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The Mean Square of the Hurwitz Zeta-Function in Short Intervals

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Abstract: The Hurwitz zeta-function $\zeta(s,\alpha)$, $s = \sigma + it$, with parameter $0 < \alpha \leqslant 1$ is a generalization of the Riemann zeta-function $\zeta(s)$ ($\zeta(s,1) = \zeta(s)$) and was introduced at the end of the 19th century. The function $\zeta(s,\alpha)$ plays an important role in investigations of the distribution of prime numbers in arithmetic progression and has applications in special function theory, algebraic number theory, dynamical system theory, other fields of mathematics, and even physics. The function $\zeta(s,\alpha)$ is the main example of zeta-functions without Euler's product (except for the cases $\alpha = 1$, $\alpha = 1/2$), and its value distribution is governed by arithmetical properties of α. For the majority of zeta-functions, $\zeta(s,\alpha)$ for some α is universal, i.e., its shifts $\zeta(s + i\tau,\alpha)$, $\tau \in \mathbb{R}$, approximate every analytic function defined in the strip $\{s : 1/2 < \sigma < 1\}$. For needs of effectivization of the universality property for $\zeta(s,\alpha)$, the interval for τ must be as short as possible, and this can be achieved by using the mean square estimate for $\zeta(\sigma + it,\alpha)$ in short intervals. In this paper, we obtain the bound $O(H)$ for that mean square over the interval $[T - H, T + H]$, with $T^{27/82} \leqslant H \leqslant T^{\sigma}$ and $1/2 < \sigma \leqslant 7/12$. This is the first result on the mean square for $\zeta(s,\alpha)$ in short intervals. In forthcoming papers, this estimate will be applied for proof of universality for $\zeta(s,\alpha)$ and other zeta-functions in short intervals.

Keywords: approximate functional equation; exponential pair; Hurwitz zeta-function; mean square of Dirichlet polynomial

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1. Introduction

Let $s = \sigma + it$ be a complex variable and $0 < \alpha \leqslant 1$ a fixed parameter. The Hurwitz zeta-function $\zeta(s,\alpha)$ was introduced in [1], and is defined, for $\sigma > 1$, by the Dirichlet series

$$\zeta(s,\alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s}.$$

Moreover, the function $\zeta(s,\alpha)$ can be analytically continued to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1. For $\alpha = 1$, the function reduces to the Riemann zeta-function $\zeta(s)$.

The Hurwitz zeta-function is an object of analytic number theory; however, it also has applications in algebraic number theory, in special function theory—for example, in [2], the infinite sum of the incomplete gamma-function was expressed in terms of the Hurwitz zeta-function—in dynamical systems, in mathematical statistics, and even in physics for the description of particle behaviour [3].

Let $\Gamma(s)$, as usual, denote the Euler gamma function. The function $\zeta(s,\alpha)$, for $\sigma > 1$, has the integral representation (see, for example, [4])

$$\zeta(s,\alpha) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{u^{s-1}\,\mathrm{e}^{-\alpha u}}{1 - \mathrm{e}^{-u}}\,\mathrm{d}u,$$

and satisfies the functional equation

$$\zeta(1-s,\alpha)=\frac{\Gamma(s)}{(2\pi)^s}\left(\mathrm{e}^{-\pi is/2}\sum_{m=1}^{\infty}\frac{\mathrm{e}^{2\pi im\alpha}}{m^s}+\mathrm{e}^{\pi is/2}\sum_{m=1}^{\infty}\frac{\mathrm{e}^{-2\pi im\alpha}}{m^s}\right),$$

and, for $\sigma<0$, the equation

$$\zeta(s,\alpha)=\frac{2\Gamma(1-s)}{(2\pi)^{1-s}}\left(\sin\frac{\pi s}{2}\sum_{m=1}^{\infty}\frac{\cos 2\pi m\alpha}{m^{1-s}}+\cos\frac{\pi s}{2}\sum_{m=1}^{\infty}\frac{\sin 2\pi m\alpha}{m^{1-s}}\right).$$

Suppose that $\chi(m)$ is a Dirichlet character modulo $q\in\mathbb{N}$, i.e., a function $\chi:\mathbb{N}\to\mathbb{C}$ satisfying the following:

1° periodic with period q: $\chi(m+q)=\chi(m)$ for all $m\in\mathbb{N}$;
2° completely multiplicative: $\chi(m_1m_2)=\chi(m_1)\chi(m_2)$ for all $m_1,m_2\in\mathbb{N}$;
3° $\chi(m)=0$ for $(m,q)>1$ ((m,q) is the greatest common divisor of m and q);
4° $\chi(m)\neq 0$ for $(m,q)=1$.

The Dirichlet L-function $L(s,\chi)$ is defined, for $\sigma>1$, by the series

$$L(s,\chi)=\sum_{m=1}^{\infty}\frac{\chi(m)}{m^s},$$

and has a meromorphic continuation to the whole complex plane. The unique simple pole is at the point $s=1$ if χ is the principal character. The function $L(s,\chi)$ is closely connected to the function $\zeta(s,\alpha)$ with the rational parameter α, namely,

$$L(s,\chi)=\frac{1}{q^s}\sum_{m=1}^{q-1}\chi(m)\zeta\left(s,\frac{m}{q}\right). \tag{1}$$

Thus, the function $\zeta(s,\alpha)$, as $\zeta(s)$, plays an important role in the theory of distribution of prime numbers. The above and other formulae and representations for $\zeta(s,\alpha)$ can be found, for example, in [4–6].

The analytic properties of the function $\zeta(s,\alpha)$ are influenced by the arithmetic of the parameter α; at least proof of some facts requires different methods. For example, this can be illustrated by zero distribution of $\zeta(s,\alpha)$. It is easily seen that

$$\zeta\left(s,\frac{1}{2}\right)=(2^s-1)\zeta(s). \tag{2}$$

This shows that $\zeta(s,1/2)\neq 0$ for $\sigma>1$. However, H. Davenport and H. Heilbronn proved [7] that for transcendental or rational $\alpha\neq 1/2$, the function $\zeta(s,\alpha)$ has zeros lying in the strip $\{s\in\mathbb{C}:1<\sigma<1+\delta\}$ for every $\delta>0$. Later, J. W. S. Cassels observed [8] by a different complicated method that this is also true for algebraic irrational α.

The function $\zeta(s,\alpha)$, as $\zeta(s)$ and some other zeta-functions, is universal in the sense that its shifts $\zeta(s+i\tau,\alpha)$, $\tau\in\mathbb{R}$, approximate a certain class of analytic functions. The universality property of $\zeta(s,\alpha)$ also is closely connected to the arithmetic of the parameter α. The cases of rational and transcendental α are the simplest ones, and the final result is known. Let $D=\{s\in\mathbb{C}:1/2<\sigma<1\}$, $\mathcal{K}$ be the class of compact subsets of D with connected complements, let $H(K)$ with $K\in\mathcal{K}$ be the class of continuous on K functions that are analytic in the interior of K, and let meas A denote the Lebesgue measure of a measurable set $A\subset\mathbb{R}$. Then, the universality of $\zeta(s,\alpha)$ is described by the following statement.

Theorem 1. *Suppose that α is a transcendental or rational number not equal to $1/2, 1$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{ \tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau,\alpha) - f(s)| < \varepsilon \right\} > 0.$$

Moreover, "$\liminf$" can be replaced by "$\lim$" for all but at most countably many $\varepsilon > 0$.

The case of transcendental α uses the fact that the set $\{\log(m+\alpha) : m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$ is linearly independent over the field of rational numbers $\mathbb{Q}$, and was discussed in [9,10] (see also [4]). The case of rational α was already known to S. M. Voronin who introduced the notion of universality [11] for zeta- and L-functions, and is based on the inverse formula

$$\zeta\left(s, \frac{a}{q}\right) = \frac{q^s}{\varphi(q)} \sum_{\chi (\operatorname{mod} q)} \chi(a) L(s,\chi)$$

of Formula (1), where $\varphi(q)$ is the Euler totient function, and summing runs over all Dirichlet characters modulo q. The second assertion of Theorem 1 was treated in [12], and was influenced by a similar result for the function $\zeta(s)$ [13].

The case of algebraic irrational α is the most complicated because there is not any information on the linear independence of $\log(m+\alpha)$, $m \in \mathbb{N}_0$, and it was solved with a certain exception in [14].

One of the most important problems of universality theorems for zeta-functions is their effectivization, which is connected to the localization of values of τ in approximating shifts. This leads to universality theorems in short intervals, i.e., in intervals of length of $o(T)$, as $T \to \infty$. Proofs of theorems of such a type are closely connected to mean square estimates for zeta-functions in short intervals. In the case of the function $\zeta(s)$, the mean square estimates in short intervals are given in [6]. Theorem 7.1 is devoted to the case of the Riemann zeta-function. Unfortunately, the function $\zeta(s,\alpha)$ has no Euler product over primes; it is not connected to the divisor function and we cannot reach for $\zeta(s,\alpha)$ a valid result for $\zeta(s)$.

The aim of this paper is to prove the following theorem.

Theorem 2. *Suppose that $\alpha \neq 1/2, 1$, and $1/2 < \sigma \leqslant 7/12$ is fixed. Then, for $T^{27/82} \leqslant H \leqslant T^{\sigma}$, uniformly in H,*

$$\int_{T-H}^{T+H} |\zeta(\sigma+it,\alpha)|^2 \, dt \ll_{\sigma,\alpha} H.$$

In view of (2), the cases $\alpha = 1/2$ and $\alpha = 1$ are included in Theorem 7.1 of [6].

To our knowledge, Theorem 2 is the first result in short intervals for the Hurwitz zeta-function. Until now, only estimates of the form

$$\int_{-T}^{T} |\zeta(\sigma+it,\alpha)|^2 \, dt \ll_{\sigma,\alpha} T$$

for $1/2 < \sigma \leqslant 1$ were known (see [4]).

For mean square estimates of zeta-functions, usually approximate functional equations are applied. Therefore, we start with an approximate functional equation for $\zeta(s,\alpha)$.

2. Approximate Functional Equation

Let $0 < \lambda \leqslant 1$ and $0 < \alpha \leqslant 1$ be fixed parameters. The Lerch zeta-function $L(\lambda, \alpha, s)$ was introduced independently by M. Lerch [15] and R. Lipschitz [16], and is defined, for $\sigma > 1$, by the Dirichlet series

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{\mathrm{e}^{2\pi i \lambda m}}{(m+\alpha)^s}.$$

Moreover, for $0 < \lambda < 1$, the function $L(\lambda, \alpha, s)$ has an analytic continuation to the whole complex plane. Obviously, $L(1, \alpha, s) = \zeta(s, \alpha)$ in this case $L(\lambda, \alpha, s)$ is a meromorphic function with the unique simple pole at the point $s = 1$ with residue 1. In [17], an approximate functional equation for $L(\lambda, \alpha, s)$ has been obtained. Let

$$\psi(a) = \frac{\cos(\pi(a^2/2 - a - 1/8))}{\cos \pi a},$$

and $\{x\}$ denote the fractional part and $[x]$ the integer part of x. Moreover, for $t \geqslant 1$,

$$y_t = \left(\frac{t}{2\pi}\right)^{1/2}, \quad q_t = [y_t], \quad u_t = [y_t - \alpha], \quad v_t = q_t - v_t.$$

Lemma 1 ([17])**.** *Suppose that* $0 < \lambda \leqslant 1$, $0 < \alpha \leqslant 1$, $0 \leqslant \sigma \leqslant 1$ *and* $t \geqslant 1$. *Then,*

$$L(\lambda, \alpha, s) = \sum_{m=0}^{u_t} \frac{\mathrm{e}^{2\pi i \lambda m}}{(m+\alpha)^s} + \left(\frac{2\pi}{t}\right)^{\sigma - 1/2 + it} \mathrm{e}^{it + \pi i/4 - 2\pi i \{\lambda\}\alpha} \sum_{m=0}^{q_t} \frac{\mathrm{e}^{-2\pi i \alpha m}}{(m+\lambda)^{1-s}}$$
$$+ \left(\frac{2\pi}{t}\right)^{\sigma/2} \mathrm{e}^{\pi i f(\lambda, \alpha, \sigma, t)} \psi(2y_t - 2q_t + v_t - \{\lambda\} - \alpha) + O(t^{(\sigma-2)/2}),$$

where

$$f(\lambda, \alpha, \sigma, t) = -\frac{t}{2\pi} \log \frac{t}{2\pi \mathrm{e}} - \frac{7}{8} + \frac{1}{2}(\alpha^2 - \{\lambda\}^2) - \alpha v_t + 2y_t(v_t + \{\lambda\} - \alpha)$$
$$- \frac{1}{2}(q_t + u_t) - \{\lambda\}(v_t + \alpha).$$

Since $\zeta(s, \alpha) = L(1, \alpha, s)$, Lemma 1 implies the following approximation functional equation for $\zeta(s, \alpha)$.

Lemma 2. *Suppose that* $0 < \alpha \leqslant 1$, $0 \leqslant \sigma \leqslant 1$ *and* $t \geqslant 1$. *Then,*

$$\zeta(s, \alpha) = \sum_{m=0}^{u_t} \frac{1}{(m+\alpha)^s} + \left(\frac{2\pi}{t}\right)^{\sigma - 1/2 + it} \mathrm{e}^{it + \pi i/4} \sum_{m=0}^{q_t} \frac{\mathrm{e}^{-2\pi i \alpha m}}{(m+1)^{1-s}}$$
$$+ \left(\frac{2\pi}{t}\right)^{\sigma/2} \mathrm{e}^{\pi i g(\alpha, \sigma, t)} \psi(2y_t - 2q_t + v_t - \alpha) + O(t^{(\sigma-2)/2}),$$

where

$$g(\alpha, \sigma, t) = -\frac{t}{2\pi} \log \frac{t}{2\pi \mathrm{e}} - \frac{7}{8} + \frac{1}{2}\alpha^2 - \alpha v_t + 2y_t(v_t - \alpha) - \frac{1}{2}(q_t + u_t).$$

Proof. The lemma follows from Lemma 1 by taking $\lambda = 1$. □

Note that an approximate functional equation for $\zeta(1/2 + it, \alpha)$ was obtained by V. V. Rane [18].

Clearly,

$$\sum_{m=0}^{q_t} \frac{e^{-2\pi i\alpha m}}{(m+1)^{1-s}} = e^{2\pi i\alpha} \sum_{m=1}^{q_t+1} \frac{e^{-2\pi i\alpha m}}{m^{1-s}}. \tag{3}$$

Recall that $x \ll_\theta y$, $x \in \mathbb{C}$, $y > 0$, means that there exists a constant $c = c(\theta) > 0$ such that $|x| \leqslant cy$. Thus, Lemma 2 and (3) show that, under the hypotheses of Lemma 2,

$$\zeta(\sigma + it, \alpha) \ll_\alpha 1 + \left| \sum_{m=1}^{q_t} \frac{1}{(m+\alpha)^{\sigma+it}} \right| + t^{1/2-\sigma} \left| \sum_{m=1}^{q_t} \frac{e^{-2\pi i\alpha m}}{m^{1-\sigma-it}} \right|. \tag{4}$$

Therefore, the estimation of $\zeta(\sigma + it, \alpha)$ reduces to mean square estimates for Dirichlet polynomials.

3. Mean Square of Dirichlet Polynomials

To prove our aim, one formula for the exponential integral will be useful (see, for example, [6], p. 492).

Lemma 3. *Suppose that $b > 0$. Then, for every $a \in \mathbb{C}$,*

$$\int_{-\infty}^{\infty} \exp\{at - bt^2\}\, dt = \left(\frac{\pi}{b}\right)^{1/2} \exp\left\{\frac{a^2}{4b}\right\}.$$

Also, we will apply the following version of the partial summation.

Lemma 4. *Let $a_m \in \mathbb{C}$ and $B = \{b_m : b_m \geqslant 0\}$. Then, for $n \in \mathbb{N}$,*

$$\left| \sum_{n<m\leqslant n_1} a_m b_m \right| \leqslant 2^r b_{\widehat{n}} \max_{n<m\leqslant n_1} \left| \sum_{n<k\leqslant m} a_k \right|,$$

where $r = 0$ and $\widehat{n} = n$ if the sequence B is non-increasing, and $r = 1$ and $\widehat{n} = n_1$ if B is non-decreasing.

Proof of the lemma is given in [6] (p. 489).

Lemma 5. *Suppose that $n < n_1 \leqslant 2n \leqslant T^{1/2}$, $T^a \leqslant H \leqslant T^\sigma$, with $a = 27/82$, $\sigma > 1/2$, and $T_1 = T_1(T) = nH^{-1} \log T$. Then, uniformly in H,*

$$\begin{aligned} I_1 &\stackrel{\text{def}}{=} \int_{T-H}^{T+H} \left| \sum_{n<m\leqslant n_1} \frac{1}{(m+\alpha)^{it}} \right|^2 dt \\ &\ll nH\log T + H \sum_{l\leqslant T_1} \max_{n<n_2\leqslant n_1-l} \left| \sum_{n<k\leqslant n_2} \exp\left\{ iT\log\left(1 + \frac{l}{k+\alpha}\right)\right\} \right|. \end{aligned}$$

Proof. For $t \in [-H, H]$, the inequality $\exp\{t^2H^{-2}\} \leqslant e$ holds. Therefore,

$$\begin{aligned} I_1 &\leqslant e \int_{-H}^{H} \left| \sum_{n<m\leqslant n_1} \frac{1}{(m+\alpha)^{it+iT}} \right|^2 \exp\{-t^2H^{-2}\}\, dt \\ &\ll \int_{-H\log T}^{H\log T} \left| \sum_{n<m\leqslant n_1} \frac{1}{(m+\alpha)^{it+iT}} \right|^2 \exp\{-t^2H^{-2}\}\, dt. \end{aligned} \tag{5}$$

We have

$$\left|\sum_{n<m\leqslant n_1}\frac{1}{(m+\alpha)^{it+iT}}\right|^2=\sum_{n<m\leqslant n_1}\frac{1}{(m+\alpha)^{it+iT}}\sum_{n<m\leqslant n_1}\frac{1}{(m+\alpha)^{-it-iT}}$$
$$=\sum_{n<m\leqslant n_1}1+\sum_{\substack{n<m\leqslant n_1\\ n<k\leqslant n_1\\ m\neq k}}\sum\left(\frac{k+\alpha}{m+\alpha}\right)^{it+iT}.$$

Hence, by (5),

$$I_1 \ll nH\log T + \left|\sum\sum_{n<m\leqslant n_1\, n<k\leqslant n_1\, m\neq k}\left(\tfrac{k+\alpha}{m+\alpha}\right)^{iT}\int_{-H\log T}^{H\log T}\exp\left\{it\log\tfrac{k+\alpha}{m+\alpha}-t^2H^{-2}\right\}\mathrm{d}t\right|. \tag{6}$$

Obviously,

$$\int_{-\infty}^{-H\log T}\exp\{-t^2H^{-2}\}\,\mathrm{d}t,\ \int_{H\log T}^{\infty}\exp\{-t^2H^{-2}\}\,\mathrm{d}t$$
$$\ll \exp\left\{-\frac{1}{2}\log T\right\}\int_{H\log T}^{\infty}\exp\left\{-\frac{1}{2}t^2\right\}\mathrm{d}t \ll \exp\left\{-\frac{1}{2}\log^2 T\right\}.$$

Since $n_1 \ll T^{1/2}$, this and (6) lead to the bound

$$I_1 \ll nH\log T + \left|\sum\sum_{n<m\leqslant n_1\, n<k\leqslant n_1\, m\neq k}\left(\tfrac{k+\alpha}{m+\alpha}\right)^{iT}\int_{-\infty}^{\infty}\exp\left\{it\log\tfrac{k+\alpha}{m+\alpha}-t^2H^{-2}\right\}\mathrm{d}t\right|. \tag{7}$$

In view of Lemma 3,

$$\int_{-\infty}^{\infty}\exp\left\{it\log\frac{k+\alpha}{m+\alpha}-t^2H^{-2}\right\}\mathrm{d}t=\sqrt{\pi}H\exp\left\{-\frac{1}{4}H^2\log^2\frac{k+\alpha}{m+\alpha}\right\}.$$

Thus, by (7), for $m > k$, we have

$$I_1 \ll nH\log T + H\left|\sum_{\substack{n<m\leqslant n_1\\ n<k\leqslant n_1\\ m>k}}\sum\exp\left\{-iT\log\frac{m+\alpha}{k+\alpha}-\frac{1}{4}H^2\log^2\frac{m+\alpha}{k+\alpha}\right\}\right|$$
$$\ll nH\log T \tag{8}$$
$$+H\left|\sum_{l\leqslant T_1}\sum_{n<k\leqslant n_1-l}\exp\left\{-iT\log\left(1+\frac{l}{k+\alpha}\right)-\frac{1}{4}H^2\log^2\left(1+\frac{l}{k+\alpha}\right)\right\}\right|$$
$$+H\left|\sum_{l\geqslant T_1}\sum_{n<k\leqslant n_1-l}\exp\left\{-\frac{1}{4}H^2\log^2\left(1+\frac{l}{k+\alpha}\right)\right\}\right|.$$

The third term of the right-hand side of (8) is estimated as

$$\ll T^{\sigma}\sum_{l\geqslant T_1}\sum_{n<k\leqslant n_1-l}\exp\left\{-H^2\frac{l^2}{20(k+\alpha)}\right\}\ll T^{\sigma}n_1\exp\left\{-\frac{1}{20}\log^2 T\right\}=o(1)$$

as $T \to \infty$. Therefore, by (8), we have

$$\begin{aligned} I_1 \ll \; & nH \log T \\ & + H \sum_{l \leqslant T_1} \sum_{n<k\leqslant n_1-l} \exp\Big\{-iT \log\Big(1+\tfrac{l}{k+\alpha}\Big) - \tfrac{1}{4}H^2 \log^2\Big(1+\tfrac{l}{k+\alpha}\Big)\Big\}. \end{aligned} \tag{9}$$

Now, Lemma 4 gives

$$\begin{aligned} & \sum_{n<k\leqslant n_1-l} \exp\left\{-iT \log\left(1+\frac{l}{k+\alpha}\right) - \frac{1}{4}H^2 \log^2\left(1+\frac{l}{k+\alpha}\right)\right\} \\ & \quad \ll \exp\left\{-\frac{1}{4}H^2 \log^2\left(1+\frac{l}{k+\alpha}\right)\right\} \\ & \quad \times \max_{n<m\leqslant n_1-l} \left| \sum_{n<k\leqslant m} \exp\left\{-iT \log\left(1+\frac{l}{k+\alpha}\right)\right\} \right| \\ & \quad \ll \max_{n<m\leqslant n_1-l} \left| \sum_{n<k\leqslant m} \exp\left\{iT \log\left(1+\frac{l}{k+\alpha}\right)\right\} \right| \end{aligned}$$

because $|z| = |\overline{z}|$, where $\overline{z}$ denotes the complex conjugate of z. This together with (9) proves the lemma. □

Lemma 6. *Under the hypotheses and notation of Lemma 5, uniformly in H,*

$$\begin{aligned} I_2 & \stackrel{\text{def}}{=} \int_{T-H}^{T+H} \left| \sum_{n<m\leqslant n_1} \frac{e^{-2\pi i \alpha m}}{m^{it}} \right|^2 dt \\ & \ll nH \log T + H \sum_{l \leqslant T_1} \max_{n<n_2\leqslant n_1-l} \left| \sum_{n<k\leqslant n_2-l} \exp\left\{iT \log\left(1+\frac{l}{k}\right)\right\} \right|. \end{aligned}$$

Proof. As in the proof of Lemma 5, we have

$$I_2 \ll \int_{-H\log T}^{H\log T} \left| \sum_{n<m\leqslant n_1} \frac{e^{-2\pi i \alpha m}}{m^{it+iT}} \right|^2 \exp\{-t^2 H^{-2}\}\, dt. \tag{10}$$

Since

$$\begin{aligned} \left| \sum_{n<m\leqslant n_1} \frac{e^{-2\pi i \alpha m}}{m^{it+iT}} \right|^2 & = \sum_{n<m\leqslant n_1} \frac{e^{-2\pi i \alpha m}}{m^{it+iT}} \sum_{n<m\leqslant n_1} \frac{e^{2\pi i \alpha m}}{m^{-it-iT}} \\ & = \sum_{n<m\leqslant n_1} 1 + \sum_{\substack{n<m\leqslant n_1 \\ n<k\leqslant n_1 \\ m\neq k}}\sum e^{-2\pi i \alpha (m-k)} \left(\frac{k}{m}\right)^{it+iT}, \end{aligned}$$

by (10), we find

$$
\begin{aligned}
I_2 \ll & nH \log T \\
& + \left| \sum_{\substack{n<m\leqslant n_1 \\ n<k\leqslant n_1 \\ m\neq k}} \sum \mathrm{e}^{-2\pi i\alpha(m-k)} \left(\frac{k}{m}\right)^{iT} \int_{-H\log T}^{H\log T} \exp\left\{ it\log\frac{k}{m} - t^2H^{-2} \right\} \mathrm{d}t \right| \\
\ll & nH \log T \\
& + \left| \sum_{\substack{n<m\leqslant n_1 \\ n<k\leqslant n_1 \\ m\neq k}} \sum \mathrm{e}^{-2\pi i\alpha(m-k)} \left(\frac{k}{m}\right)^{iT} \int_{-\infty}^{\infty} \exp\left\{ it\log\frac{k}{m} - t^2H^{-2} \right\} \mathrm{d}t \right| \\
\ll & nH \log T \\
& + \left| \sum_{\substack{n<m\leqslant n_1 \\ n<k\leqslant n_1 \\ m> k}} \sum \mathrm{e}^{-2\pi i\alpha(m-k)} \left(\frac{m}{k}\right)^{iT} \int_{-\infty}^{\infty} \exp\left\{ it\log\frac{m}{k} - t^2H^{-2} \right\} \mathrm{d}t \right|
\end{aligned}
$$

because for $k > m$, the estimated expression becomes a conjugate of the case $m > k$. Thus, by Lemma 3,

$$
\begin{aligned}
I_2 \ll & nH \log T + H \left| \sum_{\substack{n<m\leqslant n_1 \\ n<k\leqslant n_1 \\ m> k}} \sum \exp\left\{ -2\pi i\alpha(m-k) + iT\log\frac{m}{k} - \frac{1}{4}H^2\log^2\frac{m}{k} \right\} \right| \\
\ll & nH \log T \qquad\qquad (11) \\
& + H \sum_{l\leqslant T_1} \left| \exp\{-2\pi i\alpha l\} \sum_{n<k\leqslant n_1-l} \exp\left\{ iT\log\left(1+\frac{l}{k}\right) - \frac{1}{4}H^2\log^2\left(1+\frac{l}{k}\right) \right\} \right| \\
& + H \sum_{l\geqslant T_1} \sum_{n<k\leqslant n_1-l} \exp\left\{ -\frac{1}{4}H^2\log^2\left(1+\frac{l}{k}\right) \right\}.
\end{aligned}
$$

The third term in the right-hand side tends to zero as $T \to \infty$. This, (11) and Lemma 4 lead to the estimate of the lemma. □

4. Exponential Pairs

In the theory of the estimation of exponential sums, the method of exponential pairs is successfully applied. We shortly recall this method. The exponential sum is as follows:

$$
S \stackrel{\text{def}}{=} \sum_{a<m\leqslant a+h} \exp\{2\pi i f(m)\}
$$

with $a \geqslant 1$, $1 < h \leqslant a$. The notation $x \gg y$ means that $y \ll x$. Suppose that for $x \in [a, 2a]$ and $b > 1/2$,

$$
b \ll |f'(x)| \ll b, \qquad \text{and} \qquad S \ll b^{\kappa} a^{\lambda}. \tag{12}
$$

Then, the pair (κ, λ) of numbers is called the exponential pair if $0 \leqslant \kappa \leqslant 1/2 \leqslant \lambda \leqslant 1$.

For example, the pairs $(0,1)$ and $(1/2,1/2)$ are exponential pairs. Moreover, the set of exponential pairs is convex. Thus, two exponential pairs lead to a new exponential pair.

To extend the set of exponential pairs, usually, besides (12), it is required that $f(x)$ has continuous derivatives of higher order k, and

$$ba^{1-k} \ll |f^{(k)}(x)| \ll ba^{1-k}. \tag{13}$$

We will apply exponential pairs for the estimation of trigonometric sums appearing in Lemmas 5 and 6.

Let

$$V(\beta) = \sum_{n<k\leqslant n_1} \exp\{iTf(k,\beta)\}$$

with $0 \leqslant \beta \leqslant 1$, $n_1 \in [n+1, 2n]$, where

$$f(x,\beta) = T\log\left(1 + \frac{l}{x+\beta}\right), \quad 1 \leqslant l \leqslant n.$$

Lemma 7. *Let (κ, λ) be an exponential pair and $n \ll T^{1/2}$. Then,*

$$V(\beta) \ll T^{\kappa} l^{\kappa} n^{\lambda-2\kappa}.$$

Proof. By the definition of $f(x,\beta)$, we have

$$\begin{aligned}
f'(x,\beta) =& -T\left(1+\frac{l}{x+\beta}\right)^{-1}\frac{l}{(x+\beta)^2} = -\frac{Tl}{(x+\beta)^2 + l(x+\beta)} \\
=& -Tl\Big((x+\beta)^2 + l(x+\beta)\Big)^{-1}, \\
f''(x,\beta) =& Tl\Big((x+\beta)^2 + l(x+\beta)\Big)^{-2}(2(x+\beta)+l), \\
f'''(x,\beta) =& -2Tl\Big((x+\beta)^2 + l(x+\beta)\Big)^{-3}(2(x+\beta)+l)^2 \\
&+ 2Tl\Big((x+\beta)^2 + l(x+\beta)\Big)^{-2}, \\
f^{IV}(x,\beta) =& 6Tl\Big((x+\beta)^2 + l(x+\beta)\Big)^{-4}(2(x+\beta)+l)^3 \\
&- 8Tl\Big((x+\beta)^2 + l(x+\beta)\Big)^{-3}(2(x+\beta)+l) \\
&- 4Tl\Big((x+\beta)^2 + l(x+\beta)\Big)^{-3}(2(x+\beta)+l), \quad \text{etc.}
\end{aligned}$$

This gives the estimates

$$1 \ll Tln^{-2} \ll_\beta |f'(x,\beta)| \ll_\beta Tl(x+\beta)^{-2} \ll_\beta Tln^{-2}$$

and

$$Tln^{-2m+m-1} \ll_\beta |f^{(m)}(x,\beta)| \ll_\beta Tln^{-2m+m-1}, \quad m \in \mathbb{N}.$$

Thus,

$$bn^{1-m} \ll_\beta |f^{(m)}(x,\beta)| \ll_\beta bn^{1-m} \quad \text{with } b = Tln^{-2}.$$

These observations show that if (κ,λ) is an exponential pair, then

$$V(\beta) \ll_\beta (Tln^{-2})^{\kappa} n^{\lambda} \ll_\beta T^{\kappa} l^{\kappa} n^{\lambda-2\kappa}.$$

□

5. Proof of the Main Theorem

In this section, we will prove the mean square estimate for the Hurwitz zeta-function in short intervals, Theorem 2. First, we recall the mean square theorem for Dirichlet polynomials.

Lemma 8. *Let $a_1, \dots, a_M$ be arbitrary complex numbers, and $0 \leqslant \beta \leqslant 1$. Then,*

$$\int_0^T \left| \sum_{1 \leqslant m \leqslant M} \frac{a_m}{(m+\beta)^{\sigma+it}} \right|^2 \mathrm{d}t \ll T \sum_{1 \leqslant m \leqslant M} \frac{|a_m|^2}{(m+\beta)^{2\sigma}} + \sum_{1 \leqslant m \leqslant M} \frac{m|a_m|}{(m+\beta)^{2\sigma}}.$$

Proof. For $\beta = 0$, proof of the lemma can be, for example, found in [6], and is based on the Montgomery–Vaughan inequality [19]. For $\beta = \alpha$, the Montgomery–Vaughan inequality, where α is from the definition of $\zeta(s, \alpha)$, is given in [20]. □

Proof of Theorem 2. Replace the summation in (4). For $T \leqslant t \leqslant T + H$, we have

$$\begin{aligned}
\sum_{1 \leqslant m \leqslant q_t} \frac{1}{(m+\alpha)^{\sigma+it}} &= \left(\sum_{1 \leqslant m \leqslant q_T} + \sum_{q_T \leqslant m \leqslant q_t} \right) \frac{1}{(m+\alpha)^{\sigma+it}} \\
&= \sum_{1 \leqslant m \leqslant q_T} \frac{1}{(m+\alpha)^{\sigma+it}} + O(q_t - q_T) q_T^{-\sigma} \\
&= \sum_{1 \leqslant m \leqslant q_T} \frac{1}{(m+\alpha)^{\sigma+it}} + O(1).
\end{aligned}$$

The same estimates remain valid also for $T - H \leqslant t \leqslant T$. Therefore, (4) can be rewritten, for $T - H \leqslant t \leqslant T + H$, in the form

$$\zeta(\sigma + it, \alpha) \ll_\sigma 1 + \left| \sum_{1 \leqslant m \leqslant q_T} \frac{1}{(m+\alpha)^{\sigma+it}} \right| + T^{1/2-\sigma} \left| \sum_{1 \leqslant m \leqslant q_T} \frac{\mathrm{e}^{-2\pi i \alpha m}}{m^{1-\sigma-it}} \right|. \tag{14}$$

Divide the interval of summation $1 \leqslant m \leqslant q_1$ into partial intervals $[n_j, 2n_j]$, where $n_j = q_T 2^{-j}$, $j = 1, 2, \dots$. The number of intervals is $\ll \log T$. Let $\delta > 0$ be a fixed number such that $1/2 < \sigma - \delta < 7/12$, and let $\sigma_1 = \sigma - \delta$. Then, by (14),

$$\begin{aligned}
|\zeta(\sigma+it,\alpha)|^2 \ll_\alpha & 1 + \left(\sum_j \left| \sum_{n_j < m \leqslant 2n_j} \frac{1}{(m+\alpha)^{\sigma+it}} \right| \right)^2 \\
& + T^{1-2\sigma} \left(\sum_j \left| \sum_{n_j < m \leqslant 2n_j} \frac{\mathrm{e}^{-2\pi i \alpha m}}{m^{1-\sigma-it}} \right| \right)^2 \\
\ll_\sigma & 1 + \sum_j \frac{1}{(m+\alpha)^{2\delta}} \sum_j (m+\alpha)^{2\delta} \left| \sum_{n_j < m \leqslant 2n_j} \frac{1}{(m+\alpha)^{\sigma+it}} \right|^2 \\
& + T^{1-2\sigma} \sum_j \frac{1}{m^{2-2\delta}} \sum_j m^{-2+2\sigma} \left| \sum_{n_j < m \leqslant 2n_j} \frac{\mathrm{e}^{-2\pi i \alpha m}}{m^{1-\sigma-it}} \right|^2 \\
\ll_\sigma & 1 + \sum_j (m+\alpha)^{2\delta} \left| \sum_{n_j < m \leqslant 2n_j} \frac{1}{(m+\alpha)^{\sigma+it}} \right|^2 \\
& + T^{1-2\sigma} \sum_j (m+\alpha)^{-2+2\delta} \left| \sum_{n_j < m \leqslant 2n_j} \frac{\mathrm{e}^{-2\pi i \alpha m}}{m^{1-\sigma-it}} \right|^2.
\end{aligned}$$

Hence, by Lemma 4,

$$\begin{aligned}
&\int_{T-H}^{T+H} |\zeta(\sigma+it,\alpha)|^2\,\mathrm{d}t \\
&\ll_\sigma H + \sum_j \left(\frac{1}{(n_j+\alpha)^{2\sigma_1}}\right) \max_{n_j<k\leqslant 2n_j} \int_{T-H}^{T+H} \left|\sum_{n_j<m\leqslant k} \frac{1}{(m+\alpha)^{it}}\right|^2 \mathrm{d}t \\
&\quad + T^{1-2\sigma} \sum_j \left(\frac{1}{n_j^{2-2\sigma_1}}\right) \max_{n_j<k\leqslant 2n_j} \int_{T-H}^{T+H} \left|\sum_{n_j<m\leqslant k} \frac{\mathrm{e}^{-2\pi i\alpha m}}{m^{-it}}\right|^2 \mathrm{d}t.
\end{aligned} \tag{15}$$

In virtue of Lemma 8,

$$\int_{T-H}^{T+H} \left|\sum_{n_j<m\leqslant k} \frac{1}{(m+\alpha)^{it}}\right|^2 \mathrm{d}t \ll H \sum_{n_j<m\leqslant k} 1 + O\left(\sum_{n_j\leqslant m\leqslant k} m\right) \ll Hn_j + n_j^2, \tag{16}$$

and the same bound is true for the second integral in the right-hand side of (15). Now, let $\varepsilon > 0$ be a fixed arbitrary small number, and $\sum_1$ denote the summation over j such that $n_j \leqslant T^\varepsilon$. Then, taking into account (16), we find that the second term in the right-hand side of (15) can be estimated using the bounds

$$H \sum_j{}_1 \frac{n_j}{(n_j+\alpha)^{2\sigma_1}} \ll_\alpha H \sum_j{}_1 n_j^{1-2\sigma_1} \ll_\alpha H$$

and

$$\sum_j{}_1 \frac{n_j^2}{(n_j+\alpha)^{2\sigma_1}} \ll_\alpha T^{(1-\sigma_1)\varepsilon} \log T$$

because $j \ll \log T$. Thus, since $H \geqslant T^{27/82}$, we have

$$\begin{aligned}
&\sum_j{}_1 \tfrac{1}{(n_j+\alpha)^{2\sigma_1}} \max_{n_j\leqslant k\leqslant 2n_j} \quad \int_{T-H}^{T+H} \left|\sum_{n_j\leqslant m\leqslant 2n_j} \tfrac{1}{(m+\alpha)^{it}}\right|^2 \mathrm{d}t \\
&\qquad \ll_\alpha H + HH^{-1}T^{(1-\sigma_1)\varepsilon} \log T \ll_\alpha H.
\end{aligned} \tag{17}$$

Moreover, since $n_T \leqslant T^{1/2}$ and $\sigma_1 > 1/2$, we have the bound $T^{1-2\sigma} \ll n_j^{2-4\sigma_1}$, and similarly as above, we obtain, by Lemma 8,

$$\begin{aligned}
&T^{1-2\sigma} \sum{}_{1j} \tfrac{1}{n_j^{2-2\sigma_1}} \quad \int_{T-H}^{T+H} \left|\sum_{n_j\leqslant m\leqslant k} \tfrac{\mathrm{e}^{-2\pi i\alpha m}}{m^{-it}}\right| \mathrm{d}t \\
&\qquad \ll \sum{}_{1j} \tfrac{1}{n_j^{2\sigma_1}} \int_{T-H}^{T+H} \left|\sum_{n_j\leqslant m\leqslant k} \tfrac{\mathrm{e}^{-2\pi\alpha m}}{m^{-it}}\right| \mathrm{d}t.
\end{aligned} \tag{18}$$

All that remains is to find estimates for sum $\sum_j{}_2$ in (15) over j satisfying $n_j > T^\varepsilon$. For this, we will apply Lemmas 5 and 7.

By Lemma 5,

$$\begin{aligned}
&\int_{T-H}^{T+H} \left|\sum_{n_j<m\leqslant k} \tfrac{1}{(m+\alpha)^{it}}\right|^2 \mathrm{d}t \\
&\ll n_j H \log T + H \sum_{l\leqslant T_1} \max_{n_j<m\leqslant 2n_j-l} \left|\sum_{n_j\leqslant k\leqslant m} \exp\left\{iT\log\left(1+\tfrac{l}{k+\alpha}\right)\right\}\right|.
\end{aligned} \tag{19}$$

Lemma 7 asserts that

$$\sum_{n_j<k\leqslant m} \exp\left\{iT\log\left(1+\frac{l}{k+\alpha}\right)\right\} \ll_\alpha T^\kappa l^\kappa n_j^{\lambda-2\kappa},$$

where (κ, λ) is an exponential pair. This and (19) give

$$\begin{aligned}
&\log T \sum_{j}{}_2 \frac{1}{(n_j+\alpha)^{2\sigma}} \max_{n_j<k\leqslant 2n_j} \int_{T-H}^{T+H} \left| \sum_{n_j<m\leqslant k} \frac{1}{(m+\alpha)^{it}} \right|^2 \mathrm{d}t \\
&\ll_\alpha H \log^2 T \sum_{j}{}_2 n_j^{1-2\sigma} + H \log T \sum_{j}{}_2 \sum_{l\leqslant T_1} T^\kappa l^\kappa n_j^{\lambda-2\kappa} \\
&\ll_\alpha HT^{\varepsilon(1-2\sigma)} \log^3 T + HT^\kappa H^{-\kappa-1} \log^\kappa T \sum_{j}{}_2 n_j^{1+\lambda-\kappa-2\sigma} \\
&\ll_\alpha H + HT^{(1+\kappa+\lambda-2\sigma)/2} H^{-\kappa-1} \ll_\alpha H
\end{aligned} \tag{20}$$

if we take the exponential pair $(\kappa, \lambda) = (11/30, 16/30)$ (see [6]) and $1/2 < \sigma \leqslant 7/12$.

The application of Lemmas 6 and 7 shows that

$$\begin{aligned}
&T^{1-2\sigma} \log T \sum_{j}{}_2 \frac{1}{n_j^{2-2\sigma}} \max_{n_j<k\leqslant 2n_j} \int_{T-H}^{T+H} \left| \sum_{n_j<m\leqslant k} \frac{\mathrm{e}^{-2\pi i\alpha m}}{m^{-it}} \right|^2 \mathrm{d}t \\
&\ll \log T \sum_{j}{}_2 \frac{1}{n_j^{2\sigma}} \max_{n_j<k\leqslant 2n_j} \int_{T-H}^{T+H} \left| \sum_{n_j<m\leqslant k} \frac{\mathrm{e}^{-2\pi i\alpha m}}{m^{-it}} \right|^2 \mathrm{d}t \\
&\ll H \log^2 T \sum_{j}{}_2 n_j^{1-2\sigma} + H \log T \sum_{j}{}_2 \sum_{l\leqslant T_1} T^\kappa l^\kappa n_j^{\lambda-2\kappa} \ll H.
\end{aligned} \tag{21}$$

Now, (15), (17), (18), (20) and (21) yield the assertion of the theorem. □

6. Conclusions

Let $\zeta(s, \alpha)$, $s = \sigma + it$, denote the Hurwitz zeta-function,

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s}, \quad \sigma > 1,$$

with parameter $0 < \alpha \leqslant 1$. In this paper, we obtained the bound

$$\int_{T-H}^{T+H} |\zeta(\sigma+it)|^2 \, \mathrm{d}t \ll_{\sigma,\alpha} H$$

for fixed $1/2 < \sigma \leqslant 7/12$ and $T^{27/82} \leqslant H \leqslant T^{7/12}$. The latter estimate will be applied to prove the universality of $\zeta(s, \alpha)$ in short intervals. More precisely, this can prove that any analytic in the $D = \{1/2 < \sigma \leqslant 7/12\}$ function $f(s)$ can be approximated by shifts $\zeta(s + i\tau, \alpha)$, and

$$\liminf_{T\to\infty} \frac{1}{H} \mathrm{meas} \left\{ \tau \in [T, T+H] : \sup_{s\in K} |\zeta(s+i\tau, \alpha) - f(s)| < \varepsilon \right\} > 0$$

with $T^{27/82} \leqslant H \leqslant T^{7/12}$, a compact set $K \subset D$ and $\varepsilon > 0$. This is a certain step toward the effectivization of universality for the function $\zeta(s, \alpha)$. The universality of the Riemann zeta-function in short intervals was obtained in [21] (see also [22]). The main result of this paper can be generalized for more general zeta-functions, for example, for periodic and periodic Hurwitz zeta-functions. This will be focused on in our future papers.

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References

1. Hurwitz, A. Einige Eigenschaften der Dirichletschen Funktionen $F(s) = \sum \left(\frac{D}{n}\right) \frac{1}{n^s}$, die bei der Bestimmung der Klassenanzahlen binärer quadratischer Formen auftreten. *Z. Math. Phys.* **1882**, *27*, 86–101.
2. Reynolds, R.; Stauffer, A. Infinite sum of the incomplete gamma-functon expressed in terms of the Hurwitz zeta-function. *Mathematics* **2021**, *9*, 1952. [CrossRef]
3. Schwinger, J. On gauge invariance and vacuum polarization. *Phys. Rev.* **1951**, *82*, 664–779. [CrossRef]
4. Laurinčikas, A.; Garunkštis, R. *The Lerch Zeta-Function*; Kluwer Academic Publishers: Dordrecht, The Netherlands; Boston, MA, USA; London, UK, 2002.
5. Apostol, T.M. *Introduction to Analytic Number Theory*; Springer: New York, NY, USA; Berlin/Heidelberg, Germany, 1976.
6. Ivič, A. *The Riemann Zeta-Function*; John Wiley & Sons: New York, NY, USA, 1985.
7. Davenport, H.; Heilbronn, H. On the zeros of certain Dirichlet series. *J. Lond. Math. Soc.* **1936**, *11*, 181–185. [CrossRef]
8. Cassels, J.W.S. Footnote to a note of Davenport and Heilbronn. *J. Lond. Math. Soc.* **1961**, *36*, 177–184. [CrossRef]
9. Gonek, S.M. Analytic Properties of Zeta and L-Functions. Ph.D. Thesis, University of Michigan, Ann Arbor, MI, USA, 1979.
10. Bagchi, B. The Statistical Behaviour and Universality Properties of the Riemann Zeta-Function and Other Allied Dirichlet Series. Ph.D. Thesis, Indian Statistical Institute, Calcutta, India, 1981.
11. Voronin, S.M. Theorem on the "universality" of the Riemann zeta-function. *Math. USSR Izv.* **1975**, *9*, 443–453. [CrossRef]
12. Laurinčikas, A.; Meška, L. On the modification of the universality of the Hurwitz zeta-function. *Nonlinear Anal. Model. Control.* **2016**, *21*, 564–576. [CrossRef]
13. Mauclaire, J.-L. Universality of the Riemann zeta-function: Two remarks. *Ann. Univ. Sci. Budapest. Sect. Comp.* **2013**, *39*, 311–319.
14. Sourmelidis, A.; Steuding, J. On the value distribution of Hurwitz zeta-function with algebraic irrational parameter. *Constr. Approx.* **2022**, *55*, 829–860. [CrossRef]
15. Lerch, M. Note sur la fonction $K(w, x, s) = \sum_{n \geqslant 0} \exp\{2\pi i n x\}(n + w)^{-s}$. *Acta Math.* **1887**, *11*, 19–24. [CrossRef]
16. Lipschitz, R. Untersuchung einer aus vier Elementen gebildeten Reihe. *J. Reine Angew. Math.* **1889**, *105*, 127–185. [CrossRef]
17. Garunkštis, R.; Laurinčikas, A.; Steuding, J. An approximate functional equation for Lerch zeta-function. *Math. Notes* **2003**, *74*, 469–476. [CrossRef]
18. Rane, V.V. On the mean square value of Dirichlet L-Functions. *J. Lond. Math. Soc.* **1980**, *21*, 203–215. [CrossRef]
19. Montgomery, H.L.; Vaughan, R.C. Hilbert's inequality. *J. Lond. Math. Soc.* **1974**, *8*, 73–82. [CrossRef]
20. Ramachandra, K. *On the Mean-Value and Omega-Theorems for the Riemann Zeta-Function*; Tata Institute of Fundamental Research Lectures on Mathematics and Physics, 85; Springer: Berlin, Germany, 1995.
21. Laurinčikas, A. Universality of the Riemann zeta-function in short intervals. *J. Number Theory* **2019**, *204*, 279–295. [CrossRef]
22. Anderson, J.; Garunkštis, R.; Kačinskaitė, R.; Nakai, K.; Pańkowski, Ł.; Sourmelidis, A.; Steuding, R.; Steuding, J.; Wananiyakul, S. Notes on universality in short intervals and exponential shifts. *Lith. Math. J.* **2024**, *64*, 125–137. [CrossRef]