Research Article

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On generalized shifts of the Mellin transform of the Riemann zeta-function

https://doi.org/10.1515/math-2024-0055
received June 12, 2024; accepted August 5, 2024

Abstract: In this article, we consider the asymptotic behaviour of the modified Mellin transform $\mathcal{Z}(s)$, $s = \sigma + it$, of the Riemann zeta-function using weak convergence of probability measures in the space of analytic functions defined by means of shifts $\mathcal{Z}(s + i\varphi(\tau))$, where $\varphi(\tau)$ is a real increasing to $+\infty$ differentiable function with monotonically decreasing derivative satisfying a certain estimate connected to the second moment of $\mathcal{Z}(s)$. We prove in this case that the limit measure is concentrated at the point $g_0(s) \equiv 0$. This result is applied to the approximation of $g_0(s)$ by shifts $\mathcal{Z}(s + i\varphi(\tau))$.

Keywords: approximation of analytic functions, limit theorem, Mellin transform, Riemann zeta-function, weak convergence

MSC 2020: 11M06

1 Introduction

In function theory, for the investigation of complicated functions, various transforms are widely applied. Sometimes it is convenient to calculate a certain transform of the considered function and then, using the inverse formula, obtain desired information on the initial function. Therefore, investigation of various types of transforms is an important problem of function theory and has numerous theoretical and practical applications.

In this article, we are interested in some Mellin transforms. Denote by $s = \sigma + it$, $\sigma, t \in \mathbb{R}$, a complex variable. Let $g(x)x^{\sigma-1} \in L(0, \infty)$, where $L(0, \infty)$ denotes the space of integrable over $(0, \infty)$ functions. The Mellin transform $G(s)$ of $g(x)$ is defined by

$$G(s) = \int_0^\infty g(x)x^{s-1}\mathrm{d}x.$$

Putting $y = \mathrm{e}^x$ gives

$$G(\sigma + it) = \int_{-\infty}^\infty \mathrm{e}^{ixt}g(\mathrm{e}^x)\mathrm{e}^{\sigma x}\mathrm{d}x,$$

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thus, $G(\sigma + it)$ is the Fourier transform of $g(e^x)e^{\sigma x}$, i.e., Mellin transforms are a partial case of Fourier transforms. If, additionally, $g(x)$ is of bounded variation in every finite interval, then the inverse formula

$$\frac{g(x+0)+g(x-0)}{2} = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} G(s)x^{-s}ds = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\sigma-iT}^{\sigma+iT} G(s)x^{-s}ds$$

is valid.

In analytic number theory, the so-called modified Mellin transforms

$$\widehat{G}(s) = \int_1^\infty g(x)x^{-s}dx$$

are very useful for the investigation of moments of zeta-functions. Note that $\widehat{G}(s)$ has a certain advantage against $G(x)$ because in $\widehat{G}(s)$ there is not a convergence problem at the point $x = 0$. Moreover, the transforms $G(s)$ and $\widehat{G}(s)$ are closely connected. Let

$$\widehat{g}(x) = \begin{cases} g(1/x), & \text{if } 0 < x \leqslant 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then, it is easily seen that the modified Mellin transform of $g(x)$ coincides with the classical Mellin transform of the function $x^{-1}\widehat{g}(x)$ [1].

One of the main objects of analytic number theory is the Riemann zeta-function $\zeta(s)$, which, for $\sigma > 1$, is defined by

$$\zeta(s) = \sum_{m=1}^\infty \frac{1}{m^s},$$

or by the Euler product

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

over primes. Moreover, $\zeta(s)$ has analytic continuation to the whole complex plane, except for a simple pole at the point $s = 1$. To be precise, the first modified Mellin transform was the function

$$\mathcal{Z}_2(s) = \int_1^\infty \left|\zeta\left(\frac{1}{2} + ix\right)\right|^4 x^{-s}dx$$

introduced by Motohashi in [2] in connection with the fourth power moment

$$M_2(T) \stackrel{\text{def}}{=} \int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^4 dt, \quad T \to \infty,$$

(see also [3,4]). Analytic properties and estimates of the function $\mathcal{Z}_2(s)$ lead to significant results for $M_2(T)$. For example, using a method involving $\mathcal{Z}_2(s)$, the bound

$$E_2(T) \ll_\varepsilon T^{2/3+\varepsilon}$$

with every $\varepsilon > 0$ was obtained, where

$$E_2(T) = M_2(T) - TP(\log T)$$

with a polynomial P of degree 4. Here and in what follows, ε is an arbitrary fixed positive number, in different accuracies not the same, and $x \ll_\varepsilon y$, $x \in \mathbb{C}$, $y > 0$, means that there exists a constant $c = c(\varepsilon)$ such that $|x| \leqslant cy$. These examples show the importance of the modified Mellin transforms in the theory of $\zeta(s)$. In general, the modified Mellin transforms of the Riemann zeta-function are interesting analytic objects, and this is confirmed by previous studies [5–10].

In the article, we continue the study of asymptotic behaviour of the function

$$\mathcal{Z}(s) \overset{\text{def}}{=} \int_1^\infty \left|\zeta\left(\frac{1}{2}+ix\right)\right|^2 x^{-s}\mathrm{d}x, \quad \sigma > 1.$$

The analytic theory of $\mathcal{Z}(s)$ is given in [4]; Section 3. We recall some results from [4] that will be used below. Denote by γ the Euler constant and set

$$E(T) = \int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right| \mathrm{d}t - T\log\frac{T}{2\pi} - (2\gamma-1)T,$$

$$F(T) = \int_1^T E(t)\mathrm{d}t - T\pi, \qquad F_1(T) = \int_1^T F(t)\mathrm{d}t.$$

Then, [4] $\mathcal{Z}(s)$ has analytic continuation to the half-plane $\sigma > -3/4$ except for a double pole at the point $s = 1$. Moreover, the representation

$$\mathcal{Z}(s) = \frac{1}{(s-1)^2} + \frac{a}{s-1} - E(1) + \pi(s+1) + s(s+1)(s+2)\int_1^\infty F_1(x)x^{-s-3}\mathrm{d}x \tag{1}$$

with $a = 2\gamma - \log 2\pi$ holds. Also, the estimate

$$\mathcal{Z}(\sigma+it) \ll_\varepsilon t^{1-\sigma+\varepsilon}, \quad 0 \leqslant \sigma \leqslant 1,\ t \geqslant t_0 > 0,$$

and the mean square estimate

$$I_\sigma(T) \overset{\text{def}}{=} \int_1^T |\mathcal{Z}(\sigma+it)|^2\mathrm{d}t \ll_{\varepsilon,\sigma} T^{2-2\sigma+\varepsilon} \tag{2}$$

are valid. In [11], it was shown that

$$I_\sigma(T) \gg_\varepsilon T^{2-2\sigma-\varepsilon}, \quad \frac{1}{2} \leqslant \sigma \leqslant 1. \tag{3}$$

By works of Bohr-Jessen in the beginning of the last century [12–18], it is known that a chaotic behaviour of the Riemann zeta-function and other Dirichlet series is governed by probabilistic laws. Roughly speaking, this means that, for $\sigma > 1/2$, for some classes of sets A, the density

$$\frac{1}{T}\mathrm{m}\{t \in [0,T] : \zeta(\sigma+it) \in A\},$$

where mA is a certain measure on $\mathbb{R}$, has a limit as $T \to \infty$. For this, it is convenient to use the weak convergence of probability measures and its theory. Thus, let $\mathcal{B}(\mathbb{X})$ denote the Borel σ-field of the space $\mathbb{X}$ (in the general case, topological), and P_T and P be the probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. By the definition, P_T converges weakly to P as $T \to \infty$ $\left(P_T \underset{T\to\infty}{\overset{\mathrm{w}}{\to}} P\right)$ if, for every real continuous bounded function g on $\mathbb{X}$,

$$\lim_{T\to\infty}\int_{\mathbb{X}} g\mathrm{d}P_T = \int_{\mathbb{X}} g\mathrm{d}P.$$

Let $\mathcal{L}A$ stand for the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then, the modern Bohr-Jessen theorem is stated as follows: for every fixed $\sigma > 1/2$,

$$\frac{1}{T}\mathcal{L}\{t \in [0,T] : \zeta(\sigma+it) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to a certain probability measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ as $T \to \infty$.

For $\sigma = 1/2$, a certain normalization of the function $\zeta(1/2+it)$ is needed [14,15].

In [19] and [20], a similar approach was applied to the function $\mathcal{Z}(\sigma + it)$; however, as it was observed in [11], the limit measure is degenerated at the point $s = 0$. Therefore, in [11], in place of

$$\frac{1}{T}\mathcal{L}\{t \in [0, T] : \mathcal{Z}(\sigma + it) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}),$$

for $1/2 < \sigma < 1$, the probability measure

$$\frac{1}{T}\mathcal{L}\{t \in [T, 2T] : \mathcal{Z}(\sigma + i\varphi(t)) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}), \tag{4}$$

with $\varphi(t) \in W_\sigma$ was considered. The class W_σ consists of increasing to $+\infty$ differentiable functions $\varphi(t)$ with monotonically decreasing derivative $\varphi'(t)$ such that

$$\frac{I_{\sigma-\varepsilon}(\varphi(T))}{\varphi'(T)} \ll T, \quad T \to \infty. \tag{5}$$

We see that the shifts $\mathcal{Z}(\sigma + i\varphi(t))$ in (4) depend on σ because, by (5), $\varphi(t)$ is connected to σ.

Our aim is to extend the aforementioned results to the space of analytic functions. We note that limit theorems in the space of analytic functions have a certain advantage against those in the space $\mathbb{C}$ because they are closely connected to approximation of analytic functions by shifts $\mathcal{Z}(s + i\varphi(\tau))$. Let $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$. Denote by $\mathbb{H} = \mathbb{H}(D)$ the space of analytic on D functions endowed with the topology of uniform convergence on compacts. We will deal with weak convergence, as $T \to \infty$, for

$$P_{T,\mathbb{H}}(A) = \frac{1}{T}\mathcal{L}\{\tau \in [T, 2T] : \mathcal{Z}(s + i\varphi(\tau)) \in A\}, \quad A \in \mathcal{B}(\mathbb{H}).$$

Differently, from (4), the function $\varphi(\tau)$ must be the same for all $s \in D$. Therefore, we replace hypothesis (5) by

$$\sup_{\sigma\in(1/2,1)} \frac{I_\sigma(2\varphi(2T))}{T\varphi'(2T)} \ll 1, \quad T \to \infty. \tag{6}$$

Let V be the class of positive increasing to $+\infty$ differentiable functions $\varphi(\tau)$ on $[T_0, \infty)$, $T_0 \gg 1$, with decreasing derivative $\varphi'(\tau)$ satisfying (6). For example, the function $\varphi(\tau) = \exp\{(\log\log\tau)^\alpha\}$, $\alpha > 1$, $\tau \geqslant e^2$, satisfies hypotheses of class V.

This article is organized as follows. In Section 2, we discuss some types of convergence and state the main theorem. In Section 3, we prove limit lemmas in the space $\mathbb{H}$ for some objects connected to the function $\mathcal{Z}(s)$, including an absolutely convergent integral. Theorems 1 and 2 are proved in Section 4.

2 Some convergence remarks

Let $(\mathbb{X}, d)$ be a certain metric space and X a $\mathbb{X}$-valued random element defined on the probability space $(\Omega, \mathcal{A}, \mu)$. We say that the random element X is degenerated at the point $x \in \mathbb{X}$ if

$$\mu\{X \in A\} = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise}, \end{cases} \quad A \in \mathcal{B}(\mathbb{X}). \tag{7}$$

Let X_n, $n \in \mathbb{N}$, be $\mathbb{X}$-valued random elements, and $x \in \mathbb{X}$. If, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mu\{d(X_n, x) \geqslant \varepsilon\} = 0,$$

then we say that X_n converges in probability to x as $n \to \infty$ $\left(X_n \xrightarrow[n\to\infty]{\mathrm{P}} x\right)$. Moreover, X_n, as $n \to \infty$, converges to X in distribution $\left(X_n \xrightarrow[n\to\infty]{\mathcal{D}} X\right)$ if the distribution $\mu\{X_n \in A\}$, $A \in \mathcal{B}(\mathbb{X})$, as $n \to \infty$, converges weakly to the distribution $\mu\{X \in A\}$, $A \in \mathcal{B}(\mathbb{X})$.

It is well known, see Section 1.4 of [21], that $X_n \xrightarrow[n\to\infty]{\mathrm{P}} x$ is equivalent to $X_n \xrightarrow[n\to\infty]{\mathcal{D}} X$, when X has the distribution (7).

Let $P_{0,\mathbb{H}}$ be a probability measure on $(\mathbb{H}, \mathcal{B}(\mathbb{H}))$,

$$P_{0,\mathbb{H}}(A) = \begin{cases} 1, & \text{if } g_0 \in A, \\ 0, & \text{otherwise,} \end{cases}$$

i.e., the mass of $P_{0,\mathbb{H}}$ is concentrated at the point $g_0(s) \equiv 0$ of the space $\mathbb{H}$. We will consider the weak convergence $P_{T,\mathbb{H}} \xrightarrow[T\to\infty]{\mathrm{w}} P_{0,\mathbb{H}}$. For this, we recall a metric on the space $\mathbb{H}$. Let $\{K_j : j \in \mathbb{N}\} \subset D$ be a sequence of compact embedded sets satisfying the conditions:

(1)

$$\bigcup_{j=1}^{\infty} K_j = D;$$

(2) If $K \subset D$ is a compact set, then $K \subset K_j$ for some j.

We observe that such a sequence $\{K_j\}$ always exists. For example, we may take rectangles with edges parallel to the axes.

Now, for $f_1, f_2 \in \mathbb{H}$, set

$$\rho(f_1, f_2) = \sum_{j=1}^{\infty} \frac{1}{2^j} \rho_j(f_1, f_2),$$

where

$$\rho_j(f_1, f_2) = \frac{\sup_{s\in K_j} |f_1(s) - f_2(s)|}{1 + \sup_{s\in K_j} |f_1(s) - f_2(s)|}.$$

Then, ρ is the desired metric on the space $\mathbb{H}$ inducing its topology of uniform convergence on compact sets.

The aforementioned remarks imply that the relation $P_{T,\mathbb{H}} \xrightarrow[T\to\infty]{\mathrm{w}} P_{0,\mathbb{H}}$ holds if, for every $\varepsilon > 0$,

$$\lim_{T\to\infty} \frac{1}{T} \mathcal{L}\{\tau \in [T, 2T] : \rho(\mathcal{Z}(s + i\varphi(\tau)), 0) \geqslant \varepsilon\} = 0. \tag{8}$$

In view of the Chebyshev-type inequality, we have

$$\frac{1}{T} \mathcal{L}\{\tau \in [T, 2T] : \rho(\mathcal{Z}(s + i\varphi(\tau)), 0) \geqslant \varepsilon\} \leqslant \frac{1}{T\varepsilon} \int_T^{2T} \rho(\mathcal{Z}(s + i\varphi(\tau)), 0)\mathrm{d}\tau.$$

Hence, by the definition of the metric ρ, the left-hand side of (8) does not exceed

$$\frac{1}{\varepsilon} \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{1}{T} \int_T^{2T} \sup_{s\in K_j} |\mathcal{Z}(s + i\varphi(\tau))| \mathrm{d}\tau.$$

Thus, equality (8) is true in the case when, for all $j \in \mathbb{N}$,

$$\int_T^{2T} \sup_{s\in K_j} |\mathcal{Z}(s + i\varphi(\tau))| \mathrm{d}\tau = o(T), \quad T \to \infty. \tag{9}$$

If $\varphi(\tau) = \tau$, then the later estimate is true. Actually, let l_j be a simple closed contour enclosing set K_j lying in the strip D, and

$$\inf_{s\in K_j} \inf_{z\in l_j} |s - z| \gg_{l_j} 1.$$

By the Cauchy integral formula, we have

$$\mathcal{Z}(s + i\tau) \ll \int_{l_j} \frac{|\mathcal{Z}(z + i\tau)|}{|z - s|} |\mathrm{d}z|.$$

Thus,

$$\sup_{s\in K_j}|\mathcal{Z}(s+i\tau)| \ll_{l_j} \int_{l_j}|\mathcal{Z}(z+i\tau)|\,|\mathrm{d}z|.$$

Therefore, for $z=\sigma+it$, $1/2<\sigma<1$, an application of (2) gives

$$\begin{aligned}\int_T^{2T}\sup_{s\in K_j}|\mathcal{Z}(s+i\tau)|\mathrm{d}\tau &\ll_{l_j} \int_{l_j}\int_T^{2T}|\mathcal{Z}(z+i\tau)|\mathrm{d}\tau|\mathrm{d}z| \ll_{l_j} \int_{l_j}\left(T\int_T^{2T}|\mathcal{Z}(z+i\tau)|^2\mathrm{d}\tau\right)^{1/2}|\mathrm{d}z| \\ &\ll_{l_j} \int_{l_j}\left(T\int_{T-|t|}^{2T+|t|}|\mathcal{Z}(\sigma+i\tau)|^2\mathrm{d}\tau\right)^{1/2}|\mathrm{d}z| \ll_{l_j}\int_{l_j}T^{1/2}((2T+|t|)^{2-2\sigma+\varepsilon})^{1/2}|\mathrm{d}z| \\ &\ll_{l_j,\varepsilon} T^{1-\delta}, \quad \delta>0,\end{aligned}$$

because $\sigma>1/2$, and $\varepsilon>0$ is arbitrary, and we obtain (9).

We will prove the following statement.

Theorem 1. *Suppose that $\varphi\in V$, where V is the class defined in p. 4. Then, $P_{T,\mathbb{H}}$ converges weakly to $P_{0,\mathbb{H}}$ as $T\to\infty$.*

We observe that the theorem does not follow directly from the estimate for the second moment. Let $\kappa=\min_{z\in K_j}\{\mathrm{Re}z\}$. Similar to the aforementioned arguments, properties of the function $\varphi(\tau)$ imply that, for $\min_{z\in l_{j_0}}\{z\in l_{j_0}\}=\kappa-\varepsilon$,

$$\begin{aligned}\frac{1}{T}\int_T^{2T}\sup_{s\in K_j}|\mathcal{Z}(s+i\varphi(\tau))|\mathrm{d}\tau &\ll_{l_{j_0}} \int_{l_{j_0}}\left(\frac{1}{T}\int_T^{2T}|\mathcal{Z}(z+i\varphi(\tau))|^2\mathrm{d}\tau\right)^{1/2}|\mathrm{d}z| \\ &\ll_{l_{j_0}} \int_{l_{j_0}}\left(\int_T^{2T}|\mathcal{Z}(\kappa-\varepsilon+i\mathrm{Im}z+i\varphi(\tau))|^2\,\mathrm{d}\tau\right)^{1/2}|\mathrm{d}z| \\ &\ll_{l_{j_0}} \int_{l_{j_0}}\left(\frac{1}{T}\,|\mathcal{Z}(\kappa-\varepsilon+i\mathrm{Im}z+i\varphi(\tau))|^2\,\frac{1}{\varphi'(\tau)}\mathrm{d}(\varphi(\tau)+\mathrm{Im}z)\right)^{1/2}|\mathrm{d}z| \\ &\ll_{l_{j_0}} \int_{l_{j_0}}\left(\frac{1}{\varphi'(2T)}\frac{1}{T}\int_{T-|\mathrm{Im}z|}^{\varphi(2T)+|\mathrm{Im}z|}|\mathcal{Z}(\kappa-\varepsilon+iu)|^2\mathrm{d}u\right)^{1/2}|\mathrm{d}z| \\ &\ll_{l_{j_0}} \int_{l_{j_0}}\left(\frac{1}{T\varphi'(2T)}I_{\kappa-\varepsilon}(2\varphi(2T))\right)^{1/2}|\mathrm{d}z| \\ &\ll_{l_{j_0}} \left(\frac{1}{T\varphi'(2T)}I_{\kappa-\varepsilon}(2\varphi(2T))\right)^{1/2} \\ &\ll_{l_{j_0}} \sup_{\sigma\in(1/2,1)}\left(\frac{1}{T\varphi'(2T)}I_\sigma(2\varphi(2T))\right) \ll_{l_{j_0}} 1.\end{aligned}$$

However, this does not ensure that

$$\frac{1}{T}\int_T^{2T}\sup_{s\in K_j}|\mathcal{Z}(s+i\varphi(\tau))|\mathrm{d}\tau \underset{T\to\infty}{\to} 0.$$

For $A \in \mathcal{B}(\mathbb{C})$, define

$$P_{T,\mathbb{C}}(A) = \frac{1}{T}\mathcal{L}\{\tau \in [T, 2T] : \mathcal{Z}(\sigma + i\varphi(\tau)) \in A\}$$

and

$$P_{0,\mathbb{C}}(A) = \begin{cases} 1, & \text{if } 0 \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Then, we have the following.

Proposition 2. *Suppose that* $1/2 < \sigma < 1$ *is fixed. Then, the weak convergence of* $P_{T,\mathbb{H}}$ *to* $P_{0,\mathbb{H}}$ *implies that of* $P_{T,\mathbb{C}}$ *to* $P_{0,\mathbb{C}}$ *as* $T \to \infty$.

Proof. Consider the mapping $h : \mathbb{H} \to \mathbb{C}$ defined by

$$h(g) = g(\sigma), \quad g(s) \in \mathbb{H}(D), \; s = \sigma + it.$$

Moreover, we have, for $\forall A \in \mathcal{B}(\mathbb{C})$,

$$\begin{aligned} \frac{1}{T}\mathcal{L}\{\tau \in [T, 2T] : |\mathcal{Z}(\sigma + i\varphi(\tau))| \in A\} &= \frac{1}{T}\mathcal{L}\{\tau \in [T, 2T] : h(\mathcal{Z}(\sigma + i\varphi(\tau))) \in A\} \\ &= \frac{1}{T}\mathcal{L}\{\tau \in [T, 2T] : |\mathcal{Z}(\sigma + i\varphi(\tau))| \in h^{-1}A\}. \end{aligned}$$

Thus, by the definitions of $P_{T,\mathbb{H}}$ and $P_{T,\mathbb{C}}$, it follows that $P_{T,\mathbb{C}} = P_{T,\mathbb{H}}h^{-1}$, where

$$P_{T,\mathbb{H}}h^{-1}(A) = P_{T,\mathbb{C}}(h^{-1}A), \quad A \in \mathcal{B}(\mathbb{C}).$$

Therefore, the relation $P_{T,\mathbb{H}} \xrightarrow[T\to\infty]{\mathrm{w}} P_{0,\mathbb{H}}$ and the principle of preservation of weak convergence under continuous mappings, see Section 1.5 of [21], show that

$$P_{T,\mathbb{C}} \xrightarrow[T\to\infty]{\mathrm{w}} P_{0,\mathbb{H}}h^{-1}.$$

By the definition of $P_{0,\mathbb{H}}$,

$$P_{0,\mathbb{H}}h^{-1}(A) = P_{0,\mathbb{H}}(h^{-1}A) = \begin{cases} 1, & \text{if } g_0 \in h^{-1}A, \\ 0, & \text{otherwise,} \end{cases} = \begin{cases} 1, & \text{if } 0 \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Hence,

$$P_{0,\mathbb{H}}h^{-1} = P_{0,\mathbb{C}}.$$

From the latter remark, we obtain that if $P_{T,\mathbb{C}}$ does not converge weakly to $P_{0,\mathbb{C}}$, then $P_{T,\mathbb{H}}$ does not converge weakly to $P_{0,\mathbb{H}}$ as $T \to \infty$ as well. □

Theorem 1 can be applied to the approximation of the function $g_0(s)$ by shifts $\mathcal{Z}(s + i\varphi(\tau))$.

Theorem 3. *Suppose that* $\varphi \in V$. *Then, for arbitrary compact set* $K \subset D$ *and* $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T}\mathcal{L}\left\{\tau \in [T, 2T] : \sup_{s\in K}|\mathcal{Z}(s + i\varphi(\tau))| < \varepsilon\right\} > 0.$$

Moreover, the limit

$$\lim_{T\to\infty} \frac{1}{T}\mathcal{L}\left\{\tau \in [T, 2T] : \sup_{s\in K}|\mathcal{Z}(s + i\varphi(\tau))| < \varepsilon\right\}$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

As it was observed in p. 4, the function $\varphi(\tau) = \exp\{(\log\log\tau)^\alpha\}$, $\alpha > 1$, $\tau \geqslant \mathrm{e}^2$, satisfies the hypotheses of Theorems 1 and 3.

3 Limit lemmas for integrals

Proofs of all statements on weakly convergent probability measures are based on a theorem from [21] on convergence in distribution $\left(\xrightarrow{\mathcal{D}}\right)$. For convenience of application, we state this theorem as a separate lemma.

Lemma 4. [21, Theorem 4.2] *Suppose that the metric space* $(\mathbb{X}, d)$ *is separable, and* $\mathbb{X}$*-valued random elements* X_{nk} *and* Y_n, $n, k \in \mathbb{N}$, *are defined on a certain probability space* $(\Omega, \mathcal{A}, \mu)$. *Let, for all* $k \in \mathbb{N}$,

$$X_{nk} \underset{n\to\infty}{\xrightarrow{\mathcal{D}}} X_k$$

and

$$X_k \underset{k\to\infty}{\xrightarrow{\mathcal{D}}} X.$$

Moreover, for every $\varepsilon > 0$,

$$\lim_{k\to\infty} \limsup_{n\to\infty} \mu\{d(X_{nk}, Y_n) \geqslant \varepsilon\} = 0.$$

Then,

$$Y_n \underset{n\to\infty}{\xrightarrow{\mathcal{D}}} X.$$

Let $\eta > 1/2$ be a fixed number, and, for $x, y \in [1, \infty)$,

$$a(x, y) = \exp\left\{-\left(\frac{x}{y}\right)^{\eta}\right\}.$$

For brevity, use the notations

$$\zeta_1(x) = \left|\zeta\left(\frac{1}{2} + ix\right)\right|^2$$

and

$$\mathcal{Z}_y(s) = \int_1^{\infty} \zeta_1(x) a(x, y) x^{-s} \mathrm{d}x.$$

Thus, $\mathcal{Z}_y(s)$ is the modified Mellin transform of the function $\zeta_1(x)a(x, y)$. Since, by the classical estimate, $\zeta_1(x) \ll (1 + |x|)^{1/6}$ [22], and $a(x, y)$ decreases exponentially with respect to x, the integral for $\mathcal{Z}_y(s)$ is absolutely convergent in every half-plane $\sigma > \sigma_0$, $\sigma_0 < \infty$.

Let $b > 1$ be a fixed number. Our first step consists of a limit lemma for the integral

$$\mathcal{Z}_{b,y}(s) = \int_1^{b} \zeta_1(x) a(x, y) x^{-s} \mathrm{d}x.$$

Define

$$Q_{T,b,y}(A) = \frac{1}{T} \mathcal{L}\{\tau \in [T, 2T] : \mathcal{Z}_{b,y}(s + i\varphi(\tau)) \in A\}, \quad A \in \mathcal{B}(\mathbb{H}).$$

Lemma 5. *Suppose that* $\varphi \in V$. *Then, for every fixed* b *and* y, $Q_{T,b,y}$ *converges weakly to* $P_{0,\mathbb{H}}$ *as* $T \to \infty$.

Proof. It suffices to show that, for every compact set $K \subset D$,

$$\frac{1}{T} \int_T^{2T} \sup_{s\in K} |\mathcal{Z}_{b,y}(s + i\varphi(\tau))|^2 \mathrm{d}\tau = o(1), \quad T \to \infty. \tag{10}$$

An application of the Cauchy integral formula reduces the estimation of the aforementioned integral to that of

$$\frac{1}{T}\int_T^{2T} |\mathcal{Z}_{b,y}(\sigma + i\varphi(\tau))|^2 \mathrm{d}\tau,$$

with $\sigma > 1/2$. By the definition of $\mathcal{Z}_{b,y}(s)$, denoting by $\bar{z}$ the complex conjugate of $z \in \mathbb{C}$, we have

$$\begin{aligned}|\mathcal{Z}_{b,y}(\sigma + i\varphi(\tau))|^2 &= \mathcal{Z}_{b,y}(\sigma + i\varphi(\tau))\overline{\mathcal{Z}_{b,y}(\sigma + i\varphi(\tau))} \\ &= \int_1^b \zeta_1(x)a(x,y)x^{-\sigma - i\varphi(\tau)}\mathrm{d}x \int_1^b \zeta_1(x)a(x,y)x^{-\sigma + i\varphi(\tau)}\mathrm{d}x \\ &= \left(\underset{x_1 = x_2}{\int_1^b\int_1^b} + \underset{x_1 \neq x_2}{\int_1^b\int_1^b}\right)\zeta_1(x_1)\zeta_1(x_2)a(x_1,y)a(x_2,y)x_1^{-\sigma - i\varphi(\tau)}x_2^{-\sigma + i\varphi(\tau)}\mathrm{d}x_1\mathrm{d}x_2.\end{aligned}$$

Thus,

$$\frac{1}{T}\int_T^{2T} |\mathcal{Z}_{b,y}(\sigma + i\varphi(\tau))|^2 \mathrm{d}\tau = \frac{1}{T}\underset{x_1 \neq x_2}{\int_1^b\int_1^b}\left(\zeta_1(x_1)\zeta_1(x_2)a(x_1,y)a(x_2,y)(x_1x_2)^{-\sigma}\int_T^{2T}\left(\frac{x_2}{x_1}\right)^{i\varphi(\tau)}\mathrm{d}\tau\right)\mathrm{d}x_1\mathrm{d}x_2. \tag{11}$$

By the second mean value theorem,

$$\begin{aligned}\operatorname{Re}\int_T^{2T}\left(\frac{x_2}{x_1}\right)^{i\varphi(\tau)}\mathrm{d}\tau &= \int_T^{2T}\cos\left(\varphi(\tau)\log\left(\frac{x_2}{x_1}\right)\right)\mathrm{d}\tau = \left(\log\left(\frac{x_2}{x_1}\right)\right)^{-1}\int_T^{2T}\frac{1}{\varphi'(\tau)}\mathrm{d}\sin\left(\varphi(\tau)\log\left(\frac{x_2}{x_1}\right)\right) \\ &= \left(\log\left(\frac{x_2}{x_1}\right)\right)^{-1}\frac{1}{\varphi'(\tau)}\int_T^{\xi}\mathrm{d}\sin\left(\varphi(\tau)\log\left(\frac{x_2}{x_1}\right)\right) \ll \left|\left(\log\left(\frac{x_2}{x_1}\right)\right)\right|^{-1}\frac{1}{\varphi'(2T)},\end{aligned}$$

where $T \leqslant \xi \leqslant 2T$. The same estimate is valid for

$$\operatorname{Im}\int_T^{2T}\left(\frac{x_2}{x_1}\right)^{i\varphi(\tau)}\mathrm{d}\tau$$

as well. Therefore, by (11),

$$\frac{1}{T}\int_T^{2T} |\mathcal{Z}_{b,y}(\sigma + i\varphi(\tau))|^2 \mathrm{d}\tau \ll \frac{1}{T\varphi'(2T)}\underset{x_1 \neq x_2}{\int_1^b\int_1^b}\zeta_1(x_1)\zeta_1(x_2)a(x_1,y)a(x_2,y)(x_1x_2)^{-\sigma}\left|\left(\log\left(\frac{x_2}{x_1}\right)\right)\right|^{-1}\mathrm{d}x_1\mathrm{d}x_2. \tag{12}$$

In view of the lower bound (3), for $1/2 < \sigma < 1$,

$$I_\sigma(2T) \to \infty \quad \text{as} \quad T \to \infty.$$

Therefore, the definition of the class V implies that

$$\frac{1}{T\varphi'(2T)} = o(1), \quad T \to \infty.$$

This together with (12) shows that, for $1/2 < \sigma < 1$,

$$\frac{1}{T}\int_T^{2T} |\mathcal{Z}_{b,y}(\sigma + i\varphi(\tau))|^2 \mathrm{d}\tau = o(1), \quad T \to \infty.$$

Thus, (10) is true, and the lemma is proved. □

For $A \in \mathcal{B}(\mathbb{H})$, define

$$Q_{T,y}(A) = \frac{1}{T}\mathcal{L}\{\tau \in [T, 2T] : \mathcal{Z}_y(s + i\varphi(\tau)) \in A\}.$$

Lemma 6. *Suppose that $\varphi \in V$. Then, for every fixed y, $Q_{T,y}$ converges weakly to $P_{0,\mathbb{H}}$ as $T \to \infty$.*

Proof. Introduce a random variable ξ_T defined on a certain probability space $(\Omega, \mathcal{A}, \nu)$ and suppose that ξ_T is uniformly distributed in $[T, 2T]$. Define the $\mathbb{H}$-valued random element $x_{b,y} = x_{b,y}(s)$ having the distribution $P_{0,\mathbb{H}}$, and

$$x_{T,b,y} = x_{T,b,y}(s) = \mathcal{Z}_{b,y}(s + i\varphi(\xi_T)).$$

Then, by Lemma 5,

$$x_{T,b,y} \xrightarrow[T\to\infty]{\mathcal{D}} x_{b,y}. \tag{13}$$

Since the distribution of $x_{b,y}$ for all b and y is $P_{0,\mathbb{H}}$, we have

$$x_{b,y} \xrightarrow[b\to\infty]{\mathcal{D}} P_{0,\mathbb{H}}. \tag{14}$$

Define one more $\mathbb{H}$-valued random element

$$x_{T,y} = x_{T,y}(s) = \mathcal{Z}_y(s + i\varphi(\xi_T)).$$

Let $K \subset D$ be a compact set. Then, for $s \in K$,

$$\mathcal{Z}_y(s + i\varphi(\tau)) - \mathcal{Z}_{b,y}(s + i\varphi(\tau)) = \int_b^\infty \zeta_1(x)a(x, y)x^{-s-i\varphi(\tau)}\mathrm{d}\tau \ll \int_b^\infty \zeta_1(x)a(x, y)x^{-\mathrm{Re}s}\mathrm{d}x = o_y(1),$$

as $b \to \infty$. From this, we obtain that

$$\lim_{b\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_T^{2T}\sup_{s\in K}|\mathcal{Z}_y(s + i\varphi(\tau)) - \mathcal{Z}_{b,y}(s + i\varphi(\tau))|\mathrm{d}\tau \ll \lim_{b\to\infty} o_y(1) = 0.$$

Hence, for every $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{b\to\infty}\limsup_{T\to\infty}\nu\{\rho(x_{T,y}(s), x_{T,b,y}(s)) \geqslant \varepsilon\}\\
&\quad= \lim_{b\to\infty}\limsup_{T\to\infty}\frac{1}{T}\mathcal{L}\{\tau \in [T, 2T] : \rho(\mathcal{Z}_y(s + i\varphi(\tau)), \mathcal{Z}_{b,y}(s + i\varphi(\tau))) \geqslant \varepsilon\}\\
&\quad\leqslant \lim_{b\to\infty}\limsup_{T\to\infty}\frac{1}{\varepsilon T}\int_T^{2T}\sum_{j=1}^\infty 2^{-j}\sup_{s\in K_j}|\mathcal{Z}_y(s + i\varphi(\tau)) - \mathcal{Z}_{b,y}(s + i\varphi(\tau))| = 0.
\end{aligned}$$

This together with (13) and (14) shows that all hypotheses of Lemma 4 are fulfilled. Therefore,

$$x_{T,y} \xrightarrow[T\to\infty]{\mathcal{D}} P_{0,\mathbb{H}}, \tag{15}$$

and the lemma is proved. □

4 Proofs of theorems

The scheme of the proof of Theorem 1 is similar to that of used in the proof of Lemma 6; however, with full application of hypotheses for the function $\varphi(\tau)$, first, we recall the integral representation for $\mathcal{Z}_y(s)$.

Lemma 7. [19, Lemma 7] *Suppose that* $1/2 < \sigma < 1$. *Then,*

$$\mathcal{Z}_y(s) = \frac{1}{2\pi i}\int\limits_{\eta-i\infty}^{\eta+i\infty} \mathcal{Z}(s+z)g(z,y)\mathrm{d}z,$$

where

$$g(s,y) = \frac{1}{\eta}\Gamma\left(\frac{s}{\eta}\right)y^s,$$

and $\Gamma(s)$ *is the Euler gamma-function.*

Define the $\mathbb{H}$-valued random element $x_T = x_T(s)$ by

$$x_T(s) = \mathcal{Z}(s + i\varphi(\xi_T)),$$

where ξ_T, as in the proof of Lemma 6, is a random variable uniformly distributed on $[T, 2T]$.

Lemma 8. *Suppose that* $\varphi \in V$. *Then, for every* $\varepsilon > 0$,

$$\lim_{y\to\infty}\limsup_{T\to\infty} \nu\{\rho(x_T(s), x_{T,y}(s)) \geqslant \varepsilon\} = 0.$$

Proof. Using the uniform distribution of the random variable ξ_T, and the Chebyshev-type inequality, we find

$$\begin{aligned}\nu\{\rho(x_T(s), x_{T,y}(s)) \geqslant \varepsilon\} &= \frac{1}{T}\mathcal{L}\{\tau \in [T, 2T] : \rho(\mathcal{Z}(s+i\varphi(\tau)), \mathcal{Z}_y(s+i\varphi(\tau))) \geqslant \varepsilon\}\\ &\leqslant \frac{1}{\varepsilon T}\int\limits_T^{2T}\rho(\mathcal{Z}(s+i\varphi(\tau)), \mathcal{Z}_y(s+i\varphi(\tau)))\mathrm{d}\tau.\end{aligned}$$

Thus, by the definition of the metric ρ,

$$\nu\{\rho(x_T(s), x_{T,y}(s)) \geqslant \varepsilon\} \leqslant \frac{1}{\varepsilon T}\sum_{j=1}^{\infty}\frac{1}{2^j}\int\limits_T^{2T}\sup_{s\in K_j}|\mathcal{Z}(s+i\varphi(\tau)) - \mathcal{Z}_y(s+i\varphi(\tau))|\mathrm{d}\tau. \tag{16}$$

Properties of the class V show that $1/2 + 2\kappa \leqslant \sigma \leqslant 1-\kappa$, $\kappa \overset{\text{def}}{=} \kappa_j > 0$, for $s = \sigma + it \in K \overset{\text{def}}{=} K_j$. Let $\eta = 1/2 + \kappa$, and define $\eta_1 = \sigma - \kappa - 1/2$. Then, we have $\eta_1 > 0$ for all $s \in K$. Since the function $\Gamma(z)$ has a simple pole at the point $z = 0$ and the function $\mathcal{Z}(s+z)$ has a double pole at the point $z = 1-s$, the integrand in Lemma 7 has only the above poles in the strip $-\eta_1 < \mathrm{Re}z < \eta$. Therefore, Lemma 7 and the residue theorem, for all $s \in K$, yield

$$\mathcal{Z}_y(s) - \mathcal{Z}(s) = \frac{1}{2\pi i}\int\limits_{-\eta_1-i\infty}^{-\eta_1+i\infty}\mathcal{Z}(s+z)g(z,y)\mathrm{d}z + r_y(s),$$

with

$$r_y(s) = \mathrm{Res}_{z=1-s}\mathcal{Z}(s+z)g(z,y). \tag{17}$$

Hence, for all $s \in K$,

$$\begin{aligned}&\mathcal{Z}_y(s+i\varphi(\tau)) - \mathcal{Z}(s+i\varphi(\tau))\\ &= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\mathcal{Z}\left(\sigma + it - \sigma + \frac{1}{2} + \kappa + i\varphi(\tau) + iu\right)g\left(\frac{1}{2} + \kappa - \sigma + iu, y\right)\mathrm{d}u + r_y(s+i\varphi(\tau))\\ &= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\mathcal{Z}\left(\frac{1}{2} + \kappa + i\varphi(\tau) + iu\right)g\left(\frac{1}{2} + \kappa - s + iu, y\right)\mathrm{d}u + r_y(s+i\varphi(\tau))\\ &\ll \int\limits_{-\infty}^{\infty}\left|\mathcal{Z}\left(\frac{1}{2} + \kappa + i\varphi(\tau) + iu\right)\right|\sup_{s\in K}\left|g\left(\frac{1}{2} + \kappa - s + iu, y\right)\right|\mathrm{d}u + \sup_{s\in K}|r_y(s+i\varphi(\tau))|.\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{T}\int_T^{2T} \sup_{s\in K}|\mathcal{Z}_y(s+i\varphi(\tau))-\mathcal{Z}(s+i\varphi(\tau))|\mathrm{d}\tau \\
&\ll \int_{-\infty}^{\infty}\left(\frac{1}{T}\int_T^{2T}\left|\mathcal{Z}\left(\frac{1}{2}+\kappa+i\varphi(\tau)+iu\right)\right|\mathrm{d}\tau\right)\sup_{s\in K}\left|g\left(\frac{1}{2}+\kappa-s+iu,y\right)\right|\mathrm{d}u+\frac{1}{T}\int_T^{2T}\sup_{s\in K}|r_y(s+i\varphi(\tau))|\mathrm{d}\tau \\
&\stackrel{\text{def}}{=} J_1+J_2.
\end{aligned} \tag{18}$$

First, we estimate the quantity

$$A_T(u) \stackrel{\text{def}}{=} \frac{1}{T}\int_T^{2T}\left|\mathcal{Z}\left(\frac{1}{2}+\kappa+i\varphi(\tau)+iu\right)\right|\mathrm{d}\tau \leqslant \left(\frac{1}{T}\int_T^{2T}\left|\mathcal{Z}\left(\frac{1}{2}+\kappa+i\varphi(\tau)+iu\right)\right|^2\mathrm{d}\tau\right)^{1/2},$$

for all $u\in\mathbb{R}$. Using properties of the function $\varphi(\tau)$ gives

$$\begin{aligned}
A_T^2(u) &= \frac{1}{T}\int_T^{2T}\frac{1}{\varphi'(\tau)}\left|\mathcal{Z}\left(\frac{1}{2}+\kappa+i\varphi(\tau)+iu\right)\right|^2\mathrm{d}\varphi(\tau) \\
&\leqslant \frac{1}{T\varphi'(2T)}\int_T^{2T}\left|\mathcal{Z}\left(\frac{1}{2}+\kappa+i\varphi(\tau)+iu\right)\right|^2\mathrm{d}\varphi(\tau) \\
&\leqslant \frac{1}{T\varphi'(2T)}\int_{\varphi(T)-|u|}^{\varphi(2T)+|u|}\left|\mathcal{Z}\left(\frac{1}{2}+\kappa+iv\right)\right|^2\mathrm{d}v \\
&\ll_\kappa \frac{1}{T\varphi'(2T)}I_{1/2+\kappa}(\varphi(2T)+|u|).
\end{aligned}$$

If $|u|\leqslant\varphi(2T)$, then this yields

$$A_T^2(u)\ll_\kappa \frac{1}{T\varphi'(2T)}I_{1/2+\kappa}(2\varphi(2T))\ll_\kappa \sup_{\sigma\in(1/2,1)}\left(\frac{I_\sigma(2\varphi(2T))}{T\varphi'(2T)}\right)\ll_\kappa 1,$$

by the definition of class V. If $|u|>\varphi(2T)$, then, in view of (2),

$$A_T^2(u)\ll_\kappa \frac{1}{T\varphi'(2T)}I_{1/2+\kappa}(2|u|)\ll_\kappa \frac{1}{T\varphi'(2T)}|u|^{1-2\kappa+\varepsilon}.$$

Since $1/(T\varphi'(2T))=o(1)$, the latter estimates show that

$$A_T(u)\ll_\kappa (1+|u|)^{1/2}. \tag{19}$$

For the estimate of

$$\sup_{s\in K}\left|g\left(\frac{1}{2}+\kappa-s+iu,y\right)\right|,$$

we apply the bound

$$\Gamma(\sigma+it)\ll\exp\{-c|t|\},\quad c>0, \tag{20}$$

which is uniform in every finite interval of σ. Thus, by the definition of $g(z,y)$, we obtain that, for $s=\sigma+it\in K$,

$$g\left(\frac{1}{2}+\kappa-s+iu,y\right)\ll_\kappa y^{1/2+\kappa-\sigma}\exp\left\{-\frac{c}{\eta}|u-t|\right\}\ll_{\kappa,K} y^{-\kappa}\exp\{-c_1|u|\},\quad c_1>0.$$

This together with (19) implies the bound

$$J_1 \ll_{\kappa,K} y^{-\kappa} \int_{-\infty}^{\infty} (1+|u|)^{1/2} \exp\{-c_1|u|\} du \ll_{\kappa,K} y^{-\kappa}. \tag{21}$$

It remains to estimate J_2. Taking into account formula (1), we find that

$$\operatorname{Res}_{z=1-s} \mathcal{Z}(s+z) g(z,y) = g'(1-s,y) + a g(1-s,y).$$

Hence,

$$\begin{aligned} r_y(s) &= \frac{1}{\eta^2} \Gamma'\left(\frac{1-s}{\eta}\right) y^{1-s} + \frac{1}{\eta} \Gamma\left(\frac{1-s}{\eta}\right) y^{1-s} \log y + \frac{a}{\eta} \Gamma\left(\frac{1-s}{\eta}\right) y^{1-s} \\ &= \eta^{-1} y^{1-s} \Gamma(\eta^{-1}(1-s)) \left(\eta^{-1} \frac{\Gamma'}{\Gamma}(\eta^{-1}(1-s)) + a + \log y\right). \end{aligned}$$

This, (20), and the estimate for the digamma-function

$$\frac{\Gamma'}{\Gamma}(\sigma + it) \ll \log(|t| + 2),$$

lead, for $s = \sigma + it \in K$, to

$$r_y(s + \varphi(\tau)) \ll_\kappa y^{1/2-2\kappa} \log y \exp\{-c_2|t + \varphi(\tau)|\} \ll_{\kappa,K} y^{1/2-\kappa} \exp\{-c_3 \varphi(\tau)\}, \quad c_2 > 0, \ c_3 > 0.$$

Therefore,

$$J_2 \ll_{\kappa,K} \frac{y^{1/2-\kappa}}{T} \int_T^{2T} \exp\{-c_3 \varphi(\tau)\} d\tau \ll_{\kappa,K} y^{1/2-\kappa} \exp\left\{-\frac{c_3}{2} \varphi(T)\right\}. \tag{22}$$

Now, (18), (21), and (22) imply

$$\frac{1}{T} \int_T^{2T} \sup_{s \in K} |\mathcal{Z}(s + i\varphi(\tau)) - \mathcal{Z}_y(s + i\varphi(\tau))| d\tau \ll_{\kappa,K} y^{-\kappa} + y^{1/2-\kappa} \exp\left\{-\frac{c_3}{2} \varphi(T)\right\}.$$

Thus, by (16),

$$\lim_{y \to \infty} \limsup_{T \to \infty} \nu\{\rho(x_T(s), x_{T,y}(s)) \geqslant \varepsilon\} = 0,$$

and the lemma is proved. □

Proof of Theorem 1. By (15),

$$x_{T,y} \xrightarrow[T \to \infty]{\mathcal{D}} x_y, \tag{23}$$

where $x_y = x_y(s)$ is the $\mathbb{H}$-valued random element, for all $y > 0$, having the distribution $P_{0,\mathbb{H}}$. Hence,

$$x_y \xrightarrow[T \to \infty]{\mathcal{D}} P_{0,\mathbb{H}}.$$

This, (23), and Lemmas 8 and 4 show that

$$x_T \xrightarrow[T \to \infty]{\mathcal{D}} P_{0,\mathbb{H}},$$

and this relation is equivalent to the assertion of the theorem. □

We recall the equivalence of weak convergence of probability measures in terms of open and continuity sets. Recall that a set A is a continuity set of the probability measure P if $P(\partial A) = 0$, where ∂A is the boundary of A.

Lemma 9. [21, Theorem 2.1] *Let P_n, $n \in \mathbb{N}$, and P be probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. Then, the following statements are equivalent*:

(1)

$$P_n \xrightarrow[n\to\infty]{\text{w}} P;$$

(2) *For every open set $G \subset \mathbb{X}$*,

$$\liminf_{n\to\infty} P_n(G) \geqslant P(G);$$

(3) *For every continuity set A of P*,

$$\lim_{n\to\infty} P_n(A) = P(A).$$

Proof of Theorem 3. The support of the measure $P_{0,\mathbb{H}}$ is the set $\{g_0\}$, where $g_0(s) \equiv 0$ for $s \in D$. Therefore, the set

$$G_\varepsilon = \left\{ g \in \mathbb{H} : \sup_{s\in K} |g(s)| < \varepsilon \right\}$$

is an open neighbourhood of a support of the measure $P_{0,\mathbb{H}}$. Hence, by the support property,

$$P_{0,\mathbb{H}}(G_\varepsilon) > 0. \tag{24}$$

This, Theorem 1, and (1) and (2) of Lemma 9 imply

$$\liminf_{T\to\infty} P_{T,\mathbb{H}}(G_\varepsilon) \geqslant P_{0,\mathbb{H}}(G_\varepsilon) > 0,$$

and the definitions of $P_{T,H}$ and G_ε give the first assertion of the theorem.

For the proof of the second assertion, we observe that the boundaries of the sets G_{ε_1} and G_{ε_2} do not intersect for different positive ε_1 and ε_2. This remark implies that the set G_ε is a continuity set of the measure $P_{0,\mathbb{H}}$ for all but at most countably many $\varepsilon > 0$. Therefore, the second assertion of the theorem is a consequence of Theorem 1, (1) and (3) of Lemma 9, and (24). The theorem is proved. □

Acknowledgements: The authors thank referees for useful remarks.

Author contributions: Both authors contributed equally to the content of this manuscript.

Conflict of interest: The authors state no conflict of interest.

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