3.10). Using the fact that $\operatorname{Re}(z(s)) = \operatorname{Re}(\overline{z(s)})$ for any complex function $z(s)$ and $z(\overline{s}) = \overline{z(s)}$ for $z=\Gamma,\zeta,\Gamma_2$, and also that

$$\operatorname{Re}(U(s)) = -\frac{5+26\sigma}{6}\log t + \frac{t}{3}\arctan\left(\frac{t}{\sigma}\right) + O(\log t),$$

(see equation (3.12)), we obtain

$$\operatorname{Re}\left(\frac{Z_\Gamma'(s)}{Z_\Gamma(s)}\right) > -\operatorname{Re}(U(\sigma+it)) > -t\cdot\frac{\pi}{6}\left(1+O\left(\frac{\log t}{t}\right)\right) \tag{3.15}$$

for $3/4<\sigma<1$, $t>C_0>0$. □

Recall the Riemann's xi function is given by an equation

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s).$$

Similarly as $\operatorname{Re}\left(\frac{Z_\Gamma'(s)}{Z_\Gamma(s)}\right)$, we can bound the function $\operatorname{Re}\left(\frac{\zeta'(s)}{\zeta(s)}\right)$ for $1/2<\sigma<1$ and $t\geqslant 8$.

The following lemma is needed.

Lemma 3.11. *For $t>1$, $|Re(\cot(s))|<e^{-t}$.*

Proof. Writing an explicit form

$$\cot(s) = \frac{\cos\sigma\cosh t - i\sin x\sinh t}{\sin\sigma\cosh t + i\cos\sigma\sinh t}$$

and calculating its real part, we obtain

$$\frac{4|\sin\sigma\cos\sigma|}{4\sin^2\sigma+(e^t-e^{-t})^2} < e^{-t}$$

or

$$4|\sin\sigma\cos\sigma| < e^{-t}\left(4\sin^2\sigma + e^{2t} - 2 + e^{-2t}\right).$$

For $t>1$ the last inequality is easily verified by elementary means. □

Taking a logarithmic derivative of both sides of functional equation

$$\zeta(s) = 2^s \pi^{s-1} \zeta(1-s)\Gamma(1-s)\sin\frac{\pi s}{2},$$

we obtain

$$\frac{\zeta'(s)}{\zeta(s)} = \log 2\pi - \frac{\zeta'(1-s)}{\zeta(1-s)} + \frac{\pi}{2}\cot\left(\frac{\pi s}{2}\right) - \frac{\Gamma'(1-s)}{\Gamma(1-s)}. \tag{3.16}$$

From equation (3.16) and Theorem D, we see that

$$\mathrm{Re}\left(\frac{\zeta'(s)}{\zeta(s)}\right) > \log 2\pi + \frac{\pi}{2}\mathrm{Re}\cot\left(\frac{\pi(1-s)}{2}\right) - \mathrm{Re}\left(\frac{\Gamma'(s)}{\Gamma(s)}\right)$$

for $1/2 < \sigma < 1$ and $t \geqslant 8$. In view of Lemmas 3.8 and 3.11, we rewrite the last inequality in the following way

$$\mathrm{Re}\left(\frac{\zeta'(s)}{\zeta(s)}\right) > -(1+\delta)\log\left(\frac{|s|}{2\pi}\right),$$

where $\delta = \delta(t) = O(e^{-t})$.

3.2. **Proof of Theorems 3.4 and 3.5.**

First, we prove Theorem 3.4.

Proof of Theorem 3.4. The function $M(s)$ is an entire function of order two, and it has the same form of canonical product expansion (3.13) as the function $\Xi(s)$. So, for $t > t_0 > 0$, the function $|M(s)|$ is monotone decreasing with respect to $0 < \sigma < 1/2$.

Let

$$l(s) = \exp\left(\int_0^{1/2-s} v\tan \pi v \, dv\right).$$

To complete the proof, we need to show that

$$\mathrm{Re}\left(\frac{l'(s)}{l(s)}\right) > 0, \ \textit{for } 0 < \sigma < 1/2, \ t > \hat{t}_0.$$

By elementary calculation, we obtain

$$\begin{aligned}
&\mathrm{Re}\left(\frac{l'(s)}{l(s)}\right) = \mathrm{Re}\left\{\left(s-\frac{1}{2}\right)\tan\pi\left(\frac{1}{2}-s\right)\right\} \\
&= \frac{t(1-e^{-4\pi t})}{e^{-4\pi t} - 2e^{-2\pi t}\cos 2\pi\sigma + 1} + \left(\sigma - \frac{1}{2}\right)\cdot\frac{2e^{-2\pi t}\sin 2\pi\sigma}{e^{-4\pi t} - 2e^{-2\pi t}\cos 2\pi\sigma + 1} \\
&= t(1+o(1)).
\end{aligned}$$

We take $B = \max(t_0, \hat{t}_0)$, and the proof follows. $\square$

We turn to the proof of Theorem 3.5.

Proof of Theorem 3.5. Let $s = 1/2 - \Delta + it$. From functional equation (3.5), we have

$$Z_C(1/2 - \Delta + it) = f(1/2 - \Delta + it) Z_C(1/2 + \Delta - it).$$

Since $Z_C(\bar{s}) = \overline{Z_C(s)}$ and $Z_C(1/2 + \Delta + it) \neq 0$ for $0 < \Delta \leqslant 1/2$ we have

$$Z_C(1/2 - \Delta + it) = f(1/2 - \Delta + it)\overline{Z_C(1/2 + \Delta + it)}$$

and

$$\begin{aligned} \frac{|Z_C(1/2 - \Delta + it)|}{|Z_C(1/2 + \Delta + it)|} &= |f(1/2 - \Delta + it)| \\ &= \left| \exp\left(4\pi(g-1)\int_0^{-\Delta+it} v\tan(\pi v)\,dv\right)\right|. \end{aligned}$$

From last equation we see that only real part of function

$$\int_0^{-\Delta+it} v\tan(\pi v)\,dv$$

is a subject of our investigation.

Since

$$I = \tan(\pi v) = \frac{(1 - e^{2\pi i v})i}{1 + e^{2\pi i v}},$$

we have

$$\int_0^{-\Delta+it} \frac{(1 - e^{2\pi i v})iv}{1 + e^{2\pi i v}}\,dv. \tag{3.17}$$

Integrating along $\gamma(\lambda) = \lambda\left(\frac{-\Delta+it}{\Delta}\right), \lambda \in (0, \Delta)$ we get

$$I = \left(\frac{-\Delta + it}{\Delta}\right)^2 \int_0^{\Delta} \frac{\left(1 - \exp\left(2\pi i\lambda\left(\frac{-\Delta+it}{\Delta}\right)\right)\right)i\lambda}{1 + \exp\left(2\pi i\lambda\left(\frac{-\Delta+it}{\Delta}\right)\right)}\,d\lambda \tag{3.18}$$

Making a substitution

$$\lambda = \frac{1}{2\pi i}\left(\frac{\Delta}{-\Delta + it}\right)\log x$$

in the integral (3.18), we obtain

$$\begin{aligned}
&\frac{1}{2\pi}\cdot\frac{1}{2\pi i}\int_1^{e^{2\pi i(-\Delta+it)}}\frac{(1-x)\log x}{x(1+x)}\,dx\\
&=\frac{1}{2\pi}\cdot\frac{1}{2\pi i}\int_1^{e^{2\pi i(-\Delta+it)}}\frac{\log x}{x(1+x)}\,dx\\
&-\frac{1}{2\pi}\cdot\frac{1}{2\pi i}\int_1^{e^{2\pi i(-\Delta+it)}}\frac{\log x}{1+x}\,dx\\
&=\frac{1}{4\pi^2 i}\int_1^{e^{2\pi i(-\Delta+it)}}\frac{\log x}{x}\,dx-\frac{1}{2\pi^2 i}\int_1^{e^{2\pi i(-\Delta+it)}}\frac{\log x}{1+x}\,dx\\
&=\frac{i}{2}(-\Delta+it)^2+\frac{i}{2\pi^2}\int_1^{e^{2\pi i(-\Delta+it)}}\log x\,d(\log(1+x))\\
&=\frac{i}{2}(\Delta-it)^2+\frac{1}{\pi}(\Delta-it)\log\left(1+e^{2\pi i(-\Delta+it)}\right)\\
&+\frac{i}{2\pi^2}\left(\int_1^{e^{2\pi i(-\Delta+it)}}-\frac{\log(1+x)}{x}\,dx\right)\\
&=\frac{i}{2}(\Delta-it)^2+\frac{1}{\pi}(\Delta-it)\log\left(1+e^{2\pi i(-\Delta+it)}\right)\\
&+\frac{i}{24}+\frac{i}{2\pi^2}L_2(-e^{2\pi i(-\Delta+it)}),
\end{aligned}$$

where

$$L_2(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^2}.$$

We see that

$$\begin{aligned}
&\operatorname{Re}\left(\int_0^{-\Delta+it}v\tan(\pi v)\,dv\right)\\
&=\Delta t+\frac{\Delta}{\pi}\log\left|1+e^{2\pi i(-\Delta+it)}\right|+\frac{t}{\pi}\arg\left(1+e^{2\pi i(-\Delta+it)}\right)\\
&-\frac{1}{2\pi^2}\operatorname{Im}\left(L_2\left(-e^{2\pi i(-\Delta+it)}\right)\right)\\
&=\Delta t+\frac{\Delta}{2\pi}\log\left(1+2e^{-2\pi t}\cos(2\pi\Delta)+e^{-4\pi t}\right)\\
&-\frac{t}{\pi}\arctan\left(\frac{\sin(2\pi\Delta)}{e^{2\pi t}+\cos(2\pi\Delta)}\right)\\
&-\frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{\sin(\pi n(1-2\Delta))}{n^2 e^{2\pi tn}}=:R(\Delta,t)=\Delta t+o(1),
\end{aligned}$$

Function $R(\Delta,t)$ is monotone increasing with respect to t and monotone decreasing with respect to Δ only for "small" t. To find $0<t_0<t$ such that $R(\Delta,t)>0$,

$(0 < \Delta \leqslant 1/2)$ we need to solve an equation

$$R(1/2, t_0) = 0$$

or

$$\frac{t_0}{2} + \frac{1}{4\pi} \log \left(1 - 2e^{-2\pi t_0} + e^{-4\pi t_0}\right) = 0.$$

We find that

$$t_0 = \frac{1}{\pi} \log \frac{2}{\sqrt{5} - 1} = 0.15\ldots$$

If $t > t_0$, then

$$\frac{|Z(1/2 - \Delta + it)|}{|Z(1/2 + \Delta + it)|} = \left| e^{4\pi(g-1)\int_0^{-\Delta+it} v \tan \pi v \, dv} \right| = \left| e^{4\pi(g-1)\Re \int_0^{-\Delta+it} v \tan \pi v \, dv} \right|$$
$$= \left| e^{4\pi(g-1)(\Delta t + o(1))} \right| > 1.$$

This proves the theorem. $\square$

Conclusions

In this thesis, the following results are established:

- The limit law holds for the Lerch zeta-function

$$L(\lambda, \alpha, s) = \sum_{n=0}^{\infty} \frac{e^{2\pi i \lambda n}}{(n + \alpha)^s}$$

when the pair of parameters (λ, α), depending on T, tends to $(1, 1), (1, 1/2)$, $(1/2, 1), (1/2, 1/2)$. To obtain a limit law when λ tends to 1 faster than $1 - e^{T/\log T}$, we need to make certain modifications. If the limit law holds, we write

$$\lim_{T \to \infty} \nu_T \left(\frac{\log |L(\lambda, \alpha, 1/2 + it)|}{\sqrt{2^{-1} \log \log T}} < x \right) = \Phi(x).$$

When (λ, α) tends to $(0, 0), (1/2, 0), (1, 0), (0, 1/2), (0, 1)$ to obtain the limit law, we need certain modifications of $L(\lambda, \alpha, 1/2 + it)$.

- Let $N_\Phi(\sigma_1, \sigma_2, T) = N_\Phi(\sigma_1, \sigma_2, T, q, \alpha)$ denote the number of zeros of $\Phi(q, s, \alpha)$ in the region $\{s : \sigma_1 < \mathrm{Re}(s) < \sigma_2, 0 < \mathrm{Im}(s) \leqslant T\}$. Let $\sigma_0 = \sigma_0(q, \alpha)$ be a real number defined by the equality

$$\sum_{n=1}^{\infty} \frac{|q|^n}{\left(\frac{n}{\alpha} + 1\right)^{\sigma_0}} = 1.$$

One may show that $\sigma_0 \leq c = 1.73\ldots$, where $\zeta(c) = \sum_{n=1}^{\infty} n^{-c} = 2$. The number σ_0 can take any value between $-\infty$ and c. The function $\Phi(q, s, \alpha)$ has no zeros for $\sigma > \sigma_0$, and its estimate of zeros in the strip $\sigma_1 < \sigma < \sigma_2 \leq \sigma_0$ is

$$T \ll N_\Phi(\sigma_1, \sigma_2, T) \ll T,$$

where $q \in \mathbb{C}, 0 < |q| < 1$ and $0 < \alpha < 1$ is a transcendental number.

Let

$$R = \{s : \mathrm{Re}(s) > 1, 0 < \mathrm{Im}(s) \leqslant 1000\}.$$

Using MATHEMATICA, we obtain the following table of number of zeros of function $\Phi(q, s, \alpha)$ in R when q and α are fixed:

α \ q	**0.9**	**0.95**	**0.99**	**1**
0.9	2	8	34	40
0.95	4	10	37	46
0.99	14	27	41	45
1	0	0	0	0

- Let Z_Γ be the Selberg zeta-function defined on for modular group $\Gamma = SL(2, \mathbb{Z})$ and $\Xi(s) = Z_\Gamma(s) Z_{id}(s) Z_{ell}(s) Z_{par}(s)$. The function satisfies the inequality:

$$\mathrm{Re}\left(\frac{\Xi'(s)}{\Xi(s)}\right) < 0$$

for $t > C > 0$ and $0 < \sigma < 1/2$.

Even more, under the Riemann hypothesis the functions Z_C and $\Xi(s)$ satisfy inequality:

$$\mathrm{Re}\left(\frac{Z'_\Gamma(s)}{Z_\Gamma(s)}\right) < \mathrm{Re}\left(\frac{\Xi'(s)}{\Xi(s)}\right)$$

for $t > C_1 > 0$ and $0 < \sigma < 1/4$.

- Let $Z_C(s)$ be the Selberg zeta-function defined for the compact Riemann surface and

$$M(s) = Z_C(s)\exp\left(2\pi(g-1)\int_0^{1/2-s} v\tan\pi v\,dv\right).$$

The functions $Z_C(s)$ and $M(s)$ satisfy the inequality

$$\mathrm{Re}\left(\frac{Z'_C(s)}{Z_C(s)}\right) < \mathrm{Re}\left(\frac{M'(s)}{M(s)}\right) < 0,\ t > B > 0,\ 0 < \sigma < 1/2.$$

For the modulus of the function $Z_\Gamma(s)$ holds the following statement: $|Z_\Gamma(1/2-\Delta+it)| > |Z_\Gamma(1/2+\Delta+it)|$ for $0 < \Delta \leqslant 1/2$ and $t > t_0$, where $t_0 = \frac{1}{\pi}\log\frac{2}{\sqrt{5}-1} = 0.15\ldots$

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