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The Implicit Euler Scheme for FSDEs with Stochastic Forcing: Existence and Uniqueness of the Solution

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Abstract: In this paper, we focus on fractional stochastic differential equations (FSDEs) with a stochastic forcing term, i.e., to FSDE, we add a stochastic forcing term. Using the implicit scheme of Euler's approximation, the conditions for the existence and uniqueness of the solution of FSDEs with a stochastic forcing term are established. Such equations can be applied to considering FSDEs with a permeable wall.

Keywords: stochastic differential equations; stochastic forcing; fractional Brownian motion; implicit Euler scheme; p-variation; Pearson model

MSC: 60G22; 60H10; 60H05

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1. Introduction

We will consider stochastic differential equations of the following form:

$$X_t = X_0 + \Phi(X_t) - \Phi(X_0) + \int_0^t f(s, X_s)\, ds + \int_0^t g(s, X_s)\, dB_s^H, \qquad t \in [0, T], \tag{1}$$

where $\Phi : \mathbb{R} \to \mathbb{R}$ is a continuous function, $f, g : [0, T] \times \mathbb{R} \to \mathbb{R}$ are measurable functions, and $B^H = (B_t^H)_{t \geqslant 0}$, $1/2 < H < 1$, denotes a fractional Brownian motion (fBm). The stochastic integral in Equation (1) is a pathwise generalized Lebesgue–Stieltjes integral. Thus, we can use the pathwise approach to consider this FSDE. We call such an equation FSDE with a stochastic forcing term Φ.

Many authors have considered the problem of the existence and uniqueness of solutions to FSDEs without a stochastic forcing term [1–14]. The first attempt to consider the FSDE with a stochastic forcing term was made in an article by Kubilius and Medžiūnas [15]. In this article, equations with constant and strictly positive diffusion coefficients with a "soft wall" are considered. That is, the value of the function $\Phi(x)$ depends on the position of x to a fixed point, w, called the wall boundary. For example, we can take exponential forces with a wall (w) defined by a function

$$\Phi(x) = \Phi_0 \exp\{-\lambda(x - w)\}$$

characterized by the amplitude Φ_0 and decay constant λ. The term "soft wall" was introduced in reference [16]. It should be noted that the "soft wall" model has a permeable wall. The process may cross the wall, but it is affected by the force of the selected quantity in the opposite direction. The force acts weakly when the process is far from the wall. As it approaches or crosses the wall, the force acts stronger. An illustration of the behavior of trajectories for such a model was considered in [15,17]. There, we considered the fractional Vasicek process with the soft wall. FSDE (1) includes the "soft wall" model.

In reference [17] we consider the conditions for the existence and uniqueness of solutions to Equation (1) by using the implicit Picard iteration. In the final part of the proof, an error occurred due to an incorrect simplified notation. The theorem statement remains true, but we must use the implicit Euler approximation instead of the implicit Picard iteration. The new proof repeats the proof of Theorem 3 in [17] with estimates of other norms.

The proof of the existence and uniqueness of the solution of Equation (1) is based on estimates obtained in the article by Nualart and Răşcanu [10] and the conditions for the coefficients f and g are almost the same as in their article.

This paper is organized in the following way. In Section 2, we present the paper's main results. In Section 3, we define a deterministic differential equation corresponding to FSDE (1) and consider its implicit Euler approximation properties. Moreover, it contains definitions of considered spaces of functions and a priori estimates for the Lebesgue–Stieltjes integral. In Section 4, we prove the existence and uniqueness of a solution for a deterministic differential equation. In Section 5, we consider the fractional Pearson diffusion process as an example.

2. Main Result

We will assume that the coefficients f, g satisfy the following conditions:

($\mathbf{A}_1$) $g(t,x)$ is differentiable in x, and there exist some constants $0 < \beta, \delta \leq 1$, and for every $N \geq 0$, there exists $M_N > 0$ such that the following properties hold:

(i) Lipschitz continuity in x

$$|g(t,x) - g(t,y)| \leq M_0|x-y|, \qquad \forall x,y \in \mathbb{R},\ t \in [0,T];$$

(ii) Local uniform Hölder continuity of the derivative in x

$$|g'_x(t,x) - g'_x(t,y)| \leq M_N|x-y|^{\delta}, \qquad \forall x,y \in [-N,N],\ \forall t \in [0,T];$$

(iii) Hölder continuity in t

$$|g(s,x) - g(t,x)| + |g'_x(s,x) - g'_x(t,x)| \leq M_0|t-s|^{\beta}, \qquad \forall x \in \mathbb{R},\ \forall t,s \in [0,T];$$

($\mathbf{A}_2$) There exists a bounded function b_0, and for every $N \geqslant 0$, there exists $L_N > 0$ such that the following properties hold:

(i) Local uniform Lipschitz continuity in x

$$|f(t,x) - f(t,y)| \leq L_N|x-y|, \qquad \forall x,y \in [-N,N],\ \forall t \in [0,T];$$

(ii) The rate of growth

$$|f(t,x)| \leq L_0|x| + b_0(t), \qquad |b_0(t)| \leqslant L_0 \qquad \forall x \in \mathbb{R},\ \forall t \in [0,T].$$

($\mathbf{A}_3$) Assume the following:

(i) Function $D : \mathbb{R} \to \mathbb{R}$, where $D(x) := x - \Phi(x) =$ is strictly monotonic and surjective;

(ii) There is a constant $d > 0$, such that we have the following:

$$|D(x) - D(y)| \geqslant d|x-y|. \tag{2}$$

($\mathbf{A}_4$) Assume that $\Phi : \mathbb{R} \to \mathbb{R}$ is a differentiable function, and there exist some constants $0 < c < 1$, $0 < \rho \leqslant 1$, and for every $N \geq 0$, there exists $K_N > 0$ such that we have the following:

(i) $\Phi'(x) \leqslant c$, for all $x \in \mathbb{R}$,

(ii) Local uniform Hölder continuity of the derivative

$$|\Phi'_x(x) - \Phi'_x(y)| \leq K_N|x-y|^{\rho}, \qquad \forall x,y \in [-N,N].$$

Remark 1 (see Remark 8 in [15])**.** *Under assumptions* ($\boldsymbol{A}_4$)*, function D satisfies assumptions* ($\boldsymbol{A}_3$)*.*

We can now formulate our main result. We set the following:

$$\alpha_0 = \min\left\{\frac{1}{2}, \beta, \frac{\delta}{1+\delta}\right\}, \qquad \overline{\alpha}_0 = \min\left\{\frac{1}{2}, \beta, \frac{\delta}{1+\delta}, \frac{\rho}{1+\rho}\right\} \tag{3}$$

Theorem 1. *Suppose that the functions $f(t,x)$ and $g(t,x)$ satisfy the assumptions $(\boldsymbol{A}_1)$ and $(\boldsymbol{A}_2)$ with $\frac{1}{H} - 1 < \delta \leqslant 1$, $1 - H < \beta \leqslant 1$. Let $\gamma \in (\gamma_0, H)$, where $\gamma_0 = 1 - \alpha_0$. If assumptions $(\boldsymbol{A}_3)$ are satisfied, then there exists a stochastic process $X \in C^\gamma(0,T)$ satisfying FSDE (1), where $C^\gamma(0,T)$ is the space of γ-Hölder continuous functions. If assumptions $(\boldsymbol{A}_4)$ are satisfied and $\gamma \in (\overline{\gamma}_0, H)$, where $\overline{\gamma}_0 = 1 - \overline{\alpha}_0$, then there exists a unique stochastic process $X \in C^\gamma(0,T)$ satisfying FSDE (1).*

3. Deterministic Differential Equations

3.1. Preliminaries

3.1.1. Spaces of Functions and Norms

Let us introduce some function spaces that will be used to analyze solutions of (1).

We denote by $W_0^{\alpha,\infty}(0,T)$, where $0 < \alpha < 1/2$, the space of real-valued measurable functions $f : [0,T] \to \mathbb{R}$ such that we have the following:

$$\|f\|_{\alpha,\infty} := \sup_{s\in[0,T]} \left(|f(s)| + \int_0^s \frac{|f(s) - f(u)|}{(s-u)^{1+\alpha}}\, du \right) < \infty.$$

The space $W_0^{\alpha,\infty}(0,T)$ is a Banach space with respect to the norm $\|f\|_{\alpha,\infty}$, and for $\lambda \geqslant 0$, the equivalent norm is defined by the following:

$$\|f\|_{\alpha,\lambda} := \sup_{t\in[0,T]} e^{-\lambda t} \left(|f(t)| + \int_0^t \frac{|f(t) - f(s)|}{(t-s)^{1+\alpha}}\, ds \right).$$

For any $\mu \in (0,1]$, denote by $C^\mu(0,T)$ the space of μ-Hölder continuous functions $f : [0,T] \to \mathbb{R}$ equipped with a norm $\|f\|_\mu := |f|_\infty + |f|_\mu$, where we have the following:

$$|f|_\mu := \sup_{0\leqslant s<t\leqslant T} \frac{|f(t) - f(s)|}{|s-t|^\mu}, \qquad |f|_\infty := \sup_{t\in[0,T]} |f(t)|.$$

Clearly, we have $C^{1-\alpha}(0,T) \subset W_0^{\infty,\alpha}(0,T)$ for $0 < \alpha < 1/2$ and

$$\|f\|_{\alpha,\lambda} \leqslant \|f\|_{1-\alpha} \left(1 + \frac{T^{1-2\alpha}}{1-2\alpha}\right). \tag{4}$$

We denote by $W_T^{1-\alpha,\infty}(0,T)$, $0 < \alpha < 1/2$, the space of measurable functions $g :$ $[0,T] \to \mathbb{R}$ such that we have the following:

$$\|g\|_{1-\alpha,\infty,T} := \sup_{0\leqslant s<t<T} \left(\frac{|g(t) - g(s)|}{(t-s)^{1-\alpha}} + \int_s^t \frac{|g(y) - g(s)|}{(y-s)^{2-\alpha}}\, dy \right) < \infty.$$

Note that $W_T^{1-\alpha,\infty}(0,T) \subset C^{1-\alpha}(0,T)$ (see [10]).

We also denote by $W_0^{\alpha,1}(0,T)$, $0 < \alpha < 1/2$, the space of measurable functions f on $[0,T]$ such that we have the following:

$$\|f\|_{\alpha,1} := \int_0^T \frac{|f(s)|}{s^\alpha}\, ds + \int_0^T \int_0^s \frac{|f(s) - f(y)|}{|s-y|^{1+\alpha}}\, dy\, ds < \infty. \tag{5}$$

Fix $p \in (0,\infty)$. Let $\varkappa = \{\{t_0,\dots,t_n\}\colon 0 = t_0 < \cdots < t_n = T,\ n \geqslant 1\}$ denote a set of all possible partitions of $[0,T]$. For any $f : [0,T] \to \mathbb{R}$, we define the following:

$$v_p(f;[0,T]) = \sup_{\varkappa} \sum_{k=1}^{n} |f(t_k) - f(t_{k-1})|^p, \qquad V_p(f;[0,T]) = v_p^{1/p}(f;[0,T]).$$

Recall that v_p is called the p-variation of f on $[0,T]$. We denote by $\mathcal{W}_p([a,b])$ (resp. $C\mathcal{W}_p([a,b])$) the class of (resp. continuous) functions on $[0,T]$ with bounded p-variation, $p \in (0,\infty)$.

Define $V_p(f) := V_p(f;[0,T])$, which is a seminorm on $\mathcal{W}_p([0,T])$, and $V_p(f)$ is 0 if and only if f is constant. For each f, $V_p(f)$ is a non-increasing function of $p \geqslant 1$, i.e., if $1 \leqslant q < p$, then $V_p(f) \leqslant V_q(f)$. Thus, $\mathcal{W}_q([0,T]) \subseteq \mathcal{W}_p([0,T])$ if $1 \leqslant q < p < \infty$. If $f \in \mathcal{W}_p([a,b])$, then f is bounded.

Let $p \geqslant 1$ and $V_{p,\infty}(f) := V_{p,\infty}(f;[0,T]) = V_p(f) + |f|_\infty$. Then $V_{p,\infty}(f)$ is a norm, and is $\mathcal{W}_p([0,T])$ equipped with the p-variation norm is a Banach space.

3.1.2. Riemann–Stieltjes Integral

Assume that $f \in W_0^{\alpha,1}(0,T)$ and $h \in W_T^{1-\alpha,\infty}(0,T)$, where $0 < \alpha < 1/2$. The generalized Lebesgue–Stieltjes integral (see [10]) $\int_0^t f\,dh$ exists for all $t \in [0,T]$ and for any $0 \leqslant s < t \leqslant T$

$$\left|\int_s^t f dh\right| \leqslant \Lambda_\alpha(h)\left(\int_s^t \frac{|f(r)|}{(r-s)^\alpha}\,dr + \int_s^t\int_s^r \frac{|f(r)-f(y)|}{|r-y|^{\alpha+1}}\,dydr\right), \tag{6}$$

where

$$\Lambda_\alpha(h) \leqslant \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)}\,\|h\|_{1-\alpha,\infty,T}$$

and $\Gamma(\cdot)$ is a Gamma function. Furthermore, the integral $\int_0^t f\,dh$ exists if $f \in W_0^{\alpha,\infty}(0,T)$.

If $f \in C^\nu(a,b)$ and $h \in C^\mu(a,b)$ with $\nu + \mu > 1$, then the generalized Lebesgue–Stieltjes integral exists and coincides with the Riemann–Stieltjes integral (see [18]).

From Young's Stieltjes integrability theorem [19] (see p. 264) the Riemann–Stieltjes integral $\int_0^t f\,dh$ can be defined for functions having bounded p-variation on $[0,T]$ (see [20]).

Let $f \in \mathcal{W}_q([a,b])$ and $h \in \mathcal{W}_p([a,b])$ with $p > 0$, $q > 0$, $1/p + 1/q > 1$. If f and h have no common discontinuities, then the extended Riemann–Stieltjes integral $\int_a^b f\,dh$ exists, and the Love–Young inequality.

$$\left|\int_a^b f\,dh - f(y)[h(b)-h(a)]\right| \leqslant C_{p,q} V_q(f;[a,b]) V_p(h;[a,b]) \tag{7}$$

holds for any $y \in [a,b]$, where $C_{p,q} = \zeta(p^{-1}+q^{-1})$, $\zeta(s)$ denotes the Riemann zeta function, i.e., $\zeta(s) = \sum_{n\geqslant 1} n^{-s}$.

3.1.3. Estimation of the Generalized Lebesgue–Stieltjes Integrals

From now on, we fix $0 < \alpha < 1/2$. For any function $u \in W_0^{\alpha,\infty}(0,T)$ define

$$F_t^{(f)}(u) = \int_0^t f(s,u_s)\,ds, \tag{8}$$

where f satisfies the assumptions $(\mathbf{A}_2)$.

Proposition 1 (see [10])**.** *Assume that f satisfies the assumptions $(\boldsymbol{A}_2)$. If $u \in W_0^{\alpha,\infty}(0,T)$ then $F^{(f)}(u) \in C^{1-\alpha}(0,T)$ and we have the following:*

$$
\begin{aligned}
(1) \quad \|F^{(f)}(u)\|_{1-\alpha} \leqslant & c^{(1)}\left(1+|u|_{\infty}\right), \\
(2) \quad \|F^{(f)}(u)\|_{\alpha,\lambda} \leqslant & \frac{c^{(2)}}{\lambda^{1-2\alpha}}\left(1+\|u\|_{\alpha,\lambda}\right)
\end{aligned}
$$

for all $\lambda \geqslant 1$, where $c^{(i)}$, $i \in \{1,2\}$, are positive constants depending only on α, T, L_0.

If $u, v \in W_0^{\alpha,\infty}(0,T)$ are such that $|u|_\infty \leqslant N$ and $|v|_\infty \leqslant N$, then we have the following:

$$
\|F^{(f)}(u) - F^{(f)}(v)\|_{\alpha,\lambda} \leqslant \frac{c_N}{\lambda^{1-\alpha}}\|u-v\|_{\alpha,\lambda}
$$

for all $\lambda \geqslant 1$, where $c_N = C_{\alpha,T} L_N \Gamma(1-\alpha)$ depends only on α, T, and L_N from $(\boldsymbol{A}_2)$.

Given two functions, $h \in W_T^{1-\alpha,\infty}(0,T)$ and $u \in W_0^{\alpha,\infty}(0,T)$, we denote the following:

$$
G_t(u) = \int_0^t u_s\, dh_s, \qquad G_t^{(g)}(u) = \int_0^t g(s,u_s)\, dh_s, \tag{9}
$$

where g satisfies the assumptions $(\mathbf{A}_1)$ with constant $\beta > \alpha$.

Proposition 2 (see Proposition 4.1 [10]). *If $u \in W_0^{\alpha,1}(0,T)$, then we have the following:*

$$
\begin{aligned}
&\left|G_t(u)\right| + \int_0^t \frac{\left|G_t(u) - G_s(u)\right|}{(t-s)^{1+\alpha}}\, ds \\
&\quad \leqslant \Lambda_\alpha(h)\left[\int_0^t \left(\frac{c_\alpha^{(1)}}{(t-r)^{2\alpha}} + \frac{1}{r^\alpha}\right)|u(r))|\, dr + \int_0^t \int_0^r \frac{\left|u(r)-u(v)\right|}{(r-v)^{1+\alpha}}\left[(t-v)^{-\alpha} + \alpha\right] dv dr\right],
\end{aligned}
$$

where $c_\alpha^{(1)} = B(2\alpha, 1-\alpha)$, $B(\cdot,\cdot)$ is the Beta function.

Proposition 3 (see [10]). *If $u \in W_0^{\alpha,\infty}(0,T)$ then $G^{(g)}(u) \in C^{1-\alpha}(0,T)$ and*

$$
\begin{aligned}
(1) \quad \|G^{(g)}(u)\|_{1-\alpha} \leqslant & \Lambda_\alpha(h) C^{(1)}\left(1+\|u\|_{\alpha,\infty}\right), \\
(2) \quad \|G^{(g)}(u)\|_{\alpha,\lambda} \leqslant & \frac{\Lambda_\alpha(h) C^{(2)}}{\lambda^{1-2\alpha}}\left(1+\|u\|_{\alpha,\lambda}\right)
\end{aligned}
$$

for all $\lambda \geqslant 1$, where the constants $C^{(1)}$ and $C^{(2)}$ are independent of λ, u, h (they depend on T and the constants $|g(0,0)|$, M_0, α, β from $(\boldsymbol{A}_1)$).

If $u, v \in W_0^{\alpha,\infty}(0,T)$ are such that $|u|_\infty \leqslant N$ and $|v|_\infty \leqslant N$, then we have the following:

$$
\|G^{(g)}(u) - G^{(g)}(v)\|_{\alpha,\lambda} \leqslant \frac{\Lambda_\alpha(h) C_N^{(3)}}{\lambda^{1-2\alpha}}\left(1 + \Delta(u) + \Delta(v)\right)\|u-v\|_{\alpha,\lambda}
$$

for all $\lambda \geqslant 1$, where

$$
\Delta(u) = \sup_{r\in[0,T]} \int_0^r \frac{|u_r - u_s|^\delta}{|r-s|^{1+\alpha}}\, ds, \tag{10}
$$

and the constant $C_N^{(3)}$ is independent of λ, u, v, h ($C_N^{(3)}$ depends on T and the constants from $(\boldsymbol{A}_2)$).

Remark 2. *If $u \in C^{1-\alpha}(0,T)$ and $\frac{\delta}{1+\delta} > \alpha$ then*

$$
\Delta(u) \leqslant |u|_{1-\alpha} \sup_{r\in[0,T]} \int_0^r \frac{|r-s|^{(1-\alpha)\delta}}{|r-s|^{1+\alpha}}\, ds = \frac{T^{\delta-\alpha(1+\delta)}}{\delta - \alpha(1+\delta)}|u|_{1-\alpha}. \tag{11}
$$

3.1.4. Integration with Respect to fBm

The trajectories of $B^H = (B_t^H)_{t\geqslant 0}$, $0 < H < 1$, are almost surely locally γ-Hölder continuous functions for all $\gamma \in (0, H)$. To be more precise, for all $\gamma \in (0, H)$ and $T > 0$, there exists a nonnegative random variable $G_{\gamma,T}$ such that $\mathbb{E}(|G_{\gamma,T}|^p) < \infty$ for all $p \geqslant 1$, and

$$|B_t^H - B_s^H| \leqslant G_{\gamma,T}|t-s|^\gamma \qquad a.s. \tag{12}$$

for all $s, t \in [0, T]$.

The pathwise generalized Lebesgue–Stieltjes integral for one-dimensional fBm B^H can be defined as follows:

$$\int_a^b f(s)\, dB^H(s) = (-1)^\alpha \int_a^b (\mathcal{D}_{a+}^\alpha f)(s)(\mathcal{D}_{b-}^{1-\alpha} B^H)(s)\, ds \tag{13}$$

if $\mathcal{D}_{a+}^\alpha f \in L^1(a, b)$ (see [6,21] p. 225), where $\mathcal{D}_{a+}^\alpha f$ and $\mathcal{D}_{b-}^{1-\alpha} B^H$ are fractional derivatives.

For $1/2 < H < 1$, we can choose α such that $1 - H < \alpha < 1/2$. An easy computation shows that almost all trajectories of B^H belong to the space $W_T^{1-\alpha,\infty}(0, T)$. Indeed, since $H > 1 - \alpha$, then for any $H > \gamma > 1 - \alpha$, we have the following:

$$\begin{aligned}\|B^H(\omega)\|_{1-\alpha,\infty,T} \leqslant & \sup_{0\leqslant s<t<T}\left(\frac{G_{\gamma,T}(\omega)|t-s|^\gamma}{(t-s)^{1-\alpha}} + \int_s^t \frac{G_{\gamma,T}(\omega)|u-s|^\gamma}{(u-s)^{2-\alpha}}\, du\right) \\ \leqslant & G_{\gamma,T}(\omega)(1 \vee T)\left(1 + \frac{1}{\gamma - 1 + \alpha}\right).\end{aligned}$$

Thus $B^H(\omega) \in W_T^{1-\alpha,\infty}(0, T)$ for almost all ω.

If $u = \{u(t), t \in [0, T]\}$ is a stochastic process whose trajectories belong to the space $W_T^{\alpha,1}(0, T)$, then the pathwise generalized Lebesgue–Stieltjes integral $\int_a^b u(s)\, dB^H(s)$ exists and we can express it according to (13). Moreover, if the trajectories of the process u belong to the space $W_0^{\alpha,\infty}(0, T)$, then the indefinite integral $\int_0^t u(s)\, dB^H(s)$ is a Hölder continuous function of order $1 - \alpha$ (see [10]).

3.2. The Implicit Euler Approximation and Auxiliary Results

Let $0 < \alpha < 1/2$ be fixed. Let $h \in W_T^{1-\alpha,\infty}$, $h_0 = 0$. Consider the deterministic differential equation on $\mathbb{R}$, as follows:

$$x_t = x_0 + \Phi(x_t) - \Phi(x_0) + \int_0^t f(s, x_s)\, ds + \int_0^t g(s, x_s)\, dh_s, \qquad t \in [0, T], \tag{14}$$

where $x_0 \in \mathbb{R}$, the coefficients $f, g : \mathbb{R} \to \mathbb{R}$ satisfy assumptions $(\mathbf{A}_1)$ and $(\mathbf{A}_2)$, and the function $\Phi : \mathbb{R} \to \mathbb{R}$ satisfies assumption $(\mathbf{A}_3)$.

Let $\pi^n = \{t_k^n = \frac{k}{n}T, 1 \leqslant k \leqslant n\}$ be a sequence of uniform partitions of the interval $[0, T]$, and let $\Delta_n = t_k^n - t_{k-1}^n$, $1 \leqslant k \leqslant n$, $\Delta_n < 1$. We define the implicit Euler approximations for the Equation (14) as follows:

$$\begin{aligned}y^n(t_{k+1}^n) - \Phi(y^n(t_{k+1}^n)) =& y^n(t_k^n) - \Phi(y^n(t_k^n)) + f(t_k^n, y^n(t_k^n))\Delta_n \\ &+ g(t_k^n, y^n(t_k^n))\big(h(t_{k+1}^n) - h(t_k^n)\big), \qquad y^n(0) = x_0,\end{aligned} \tag{15}$$

and their continuous interpolations are as follows:

$$y^n(t) - \Phi(y_t^n) = x_0 - \Phi(x_0) + \int_0^t f(\tau_s^n, y^n(\tau_s^n))\, ds + \int_0^t g(\tau_s^n, y^n(\tau_s^n))\, dh_s, \tag{16}$$

where $\tau_s^n = t_{k-1}^n$ and $y^n(\tau_s^n) = y^n(t_{k-1}^n)$ if $s \in [t_{k-1}^n, t_k^n)$, $1 \leqslant k \leqslant n$.

We rewrite implicit Euler approximations (15) and (16) in a more compact way, as follows:

$$D(y^n(t^n_{k+1})) = D(y^n(t^n_k)) + f(t^n_k, y^n(t^n_k))\Delta_n + g(t^n_k, y^n(t^n_k))\big(h(t^n_{k+1}) - h(t^n_k)\big) \tag{17}$$

with $y^n(0) = x_0$ and

$$D(y^n_t) = D(x_0) + F^{(f,\tau^n)}_t(y^n) + G^{(g,\tau^n)}_t(y^n), \qquad y^n(0) = x_0. \tag{18}$$

where

$$F^{(f,\tau^n)}_t(y^n) = \int_0^t f(\tau^n_s, y^n(\tau^n_s))\, ds, \qquad G^{(g,\tau^n)}_t(y^n) = \int_0^t g(\tau^n_s, y^n(\tau^n_s))\, dh_s.$$

The implicit Euler approximations scheme (17) is correctly defined. From the recursive expression (17), we calculate $D(y^n(t^n_{k+1}))$. The properties of the function $D(x)$ give us a single value of $y^n(t^n_{k+1})$. Since $D(y^n_t)$ is a continuous function, then y^n is a continuous function. Indeed, since $D^{-1}(x)$ and $D(y^n_t)$ are continuous functions, then y^n is a continuous function.

Now, we consider the properties of the implicit Euler approximation.

Lemma 1. *Let Assumption* (A_3) *be satisfied. Then* y^n, $F^{(f,\tau^n)}(y^n)$, $G^{(g,\tau^n)}(y^n) \in C^{1-\alpha}(0,T)$ *for any fixed* $n \geqslant 1$.

Proof. We first note that the functions $f(\tau^n, y^n(\tau^n))$ and $g(\tau^n, y^n(\tau^n))$ have bounded variations on $[0,T]$ for a fixed n. Thus, for the fixed n, they are bounded and have p-bounded variation, where $p = (1-\alpha)^{-1}$. From now on, we assume that $p = (1-\alpha)^{-1}$.

First, observe that for $s < t$

$$\big|F^{(f,\tau^n)}_t(y^n) - F^{(f,\tau^n)}_s(y^n)\big| = \left|\int_s^t f(\tau^n_u, y^n(\tau^n_u))du\right| \leqslant \big|f(\tau^n, y^n(\tau^n))\big|_\infty (t-s).$$

Thus, $F^{(f,\tau^n)}(y^n) \in C^1(0,T) \subset C^{1-\alpha}(0,T)$ for fixed $n \geqslant 1$.

Now consider $G^{(g,\tau^n)}(y^n)$. Since $h \in W^{1-\alpha,\infty}_T(0,T) \subset C^{1-\alpha}(0,T)$ then $h \in C\mathcal{W}_p([0,T])$ and

$$V_p(h;[s,t]) \leqslant |h|_{1-\alpha}(t-s)^{1-\alpha}. \tag{19}$$

Assume that $s \in [t^n_k, t^n_{k+1})$ for some $0 \leqslant k \leqslant n-1$ and $t \leqslant t^n_{k+1} \leqslant T$. Then

$$\left|\int_s^t g(\tau^n_u, y^n(\tau^n_u))\, dh_u\right| = \big|g(\tau^n, y^n(\tau^n))\big|_\infty |h|_{1-\alpha}(t-s)^{1-\alpha}. \tag{20}$$

If $t > t^n_{k+1}$, then from (20) and the Love–Young inequality, we obtain the following:

$$\begin{aligned}\big|G^{(g,\tau^n)}_t(y^n) - G^{(g,\tau^n)}_s(y^n)\big| &\leqslant \left|\int_s^{t^n_{k+1}} g(\tau^n_u, y^n(\tau^n_u))\, dh_u\right| + \left|\int_{t^n_{k+1}}^t g(\tau^n_u, y^n(\tau^n_u))\, dh_u\right| \\ &\leqslant \left|\int_s^{t^n_{k+1}} g(\tau^n_u, y^n(\tau^n_u))\, dh_u\right| + C_{1,p}V_{1,\infty}\big(g(\tau^n, y^n(\tau^n)); [t^n_{k+1}, t]\big)V_p\big(h; [t^n_{k+1}, t]\big) \\ &\leqslant 2C_{1,p}V_{1,\infty}\big(g(\tau^n, y^n(\tau^n)); [0,T]\big)|h|_{1-\alpha}(t-s)^{1-\alpha}.\end{aligned} \tag{21}$$

From Assumption $(\mathbf{A}_3)$, we have the following:

$$\begin{aligned}\big|y^n(t) - y^n(s)\big| &\leqslant d^{-1}\big|D(y^n(t)) - D(y^n(s))\big| \\ &\leqslant d^{-1}\big(\big|F^{(f,\tau^n)}_t(y^n) - F^{(f,\tau^n)}_s(y^n)\big| + \big|G^{(g,\tau^n)}_t(y^n) - G^{(g,\tau^n)}_s(y^n)\big|\big).\end{aligned} \tag{22}$$

Since $|y^n|_\infty$ is bounded for any fixed $n \geqslant 1$, then $y^n \in C^{1-\alpha}(0,T)$ for any fixed $n \geqslant 1$. $\square$

The next lemma enables us to apply the estimate (6) to the integral $G^{(g,\tau^n)}(y^n)$.

Lemma 2. *Let Assumptions* $(\boldsymbol{A}_1)$ *and* $(\boldsymbol{A}_3)$ *be satisfied. If* $y^n \in W_0^{\alpha,1}(0,T)$ *for any fixed* $n \geqslant 1$, *then* $g(\tau^n, y^n(\tau^n)) \in W_0^{\alpha,1}(0,T)$ *for any fixed* $n \geqslant 1$ *and* $\beta > \alpha$.

Proof. From Lemma 1, it follows that $y^n \in W_0^{\alpha,1}(0,T)$ for any fixed $n \geqslant 1$. We prove this. The following is evident:

$$\int_0^t \frac{|g(\tau_s^n, y^n(\tau_s^n))|}{s^\alpha}\,ds \leqslant |g(\tau^n, y^n(\tau^n))|_\infty (1-\alpha)^{-1} t^{1-\alpha}.$$

Now, we estimate the second term of the norm (5). From Lemma 1, it follows that there exists a constant C_n, depending on n, such that we have the following:

$$|y^n(\tau_s^n) - y^n(\tau_u^n)| \leqslant C_n (\tau_s^n - \tau_u^n)^{1-\alpha}.$$

Note that $\tau_s^n = \tau_u^n$ for $\tau_s^n \leqslant u < s$. Therefore, we have

$$\begin{aligned}
&\int_0^t \int_0^s \frac{|g(\tau_s^n, y^n(\tau_s^n)) - g(\tau_u^n, y^n(\tau_u^n))|}{(s-u)^{1+\alpha}}\,duds = \int_0^t \int_0^{\tau_s^n} \frac{|g(\tau_s^n, y^n(\tau_s^n)) - g(\tau_u^n, y^n(\tau_u^n))|}{(s-u)^{1+\alpha}}\,duds \\
&\leqslant M_0 \int_0^t \int_0^{\tau_s^n} \frac{|\tau_s^n - \tau_u^n|^\beta}{(s-u)^{1+\alpha}}\,duds + M_0 \int_0^t \int_0^{\tau_s^n} \frac{|y^n(\tau_s^n) - y^n(\tau_u^n)|}{(s-u)^{1+\alpha}}\,duds \\
&\leqslant M_0 \int_0^t \int_0^{\tau_s^n} \frac{|\tau_s^n - \tau_u^n|^\beta}{(s-u)^{1+\alpha}}\,duds + M_0 C_n \int_0^t \int_0^{\tau_s^n} \frac{|\tau_s^n - \tau_u^n|^{1-\alpha}}{(s-u)^{1+\alpha}}\,duds.
\end{aligned}$$

Since $\tau_s^n - \tau_u^n \leqslant s - u + \Delta_n$ and (see [22] p. 494)

$$\begin{aligned}
\int_0^t (s-\tau_s^n)^{-\alpha}\,ds &\leqslant \int_0^T (s-\tau_s^n)^{-\alpha}\,ds = \sum_{k=1}^{n-1} \int_{t_k^n}^{t_{k+1}^n} (s-\tau_s^n)^{-\alpha}\,ds \\
&= \sum_{k=1}^{n-1} \int_{t_k^n}^{t_{k+1}^n} (s-t_k^n)^{-\alpha}\,ds \leqslant n(1-\alpha)^{-1}\Delta_n^{1-\alpha} = (1-\alpha)^{-1} T \Delta_n^{-\alpha} \qquad (23)
\end{aligned}$$

then

$$\begin{aligned}
M_0 \int_0^t \int_0^{\tau_s^n} \frac{|\tau_s^n - \tau_u^n|^\beta}{(s-u)^{1+\alpha}}\,duds &\leqslant M_0 \int_0^t \int_0^{\tau_s^n} \frac{|s-u|^\beta + \Delta_n^\beta}{(s-u)^{1+\alpha}}\,duds \\
&\leqslant M_0(\beta-\alpha)^{-1} \int_0^t s^{\beta-\alpha}\,ds + M_0 \alpha^{-1} \Delta_n^\beta \int_0^t (s-\tau_s^n)^{-\alpha}\,ds \\
&\leqslant M_0(\beta-\alpha)^{-1} T^{\beta-\alpha+1} + M_0 \alpha^{-1}(1-\alpha)^{-1} T. \qquad (24)
\end{aligned}$$

By a similar argument, we obtain the following:

$$M_0 C_n \int_0^t \int_0^{\tau_s^n} \frac{|\tau_s^n - \tau_u^n|^{1-\alpha}}{(s-u)^{1+\alpha}}\,duds \leqslant M_0 C_n (1-2\alpha)^{-1} T^{2-2\alpha} + M_0 C_n \alpha^{-1}(1-\alpha)^{-1} T.$$

Consequently, $g(\tau^n, y^n(\tau^n))$ has the claimed property. □

The following result is crucial to prove our main results.

Proposition 4. *Let* $1 - H < \alpha < \alpha_0$ *and the functions* $f(s,x)$ *and* $g(s,x)$ *satisfy assumptions* $(\boldsymbol{A}_1)$ *(i), (iii), and* $(\boldsymbol{A}_2)$ *(ii), respectively. Moreover, let assumption* $(\boldsymbol{A}_3)$ *hold. Then there exists a constant C such that we have the following:*

$$\sup_n \|y^n\|_{\alpha,\infty} \leqslant C.$$

Proof. Set

$$\|u\|_{\infty,\alpha,t} := \sup_{s\in[0,t]} \|u\|_{\alpha,s}, \quad \|u\|_{\alpha,s} := |u(s)| + |u|_{\alpha,s},$$
$$|u|_{\alpha,s} := \int_0^s \frac{|u(s)-u(r)|}{(s-r)^{1+\alpha}}\,dr, \quad \|u\|_{\infty,t} := \sup_{s\in[0,t]} |u(s)|.$$

It is easy to check that

$$\|y^n\|_{\alpha,t} \leqslant |x_0| + d^{-1}\big(\|F^{(f,\tau^n)}(y^n)\|_{\alpha,t} + \|G^{(g,\tau^n)}(y^n)\|_{\alpha,t}\big). \tag{25}$$

Indeed, we note that

$$\begin{aligned}
|y^n_t| - |x_0| + \int_0^t \frac{|y^n_t - y^n_s|}{|t-s|^{1+\alpha}}\,ds &\leqslant |y^n_t - x_0| + \int_0^t \frac{|y^n_t - y^n_s|}{|t-s|^{1+\alpha}}\,ds \\
&\leqslant d^{-1}\left(|D(y^n_t) - D(x_0)| + \int_0^t \frac{|D(y^n_t) - D(y^n_s)|}{|t-s|^{1+\alpha}}\,ds\right) \\
&\leqslant d^{-1}\Bigg(|F^{(f,\tau^n)}_t(y^n)| + |G^{(g,\tau^n)}_t(y^n)| \\
&\quad + \int_0^t \frac{|F^{(f,\tau^n)}_t(y^n) - F^{(f,\tau^n)}_s(y^n)| + |G^{(g,\tau^n)}_t(y^n) - G^{(g,\tau^n)}_s(y^n)|}{|t-s|^{1+\alpha}}\,ds\Bigg).
\end{aligned} \tag{26}$$

From Lemma 1, we have that for any fixed $n \geqslant 1$, the norms $\|y^n\|_{\infty,\alpha,t}$, $\|F^{(f,\tau^n)}(y^n)\|_{\infty,\alpha,t}$, $\|G^{(g,\tau^n)}(y^n)\|_{\infty,\alpha,t}$ are finite.

The proof of the estimate $\|F^{(f,\tau^n)}(y^n)\|_{\alpha,t}$ is similar to the proof of the estimate (4.27) in [10]

$$\begin{aligned}
\|F^{(f,\tau^n)}(y^n)\|_{\alpha,t} &\leqslant \int_0^t |f(\tau^n_s, y^n(\tau^n_s))|\,ds + \int_0^t (t-s)^{-\alpha-1} \int_s^t |f(\tau^n_s, y^n(\tau^n_s))|\,ds \\
&\leqslant C_{\alpha,T} \int_0^t \frac{|f(\tau^n_s, y^n(\tau^n_s))|}{(t-s)^\alpha}\,ds \\
&\leqslant C_{\alpha,T} L_0 \left[\int_0^t \frac{|y^n(\tau^n_s)|}{(t-s)^\alpha}\,ds + (1-\alpha)^{-1} T^{1-\alpha}\right] \\
&\leqslant C_{\alpha,T} L_0 \left[\int_0^t \frac{\|y^n\|_{\infty,\alpha,s}}{(t-s)^\alpha}\,ds + (1-\alpha)^{-1} T^{1-\alpha}\right],
\end{aligned} \tag{27}$$

where $C_{\alpha,T} = T^\alpha + \frac{1}{\alpha}$.

Since $\alpha < \alpha_0$, we have $\beta > \alpha$. From Lemma 2, it follows that for the integral $G^{(g,\tau^n)}(y^n)_t$, we can apply the estimate (6). Applying Proposition 2, we obtain the following:

$$\begin{aligned}
\|G^{(g,\tau^n)}_t(y^n)\|_{\alpha,t} \leqslant &\Lambda_\alpha(h)\Bigg(\int_0^t \left(\frac{c^{(1)}_\alpha}{(t-r)^{2\alpha}} + \frac{1}{r^\alpha}\right)|g(\tau^n_r, y^n(\tau^n_r))|\,dr \\
&+ \int_0^t \int_0^r \frac{|g(\tau^n_r, y^n(\tau^n_r)) - g(\tau^n_v, y^n(\tau^n_v))|}{(r-v)^{1+\alpha}}\,\big[(t-v)^{-\alpha} + \alpha\big]\,dv dr\Bigg),
\end{aligned} \tag{28}$$

where $c^{(1)}_\alpha = B(2\alpha, 1-\alpha)$, $B(\cdot,\cdot)$ is the Beta function.

Note that

$$\begin{aligned}&\left|g(\tau_r^n, y^n(\tau_r^n)) - g(\tau_r^n, y^n(\tau_r^n))\right| \\ &\quad\leqslant M_0|\tau_r^n - \tau_v^n|^\beta + M_0\left|y^n(\tau_r^n) - y^n(\tau_v^n)\right| \\ &\quad\leqslant M_0|\tau_r^n - \tau_v^n|^\beta + M_0\left[\left|y^n(\tau_r^n) - y_r^n\right| + |y_r^n - y_v^n| + \left|y_v^n - y^n(\tau_v^n)\right|\right].\end{aligned} \tag{29}$$

Since

$$\begin{aligned}\left|g(\tau_r^n, y^n(\tau_r^n))\right| \leqslant&\left|g(\tau_r^n, y^n(\tau_r^n)) - g(0,0)\right| + |g(0,0)| \\ \leqslant&\left|g(\tau_r^n, y^n(\tau_r^n)) - g(0, y^n(\tau_r^n))\right| + \left|g(0, y^n(\tau_r^n)) - g(0,0)\right| + |g(0,0)| \\ \leqslant& M_0(\tau_r^n)^\beta + M_0|y^n(\tau_r^n)| + |g(0,0)| \leqslant M_0T^\beta + M_0|y^n|_{\infty,r} + |g(0,0)|,\end{aligned} \tag{30}$$

it follows that

$$\begin{aligned}\left|y_r^n - y^n(\tau_r^n)\right| \leqslant& d^{-1}\left|D(y_r^n) - D(y^n(\tau_r^n))\right| \\ =& d^{-1}\left|f(\tau_r^n, y^n(\tau_r^n))(r - \tau_r^n) + g(\tau_r^n, y^n(\tau_r^n))(h(r) - h(\tau_r^n))\right| \\ \leqslant& \lambda(\alpha)(1 + |y^n|_{\infty,r})(r - \tau_r^n)^{1-\alpha},\end{aligned} \tag{31}$$

where

$$\lambda(\alpha) = d^{-1}\max\left\{\left(L_0 + (M_0T^\beta + |g(0,0)|)|h|_{1-\alpha}\right), \left(L_0 + M_0|h|_{1-\alpha}\right)\right\}.$$

From (28), (29), and inequality

$$\alpha + (t-v)^{-\alpha} \leqslant T^\alpha\left(r^{-\alpha} + (t-v)^{-2\alpha}\right)$$

we have the following:

$$\begin{aligned}\|G^{(g,\tau^n)}(y^n)\|_{\alpha,t} \leqslant& \Lambda_\alpha(h)M_0\int_0^t\int_0^{\tau_r^n}\frac{|\tau_r^n - \tau_v^n|^\beta}{(r-v)^{1+\alpha}}\left[(t-v)^{-\alpha} + \alpha\right]dvdr \\ &+ \Lambda_\alpha(h)c_{\alpha,T}\int_0^t\left((t-r)^{-2\alpha} + r^{-\alpha}\right)\Bigg(\left|g(\tau_r^n, y^n(\tau_r^n))\right| \\ &+ M_0\int_0^{\tau_r^n}\frac{\left|y^n(\tau_r^n) - y_r^n\right| + |y_r^n - y_v^n| + \left|y_v^n - y^n(\tau_v^n)\right|}{(r-v)^{1+\alpha}}dv\Bigg)dr,\end{aligned} \tag{32}$$

where $c_{\alpha,T} = \max\{c_\alpha^{(1)}, 1\} + T^\alpha$, and

$$c_{\alpha,T} \leqslant \frac{1}{\alpha(1-\alpha)} + T^\alpha.$$

First, observe that

$$\begin{aligned}&\int_0^t\int_0^{\tau_r^n}\frac{|\tau_r^n - \tau_v^n|^\beta}{(r-v)^{1+\alpha}}\left[(t-v)^{-\alpha} + \alpha\right]dvdr \leqslant \int_0^t\left[(t-r)^{-\alpha} + \alpha\right]\int_0^{\tau_r^n}\frac{|r-v|^\beta + \Delta_n^\beta}{(r-v)^{1+\alpha}}dvdr \\ &\leqslant (\beta-\alpha)^{-1}\int_0^t\left[(t-r)^{-\alpha} + \alpha\right]r^{\beta-\alpha}dr + \alpha^{-1}\Delta_n^\beta\int_0^t\left[(t-r)^{-\alpha} + \alpha\right](r - \tau_r^n)^{-\alpha}dr \\ &\leqslant \left[(\beta-\alpha)^{-1}t^{\beta-\alpha} + \alpha^{-1}\Delta_n^{\beta-\alpha}\right]\left((1-2\alpha)^{-1} + 1\right)(1 \vee T) \leqslant c_{\alpha,\beta,T}^{(1)}.\end{aligned}$$

where

$$c_{\alpha,\beta,T}^{(1)} := \left[(\beta-\alpha)^{-1}T^{\beta-\alpha} + \alpha^{-1}\right]\left((1-2\alpha)^{-1} + 1\right)(1 \vee T).$$

Furthermore, from (30), we have the following:

$$
\begin{aligned}
&\int_0^t \left[(t-r)^{-2\alpha}+r^{-\alpha}\right]|g(\tau_r^n, y^n(\tau_r^n))|\,dr \\
&\leqslant \left(M_0T^\beta+|g(0,0)|\right)\int_0^t \left[(t-r)^{-2\alpha}+r^{-\alpha}\right]dr + M_0\int_0^t \left((t-r)^{-2\alpha}+r^{-\alpha}\right)|y^n|_{\infty,r}\,dr \\
&\leqslant \frac{2}{1-2\alpha}\left(M_0T^\beta+|g(0,0)|\right)(t\vee 1)+M_0\int_0^t \left[(t-r)^{-2\alpha}+r^{-\alpha}\right]|y^n|_{\infty,r}\,dr \\
&\leqslant c^{(2)}_{\alpha,\beta,T}\left(1+\int_0^t \left[(t-r)^{-2\alpha}+r^{-\alpha}\right]\|y^n\|_{\infty,r}\,dr\right)
\end{aligned}
$$

and

$$
\int_0^t \left[(t-r)^{-2\alpha}+r^{-\alpha}\right]\left(\int_0^{\tau_r^n}\frac{|y_r^n-y_v^n|}{(r-v)^{1+\alpha}}\,dv\right)dr \leqslant \int_0^t \left[(t-r)^{-2\alpha}+r^{-\alpha}\right]\|y^n\|_{\alpha,r}\,dr,
$$

where

$$
c^{(2)}_{\alpha,\beta,T}=\max\left\{\frac{2}{1-2\alpha}\left(M_0T^\beta+|g(0,0)|\right)(T\vee 1), M_0\right\}.
$$

Applying (31), we obtain the following:

$$
\begin{aligned}
&\int_0^t \left[(t-r)^{-2\alpha}+r^{-\alpha}\right]\int_0^{\tau_r^n}\frac{\left|y^n(\tau_r^n)-y^n(r)\right|}{(r-v)^{1+\alpha}}\,dvdr \\
&\leqslant \lambda(\alpha)\int_0^t \left[(t-r)^{-2\alpha}+r^{-\alpha}\right]\int_0^{\tau_r^n}\frac{(1+\|y^n\|_{\infty,r})(r-\tau_r^n)^{1-\alpha}}{(r-v)^{1+\alpha}}\,dvdr \\
&\leqslant \lambda(\alpha)\alpha^{-1}(1-\alpha)^{-1}T\Delta_n^{1-2\alpha}\int_0^t \left[(t-r)^{-2\alpha}+r^{-\alpha}\right](1+\|y^n\|_{\infty,r})\,dr \\
&\leqslant \lambda(\alpha)\alpha^{-1}(1-\alpha)^{-1}T\left(\frac{2(T\vee 1)}{1-2\alpha}+\int_0^t \left[(t-r)^{-2\alpha}+r^{-\alpha}\right]\|y^n\|_{\infty,r}\,dr\right) \\
&\leqslant c^{(3)}_{\alpha,T}\left(1+\int_0^t \left[(t-r)^{-2\alpha}+r^{-\alpha}\right]\|y^n\|_{\infty,r}\,dr\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^t \left[(t-r)^{-2\alpha}+r^{-\alpha}\right]\int_0^{\tau_r^n}\frac{\left|y^n(v)-y^n(\tau_v^n)\right|}{(r-v)^{1+\alpha}}\,dvdr \\
&\leqslant \lambda(\alpha)\int_0^t \left[(t-r)^{-2\alpha}+r^{-\alpha}\right]\int_0^{\tau_r^n}\frac{(1+\|y^n\|_{\infty,v})(v-\tau_v^n)^{1-\alpha}}{(r-v)^{1+\alpha}}\,dvdr \\
&\leqslant \lambda(\alpha)\int_0^t \left[(t-r)^{-2\alpha}+r^{-\alpha}\right](1+\|y^n\|_{\infty,r})\left(\int_0^{\tau_r^n}\frac{(v-\tau_v^n)^{1-\alpha}}{(r-v)^{1+\alpha}}\,dv\right)dr \\
&\leqslant \lambda(\alpha)\Delta_n^{1-\alpha}\int_0^t \left[(t-r)^{-2\alpha}+r^{-\alpha}\right](1+\|y^n\|_{\infty,r})\left(\int_0^{\tau_r^n}\frac{1}{(r-v)^{1+\alpha}}\,dv\right)dr \\
&\leqslant \lambda(\alpha)\alpha^{-1}(1-\alpha)^{-1}T\Delta_n^{1-2\alpha}\int_0^t \left[(t-r)^{-2\alpha}+r^{-\alpha}\right](1+\|y^n\|_{\infty,r})\,dr \\
&\leqslant c^{(3)}_{\alpha,T}\left(1+\int_0^t \left[(t-r)^{-2\alpha}+r^{-\alpha}\right]\|y^n\|_{\infty,r}\,dr\right),
\end{aligned}
$$

where

$$
c^{(3)}_{\alpha,T}=\lambda(\alpha)\alpha^{-1}(1-\alpha)^{-1}T\max\left\{\frac{2(T\vee 1)}{1-2\alpha},1\right\}.
$$

Consequently,

$$
\|G^{(g,\tau^n)}(y^n)\|_{\alpha,t}\leqslant C_1+C_2\int_0^t \left((t-r)^{-2\alpha}+r^{-\alpha}\right)\|y^n\|_{\infty,\alpha,r}\,dr, \tag{33}
$$

where

$$C_1 = \Lambda_\alpha(h)\left[M_0 c^{(1)}_{\alpha,\beta,T} + c_{\alpha,T}\left(c^{(2)}_{\alpha,\beta,T} + 2M_0 c^{(3)}_{\alpha,T}\right)\right], \qquad C_2 = \Lambda_\alpha(h) c_{\alpha,T}\left(c^{(2)}_{\alpha,\beta,T} + 2M_0 c^{(3)}_{\alpha,T} + 1\right).$$

From (25), (27), and (33), we have the following:

$$\begin{aligned}\|y^n\|_{\infty,\alpha,t} \leqslant & |x_0| + d^{-1} C_{\alpha,T} L_0 \int_0^t \frac{\|y^n\|_{\infty,\alpha,s}}{(t-r)^\alpha}\,dr + d^{-1}(C_0 + C_1) \\ & + d^{-1} C_2 \int_0^t \left((t-r)^{-2\alpha} + r^{-\alpha}\right)\|y^n\|_{\infty,\alpha,r}\,dr,\end{aligned}$$

where

$$C_0 = C_{\alpha,T} L_0 (1-\alpha)^{-1} T^{1-\alpha}.$$

Note that for $r < t$ we have the following:

$$\begin{aligned}(t-r)^{-\alpha} &\leqslant \frac{t^{2\alpha}(t-r)^\alpha}{r^{2\alpha}(t-r)^{2\alpha}} \leqslant \frac{t^{3\alpha}}{r^{2\alpha}(t-r)^{2\alpha}} \leqslant T^\alpha \frac{t^{2\alpha}}{r^{2\alpha}(t-r)^{2\alpha}}, \\ r^{-\alpha} + (t-r)^{-2\alpha} &\leqslant \frac{r^\alpha t^{2\alpha}}{r^{2\alpha}(t-r)^{2\alpha}} + \frac{r^{2\alpha}}{r^{2\alpha}(t-r)^{2\alpha}} \leqslant (T^\alpha + 1)\frac{t^{2\alpha}}{r^{2\alpha}(t-r)^{2\alpha}}.\end{aligned}$$

Thus,

$$\|y^n\|_{\infty,\alpha,t} \leqslant |x_0| + d^{-1}\left(C_{\alpha,T} L_0 + C_2\right)(T^\alpha + 1) t^{2\alpha} \int_0^t \frac{\|y^n\|_{\infty,\alpha,r}}{r^{2\alpha}(t-r)^{2\alpha}}\,dr + d^{-1}(C_0 + C_1)$$

and from Lemma 7.6 in [10]

$$\|y^n\|_{\infty,\alpha,t} \leqslant a d_\alpha \exp\left\{c_\alpha t b^{1/(1-2\alpha)}\right\}$$

where c_α and d_α are positive constants depending only on α,

$$a = |x_0| + d^{-1}(C_0 + C_1) \qquad b = d^{-1}\left(C_{\alpha,T} L_0 + C_2\right)(T^\alpha + 1).$$

□

Now, we can strengthen the result of Lemma 1.

Proposition 5. *Under the assumptions of Proposition 4, we obtain* $\sup_n \|y^n\|_{1-\alpha} < \infty$.

Proof. Recall that from Lemma 1, we have y^n, $F^{(f,\tau^n)}(y^n)$, $G^{(g,\tau^n)}(y^n) \in C^{1-\alpha}(0,T)$ for any fixed $n \geqslant 1$. Thus, for any fixed $n \geqslant 1$, we have the following:

$$\|y^n\|_{1-\alpha} \leqslant |x_0| + d^{-1}\left[\|F^{(f,\tau^n)}(y^n)\|_{1-\alpha} + \|G^{(g,\tau^n)}(y^n)\|_{1-\alpha}\right].$$

Indeed, similar to how we proved (26), we have the following:

$$\begin{aligned}|y^n_t| - |x_0| + \frac{|y^n_t - y^n_s|}{|t-s|^{1-\alpha}} \leqslant & d^{-1}\Bigg(|F^{(f,\tau^n)}_t(y^n)| + |G^{(g,\tau^n)}_t(y^n)| \\ & + \frac{|F^{(f,\tau^n)}_t(y^n) - F^{(f,\tau^n)}_s(y^n)| + |G^{(g,\tau^n)}_t(y^n) - G^{(g,\tau^n)}_s(y^n)|}{(t-s)^{1-\alpha}}\Bigg).\end{aligned}$$

First, observe that for $s < t$

$$\left|F^{(f,\tau^n)}_t(y^n) - F^{(f,\tau^n)}_s(y^n)\right| \leqslant L_0\left(1 + |y^n|_\infty\right)(t-s).$$

Thus,

$$\left\|F^{(f,\tau^n)}(y^n)\right\|_{1-\alpha} \leqslant L_0(1+|y^n|_\infty)T + L_0(1+|y^n|_\infty)T^\alpha = (T+T^\alpha)L_0(1+|y^n|_\infty)$$

and the boundedness of the norm $\|F^{(f,\tau^n)}(y^n)\|_{1-\alpha}$ follows from Proposition 4.

From (6), it follows that

$$\begin{aligned}&\left|G_t^{(g,\tau^n)}(y^n) - G_s^{(g,\tau^n)}(y^n)\right| \\ &\quad \leqslant \Lambda_\alpha(h)\left(\int_s^t \frac{|g(\tau_r^n, y^n(\tau_r^n))|}{(r-s)^\alpha}\,dr + \int_s^t \int_s^{\tau_r^n} \frac{|g(\tau_r^n, y^n(\tau_r^n)) - g(\tau_v^n, y^n(\tau_v^n))|}{(r-v)^{1+\alpha}}\,dvdr\right). \end{aligned} \tag{34}$$

Furthermore, from (30), we obtain the following:

$$\int_s^t \frac{|g(\tau_r^n, y^n(\tau_r^n))|}{(r-s)^\alpha}\,dr \leqslant \left[M_0(T^\beta + |y^n|_{\infty,r}) + |g(0,0)|\right](1-\alpha)^{-1}(t-s)^{1-\alpha}.$$

We obtain the proof of the estimate for the second term in (34) in the same way as presented in Lemma 2. We first compute the following integral:

$$\int_s^t (r-\tau_r^n)^{-\alpha}\,dr.$$

Assume that $t_{i-1}^n \leqslant s < t_i^n$, $\tau_t^n = t_k^n$, and $k \geqslant i$. Similar to proofs (23) and (24), we have the following:

$$\begin{aligned}\int_s^t (r-\tau_r^n)^{-\alpha}\,dr &= \int_{t_k^n}^t (r-t_k^n)^{-\alpha}\,dr + \sum_{j=i}^{k-1}\int_{t_j^n}^{t_{j+1}^n}(r-t_j^n)^{-\alpha}\,dr + \int_s^{t_i^n}(r-t_{i-1}^n)^{-\alpha}\,dr \\ &= (1-\alpha)^{-1}\left[(t-t_k^n)^{1-\alpha} + \sum_{j=i}^{k-1}\Delta_n^{1-\alpha} + [(t_i^n - t_{i-1}^n)^{1-\alpha} - (s-t_{i-1}^n)^{1-\alpha}]\right] \\ &\leqslant (1-\alpha)^{-1}\left[(t-s)^{1-\alpha} + (t_k^n - t_i^n)\Delta_n^{-\alpha} + (t_i^n - s)^{1-\alpha}\right] \\ &\leqslant (1-\alpha)^{-1}[2(t-s)^{1-\alpha} + (t-s)\Delta_n^{-\alpha}]\end{aligned}$$

and

$$\begin{aligned}\int_s^t\int_s^{\tau_r^n}\frac{|\tau_r^n-\tau_v^n|^\beta}{(r-v)^{1+\alpha}}\,dvdr &\leqslant \int_s^t\int_s^{\tau_r^n}\frac{|r-v|^\beta+\Delta_n^\beta}{(r-v)^{1+\alpha}}\,dvdr \\ &\leqslant (\beta-\alpha)^{-1}\Delta_n^{\beta-\alpha}(t-s) + \Delta_n^\beta\alpha^{-1}\int_s^t(r-\tau_r^n)^{-\alpha}\,dr \\ &\leqslant (\beta-\alpha)^{-1}(t-s) + \Delta_n^\beta\alpha^{-1}(1-\alpha)^{-1}[2(t-s)^{1-\alpha} + (t-s)\Delta_n^{-\alpha}] \\ &\leqslant [(\beta-\alpha)^{-1} + \alpha^{-1}(1-\alpha)^{-1}](t-s) + 2\alpha^{-1}(1-\alpha)^{-1}(t-s)^{1-\alpha} \\ &\leqslant \left[(\beta-\alpha)^{-1}T^\alpha + \alpha^{-1}(1-\alpha)^{-1}(T^\alpha+2)\right](t-s)^{1-\alpha}.\end{aligned}$$

Applying (31), we obtain the following:

$$\begin{aligned}&\int_s^t\int_s^{\tau_r^n}\frac{|y^n(\tau_r^n)-y_r^n|}{(r-v)^{1+\alpha}}\,dvdr \leqslant \lambda(\alpha)\int_s^t\int_s^{\tau_r^n}\frac{(1+\|y^n\|_{\infty,r})(r-\tau_r^n)^{1-\alpha}}{(r-v)^{1+\alpha}}\,dvdr \\ &\quad\leqslant \lambda(\alpha)\alpha^{-1}\Delta_n^{1-2\alpha}\int_s^t(1+\|y^n\|_{\infty,r})\,dr \leqslant \lambda(\alpha)\alpha^{-1}(1+\|y^n\|_{\infty,t})(t-s)\end{aligned}$$

and

$$\int_s^t \int_s^{\tau_r^n} \frac{|y^n(v) - y^n(\tau_v^n)|}{(r-v)^{1+2\alpha}} \, dvdr \leqslant \lambda(\alpha) \int_s^t \int_s^{\tau_r^n} \frac{(1 + \|y^n\|_{\infty,v})(v - \tau_v^n)^{1-\alpha}}{(r-v)^{1+\alpha}} \, dvdr$$
$$\leqslant \lambda(\alpha)\Delta_n^{1-\alpha}(1 + \|y^n\|_{\infty,t}) \int_s^t \int_s^{\tau_r^n} \frac{1}{(r-v)^{1+\alpha}} \, dvdr$$
$$\leqslant \lambda(\alpha)\Delta_n^{1-\alpha}(1 + \|y^n\|_{\infty,t})\alpha^{-1}(1-\alpha)^{-1}[2(t-s)^{1-\alpha} + (t-s)\Delta_n^{-\alpha}]$$
$$\leqslant \lambda(\alpha)\alpha^{-1}(1-\alpha)^{-1}(2 + T^\alpha)(1 + \|y^n\|_{\infty,t})(t-s)^{1-\alpha}.$$

Finally,

$$\int_s^t \int_s^{\tau_r^n} \frac{|y^n(r) - y^n(v)|}{(r-v)^{1+\alpha}} \, dvdr \leqslant \|y^n\|_{\alpha,\infty}(t-s).$$

Thus the norm $\|G^{(g,\tau^n)}(y^n)\|_{1-\alpha}$ is bounded for all n and the proof is complete. □

4. Existence and Uniqueness of the Solution

We find conditions when the deterministic differential Equation (14) has a unique solution.

Theorem 2. *Let $1 - H < \alpha < \alpha_0$ and the functions $f(s,x)$ and $g(s,x)$ satisfy assumptions $(\boldsymbol{A}_1)$ and $(\boldsymbol{A}_2)$, respectively, where α_0 is defined in (3). If assumptions $(\boldsymbol{A}_3)$ are satisfied, then for any $\widehat{\alpha}$, such that $\alpha < \widehat{\alpha} < \frac{\delta}{1+\delta}$, Equation (14) has a solution $x \in C^{1-\widehat{\alpha}}(0,T)$. If assumptions $(\boldsymbol{A}_4)$ are satisfied and $\alpha < \widehat{\alpha} < \frac{\rho}{1+\rho}$, then Equation (14) has a unique solution $x \in C^{1-\widehat{\alpha}}(0,T)$.*

Proof. *Existence of the solution.* From Proposition 5 and assumption $\widehat{\alpha} > \alpha$, we have that the sequence of functions (y^n) is relatively compact in $C^{1-\widehat{\alpha}}(0,T)$.

Thus, we can choose a subsequence y^{n_k}, which converges in $C^{1-\widehat{\alpha}}(0,T)$ to a limit $x \in C^{1-\widehat{\alpha}}(0,T)$, i.e.,

$$\|y^{n_k} - x\|_{1-\widehat{\alpha}} \underset{n_k \to \infty}{\longrightarrow} 0. \tag{35}$$

We show that x is a solution to Equation (14). For simplicity of notation, we write n instead of n_k. Recall the following:

$$D(y_t^n) = D(x_0) + \int_0^t f(\tau_s^n, y^n(\tau_s^n)) \, ds + \int_0^t g(\tau_s^n, y^n(\tau_s^n)) \, dh_s.$$

Thus,

$$\left| D(x_\cdot) - D(x_0) - \int_0^\cdot f(s, x_s) \, ds - \int_0^\cdot g(s, x_s) \, dh_s \right|_\infty$$
$$\leqslant |D(x_\cdot) - D(y_\cdot^n)|_\infty + \left| \int_0^\cdot [f(\tau_s^n, y^n(\tau_s^n)) - f(s, x_s)] \, ds \right|_\infty$$
$$+ \left| \int_0^\cdot [g(\tau_s^n, y^n(\tau_s^n)) - g(s, x_s)] \, dh_s \right|_\infty. \tag{36}$$

Since $|x - y^n|_\infty \to 0$ and function D is continuous, the first term converges to zero. It remains to be proven that the second and third terms also converge to zero.

First, observe that there exists a constant, N, such that $\sup_n \|y^n\|_{1-\widehat{\alpha}} \leqslant N$ and $\|x\|_{1-\widehat{\alpha}} \leqslant N$. It follows from Proposition 5 and (35).

Next, we estimate the second term in (36). Since $y^n, x \in W_0^{\widehat{\alpha},\infty}(0,T)$, then from Proposition 1, it follows that $F^{(f)}(y^n), F^{(f)}(x) \in W_0^{\widehat{\alpha},\infty}(0,T)$. Applying (4) to any $\lambda \geqslant 0$, we obtain the following:

$$
\begin{aligned}
\left|F^{(f)}(y^n) - F^{(f)}(x)\right|_\infty &\leqslant e^{\lambda T}\|F^{(f)}(y^n) - F^{(f)}(x)\|_{\widehat{\alpha},\lambda} \\
&\leqslant e^{\lambda T}\frac{c_N}{\lambda^{1-\widehat{\alpha}}}\left(1+\frac{T^{1-2\widehat{\alpha}}}{1-2\widehat{\alpha}}\right)\|x-y^n\|_{1-\widehat{\alpha}} \underset{n\to\infty}{\longrightarrow} 0.
\end{aligned}
$$

To estimate the third term in (36), we note that $h \in W_T^{1-\widehat{\alpha},\infty}(0,T)$ for $\widehat{\alpha} > \alpha$ and $G^{(g)}(y^n), G^{(g)}(x) \in W_0^{\widehat{\alpha},\infty}(0,T)$. Applying Proposition 3, Remark 2, and (4), we obtain the following:

$$
\begin{aligned}
&\left|G^{(g)}(y^n) - G^{(g)}(x)\right|_\infty \leqslant e^{\lambda T}\|G^{(g)}(y^n) - G^{(g)}(x)\|_{\widehat{\alpha},\lambda} \leqslant \frac{\Lambda_{\widehat{\alpha}}(h)C_N^{(3)}e^{\lambda T}}{\lambda^{1-2\widehat{\alpha}}}\left(1+C_N^{(4)}\right)\|x-y^n\|_{\widehat{\alpha},\lambda} \\
&\leqslant \frac{\Lambda_{\widehat{\alpha}}(h)C_N^{(3)}e^{\lambda T}}{\lambda^{1-2\widehat{\alpha}}}\left(1+C_N^{(4)}\right)\left(1+\frac{T^{1-2\widehat{\alpha}}}{1-2\widehat{\alpha}}\right)\|x-y^n\|_{1-\widehat{\alpha}} \underset{n\to\infty}{\longrightarrow} 0,
\end{aligned}
$$

where

$$
C_N^{(4)} := \frac{T^{\delta-\widehat{\alpha}(1+\delta)}}{\delta-\widehat{\alpha}(1+\delta)}\left(\sup_n \|y^n\|_{1-\widehat{\alpha}} + \|x\|_{1-\widehat{\alpha}}\right) \leqslant 2N\,\frac{T^{\delta-\widehat{\alpha}(1+\delta)}}{\delta-\widehat{\alpha}(1+\delta)}\,.
$$

From the definition of $\widehat{\alpha}$, it follows that $\delta - \widehat{\alpha}(1+\delta) > 0$. The proof is completed. □

To prove the uniqueness of the solution, we need the following result:

Lemma 3 (see Lemma 7.1 in [10])**.** *Let Φ be a function satisfying assumptions $(\boldsymbol{A}_4)$. Then for all $N > 0$ and $|x_r|$, $|x_v|$, $|\widetilde{x}_r|$, $|\widetilde{x}_v| \leqslant N$, we have the following:*

$$
\begin{aligned}
&\left|\Phi(x_r) - \Phi(x_v)) - (\Phi(\widetilde{x}_r) - \Phi(\widetilde{x}_v))\right| \\
&\quad \leqslant c|(x_r - \widetilde{x}_r) - (x_v - \widetilde{x}_v)| + K_N|x_v - \widetilde{x}_v|\cdot[|x_r - x_v|^\rho + |\widetilde{x}_r - \widetilde{x}_v|^\rho].
\end{aligned}
$$

Proof. By the mean value theorem, we can write the following:

$$
\begin{aligned}
&(\Phi(x_r) - \Phi(x_v)) - (\Phi(\widetilde{x}_r) - \Phi(\widetilde{x}_v)) \\
&\quad = \left[(x_r - \widetilde{x}_r) - (x_v - \widetilde{x}_v)\right]\int_0^1 \Phi'\big(\theta x_r + (1-\theta)\widetilde{x}_r\big)d\theta \\
&\qquad + (x_v - \widetilde{x}_v)\int_0^1 \left[\Phi'\big(\theta x_r + (1-\theta)\widetilde{x}_r\big) - \Phi'\big(\theta x_v + (1-\theta)\widetilde{x}_v\big)\right]d\theta\,.
\end{aligned}
$$

From the conditions of the lemma, we obtain the statement of the lemma. □

Uniqueness of the solution. Let x and $\widetilde{x}$ be two solutions belonging to $C^{1-\widehat{\alpha}}(0,T)$. Then there exists N, such that $\|x\|_{1-\widehat{\alpha}} \leqslant N$ and $\|\widetilde{x}\|_{1-\widehat{\alpha}} \leqslant N$. Furthermore, $x, \widetilde{x} \in W_0^{\widehat{\alpha},\infty}(0,T)$ and $F^{(f)}(x), F^{(f)}(\widetilde{x}), G^{(g)}(x), G^{(g)}(\widetilde{x}) \in W_0^{\widehat{\alpha},\infty}(0,T)$ (see Propositions 1 and 3) and $\Phi(x), \Phi(\widetilde{x}) \in W_0^{\widehat{\alpha},\infty}(0,T)$. Thus, we have the following:

$$
\|x-\widetilde{x}\|_{\widehat{\alpha},\lambda} \leqslant \left\|\Phi(x) - \Phi(\widetilde{x})\right\|_{\widehat{\alpha},\lambda} + \left\|F^{(f)}(x) - F^{(f)}(\widetilde{x})\right\|_{\widehat{\alpha},\lambda} + \left\|G^{(g)}(x) - G^{(g)}(\widetilde{x})\right\|_{\widehat{\alpha},\lambda}.
$$

We will obtain estimates of the second and third terms from Propositions 1 and 3. It remains to evaluate the first term. From Lemma 3, we have the following:

$$
\begin{aligned}
&e^{-\lambda t}\left(\left|\Phi(x_t)-\Phi(\widetilde{x}_t)\right|+\int_0^t \frac{\left|(\Phi(x_t)-\Phi(\widetilde{x}_t))-(\Phi(x_s)-\Phi(\widetilde{x}_s))\right|}{(t-s)^{1+\widehat{\alpha}}}\,ds\right)\\
&\leqslant ce^{-\lambda t}\left(|x_s-\widetilde{x}_s|+\int_0^t \frac{|(x_t-\widetilde{x}_t)-(x_s-\widetilde{x}_s)|}{(t-s)^{1+\widehat{\alpha}}}\,ds\right)\\
&\quad+K_N e^{-\lambda t}\int_0^t \frac{|x_s-\widetilde{x}_s|\cdot[|x_t-x_s|^{\rho}+|\widetilde{x}_t-\widetilde{x}_s|^{\rho}]}{(t-s)^{1+\widehat{\alpha}}}\,ds\\
&\leqslant c\|x-\widetilde{x}\|_{\widehat{\alpha},\lambda}+K_N\sup_{0\leqslant t\leqslant T}\int_0^t \frac{e^{-\lambda(t-s)}e^{-\lambda s}|x_s-\widetilde{x}_s|\cdot[|x_t-x_s|^{\rho}+|\widetilde{x}_t-\widetilde{x}_s|^{\rho}]}{(t-s)^{1+\widehat{\alpha}}}\,ds\\
&\leqslant c\|x-\widetilde{x}\|_{\widehat{\alpha},\lambda}+K_N\|x-\widetilde{x}\|_{\alpha,\lambda}\left(|x|_{1-\widehat{\alpha}}+|\widetilde{x}|_{1-\widehat{\alpha}}\right)\int_0^t e^{-\lambda(t-s)}(t-s)^{-1-\widehat{\alpha}+\rho(1-\widehat{\alpha})}\,ds\\
&\leqslant c\|x-\widetilde{x}\|_{\widehat{\alpha},\lambda}+2NK_N\|x-\widetilde{x}\|_{\widehat{\alpha},\lambda}\lambda^{\widehat{\alpha}-\rho(1-\widehat{\alpha})}\int_0^\infty y^{-1-\widehat{\alpha}+\rho(1-\widehat{\alpha})}e^{-y}\,dy\\
&\leqslant c\|x-\widetilde{x}\|_{\widehat{\alpha},\lambda}+2NK_N\lambda^{\widehat{\alpha}-\rho(1-\widehat{\alpha})}\Gamma(\rho(1-\widehat{\alpha})-\widehat{\alpha})\|x-\widetilde{x}\|_{\widehat{\alpha},\lambda}
\end{aligned}
$$

for $\rho > \frac{\widehat{\alpha}}{1-\widehat{\alpha}}$. Thus,

$$
\begin{aligned}
\|x-\widetilde{x}\|_{\widehat{\alpha},\lambda} \leqslant\, & c\|x-\widetilde{x}\|_{\widehat{\alpha},\lambda}+2NK_N\lambda^{\widehat{\alpha}-\rho(1-\widehat{\alpha})}\Gamma(\rho(1-\widehat{\alpha})-\widehat{\alpha})\|x-\widetilde{x}\|_{\widehat{\alpha},\lambda}\\
&+\frac{c_N}{\lambda^{1-\widehat{\alpha}}}\|x-\widetilde{x}\|_{\alpha,\lambda}+\frac{\Lambda_{\widehat{\alpha}}(h)C_N^{(3)}}{\lambda^{1-2\widehat{\alpha}}}\left(1+C_N^{(4)}\right)\|x-y^n\|_{\widehat{\alpha},\lambda}.
\end{aligned}
$$

For any $\varepsilon > 0$, $c+\varepsilon < 1$, we can choose a sufficiently large λ, such that we have the following:

$$
2K_N N\lambda^{\widehat{\alpha}-\rho(1-\widehat{\alpha})}\Gamma(\delta(1-\widehat{\alpha})-\widehat{\alpha})+\frac{c_N}{\lambda^{1-\widehat{\alpha}}}+\frac{\Lambda_{\widehat{\alpha}}(h)C_N^{(3)}}{\lambda^{1-2\widehat{\alpha}}}\left(1+C_N^{(4)}\right)<\varepsilon.
$$

Thus, $|x-\widetilde{x}|_\infty = 0$ and, consequently, $x=\widetilde{x}$.

Proof of Theorem 1. Fix $\gamma \in (\gamma_0, H)$. Let $\varepsilon > 0$ be such that $\gamma+\varepsilon < H$. Denote $\alpha = 1-\gamma-\varepsilon$ and $\widehat{\alpha} = 1-\gamma$. Then $1-H < \alpha < \widehat{\alpha} < \alpha_0 = 1-\gamma_0$. Since

$$
\gamma > \gamma_0 = 1-\alpha_0, \qquad \alpha_0 \leqslant \beta, \qquad \text{and} \qquad \alpha_0 \leqslant \frac{\delta}{1+\delta}
$$

then $\alpha_0 > 1-\gamma$, $\beta > 1-\gamma > 1-H$, and $\frac{\delta}{1+\delta} > 1-\gamma$. Note that from the inequality $\frac{\delta}{1+\delta} > 1-\gamma = \widehat{\alpha}$ it follows that $\delta > \gamma^{-1}-1 > H^{-1}-1$.

If $\gamma \in (\overline{\gamma}_0, H)$ then $\overline{\alpha}_0 \leqslant \frac{\rho}{1+\rho}$ and $\frac{\rho}{1+\rho} > \widehat{\alpha}$. This completes the proof. □

5. Example of a Fractional Pearson Diffusion with a Stochastic Force

Consider the Pearson diffusion process with a stochastic force, as follows:

$$
D(X_t) = D(x_0) + \int_0^t \alpha(X_s)\,ds + \int_0^t \sigma(X_s)\,dB_s^H, \qquad t \geq 0, \tag{37}
$$

where

$$
\alpha(x) = b - ax, \qquad \sigma(x) = \sqrt{\sigma_0+\sigma_1 x+\sigma_2 x^2}.
$$

Assume that the coefficients σ_i, $i = 0,1,2$, are such that $\sigma_2 > 0$ and $\sigma_1^2 - 4\sigma_2\sigma_0 < 0$. Then $\sigma(x) > 0$.

For the existence of a unique solution to problem (37), it is necessary to check the conditions of Theorem 1. Note the following:

$$
|\alpha'(x)| \leqslant |a|, \qquad \sigma'(x) = \frac{\sigma_1+2\sigma_2 x}{2\sigma(x)}, \qquad 0 < \sigma''(x) = \frac{4\sigma_2\sigma_0-\sigma_1^2}{4\sigma^3(x)} \leqslant \frac{4\sigma_2\sigma_0-\sigma_1^2}{4\sigma^3(x_0)},
$$

where $x_0 = -\frac{\sigma_1}{2\sigma_2}$ is a critical point of the function $\sigma(x)$.

An easy computation shows the following:

$$\sigma^2(x) \geqslant \sigma_2\Big(x + \frac{\sigma_1}{2\sigma_2}\Big)^2 = \frac{1}{4\sigma_2}(2\sigma_2 x + \sigma_1)^2$$

and

$$(\sigma'(x))^2 \leqslant \frac{4\sigma_2(\sigma_1 + 2\sigma_2 x)^2}{4(2\sigma_2 x + \sigma_1)^2} = \sigma_2, \qquad |\sigma'(x)| \leqslant \sqrt{\sigma_2}\,.$$

Thus, the Pearson diffusion process with a stochastic force has a unique solution under the above conditions.

6. Discussion

The mathematical literature has extensively analyzed stochastic differential equations (SDEs) driven by a fractional Brownian motion. Most of these efforts have been motivated by problems arising in the financial applications of SDEs, such as option pricing, stochastic volatility, and interest rate modeling. However, there are few results concerning SDEs with boundary conditions. Typically, SDEs involving reflection at the boundary are considered. Our focus is on introducing and solving stochastic differential equations that are subject to a force allowing a process to cross a boundary while preventing it from moving far from it. Examining such a model can be interpreted as studying the influence of the environment on the behavior of the process. These types of processes can be applied in the natural sciences. This work represents an initial attempt to consider such processes. Introducing two- or three-dimensional SDEs with such a force would be of great practical interest.

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