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ASYMPTOTIC CONDITIONS AT INFINITY FOR THE TIME-PERIODIC STOKES PROBLEM SET IN DOMAINS WITH CYLINDRICAL OUTLETS TO INFINITY

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VILNIAUS UNIVERSITETAS

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LAIKO ATŽVILGIU PERIODINIO STOKSO UŽDAVINIO SRITYSE SU CILINDRINIAIS IŠĖJIMAIS ASIMPTOTINĖS SĄLYGOS BEGALYBĖJE

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Contents

Introduction

The Navier-Stokes equations

The motion of the liquid substances in some domain $\Omega \subset \mathbf{R}^n$, $n = 2, 3$ can be described (see, e.g., [6]) by the following system of partial differential equations relating the velocity field $\mathbf{v} = (v_1(x,t), \dots, v_n(x,t))$ and the pressure $p = p(x,t)$ of the incompressible fluid with the external forcing term $\mathbf{f} = (f_1(x,t), \dots, f_n(x,t))$:

$$\begin{cases} \rho \mathbf{v}_t - \nu \Delta \mathbf{v} + \rho \left(\mathbf{v} \cdot \nabla \right) \mathbf{v} + \nabla p & = & \rho \mathbf{f}, \\ \operatorname{div} \mathbf{v} & = & 0. \end{cases} \tag{0.1.1}$$

Here the constants $\rho > 0$ and $\nu > 0$ denotes the density and the viscosity coefficient of the fluid. Without loss of generality we assume that $\rho = 1$. The first equation in (0.1.1) expresses the conservation of momentum for a selected portion of the fluid. The second one, called the mass continuity equation, represent the fact that the fluid under consideration is incompressible. We notice that the last equation can be also written as

$$-\nabla \cdot \mathbf{v} = \frac{\partial v_1}{\partial x_1} + \cdots + \frac{\partial v_n}{\partial x_n} = 0.$$

Since 1822, when the system (0.1.1) was proposed by the French engineer Claude-Louis Navier and later justified by the English mathematician George Gabriel Stokes, these equations attracted attention of many scientists in physics and mathematics. The Navier-Stokes model for the fluid dynamics is widely used in many practical fields, e.g., in engineering, biological systems, weather forecasting, oceanology and even in creating computer games, where the certain version of system (0.1.1) is used to create realistic fluid-like effects such as swirling smokes (see [95]). On the other hand, despite the long history of the equations and attention of many prominent mathematicians (see, e.g., [37], [24], [96]) there are many unsolved problems regarding the Navier-Stokes equations. For example, the famous *existence and regularity problem for the three-dimensional non-stationary Navier-Stokes equations*, included in 2000 by the

Clay Mathematics Institute into the list of seven Millennium Prize problems, or the well-known Leray problem concerning the flows in domains with multiply connected boundaries which has been open for more than 80 years[i].

Evolutionary problems are most important for the applications. In particular, Navier-Stokes equations set in domains with cylindrical outlets to infinity (systems of pipes) are used to model the flow in oil pipelines, blood motion in blood-vessels, etc. Having in mind medical applications, most interesting become pulsating flows, i.e., periodic or almost periodic in time (see, e.g., [8], [30], [61] and references therein). However, the investigation of such flows started a decade ago and relatively little is known about the solutions of the time-periodic or the general non-steady Navier-Stokes equations set in systems of pipes.

The time-periodic Stokes problem set in domains with cylindrical outlets to infinity

In the thesis we consider the time-periodic boundary value problem for the linear version of (0.1.1), i.e., we consider the following Stokes-problem:

$$\begin{cases} \partial_t \mathbf{v} - \nu\Delta\mathbf{v} + \nabla p = \mathbf{f}, & (x,t) \in \Omega \times (0,2\pi), \\ -\nabla\cdot\mathbf{v} = \mathbf{0}, & (x,t) \in \Omega \times (0,2\pi), \\ \mathbf{v} = \mathbf{0}, & (x,t) \in \partial\Omega \times (0,2\pi), \\ \mathbf{v}(x,0) = \mathbf{v}(x,2\pi), & x \in \Omega. \end{cases} \tag{0.1.2}$$

We assume that the flow domain Ω coincides outside some ball with a system of semi-infinite cylinders Ω_+^j of constant bounded cross-sections $\omega^j \subset \mathbb{R}^2$, $j = 1, \dots, J$ (see Figure 1). Moreover, we assume that the external force $\mathbf{f}$ is time-periodic[ii] and, therefore, impose the time-periodicity condition $(0.1.2_4)$[iii]. Since the fluid is viscous and does not move at the boundary $\partial\Omega$, we impose for the velocity field $\mathbf{v}$ the homogeneous Dirichlet condition $(0.1.2_3)$ also called the non-slip condition.

The whole domain Ω is treated in coordinates $x = (x_1, x_2, x_3)$, while in every semi-infinite cylinder we set the local Cartesian coordinate system $x^j = (x_1^j, x_2^j, x_3^j)$ in such a way that $\Omega_+^j = \omega^j \times (0,\infty)$. The domain obtained from Ω after the outlets

[i]We notice that the Leray's problem in $2D$ case has been recently solved in [36].

[ii]Without loss of generality we may assume that the period is 2π, i.e., $\mathbf{f}(x,0) = \mathbf{f}(x,2\pi)$ for all $x \in \Omega$.

[iii]Here and in the rest of the thesis, the subscript index i in the label of formula, stands for the i^{th} line in the considered formula.

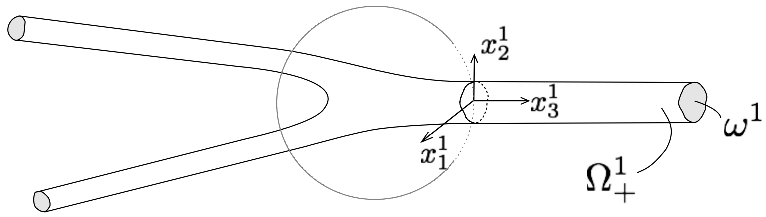

Figure 1: Domain Ω in the case $J = 3$.

to infinity are truncated at the distances $x_3^j = L$ will be denoted by Ω_L, namely

$$\Omega_L = \Big\{ x \in \Omega : \text{ if } x \in \Omega_+^j \text{ then } x_3^j < L, j = 1, \dots, J \Big\}. \qquad (0.1.3)$$

In the following we also use the notation $y^j = (x_1^j, x_2^j)$, $z^j = x_3^j$.

State of the art

In 1976 Heywood investigated the Stokes and Navier-Stokes equations set in the aperture domain[iv] (see [28]). He showed that the flow of the viscous incompressible fluid in such domain is not necessary uniquely determined by the applied external forces and the initial and boundary values of the solution, i.e., in some cases there may exist an infinite family of solutions. Therefore one shall impose (next to the standard initial and boundary conditions) some functionals of the unknown functions. It was proposed in [28] (see also [71], [89]) to specify the solution of the Stokes system by prescribing either the flux through the aperture M, or the difference between the limits $p_+ = \lim\limits_{|x|\to\infty,\, x_3>0} p(x)$ and $p_- = \lim\limits_{|x|\to\infty,\, x_3<0} p(x)$ of the pressure function, i.e., the pressure drop.

After the appearance of the paper [28] much effort was given and the large progress was made in studies of the Navier-Stokes equations set in domains that become infinite in one or several directions, i.e., in domains with outlets to infinity. Examples of such domains are the infinite layers (thin and expanding at infinity), the exterior domains, the domains with paraboloidal and, of course, cylindrical outlets to infinity.

At first, the attention was turned to the solvability of the steady and non-steady Navier-Stokes equations in the classes of functions having finite energy dissipation, i.e., possessing bounded Dirichlet integrals $\int_\Omega |\nabla \mathbf{v}|^2 dx$. It was proved in [38], [39], [89], [94] that looking for the solution of this type it is necessary to impose additional conditions (fluxes or the pressure drops) in each outlet which expands at infinity

[iv] The aperture domain is the union $\mathbb{R}^3_- \cup M \cup \mathbb{R}^3_+$ of the half-spaces $\mathbb{R}^3_\pm$ joined by the bounded "aperture" $M \subset \mathbb{R}^2$ lying on the hyperplane $\{x \in \mathbb{R}^3 : x_3 = 0\}$

"sufficiently fast".

The next step was the analysis of the Navier-Stokes equations in domains with "narrow" outlets to infinity, for example, cylindrical ones. The advance in this direction was possible due to technique of special integral estimates[v], so called "technique of the Saint-Venant principle", developed and applied for the steady Navier-Stokes problem by Ladyzhenskaya and Solonnikov in [40]. The solvability of time-dependent Navier-Stokes problems, either for small data or for small time intervals, was proved in [41], [42], [90], [92], [93].

Such questions as the regularity properties, uniqueness or asymptotic behaviour of the solution were also extensively investigated. We refer the reader to papers [51], [52], [53], [55], [75], [85] where the steady Stokes and Navier-Stokes problems were considered in layer-like domains; to papers [12], [13], [15], [18], [26], [48], [60], [20]–[23] for results concerning steady flows in the aperture domain and in the slightly curved channels. We also mention numerous papers [1], [49], [50], [72]-[74], [81]-[84], [91] devoted to analysis of problems set in strip-like domains or domains with periodically varying cross-sections. A lot of progresses have been made in studies of asymptotic behaviour of solutions to the steady problems set in infinite cylindrical domains (see, e.g., [16], [17], [47]) or in the finite tube structures (see [10], [14], [62]-[65]).

In the case of domains with cylindrical outlets an important role is played by the Poiseuille flow – the exact solution to the homogeneous Stokes system set in an infinite cylinder $\Pi = \omega \times \mathbb{R}$ ($\omega \subset \mathbb{R}^n$, $n \geq 1$). In the 3D case the Poiseuille flow has a velocity field $\mathbf{v}_p = (0, 0, v_p(x_1, x_2, t))$ directed along the axis of Π and a pressure function $p_p = p_p(x_3, t)$, which is linear with respect to the space variable x_3. This vector-field $(\mathbf{v}_p, p_p)$ is usually used to describe the asymptotic behaviour of solutions to the Stokes and Navier-Stokes problems set in systems of pipes. The Poiseulle flow can be determined by prescribing either the pressure gradient ∇p_p or the flow-rate $\phi = \int_\omega v_p dy$. When the flow is steady, these two quantities are proportional according to the Poiseuille Law. However in the case of a time-dependent flow the relation between the flow-rate and the pressure gradient becomes non-local.

The existence of the time-periodic Poiseuille solution with additionally prescribed time-periodic flow rate $\phi = \phi(t)$ was proved by Beirao da Veiga in 2005 (see [7]). In [27] the relation between the Fourier coefficients of ∇p_p and ϕ was derived. The time-periodic Stokes problem in general domain having cylindrical outlets was stud-

[v]Standard energy estimates method became insufficient in this case since the incompressible fluid in the "narrow" outlet can be "transported" to infinity only by the vector-field $\mathbf{v}$ with the infinite Dirichlet integral.

ied in [33]. It was proved there that prescribing the time-periodic flow-rates in every semi-infinite cylinder Ω^j_+, $j = 1, \dots, J$, ensures the existence of a time-periodic solution which tends in every outlet to the corresponding Poiseuille flow. The analogous results in the non-linear setting were obtained also for the general non-steady case. It was proved (see [25], [34], [76]-[78] and Chapter 8 in [80]) that the solutions of the non-stationary Stokes and Navier-Stokes problems set in the domain with cylindrical outlets to infinity tends in each cylinder to the corresponding non-stationary Poiseuille-type solution. Also we would like to mention the paper [9], where the existence of the almost time-periodic Poiseuille solution was proved and the almost time-periodic flows in the two pipes system was considered. Finally, we refer the reader to [66]-[69] where the asymptotic properties of the non-steady flows in tubes with elastic walls are considered (we notice that in these papers the Navier-Stokes equations are coupled with the equations describing the movement of the elastic wall of the blood-vessel).

We would like to emphasize that all above mentioned results are related to the two types of asymptotic conditions at infinity: either prescribed fluxes, or prescribed pressure drops. However, these conditions do not cover all possible physical phenomena which can occur in reality. For example, the flux or the pressure drop conditions are not suitable if one is interested in values of the total pressure at the ends of the outlets. It can happen also that the flow-rates are known only in part of cylinders, while the flow in the rest of cylinders is controlled by some devices, e.g., plugs, membranes, etc. In this case it is natural to ask what will be the flow-rates in these outlets, how does the flux distribution depend on the geometry of the domain or the devices attached to the ends of the outlets. Moreover, there are phenomena that shall be modelled using the variational inequalities, for example, the flow controlled by the check-valve which is open/closed when the pressure reaches some limit value.

Theory of asymptotic conditions at infinity for elliptic problems was developed by S.A. Nazarov and co-authors in [56]–[59]. Methods proposed in the book [59] were applied in [54] for analysis of the steady Stokes and Navier-Stokes equations set in domains with cylindrical outlets to infinity, where these problems were considered in weighted Sobolev spaces. Namely, the solutions $\mathbf{u} = (\mathbf{v}, p)$ having the special structure

$$\mathbf{u} = \sum_{j=1}^{J} \chi^j (a^j \mathbf{u}^j_0 + b^j \mathbf{u}^j_1) + \tilde{\mathbf{u}} \tag{0.1.4}$$

were considered. In the last formula, a^j and b^j are constants, χ^j is a smooth cut-off function with $\mathrm{supp}(\chi^j) \subset \Omega^j_+$ and such that $\chi^j(x) = 1$, for $x^j_3 \geq 1$, $j = 1, \dots, J$. The vector-field $\tilde{\mathbf{u}} = (\tilde{\mathbf{v}}, \tilde{p})$ decays in each outlet Ω^j_+, $j = 1, \dots, J$, exponentially as $x^j_3 \to \infty$. The main terms $\mathbf{u}^j_0$ and $\mathbf{u}^j_1$ in (0.1.4) are special solutions to the homogeneous Stokes problem set in the single cylinder $\Omega^j = \omega^j \times \mathbb{R}$. The solution $\mathbf{u}$ becomes unique if one fixes coefficients a^j and b^j, $j = 1, \dots, J$, in expression (0.1.4). We would like to emphasize that these quantities have special physical meaning – constants $b^1, \dots, b^J$ are equal to flow-rates in the corresponding cylinders, while constants $a^1, \dots, a^J$ form the part of the pressure p in every outlet. As a consequence, there appear natural limitations to selection of coefficients in (0.1.4). For example, due to the incompressibility of a fluid the sum $b^1 + \dots + b^J$ must be equal to zero.

Having in mind that for the time-periodic Stokes problem (0.1.2) the only known correct asymptotic conditions at infinity are of the flow-rate type or of the pressure drop type, we have formulated the following research objectives.

Aims and objectives of the thesis

The purpose of our thesis is the analysis of the time-periodic Stokes system set in domains with cylindrical outlets to infinity. Our aim is:

- to propose the methods of imposing general asymptotic conditions at infinity for the time-periodic Stokes problem set in the system of infinite cylinders;
- to construct some classes of physically reasonable asymptotic conditions at infinity that ensure existence and uniqueness of the solution to the time-periodic Stokes problem and are different from prescribtion of the flow-rates or the pressure drops only.

Methodology

Our research is based at the high extent on the methodology of setting the conditions at infinity for the steady Stokes and Navier-Stokes systems (see [54] and the book [59]). We benefit from the use the operator theory and theory of general elliptic and parabolic equations. Existence and uniqueness of the solutions to the problems considered in the thesis is shown using the Fourier transforms, energy estimate methods, the operator theory. The behaviour of solutions to the problems set in unbounded domains is described using the approach of weighted Sobolev spaces[vi].

[vi] We recall that the Sobolev space $H^m(G)$, $G \subset \mathbb{R}^n$ is a vector space of functions equipped with the norm

$$\|u\|_{H^m(G)} = \sum_{|\alpha| \le m} \|D^\alpha_x u\|_{L^2(G)},$$

Structure of the thesis

The thesis consists of the following parts – Introduction, two main Chapters, Conclusions, two Appendices and Bibliography. Introduction provides the reader with the formulation and state of the art of the problem, also contains the necessary information related to the dissemination of results presented in the dissertation.

In Chapter 1 we reduce the time-periodic Stokes problem into the sequence of elliptic Stokes-type problems for the Fourier coefficients of the time-periodic solution. For each of these problems questions of existence and uniqueness of the solutions from certain weighted Sobolev spaces are discussed in Subsection 1.1. The special asymptotic representation of the solution with unbounded Dirichlet integral is presented in this Subsection. In Subsection 1.2 we show that all problems can be treated in the same weighted function space, i.e., that we may use the same weight function for all Stokes-type problems. The generalized Green formula is derived in Subsection 1.3. Two following Subsections are devoted to the construction of the basis in the set of solutions to the homogeneous Stokes-type problems. In Subsection 1.6 we consider Stokes-type problems supplied with the general conditions at infinity and prove the Fredholm type theorem concerning the solvability of these problems. The class of matrices, necessary to model certain pressure related conditions, is presented in Subsection 1.7.

The second Chapter is devoted to analysis of the time-periodic Stokes problem. The results obtained in Chapter 1 are used here to construct the special class of the time-periodic solutions (see Subsections 2.1–2.3). The generalized Greens formula is derived and the asymptotic conditions at infinity are described in Subsection 2.4. The theorem concerning the existence and uniqueness of the time-periodic solution is proved. In Subsection 2.5 we present several different versions of the Green formula and the conditions at infinity corresponding to these Green formulas. Examples of particular conditions, that enable to select a unique solution and are different from prescribing only flow-rates, are provided in Section 2.5.

Finally, several technical questions related to the Stokes-type problems are considered in two Appendices.

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where $D_x^\alpha = \dfrac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$. For the reader's convenience we define new function spaces in the sections where they are used.

Statement of Originality

All results obtained in the dissertation are new. To our best knowledge, the methods allowing to impose general conditions at infinity for the time-periodic Stokes problem (set in domains with several cylindrical outlets) have been proposed for the first time.

Publications

The results presented in thesis were published in the scientific papers listed below:

- M. Skujus, On the Green's formula for a Stokes type problem, *Lietuvos Matematikos Rinkinys. LMD darbai*, **48/49**, 2008, p. 72-77.
- M. Skujus, On the time-periodic Stokes problem set in domains with cylindrical outlets to infinity, *Asymptotic Analysis*, **81**(2), 2013, 93-119.
- M. Skujus, Asymptotic conditions at infinity for the time-periodic Stokes problem in a system of pipes, *to appear in Analysis and Applications.*

Dissemination

The results of the thesis were presented in the seminar of the Department of Differential Equations and Numerical Mathematics (VU) and in the following international conferences:

- "Diferential equations and their applications", Panevėžys, Lithuania, September 10–12, 2009.
- "Regularity aspects of PDEs", Bedlewo, Poland, September 5–11, 2010.
- "Parabolic and Navier-Stokes equations", Bedlewo, Poland, September 3–7, 2012.
- "Applied Mathematics and Scientific Computing", Sibenik, Croatia, June 10–14, 2013.
- "EQUADIFF 13", Prague, Czech Republic, August 26–30, 2013.

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- "**Boundary value problems for the Navier-Stokes equations set in domains with non-compact boundaries**". The research project supported by the Research Council of Lithuania under the agreement No. MIP-2011/030. Duration: June, 2011 – December, 2012. Project leader: prof. K. Pileckas.

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Chapter 1

Stokes-type problems

1.1 Asymptotics of the solution

Looking for the solution of problem (0.1.2) in the form

$$\begin{aligned} \mathbf{v}(x,t) &= \frac{\mathbf{v}_{c0}}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}\left\{\mathbf{v}_{ck}(x)\cos kt + \mathbf{v}_{sk}(x)\sin kt\right\}, \\ p(x,t) &= \frac{p_{c0}}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}\left\{p_{ck}(x)\cos kt + p_{sk}(x)\sin kt\right\}, \end{aligned} \tag{1.1.1}$$

and substituting series (1.1.1) into equations (0.1.2) we get for the coefficients $\mathbf{v}_{ck}$, p_{ck}, $\mathbf{v}_{sk}$, p_{sk} a sequence of the Stokes-type elliptic problems

$$\left\{ \begin{array}{rcll} -\nu\Delta\mathbf{v}_{ck} + \nabla p_{ck} + k\mathbf{v}_{sk} &=& \mathbf{f}_{ck}, & x\in\Omega, \\ -\nabla\cdot\mathbf{v}_{ck} &=& 0, & x\in\Omega, \\ -\nu\Delta\mathbf{v}_{sk} + \nabla p_{sk} - k\mathbf{v}_{ck} &=& \mathbf{f}_{sk}, & x\in\Omega, \quad k=0,1,\ldots \\ -\nabla\cdot\mathbf{v}_{sk} &=& 0, & x\in\Omega, \\ \mathbf{v}_{ck}=\mathbf{0},\ \mathbf{v}_{sk} &=& \mathbf{0}, & x\in\partial\Omega. \end{array} \right. \tag{1.1.2}$$

Here $\mathbf{f}_{c0}/(2\pi), \mathbf{f}_{ck}/\pi, \mathbf{f}_{sk}/\pi, k = 1,2,\ldots,$ are the Fourier coefficients of the function $\mathbf{f} = \mathbf{f}(x,t)$[i]. For $k=0$ the system (1.1.1) splits into the steady-state Stokes systems for coefficients $(\mathbf{v}_{c0}, p_{c0})$ and $(\mathbf{v}_{s0}, p_{s0})$, i.e., into the problems considered in [54].

Denote $\mathbf{u}_k = (\mathbf{v}_{ck}, p_{ck}, \mathbf{v}_{sk}, p_{sk})$, $\mathbf{v}_k = (\mathbf{v}_{ck}, \mathbf{v}_{sk})$, $\mathbf{f}_k = (\mathbf{f}_{ck}, 0, \mathbf{f}_{sk}, 0)$ and rewrite

[i] At the moment we do not specify function spaces for the solution $(\mathbf{v}, p)$ and the data of the problem. We assume that the function $\mathbf{f}$ is regular enough and, therefore, it's Fourier series exists.

problem (1.1.2) in a short form

$$\left\{\begin{array}{rcll} \mathbf{S}_k(\nabla_x)\mathbf{u}_k & = & \mathbf{f}_k, & x\in\Omega, \\ \mathbf{v}_k & = & \mathbf{0}, & x\in\partial\Omega. \end{array}\right. \tag{1.1.3}$$

First we consider problem (1.1.1) set in the cylinder $\Pi^j=\{x^j\in\mathbb{R}^3 : y^j\in\omega^j, z^j\in\mathbb{R}\}$ (bellow we omit the index j):

$$\left\{\begin{array}{rcll} \mathbf{S}_k(\nabla_y,\partial_z)\mathbf{u}_k & = & \mathbf{f}_k, & (y,z)\in\Pi, \\ \mathbf{V}_k & = & \mathbf{0}, & x\in\partial\Pi. \end{array}\right.$$

Applying the Fourier transform (with respect to the variable z) to the above problem we get the boundary-value problem on the cross-section ω:

$$\left\{\begin{array}{rcll} \mathbf{S}_k(\nabla_y,i\lambda)\widehat{\mathbf{u}}_k(y) & = & \widehat{\mathbf{f}}_k(y), & y\in\omega, \\ \widehat{\mathbf{v}}_k(y) & = & \mathbf{0}, & y\in\partial\omega, \end{array}\right. \tag{1.1.4}$$

where $\widehat{\mathbf{u}}_k=\widehat{\mathbf{u}}_k(y)$, $\widehat{\mathbf{v}}_k=\widehat{\mathbf{v}}_k(y)$ and $\widehat{\mathbf{f}}_k=\widehat{\mathbf{f}}_k(y)$ are the Fourier transforms of $\mathbf{u}_k=\mathbf{u}_k(x)$, $\mathbf{v}_k=\mathbf{v}_k(x)$ and $\mathbf{f}_k=\mathbf{f}_k(x)$, respectively. Problem (1.1.4) may be identified with a family of mappings

$$\begin{aligned} \widehat{\mathbf{u}}_k \to \mathbf{A}_k(\lambda)\widehat{\mathbf{u}}_k \equiv \left(\mathbf{S}_k(\nabla_y,i\lambda)\widehat{\mathbf{u}}_k, \widehat{\mathbf{v}}_k|_{\partial\Omega}\right), \\ \mathbf{A}_k(\lambda):\mathcal{D}^lH(\omega)\to\mathcal{R}^lH(\omega), \end{aligned} \tag{1.1.5}$$

where

$$\begin{aligned} \mathcal{D}^lH(\omega) &= \left(H^l(\omega)\right)^3\times H^{l-1}(\omega)\times\left(H^l(\omega)\right)^3\times H^{l-1}(\omega), \\ \mathcal{R}^lH(\omega) &= \left(H^{l-2}(\omega)\right)^3\times H^{l-1}(\omega)\times H^{l-1/2}(\partial\omega) \\ &\quad\times\left(H^{l-2}(\omega)\right)^3\times H^{l-1}(\omega)\times H^{l-1/2}(\partial\omega). \end{aligned}$$

Here $H^l(\omega)$ and $H^{l-1/2}(\partial\omega)$ denotes the Sobolev spaces with $l\geq 2$.

The rest of the Section is based on the classical results for elliptic partial differential equations (see, e.g., the monograph of Lions, Magenes [44], papers of Agmon, Douglis and Nirenberg [2], [3], Solonnikov [88] and Agranovich, Vishik [4]). As well we use the theory concerning the elliptic PDE's set in cylindrical domains presented in the papers of Pazy [70], Kondratiev [35] and numerous works of Maz'ya, Nazarov, Plamenevsky and co-authors (see, e.g., the monographs [45], [46], [59] and references therein). These well known results are provided without proofs[ii], while the facts that are peculiar for the Stokes-type problems (1.1.2) are examined in details.

Let $\mathbf{B}(\lambda)$ be the operator bundle (depending on the complex parameter λ) in a Hilbert space H, and let the number $\lambda_0 \in \mathbb{C}$ be such that there exists a non-trivial vector-function $\mathbf{u}^{(0)}$ satisfying the equation $\mathbf{B}(\lambda_0)\mathbf{u}^{(0)} = 0$. Then λ_0 and $\mathbf{u}^{(0)}$ are called the eigenvalue and the eigenvector of $\mathbf{B}(\lambda)$. The associated vectors (also called the generalized eigenvectors) $\mathbf{u}^{(1)}, \dots, \mathbf{u}^{(m)}$ are defined by the equation

$$\sum_{n=0}^{l} \frac{1}{n!} \frac{\partial^n \mathbf{B}}{\partial \lambda^n}(\lambda_0)\mathbf{u}^{(l-n)} = 0, \quad l = 0, 1, \dots, m. \tag{1.1.6}$$

Theorem 1.1.1. *1. The only real eigenvalue of the operator $\mathbf{A}_k(\lambda)$ defined by (1.1.5) is $\lambda = 0$. There are two eigenvectors corresponding to $\lambda = 0$:*

$$\mathbf{u}_{ck}^{(0)} = (0, 0, 0, -1, 0, 0, 0, 0) \quad \text{and} \quad \mathbf{u}_{sk}^{(0)} = (0, 0, 0, 0, 0, 0, 0, -1).$$

2. The vector $\mathbf{u}_{ck}^{(1)} = (0, 0, i\varphi_k, 0, 0, 0, -i\psi_k, 0)$ is the associated vector to the eigenvector $\mathbf{u}_{ck}^{(0)}$ while the vector $\mathbf{u}_{sk}^{(1)} = (0, 0, i\psi_k, 0, 0, 0, i\varphi_k, 0)$ is the associated vector to $\mathbf{u}_{sk}^{(0)}$. Here the pair of functions $\varphi_k, \psi_k \in H^2(\omega)$ is the solution to the following problem[iii]

$$\begin{cases} k\psi_k + \nu\Delta\varphi_k = -1, & y \in \omega, \\ k\varphi_k - \nu\Delta\psi_k = 0, & y \in \omega, \\ \varphi_k = 0, \ \ \psi_k = 0, & y \in \partial\omega. \end{cases} \tag{1.1.7}$$

3. The Jordan chains $\left\{\mathbf{u}_{ck}^{(0)}, \mathbf{u}_{ck}^{(1)}\right\}$, $\left\{\mathbf{u}_{sk}^{(0)}, \mathbf{u}_{sk}^{(1)}\right\}$ cannot be prolonged, i.e. there are no associated vectors of the second order.

Proof. 1. The equation $\mathbf{A}_k(\lambda)\mathbf{u}_k^{(0)} = \mathbf{0}$ can be written as the boundary value problem in the domain ω (omitting the index k):

[ii] We refer the reader to the monographs [45] and [59] for more detailed analysis of elliptic problems in domains with cylindrical outlets to infinity.

[iii] The elliptic problem (1.1.7) is uniquely solvable in $(\mathring{H}^1(\omega))^2$. Moreover, if $\partial\omega \in C^2$ then the solution (φ_k, ψ_k) belongs to $(H^2(\omega))^2$. For more details see Appendix A.

$$
\begin{cases}
-\nu\Delta_y \mathrm{u}_1^{(0)} + \nu\lambda^2 \mathrm{u}_1^{(0)} + \dfrac{\partial}{\partial x_1}\mathrm{u}_4^{(0)} + k\mathrm{u}_5^{(0)} & = & 0,\\
-\nu\Delta_y \mathrm{u}_2^{(0)} + \nu\lambda^2 \mathrm{u}_2^{(0)} + \dfrac{\partial}{\partial x_2}\mathrm{u}_4^{(0)} + k\mathrm{u}_6^{(0)} & = & 0,\\
-\nu\Delta_y \mathrm{u}_3^{(0)} + \nu\lambda^2 \mathrm{u}_3^{(0)} + i\lambda\mathrm{u}_4^{(0)} + k\mathrm{u}_7^{(0)} & = & 0,\\
-\dfrac{\partial}{\partial x_1}\mathrm{u}_1^{(0)} - \dfrac{\partial}{\partial x_2}\mathrm{u}_2^{(0)} - i\lambda\mathrm{u}_3^{(0)} & = & 0,\\
-\nu\Delta_y \mathrm{u}_5^{(0)} + \nu\lambda^2 \mathrm{u}_5^{(0)} + \dfrac{\partial}{\partial x_1}\mathrm{u}_8^{(0)} - k\mathrm{u}_1^{(0)} & = & 0,\\
-\nu\Delta_y \mathrm{u}_6^{(0)} + \nu\lambda^2 \mathrm{u}_6^{(0)} + \dfrac{\partial}{\partial x_2}\mathrm{u}_8^{(0)} - k\mathrm{u}_2^{(0)} & = & 0,\\
-\nu\Delta_y \mathrm{u}_7^{(0)} + \nu\lambda^2 \mathrm{u}_7^{(0)} + i\lambda\mathrm{u}_8^{(0)} - k\mathrm{u}_3^{(0)} & = & 0,\\
-\dfrac{\partial}{\partial x_1}\mathrm{u}_5^{(0)} - \dfrac{\partial}{\partial x_2}\mathrm{u}_6^{(0)} - i\lambda\mathrm{u}_7^{(0)} & = & 0,\\
\mathrm{u}_l^{(0)}|_{\partial\omega} & = & 0, \quad l = 1,2,3,5,6,7.
\end{cases}
\tag{1.1.8}
$$

Here Δ_y is the Laplace operator with respect to variables $y = (x_1, x_2)$. Multiplying $(1.1.8_1)$, $(1.1.8_2)$, $(1.1.8_3)$, $(1.1.8_5)$, $(1.1.8_6)$, $(1.1.8_7)$ by the vector-fields $\overline{\mathrm{u}}_1^{(0)}$, $\overline{\mathrm{u}}_2^{(0)}$, $\overline{\mathrm{u}}_3^{(0)}$, $\overline{\mathrm{u}}_5^{(0)}$, $\overline{\mathrm{u}}_6^{(0)}$, $\overline{\mathrm{u}}_7^{(0)}$, respectively, summing the obtained equalities and then integrating by parts in ω, we derive

$$
\nu \sum_{l=1, l\neq 4}^{7} \int_\omega \left(\left|\nabla \mathrm{u}_l^{(0)}\right|^2 + \lambda^2 \left|\mathrm{u}_l^{(0)}\right|^2 \right) dy + k \sum_{l=1}^{3} \int_\omega \left(\mathrm{u}_{l+4}^{(0)} \overline{\mathrm{u}}_l^{(0)} - \mathrm{u}_l^{(0)} \overline{\mathrm{u}}_{l+4}^{(0)} \right) dy
$$
$$
+ \int_\omega \left(\left(i\lambda \overline{\mathrm{u}}_3^{(0)} - \frac{\partial \overline{\mathrm{u}}_1^{(0)}}{\partial x_1} - \frac{\partial \overline{\mathrm{u}}_2^{(0)}}{\partial x_2} \right) \mathrm{u}_4^{(0)} + \left(i\lambda \overline{\mathrm{u}}_7^{(0)} - \frac{\partial \overline{\mathrm{u}}_5^{(0)}}{\partial x_1} - \frac{\partial \overline{\mathrm{u}}_6^{(0)}}{\partial x_2} \right) \mathrm{u}_8^{(0)} \right) dy
$$
$$
-\nu \sum_{l=1, l\neq 4}^{7} \int_{\partial\omega} \frac{\partial \mathrm{u}_l^{(0)}}{\partial \mathbf{n}} \overline{\mathrm{u}}_l^{(0)} dS_y + \sum_{l=1}^{2} \int_{\partial\omega} \left(\overline{\mathrm{u}}_l^{(0)} n_l \mathrm{u}_4^{(0)} + \overline{\mathrm{u}}_{l+4}^{(0)} n_l \mathrm{u}_8^{(0)} \right) dS_y = 0,
$$

here $\mathbf{n} = (n_1, n_2, n_3)$ is an outward normal vector to the surface $\partial\omega$. According to $(1.1.8_4)$, $(1.1.8_8)$ and $(1.1.8_9)$ the last two lines in the above relation are equal to zero. Hence,

$$
\nu \sum_{l=1}^{3} \int_\omega \left(\left|\nabla \mathrm{u}_l^{(0)}\right|^2 + \left|\nabla \mathrm{u}_{l+4}^{(0)}\right|^2 + \lambda^2 \left|\mathrm{u}_l^{(0)}\right|^2 + \lambda^2 \left|\mathrm{u}_{l+4}^{(0)}\right|^2 \right) dy
$$
$$
+2ki \sum_{l=1}^{3} \int_\omega \left(Re\mathrm{u}_l^{(0)} Im\mathrm{u}_{l+4}^{(0)} - Re\mathrm{u}_{l+4}^{(0)} Im\mathrm{u}_l^{(0)} \right) dy = 0,
$$

where $Re\, u$ and $Im\, u$ denote the real and the imaginary parts of the complex function u. Since by the assumption λ is real, we obtain from the last formula, after the separation of the real and the imaginary parts, the identity

$$\nu \sum_{l=1}^{3} \int_\omega \left(\left|\nabla \mathrm{u}_l^{(0)}\right|^2 + \left|\nabla \mathrm{u}_{l+4}^{(0)}\right|^2 + \lambda^2 \left|\mathrm{u}_l^{(0)}\right|^2 + \lambda^2 \left|\mathrm{u}_{l+4}^{(0)}\right|^2 \right) dy = 0.$$

Thus $\mathrm{u}_l^{(0)} = \mathrm{u}_{l+4}^{(0)} = 0$, $l = 1, 2, 3$. Assume that $\lambda \neq 0$. Then equations (1.1.8$_3$), (1.1.8$_7$) yield $\mathrm{u}_4^{(0)} = \mathrm{u}_8^{(0)} = 0$. Hence, only $\lambda_0 = 0$ could be an eigenvalue. It is easy to compute that for $\lambda_0 = 0$ the only linearly independent eigenvalues are

$$\mathbf{u}_{ck}^{(0)} = (0, 0, 0, -1, 0, 0, 0, 0), \quad \mathbf{u}_{sk}^{(0)} = (0, 0, 0, 0, 0, 0, 0, -1).$$

2. For the associated vector $\mathbf{u}_{ck}^{(1)} = (\mathrm{u}_{c1}^{(1)}, \ldots, \mathrm{u}_{c8}^{(1)})$ (when $\lambda_0 = 0$) we get the following boundary value problem (the index ck is omitted):

$$\begin{cases} -\nu\Delta_y \mathrm{u}_1^{(1)} + \dfrac{\partial}{\partial x_1}\mathrm{u}_4^{(1)} + k\mathrm{u}_5^{(1)} &= 0, \\ -\nu\Delta_y \mathrm{u}_2^{(1)} + \dfrac{\partial}{\partial x_2}\mathrm{u}_4^{(1)} + k\mathrm{u}_6^{(1)} &= 0, \\ -\nu\Delta_y \mathrm{u}_3^{(1)} + k\mathrm{u}_7^{(1)} &= i, \\ -\dfrac{\partial}{\partial x_1}\mathrm{u}_1^{(1)} - \dfrac{\partial}{\partial x_2}\mathrm{u}_2^{(1)} &= 0, \\ -\nu\Delta_y \mathrm{u}_5^{(1)} + \dfrac{\partial}{\partial x_1}\mathrm{u}_8^{(1)} - k\mathrm{u}_1^{(1)} &= 0, \\ -\nu\Delta_y \mathrm{u}_6^{(1)} + \dfrac{\partial}{\partial x_2}\mathrm{u}_8^{(1)} - k\mathrm{u}_2^{(1)} &= 0, \\ -\nu\Delta_y \mathrm{u}_7^{(1)} - k\mathrm{u}_3^{(1)} &= 0, \\ -\dfrac{\partial}{\partial x_1}\mathrm{u}_5^{(1)} - \dfrac{\partial}{\partial x_2}\mathrm{u}_6^{(1)} &= 0, \\ \mathrm{u}_l^{(1)}|_{\partial\omega} &= 0, \quad l = 1, 2, 3, 5, 6, 7. \end{cases}$$

One can straightforwardly verify that $\mathbf{u}_{ck}^{(1)} = (0, 0, i\varphi_k, 0, 0, 0, -i\psi_k, 0)$ is a solution of the above problem. Analogously could be proved that the vector $\mathbf{u}_{sk}^{(1)} = (0, 0, i\psi_k, 0, 0, 0, i\varphi_k, 0)$ is associated for the eigenvector $\mathbf{u}_{sk}^{(0)}$.

3. From (1.1.6) we get the relation

$$\mathbf{A}_k(\lambda_0)\mathbf{u}_{ck}^{(2)} + \frac{\partial \mathbf{A}_k}{\partial \lambda}(\lambda_0)\mathbf{u}_{ck}^{(1)} + \frac{\partial^2 \mathbf{A}_k}{\partial \lambda^2}(\lambda_0)\mathbf{u}_{ck}^{(0)} = \mathbf{0}.$$

One can straightforwardly verify that $\frac{\partial \mathbf{A}_k}{\partial \lambda}(\lambda_0)\mathbf{u}_{ck}^{(1)} = (0,0,\phi_k,0,0,0,-\psi_k,0)$ and $\frac{\partial^2 \mathbf{A}_k}{\partial \lambda^2}(\lambda_0)\mathbf{u}_{ck}^{(0)} = \mathbf{0}$. Therefore the first two coordinates of the associated vector $\mathbf{u}_{ck}^{(2)}$ should be equal to zero on $\partial\omega$ and should satisfy in the domain ω the divergence equation (the index ck is omitted)

$$-\frac{\partial}{\partial x_1}\mathrm{u}_1^{(2)} - \frac{\partial}{\partial x_2}\mathrm{u}_2^{(2)} = -\varphi_k.$$

Integrating by parts in ω, and using system (1.1.7) we get the contradiction:

$$\begin{aligned} 0 &= \int_\omega \left(-\frac{\partial}{\partial x_1}\mathrm{u}_1^{(2)} - \frac{\partial}{\partial x_2}\mathrm{u}_2^{(2)}\right) d\,y = -\int_\omega \varphi_k d\,y \\ &= \int_\omega (\nu\Delta\varphi_k + k\psi_k)\,\varphi_k d\,y = -\nu\int_\omega \left(|\nabla\varphi_k|^2 + |\nabla\psi_k|^2\right) d\,y < 0 \end{aligned} \tag{1.1.9}$$

which proves the third claim of the theorem. The same arguments hold true for the Jordan chain $\left\{\mathbf{u}_{sk}^{(0)}, \mathbf{u}_{sk}^{(1)}\right\}$. □

It is well known (e.g., Lemma 3.1.2 in [59]) that the functions

$$\mathbf{u}^s(y,z) = e^{i\lambda z}\sum_{k=0}^{s}\frac{1}{k!}(iz)^k\mathbf{u}^{(s-k)}, \quad s = 0,1,\ldots,m, \tag{1.1.10}$$

where $\{\mathbf{u}^{(0)},\, \mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(m)}\}$ is the Jordan chain corresponding to eigenvalue λ of the operator bundle $\mathbf{A}_k(\lambda)$, satisfy the equation

$$\mathbf{A}_k(\lambda)\mathbf{u} = 0.$$

Hence, using the Jordan chain $\left\{\mathbf{u}_{ck}^{(0)}, \mathbf{u}_{ck}^{(1)}\right\}$ we get two solutions of the homogeneous Stokes-type problem (1.1.2) set in the cylinder $\omega \times \mathbb{R}$. The first one is

$$\mathbf{u}_{ck}^0 = (0,0,0,1,0,0,0,0),$$

while the second one is equal to

$$\mathbf{u}_{ck}^{(1)} + iz\mathbf{u}_{ck}^{(0)} = (0,0,i\varphi_k,-iz,0,0,-i\psi_k,0).$$

Multiplying the last vector by $-i$ we obtain the vector-field

$$\mathbf{u}_{ck}^1 = (0,0,\varphi_k,-z,0,0,-\psi_k,0).$$

Analogously, the Jordan chain $\left\{\mathbf{u}_{sk}^{(0)}, \mathbf{u}_{sk}^{(1)}\right\}$ generates another two solutions to the homogeneous Stokes-type problem:

$$\mathbf{u}_{sk}^{0} = (0,0,0,0,0,0,0,1), \quad \mathbf{u}_{sk}^{1} = (0,0,\psi_k,0,0,0,\varphi_k,-z).$$

Applying this procedure for the cylinders $\Omega^j = \omega^j \times \mathbb{R}$, $j = 1, \dots, J$, and taking into account the notation $z^j = x_3^j$, we obtain in every outlet to infinity four solutions to the homogeneous problem (1.1.2):

$$\mathbf{u}_{ck}^{j0} = (0,0,0,1,0,0,0,0), \quad \mathbf{u}_{ck}^{j1} = (0,0,\varphi_k^j,-x_3^j,0,0,-\psi_k^j,0), \tag{1.1.11}$$

$$\mathbf{u}_{sk}^{j0} = (0,0,0,0,0,0,0,1), \quad \mathbf{u}_{sk}^{j1} = (0,0,\psi_k^j,0,0,0,\varphi_k^j,-x_3^j). \tag{1.1.12}$$

Consider now problem (1.1.2) in the whole domain Ω. Denote by $H_\beta^m(\Omega)$ a weighted Sobolev space which is the closure of $C_0^\infty(\overline{\Omega})$[iv] with respect to the norm

$$\|u\|^2_{H_\beta^m(\Omega)} = \sum_{|\alpha|\le m} \int_\Omega \rho_\beta(x) \left|D_x^\alpha u(x)\right|^2 \mathrm{d}x,$$

here the weight function ρ_β is defined as

$$\rho_\beta(x) = \begin{cases} 1, & x \in \Omega \setminus \cup_{j=1}^J \Omega_+^j, \\ e^{2\beta x_3^j}, & x \in \Omega_+^j,\ j = 1, \dots, J. \end{cases}$$

As usually we denote $L_\beta^2(\Omega) = H_\beta^0(\Omega)$. If $\beta > 0$, elements of the space $H_\beta^m(\Omega)$ decay exponentially as x_3^j tends to infinity, and they may grow, if $\beta < 0$.

Define the following weighted function spaces

$$\begin{aligned} \mathcal{D}_\beta^l H(\Omega) &= \left(H_\beta^l(\Omega)\right)^3 \times H_\beta^{l-1}(\Omega) \times \left(H_\beta^l(\Omega)\right)^3 \times H_\beta^{l-1}(\Omega), \\ \mathcal{R}_\beta^l H(\Omega) &= \left(H_\beta^{l-2}(\Omega)\right)^3 \times H_\beta^{l-1}(\Omega) \times \left(H_\beta^{l-2}(\Omega)\right)^3 \times H_\beta^{l-1}(\Omega). \end{aligned} \tag{1.1.13}$$

and the operator

$$\mathbf{A}_{\beta,k}^l : \mathcal{D}H_\beta^l(\Omega) \to \mathcal{R}H_\beta^l(\Omega). \tag{1.1.14}$$

According to results for general elliptic equations (see [4]), the mapping (1.1.5) is an isomorphism for all $\lambda \in \mathbb{C}$, except the countable set Λ of isolated points. The set Λ consists from eigenvalues of the operator bundle $\mathbf{A}_k(\lambda)$ and is located, with exception of finite number of points, in the two-sided sector $\{\lambda \in \mathbb{C} : |Re\lambda| < C|Im\lambda|\}$ (the

[iv] $C_0^\infty(\overline{\Omega})$ is a class of infinitely differentiable functions with compact supports in $\overline{\Omega}$.

constant C is independent of λ). This result was used in [35] to prove analogous statement for the general elliptic problems set in the single infinite cylinder (see also Theorem 3.1.2 in [45]). Namely, let $\xi_\gamma = e^{2\gamma x_3^j}$ denotes the weight-function defined in the infinite cylinder $\Omega^j = \omega^j \times \mathbb{R}$, and let the weighted Sobolev spaces $\mathcal{D}^l_\gamma H(\Omega^j)$ and $\mathcal{R}^l_\gamma H(\Omega^j)$ be defined analogously as (1.1.13) (by using the weight function ξ_γ instead of ρ_β). Then the following holds:

Theorem 1.1.2. *Assume that the line $\mathbb{R} + i\gamma = \{\lambda \in \mathbb{C} : Im\lambda = \gamma\}$ contain no eigenvalues of the operator bundle $\mathbf{A}^{(j)}_k(\lambda)$ (here j refers to the number of the outlet $\Omega^j, j = 1, \dots, J$). Then for arbitrary $\mathbf{f}_k \in \mathcal{R}^l_\gamma H(\Omega^j)$ problem (1.1.2) set in the cylinder Ω^j is uniquely solvable in $\mathcal{D}^l_\gamma H(\Omega^j)$ and the estimate*

$$\|\mathbf{u}_k\|_{\mathcal{D}^l_\gamma H(\Omega^j)} \leq \|\mathbf{f}_k\|_{\mathcal{R}^l_\gamma H(\Omega^j)}$$

holds. If on the line $\mathbb{R}+i\gamma$ an eigenvalue of $\mathbf{A}^{(j)}_k(\lambda)$ lies, then the range of the operator is not closed.

Consider problem (1.1.2) set in the domain Ω with several outlets to infinity and the corresponding operator (1.1.14). Below we define another Stokes-type problem (see (1.1.16)) which is formally adjoint to (1.1.2). These two problems differs by signs of the terms $k\mathbf{v}_{ck}$ and $k\mathbf{v}_{sk}$. Consequently, the operator $\mathbf{A}^l_{k,\beta}$ is not formally self-adjoint. However, it can be shown that the operator $\mathbf{A}^l_{\beta,k}$ is Fredholm[v] if and only if the line $\mathbb{R}+i\beta$ does not contain eigenvalues of the operator bundles $\mathbf{A}^{(1)}, \dots, \mathbf{A}^{(J)}$[vi].

According to the Part (1) in Theorem 1.1.1, $\lambda = 0$ is the only real eigenvalue of the operator bundles $\mathbf{A}^{(1)}_k(\lambda), \dots, \mathbf{A}^{(J)}_k(\lambda)$ for all $k = 0, 1, \dots$. Since the eigenvalues of $\mathbf{A}^{(j)}_k(\lambda)$ for each $j = 1, \dots, J$ are isolated, there exists a positive constant β^0_k such that the strips $\{\lambda \in \mathbb{C} : 0 < |Im\lambda| < \beta^0_k\}$ are free of the eigenvalues of the operator bundles $\mathbf{A}^{(j)}_k(\lambda)$, $j = 1, \dots, J$. Consequently, for all $0 < \beta < \beta^0_k$ the operators

$$\mathbf{A}^l_{\beta,k} : \mathcal{D}H^l_\beta(\Omega) \to \mathcal{R}H^l_\beta(\Omega), \quad \mathbf{A}^l_{-\beta,k} : \mathcal{D}H^l_{-\beta}(\Omega) \to \mathcal{R}H^l_{-\beta}(\Omega)$$

are of the Fredholm type. In Section 1.2 we show that it is possible to select one bound β^0, independent of k, for all Stokes-type problems (1.1.2). At the moment for

[v] We recall that the operator $A : B_1 \to B_2$ between two Banach spaces is called a Fredholm operator if its range ImA is closed and the subspaces $\ker A$ and coker$A$ have finite dimensions.

[vi] This fact can be proved by repeating the same steps as in the proofs of Theorems 4.1.2, 5.1.4 in [59], where the self-adjoint Stokes problem is studied. The proof that the operator corresponding to the steady Stokes system is Fredholm, is based on the local estimates (see Theorem 4.1.2 in [59]). Exploiting the structure of the equations (1.1.2) one can derive the same estimates in the case of the Stokes-type problem.

every $k = 0, 1, \dots$ we fix some β form the interval $(0, \beta_k^0)$.

Formally adjoint problem

It is well known that the elliptic problems with the Fredholm operators are solvable if and only if their data satisfy certain compatibility conditions, which are described in terms of orthogonality of the data to solutions of the homogeneous formally adjoint problems. These problems are defined via so-called Green's formula (see, e.g., [44]).

Assume for a while that functions $\mathbf{u}_k = (\mathbf{v}_{ck}, p_{ck}, \mathbf{v}_{sk}, p_{sk})$ and $\mathbf{U}_k = (\mathbf{V}_{ck}, P_{ck}, \mathbf{V}_{sk}, P_{sk})$ are smooth (of class $C_0^\infty(\overline{\Omega})$, for example). Multiplying $\mathbf{S}_k\mathbf{u}_k$ (the left-hand side of the equations in problem (1.1.2)) by $\mathbf{U}_k$ and integrating by parts one gets Green's formula

$$
\begin{aligned}
&(-\nu\Delta\mathbf{v}_{ck} + \nabla p_{ck} + k\mathbf{v}_{sk}, \mathbf{V}_{ck})_\Omega + (-\nabla\cdot\mathbf{v}_{ck}, P_{ck})_\Omega \\
&+(-\nu\Delta\mathbf{v}_{sk} + \nabla p_{sk} - k\mathbf{v}_{ck}, \mathbf{V}_{sk})_\Omega + (-\nabla\cdot\mathbf{v}_{sk}, P_{sk})_\Omega \\
&+(\mathbf{v}_{ck}, \mathbf{n}P_{ck} - \nu\partial_{\mathbf{n}}\mathbf{V}_{ck})_{\partial\Omega} + (\mathbf{v}_{sk}, \mathbf{n}P_{sk} - \nu\partial_{\mathbf{n}}\mathbf{V}_{sk})_{\partial\Omega} \\
&-(\mathbf{v}_{ck}, -\nu\Delta\mathbf{V}_{ck} + \nabla P_{ck} - k\mathbf{V}_{sk})_\Omega - (p_{ck}, -\nabla\cdot\mathbf{V}_{ck})_\Omega \\
&-(\mathbf{v}_{sk}, -\nu\Delta\mathbf{V}_{sk} + \nabla P_{sk} + k\mathbf{V}_{ck})_\Omega - (p_{sk}, -\nabla\cdot\mathbf{V}_{sk})_\Omega \\
&-(\mathbf{n}p_{ck} - \nu\partial_{\mathbf{n}}\mathbf{v}_{ck}, \mathbf{V}_{ck})_{\partial\Omega} - (\mathbf{n}p_{sk} - \nu\partial_{\mathbf{n}}\mathbf{v}_{sk}, \mathbf{V}_{sk})_{\partial\Omega} = 0.
\end{aligned}
\tag{1.1.15}
$$

Here $(\cdot,\cdot)_G$ stands for the inner product in $L^2(G)$, $\mathbf{n}$ and $\partial_{\mathbf{n}}$ denotes the outer normal vector to the surface $\partial\Omega$ and the normal derivative, respectively. The Green formula (1.1.15) defines for problem (1.1.2) the following formally adjoint problem:

$$
\left\{
\begin{array}{rcll}
-\nu\Delta\mathbf{V}_{ck} + \nabla P_{ck} - k\mathbf{V}_{sk} & = & \mathbf{F}_{ck}, & x \in \Omega, \\
-\nabla\cdot\mathbf{V}_{ck} & = & 0, & x \in \Omega, \\
-\nu\Delta\mathbf{V}_{sk} + \nabla P_{sk} + k\mathbf{V}_{ck} & = & \mathbf{F}_{sk}, & x \in \Omega, \\
-\nabla\cdot\mathbf{V}_{sk} & = & 0, & x \in \Omega, \\
\mathbf{V}_{ck} = \mathbf{0},\ \mathbf{V}_{sk} & = & \mathbf{0}, & x \in \partial\Omega.
\end{array}
\right.
\tag{1.1.16}
$$

This problem has similar properties as problem (1.1.2). For example, in the same way, one can show that in every cylinder Ω^j the homogeneous system (1.1.16) has four linearly independent solutions:

$$
\mathbf{U}_{ck}^{j0} = (0,0,0,1,0,0,0,0), \quad \mathbf{U}_{ck}^{j1} = (0,0,\varphi_k^j, -x_3^j, 0, 0, \psi_k^j, 0), \tag{1.1.17}
$$

$$\mathbf{U}_{sk}^{j0} = (0,0,0,0,0,0,0,1), \quad \mathbf{U}_{sk}^{j1} = (0,0,\psi_k^j,0,0,0,-\varphi_k^j,x_3^j), \tag{1.1.18}$$

where functions φ_k^j and ψ_k^j are defined by (1.1.7).

Let us define the vector fields $\mathbf{V}_k = (\mathbf{V}_{ck}, \mathbf{V}_{sk})$, $\mathbf{F}_k = (\mathbf{F}_{ck}, 0, \mathbf{F}_{sk}, 0)$ and denote problem (1.1.16) by

$$\mathbf{S}_k^* \mathbf{U}_k = \mathbf{F}_k, \quad x \in \Omega, \qquad \mathbf{V}_k = \mathbf{0}, \quad x \in \partial\Omega.$$

Now we can rewrite formula (1.1.15) in the following form:

$$\begin{aligned} &(\mathbf{S}_k \mathbf{u}_k, \mathbf{U}_k)_\Omega + (\mathbf{v}_k, \mathbf{n}P_k - \nu \partial_{\mathbf{n}} \mathbf{V}_k)_{\partial\Omega} \\ &= (\mathbf{u}_k, \mathbf{S}_k^* \mathbf{U}_k)_\Omega + (\mathbf{n}p_k - \nu \partial_{\mathbf{n}} \mathbf{v}_k, \mathbf{V}_k)_{\partial\Omega}. \end{aligned} \tag{1.1.19}$$

Remark 1.1.3. Green's formula (1.1.19) holds also if one of the functions, $\mathbf{u}_k$ or $\mathbf{U}_k$, grows, provided that the other function decays at infinity fast enough. For example, this is the case when $\mathbf{u}_k \in \mathcal{D}^2_{-\beta} H(\Omega)$ and $\mathbf{U}_k \in \mathcal{D}^2_{\beta} H(\Omega)$ or $\mathbf{u}_k \in \mathcal{D}^2_{\beta} H(\Omega)$ and $\mathbf{U}_k \in \mathcal{D}^2_{-\beta} H(\Omega)$.

Using this Remark we prove the following statement (see Theorem 3.2 in [54] where the similar result for the Stokes problem is proved):

Theorem 1.1.4. *(a) If $\beta > 0$, then the operator $\mathbf{A}^l_{k,\beta}$ is a monomorphism, i.e.,*

$$\dim \ker \mathbf{A}^l_{k,\beta} = 0.$$

(b) If $\beta < 0$, then $\mathbf{A}^l_{k,\beta}$ is an epimorphism, i.e.,

$$\dim \operatorname{coker} \mathbf{A}^l_{k,\beta} = 0.$$

Proof. (a) Assume that $\mathbf{u}_k = (\mathbf{v}_{ck}, p_{ck}, \mathbf{v}_{sk}, p_{sk}) \in \ker \mathbf{A}^l_{k,\beta}$, i.e., satisfies the homogeneous system

$$\begin{cases} -\nu \Delta \mathbf{v}_{ck} + \nabla p_{ck} + k \mathbf{v}_{sk} &= \mathbf{0}, \qquad x \in \Omega, \\ -\nabla \cdot \mathbf{v}_{ck} &= 0, \qquad x \in \Omega, \\ -\nu \Delta \mathbf{v}_{sk} + \nabla p_{sk} - k \mathbf{v}_{ck} &= \mathbf{0}, \qquad x \in \Omega, \\ -\nabla \cdot \mathbf{v}_{sk} &= 0, \qquad x \in \Omega, \\ \mathbf{v}_{ck} = \mathbf{0}, \ \mathbf{v}_{sk} &= \mathbf{0}, \qquad x \in \partial\Omega. \end{cases} \tag{1.1.20}$$

Multiplying equations $(1.1.20_1)$ and $(1.1.20_3)$ by $\mathbf{v}_{ck}$ and $\mathbf{v}_{sk}$, respectively, integrating by parts and summing the obtained expressions we derive:

$$\nu(\nabla \mathbf{v}_{ck}, \nabla \mathbf{v}_{ck})_\Omega + \nu(\nabla \mathbf{v}_{sk}, \nabla \mathbf{v}_{sk})_\Omega - (p_{ck}, \nabla \cdot \mathbf{v}_{ck})_\Omega - (p_{sk}, \nabla \cdot \mathbf{v}_{sk})_\Omega$$
$$+(\mathbf{n}p_{ck} - \nu \frac{\partial \mathbf{v}_{ck}}{\partial \mathbf{n}}, \mathbf{v}_{ck})_{\partial\Omega} + (\mathbf{n}p_{sk} - \nu \frac{\partial \mathbf{v}_{sk}}{\partial \mathbf{n}}, \mathbf{v}_{sk})_{\partial\Omega} = 0.$$

Taking into account equations ($1.1.20_2$), ($1.1.20_4$) and boundary conditions ($1.1.20_5$), we get from the last identity, the relation

$$\int_\Omega |\nabla \mathbf{v}_{ck}(x)|^2 + |\nabla \mathbf{v}_{sk}(x)|^2 dx = 0.$$

Therefore $\mathbf{v}_{ck} = \text{const}$, $\mathbf{v}_{sk} = \text{const}$. Since the velocity fields are zero on the boundary, we conclude that $\mathbf{v}_{ck} \equiv \mathbf{0}$ and $\mathbf{v}_{sk} \equiv \mathbf{0}$. Moreover, form equations ($1.1.20_1$) and ($1.1.20_3$) we obtain

$$\nabla p_{ck} = -\nu \Delta \mathbf{v}_{ck} - k\mathbf{v}_{sk} \equiv \mathbf{0}, \quad \nabla p_{sk} = -\nu \Delta \mathbf{v}_{sk} + k\mathbf{v}_{ck} \equiv \mathbf{0}.$$

The last identities yield $p_{ck} = \text{const} = c_1$, $p_{sk} = \text{const} = c_2$. Recall that p_{ck} and p_{sk} belong to the space $H^{l+1}_\beta(\Omega)$, i.e., the $L^2(\Omega)$ norms of $(\rho_\beta)^{1/2}p_{ck}, (\rho_\beta)^{1/2}p_{sk} \in L^2_\beta(\Omega)$ are finite. Since $\beta > 0$, i.e., the weight function ρ_β grows at infinity, the constants c_1, c_2 must be zero. Hence, the kernel of the operator $\mathbf{A}^l_{k,\beta}$ is empty.

(b) Green's formula (1.1.19) shows that

$$\text{coker}\mathbf{A}^l_{k,\beta} = \{(\mathbf{U}_k, \mathbf{n}P_k - \nu\partial_\mathbf{n}\mathbf{V}_k|_{\partial\Omega}) : \mathbf{U}_k \in \ker\left(\mathbf{A}^l_{k,-\beta}\right)^*\},$$

where $\left(\mathbf{A}^l_{k,-\beta}\right)^* : \mathcal{D}^l_{-\beta}H(\Omega) \to \mathcal{R}^l_{-\beta}H(\Omega)$ is the operator corresponding to the formally adjoint problem (1.1.16). By assumption of the part (b) the power β is negative. Therefore the set $\left(\mathbf{A}^l_{k,-\beta}\right)^*$ consists from the decaying solutions of the homogeneous system (1.1.16). In the same way as in the Part (a) one can show that this problem has only the trivial solution in the class of decaying functions. Therefore the dimension of the subspace $\ker\left(\mathbf{A}^l_{k,-\beta}\right)^*$ and, consequently, of the subspace $\text{coker}\mathbf{A}^l_{k,\beta}$, is equal to zero. □

The index of the Fredholm operator $\mathbf{A}$ is defined as (see, e.g., [5])

$$\text{Ind}\,\mathbf{A} = \dim\ker\mathbf{A} - \dim\text{coker}\,\mathbf{A}.$$

Theorems 4.3.3, 5.1.4 in [59] relates the indexes of the operators $\mathbf{A}^l_{k,\gamma}$ and $\mathbf{A}^l_{k,\delta}$ with the total multiplicity κ of the eigenvalues of the operator bundles $\mathbf{A}_k^{(1)}(\lambda), \dots, \mathbf{A}_k^{(J)}(\lambda)$ lying between the lines $\mathbb{R} + i\gamma$ and $\mathbb{R} + i\delta$, namely, the following relation holds:

$$\mathrm{Ind}\mathbf{A}^l_{k,\gamma} = \mathrm{Ind}\mathbf{A}^l_{k,\delta} + \kappa.$$

We recall that for every $k = 0, 1, \dots$, between the lines $\mathbb{R} - i\beta_k^0$ and $\mathbb{R} + i\beta_k^0$ there is only one real eigenvalue $\lambda = 0$ of the operators $\mathbf{A}_k^{(1)}(\lambda), \dots, \mathbf{A}_k^{(J)}(\lambda)$. Moreover, according to the Theorem 1.1.1, for each $j = 1, \dots, J$ the multiplicity of this eigenvalue is 4. Therefore in the case of the operators $\mathbf{A}^l_{k,\beta}$ and $\mathbf{A}^l_{k,-\beta}$

$$\mathrm{Ind}\mathbf{A}^l_{k,-\beta} = \mathrm{Ind}\mathbf{A}^l_{k,\beta} + 4J.$$

Taking into account the definition of the index and Theorem 1.1.4, we get that

$$\mathrm{Ind}\mathbf{A}^l_{k,-\beta} = \dim\ker\mathbf{A}^l_{k,-\beta}, \quad \mathrm{Ind}\mathbf{A}^l_{k,\beta} = -\dim\mathrm{coker}\mathbf{A}^l_{k,\beta}.$$

Since the dimensions of $\mathrm{coker}\mathbf{A}^l_{k,\beta}$ and $\ker\mathbf{A}^l_{k,-\beta}$ coincides, we obtain the relations

$$\mathrm{Ind}\mathbf{A}^l_{k,-\beta} = 2J, \quad \mathrm{Ind}\mathbf{A}^l_{k,\beta} = -2J.$$

As a consequence of these relations and Theorem 1.1.4 we conclude that the Fredholm operators $\mathbf{A}^l_{\beta,k}$ and $\mathbf{A}^l_{-\beta,k}$ possess the following properties

$$\begin{aligned} \dim\ker\mathbf{A}^l_{\beta,k} &= \dim\mathrm{coker}\mathbf{A}^l_{-\beta,k} = 0, \\ \dim\ker\mathbf{A}^l_{-\beta,k} &= \dim\mathrm{coker}\mathbf{A}^l_{\beta,k} = 2J, \end{aligned} \tag{1.1.21}$$

i.e., in the class of exponentially decaying functions problem (1.1.2) is solvable if and only if the right-hand side fulfils $2J$ compatibility conditions, and the solution is unique. In the class of growing functions problem (1.1.2) is solvable for every data $\mathbf{f} \in \mathcal{R}^l_{-\beta}H(\Omega)$, but of course not uniquely.

Assume that the lines $\mathbb{R} + i\gamma$ and $\mathbb{R} + i\delta$ are free of eigenvalues of the operators $\mathbf{A}^{(1)}(\lambda), \dots, \mathbf{A}^{(J)}(\lambda)$. Then, according to the general theory of elliptic equations (see [35], Subsection 3.2.2 in [45]) for the solutions $\mathbf{u}_{k,\gamma} \in \mathcal{D}^l_\gamma H(\Omega)$ and $\mathbf{u}_{k,\delta} \in \mathcal{D}^l_\delta H(\Omega)$ of problem (1.1.2) the following relation

$$\mathbf{u}_{k,\gamma} = \sum_\lambda \sum_{l=0}^{N_\lambda} c^l_{k,\lambda}\mathbf{u}^l_{k,\lambda} + \mathbf{u}_{k,\delta}$$

holds. In this relation the sum is taken over all solutions to the homogeneous problem corresponding to the eigenvalues λ such that $\gamma < Im\lambda < \delta$. The constants $c^l_{k,\lambda}$ are defined using certain functionals on data of the problem (see [35], [45]).

Consider system (1.1.2). Formulas (1.1.11), (1.1.12) describe the solutions (corresponding to the eigenvalue $\lambda = 0$) to the homogeneous Stokes-type problem. Therefore using Theorem 3.2.4 in [45] we may express the "growing" at infinity solution to problem (1.1.2) in terms of the functions (1.1.11), (1.1.12) and a vector-field, which decays at infinity exponentially. Namely, the following statement holds:

Theorem 1.1.5. *If* $\mathbf{u}_k = (\mathbf{v}_{ck}, p_{ck}, \mathbf{v}_{sk}, p_{sk}) \in \mathcal{D}^l_{-\beta}H(\Omega)$, $0 < \beta < \beta^0_k$, *is the solution to problem* (1.1.2) *with the right-hand side* $\mathbf{f}_k \in \mathcal{R}^l_\beta H(\Omega)$, *then*

$$\mathbf{u}_k = \sum_{j=1}^{J} \chi^j \left\{ a^j_{ck}\mathbf{u}^{j0}_{ck} + a^j_{sk}\mathbf{u}^{j0}_{sk} + b^j_{ck}\mathbf{u}^{j1}_{ck} + b^j_{sk}\mathbf{u}^{j1}_{sk} \right\} + \tilde{\mathbf{u}}_k. \tag{1.1.22}$$

Here $a^j_{ck}, a^j_{sk}, b^j_{ck}, b^j_{sk} \in \mathbb{R}$, $\tilde{\mathbf{u}}_k \in \mathcal{D}^l_\beta H(\Omega)$ *and* χ^j *is a smooth cut-off function such that* $\mathrm{supp}(\chi_j) \subseteq \Omega^j_+$ *and* $\chi^j(x) = 1$ *for* $x^j_3 \geq 1$, $j = 1, \dots, J$.

Remark 1.1.6. Recall that the vector-fields $\mathbf{u}_k = (\mathbf{v}_{ck}, p_{ck}, \mathbf{v}_{sk}, p_{sk})$, $k = 0, 1, \dots$ are composed from the Fourier coefficient of series (1.1.1). Taking into account the structure of the functions $\mathbf{u}^{j0}_{ck}$, $\mathbf{u}^{j0}_{sk}$, $\mathbf{u}^{j1}_{ck}$ and $\mathbf{u}^{j1}_{sk}$ (see (1.1.11), (1.1.12)) we notice that the coefficients in (1.1.1) admit the following representations:

$$\begin{aligned} \mathbf{v}_{ck} &= \left(0, 0, \sum_{j=1}^{J} \chi^j \left\{ b^j_{ck}\phi^j_k + b^j_{sk}\psi^j_k \right\}\right) + \tilde{\mathbf{v}}_{ck}, \\ \mathbf{v}_{sk} &= \left(0, 0, \sum_{j=1}^{J} \chi^j \left\{ b^j_{sk}\phi^j_k - b^j_{ck}\psi^j_k \right\}\right) + \tilde{\mathbf{v}}_{sk}, \end{aligned} \tag{1.1.23}$$

$$p_{ck} = \sum_{j=1}^{J} \chi^j \left\{ a^j_{ck} - b^j_{ck}x^j_3 \right\} + \tilde{p}_{ck}, \quad p_{sk} = \sum_{j=1}^{J} \chi^j \left\{ a^j_{sk} - b^j_{sk}x^j_3 \right\} + \tilde{p}_{sk}. \tag{1.1.24}$$

Remark 1.1.7. In the same way as above we can show that any function $\mathbf{U}_k \in \mathcal{D}^l_{-\beta}H(\Omega)$, which has a velocity part $\mathbf{V}_k|_{\partial\Omega} = \mathbf{0}$ and solves the formally adjoint problem (1.1.16) for some $\mathbf{F}_k \in \mathcal{R}^l_\beta H(\Omega)$, may be expressed as

$$\mathbf{U}_k = \sum_{j=1}^{J} \chi^j \left\{ A^j_{ck}\mathbf{U}^{j0}_{ck} + A^j_{sk}\mathbf{U}^{j0}_{sk} + B^j_{ck}\mathbf{U}^{j1}_{ck} + B^j_{sk}\mathbf{U}^{j1}_{sk} \right\} + \widetilde{\mathbf{U}}_k. \tag{1.1.25}$$

Here $\widetilde{\mathbf{U}}_k \in \mathcal{D}^l_\beta H(\Omega)$, $A^j_{ck}, A^j_{sk}, B^j_{ck}, B^j_{sk} \in \mathbb{R}$, while the functions $\mathbf{U}^{j0}_{ck}$, $\mathbf{U}^{j0}_{sk}$, $\mathbf{U}^{j1}_{ck}$, $\mathbf{U}^{j1}_{sk}$ are defined by (1.1.17) and (1.1.18).

1.2 Estimates for the decaying term

In this section we show that it is possible to choose the same exponent β in the definition of the weight function ρ_β for all Stokes-type problems (1.1.2), i.e., independently of k. Let $\mathbf{u}_k = (\mathbf{v}_{ck}, p_{ck}, \mathbf{v}_{sk}, p_{sk}) \in \mathcal{D}^l_\beta H(\Omega)$ be the solution of (1.1.2) with data from $\mathcal{R}^l_\beta H(\Omega)$. Let us fix some positive β and define the step-weight function

$$\rho^{(r)}_\beta(x) = \begin{cases} \rho_\beta(x), & x^j_3 < r, \quad j = 1, \dots, J, \\ \rho_\beta(r), & x^j_3 \geq r, \quad j = 1, \dots, J. \end{cases}$$

Notice that the function $\rho^{(r)}_\beta$ is constant for $x^j_3 \geq r$, $j = 1, \dots, J$. The product $\rho^{(r)}_\beta \mathbf{v}_{ck}$ and $\rho^{(r)}_\beta \mathbf{v}_{sk}$ (of the weight function and the vectors with zero divergence) are no more divergence-free. In order to construct the divergence-free vector-fields, we shall use the following

Lemma 1.2.1. (see Lemma 1.13 in [80]) Let $\mathbf{v} \in (\mathring{H}^1(\Omega))^3$, $\nabla \cdot \mathbf{v} = 0$ and

$$\int_{\omega^j} \mathbf{v} \cdot \mathbf{n}\, dS = 0, \quad j = 1, \dots, J.$$

Then there exists a vector-field $\mathbf{w}^{(r)} \in (\mathring{H}^1(\Omega))^3$ such that $\operatorname{supp} \mathbf{w}^{(r)} \subset \cup_{j=1}^J \Omega^j_r$ and

$$\nabla \cdot \mathbf{w}^{(r)} = -\nabla \cdot \left(\rho^{(r)}_\beta \mathbf{v} \right), \quad x \in \Omega.$$

Moreover, there holds the estimate

$$\int_\Omega \rho^{(r)}_{-\beta}(x) |\nabla \mathbf{w}^{(r)}(x)|^2 dx \leq c\beta^2 \int_\Omega \rho^{(r)}_\beta(x) |\mathbf{v}(x)|^2 dx,$$

with the constant c independent of r, β and $\mathbf{v}$.

We also need the weighted Poincaré inequality (see Lemma 1.9 in [80]):

$$\int_\Omega \rho_\gamma(x) |v(x)|^2 dx \leq c \int_\Omega \rho_\gamma(x) |\nabla v(x)|^2 dx, \tag{1.2.1}$$

which holds for every γ and each function v equal to zero on $\partial\Omega$ and such that $(\rho_\gamma)^{1/2} v \in H^1(\Omega)$. The constant c is independent of r, γ and v. Moreover, in (1.2.1) we can take $\rho^{(r)}_\gamma$ instead of ρ_γ.

Since $\mathbf{u}_k \in \mathcal{D}H^l_\beta(\Omega)$ for some $0 < \beta < \beta^0_k$, functions $\mathbf{v}_{ck}$ and $\mathbf{v}_{sk}$ belong to the space $(\mathring{H}^1(\Omega))^3$ and

$$\int_{\omega^j} \mathbf{v}_{ck} \cdot \mathbf{n} dS = 0, \quad \int_{\omega^j} \mathbf{v}_{sk} \cdot \mathbf{n} dS = 0, \quad j = 1, \dots, J. \tag{1.2.2}$$

According to Lemma 2.3, there exist compactly supported vector-functions $\mathbf{w}_{ck}^{(r)}$, $\mathbf{w}_{sk}^{(r)} \in (\mathring{H}^1(\Omega))^3$ such that

$$\begin{aligned} \nabla \cdot \mathbf{w}_{ck}^{(r)} &= -\nabla \cdot (\rho_\beta^{(r)} \mathbf{v}_{ck}) = -\nabla \rho_\beta^{(r)} \cdot \mathbf{v}_{ck}, \quad x \in \Omega, \\ \nabla \cdot \mathbf{w}_{sk}^{(r)} &= -\nabla \cdot (\rho_\beta^{(r)} \mathbf{v}_{sk}) = -\nabla \rho_\beta^{(r)} \cdot \mathbf{v}_{sk}, \quad x \in \Omega, \end{aligned}$$

and the following estimates

$$\begin{aligned} \|(\rho_{-\beta}^{(r)})^{1/2} \nabla \mathbf{w}_{ck}^{(r)}\|_{L^2(\Omega)} \le c\beta \|(\rho_\beta^{(r)})^{1/2} \mathbf{v}_{ck}\|_{L^2(\Omega)} \le c\beta \|(\rho_\beta^{(r)})^{1/2} \nabla \mathbf{v}_{ck}\|_{L^2(\Omega)}, \\ \|(\rho_{-\beta}^{(r)})^{1/2} \nabla \mathbf{w}_{sk}^{(r)}\|_{L^2(\Omega)} \le c\beta \|(\rho_\beta^{(r)})^{1/2} \mathbf{v}_{sk}\|_{L^2(\Omega)} \le c\beta \|(\rho_\beta^{(r)})^{1/2} \nabla \mathbf{v}_{sk}\|_{L^2(\Omega)} \end{aligned} \tag{1.2.3}$$

holds.

Theorem 1.2.2. *Let $\partial\Omega \in C^2$ and let $\mathbf{f}_{ck}$, $\mathbf{f}_{sk} \in (L^2_\beta(\Omega))^3$. Suppose that the exponent β in the weight-function ρ_β satisfies the condition $\beta < \beta^*$, where the number β^* is sufficiently small (see details in the proof). Then the velocity fields $\mathbf{v}_{ck}$ and $\mathbf{v}_{sk}$ of the solution $(\mathbf{v}_{ck}, p_{ck}, \mathbf{v}_{sk}, p_{sk})$ to problem* (1.1.2) *belong to $(H^2_\beta(\Omega))^3$, the pressure gradients ∇p_{ck} and ∇p_{sk} belong to $(L^2_\beta(\Omega))^3$ and the estimate*

$$\begin{aligned} \|\mathbf{v}_{ck}\|_{H^2_\beta(\Omega)} + \|\mathbf{v}_{sk}\|_{H^2_\beta(\Omega)} + \|\nabla p_{ck}\|_{L^2_\beta(\Omega)} + \|\nabla p_{sk}\|_{L^2_\beta(\Omega)} \\ \le c\|\mathbf{f}_{ck}\|_{L^2_\beta(\Omega)} + \|\mathbf{f}_{sk}\|_{L^2_\beta(\Omega)}. \end{aligned} \tag{1.2.4}$$

holds. Moreover the pressure functions p_{ck} and p_{sk} tend in each outlet Ω^j_+ to constants p^j_{ck} and p^j_{sk}, respectively, and for $0 < \beta' < \beta < \beta^*$ the following estimates

$$\begin{aligned} \int_{\Omega^j_+} e^{2\beta' x_3^j} |p_{ck}(x'^j, x_3^j) - p^j_{ck}|^2 dx'^j dx_3^j \le c \int_\Omega \rho_\beta \left(|\mathbf{f}_{ck}(x)|^2 + |\mathbf{f}_{sk}(x)|^2 \right) dx, \\ \int_{\Omega^j_+} e^{2\beta' x_3^j} |p_{sk}(x'^j, x_3^j) - p^j_{sk}|^2 dx'^j dx_3^j \le c \int_\Omega \rho_\beta \left(|\mathbf{f}_{ck}(x)|^2 + |\mathbf{f}_{sk}(x)|^2 \right) dx \end{aligned} \tag{1.2.5}$$

hold. Constants c in estimates (1.2.4), (1.2.5) *are independent of k.*

Proof. First multiply equation $(1.1.2_1)$ by $k(\rho_\beta^{(r)} \mathbf{v}_{ck} + \mathbf{w}_{ck}^{(r)})$ and equation $(1.1.2_3)$ by $k(\rho_\beta^{(r)} \mathbf{v}_{sk} + \mathbf{w}_{sk}^{(r)})$, and then multiply $(1.1.2_1)$ by $k(\rho_\beta^{(r)} \mathbf{v}_{sk} + \mathbf{w}_{sk}^{(r)})$ and $(1.1.2_3)$ by $-k(\rho_\beta^{(r)} \mathbf{v}_{ck} + \mathbf{w}_{ck}^{(r)})$, add the obtained equalities, and integrate them over the domain

Ω. After the integration by parts we get the relation

$$\nu k\int_\Omega \nabla \mathbf{v}_{ck}\cdot\nabla(\rho_\beta^{(r)}\mathbf{v}_{ck}+\mathbf{w}_{ck}^{(r)})dx+\nu k\int_\Omega \nabla \mathbf{v}_{sk}\cdot\nabla(\rho_\beta^{(r)}\mathbf{v}_{sk}+\mathbf{w}_{sk}^{(r)})dx$$

$$+\nu k\int_\Omega \nabla \mathbf{v}_{ck}\cdot\nabla(\rho_\beta^{(r)}\mathbf{v}_{sk}+\mathbf{w}_{sk}^{(r)})dx-\nu k\int_\Omega \nabla \mathbf{v}_{sk}\cdot\nabla(\rho_\beta^{(r)}\mathbf{v}_{ck}+\mathbf{w}_{ck}^{(r)})dx$$

$$+k^2\int_\Omega \mathbf{v}_{ck}\cdot(\rho_\beta^{(r)}\mathbf{v}_{ck}+\mathbf{w}_{ck}^{(r)})dx+k^2\int_\Omega \mathbf{v}_{sk}\cdot(\rho_\beta^{(r)}\mathbf{v}_{sk}+\mathbf{w}_{sk}^{(r)})dx.$$

$$+k^2\int_\Omega(\mathbf{v}_{sk}\cdot\mathbf{w}_{ck}^{(r)}-\mathbf{v}_{ck}\cdot\mathbf{w}_{sk}^{(r)})dx=\int_\Omega(\mathbf{f}_{ck}-\mathbf{f}_{sk})\cdot k(\rho_\beta^{(r)}\mathbf{v}_{ck}+\mathbf{w}_{ck}^{(r)})dx$$

$$+\int_\Omega(\mathbf{f}_{ck}+\mathbf{f}_{sk})\cdot k(\rho_\beta^{(r)}\mathbf{v}_{sk}+\mathbf{w}_{sk}^{(r)})dx$$

In the left-hand side of the last identity we leave only the sum

$$\nu k\int_\Omega \rho_\beta^{(r)}\left(|\nabla\mathbf{v}_{ck}|^2+|\nabla\mathbf{v}_{sk}|^2\right)dx+k^2\int_\Omega \rho_\beta^{(r)}\left(|\mathbf{v}_{ck}|^2+|\mathbf{v}_{sk}|^2\right)dx,$$

putting all other terms on the right-hand side. We estimate the obtained right-hand side using Hölder's and Young's inequalities, the weighted Poincaré inequality (1.2.1) and estimates (1.2.3):

$$\left|\int_\Omega(\mathbf{f}_{ck}-\mathbf{f}_{sk})\cdot k(\rho_\beta^{(r)}\mathbf{v}_{ck}+\mathbf{w}_{sk}^{(r)})dx\right|$$

$$\le\frac{1}{\varepsilon}\int_\Omega\rho_\beta^{(r)}\left(|\mathbf{f}_{ck}|^2+|\mathbf{f}_{sk}|^2\right)dx+c_1\varepsilon k^2\left(\int_\Omega\rho_\beta^{(r)}|\mathbf{v}_{ck}|^2dx+\int_\Omega\rho_{-\beta}^{(r)}|\mathbf{w}_{ck}^{(r)}|^2dx\right)$$

$$\le\frac{1}{\varepsilon}\int_\Omega\rho_\beta^{(r)}\left(|\mathbf{f}_{ck}|^2+|\mathbf{f}_{sk}|^2\right)dx+c_2\varepsilon k^2\left(\int_\Omega\rho_\beta^{(r)}|\mathbf{v}_{ck}|^2dx+\int_\Omega\rho_{-\beta}^{(r)}|\nabla\mathbf{w}_{ck}^{(r)}|^2dx\right)$$

$$\le\frac{1}{\varepsilon}\int_\Omega\rho_\beta\left(|\mathbf{f}_{ck}|^2+|\mathbf{f}_{sk}|^2\right)dx+c_3\varepsilon k^2\int_\Omega\rho_\beta^{(r)}|\mathbf{v}_{ck}|^2dx,$$

$$\nu k\left|\int_\Omega\nabla\mathbf{v}_{ck}\cdot(\nabla\rho_\beta^{(r)}\cdot\mathbf{v}_{ck}+\nabla\mathbf{w}_{ck}^{(r)})dx\right|\le c\nu\beta k\int_\Omega\rho_\beta^{(r)}|\nabla\mathbf{v}_{ck}||\mathbf{v}_{ck}|dx$$

$$+\nu k\int_\Omega|\nabla\mathbf{v}_{ck}||\nabla\mathbf{w}_{ck}^{(r)}|dx\le c_4\nu\beta k\int_\Omega\rho_\beta^{(r)}|\nabla\mathbf{v}_{ck}|^2dx+\nu k\left(\int_\Omega\rho_\beta^{(r)}|\nabla\mathbf{v}_{ck}|^2dx\right)^{1/2}$$

$$\times\left(\int_\Omega\rho_{-\beta}^{(r)}|\nabla\mathbf{w}_{ck}^{(r)}|^2dx\right)^{1/2}\le c_5\nu\beta k\int_\Omega\rho_\beta^{(r)}|\nabla\mathbf{v}_{ck}|^2dx,$$

$$k^2\left|\int_\Omega\mathbf{v}_{ck}\cdot\mathbf{w}_{ck}^{(r)})dx\right|\le k^2\left(\int_\Omega\rho_\beta^{(r)}|\mathbf{v}_{ck}|^2dx\right)^{1/2}\left(\int_\Omega\rho_{-\beta}^{(r)}|\mathbf{w}_{ck}^{(r)}|^2dx\right)^{1/2}$$

$$\le c_6\beta k^2\int_\Omega\rho_\beta^{(r)}|\mathbf{v}_{ck}|^2dx.$$

The rest terms could be estimated analogously. As a result we obtain the inequality

$$\begin{aligned}
&\nu k \int_\Omega \rho_\beta^{(r)} \left(|\nabla \mathbf{v}_{ck}|^2 + |\nabla \mathbf{v}_{sk}|^2\right) dx + k^2 \int_\Omega \rho_\beta^{(r)} \left(|\mathbf{v}_{ck}|^2 + |\mathbf{v}_{sk}|^2\right) dx \\
\leq &\frac{1}{\varepsilon} \int_\Omega \rho_\beta \left(|\mathbf{f}_{ck}|^2 + |\mathbf{f}_{sk}|^2\right) dx + c_7 \nu \beta k \int_\Omega \rho_\beta^{(r)} \left(|\nabla \mathbf{v}_{ck}|^2 + |\nabla \mathbf{v}_{sk}|^2\right) dx \\
&+ c_8(\varepsilon + \beta) k^2 \int_\Omega \rho_\beta^{(r)} \left(|\mathbf{v}_{ck}|^2 + |\mathbf{v}_{sk}|^2\right) dx.
\end{aligned} \tag{1.2.6}$$

Taking $\varepsilon \leq 1/(4c_8)$ and assuming that $\beta \leq \beta^* = \min\{1/(2c_7), 1/(4c_8)\}$, from (1.2.6) follows that

$$\begin{aligned}
&\frac{\nu k}{2} \int_\Omega \rho_\beta^{(r)} \left(|\nabla \mathbf{v}_{ck}|^2 + |\nabla \mathbf{v}_{sk}|^2\right) dx + \frac{k^2}{2} \int_\Omega \rho_\beta^{(r)} \left(|\mathbf{v}_{ck}|^2 + |\mathbf{v}_{sk}|^2\right) dx \\
&\qquad \leq c_9 \int_\Omega \rho_\beta \left(|\mathbf{f}_{ck}|^2 + |\mathbf{f}_{sk}|^2\right) dx,
\end{aligned}$$

where the constant c_9 is independent of k. Since in the last inequality the right-hand side is independent of r, we may pass to the limit as $r \to \infty$:

$$\begin{aligned}
&\frac{\nu k}{2} \int_\Omega \rho_\beta \left(|\nabla \mathbf{v}_{ck}|^2 + |\nabla \mathbf{v}_{sk}|^2\right) dx + \frac{k^2}{2} \int_\Omega \rho_\beta \left(|\mathbf{v}_{ck}|^2 + |\mathbf{v}_{sk}|^2\right) dx \\
&\qquad \leq c \int_\Omega \rho_\beta \left(|\mathbf{f}_{ck}|^2 + |\mathbf{f}_{sk}|^2\right) dx.
\end{aligned} \tag{1.2.7}$$

Consider now $\mathbf{v}_{ck}$ and $\mathbf{v}_{sk}$ in (1.1.2) as solutions to the following Stokes problems:

$$\begin{cases}
-\nu \Delta \mathbf{v}_{ck} + \nabla p_{ck} &= \mathbf{f}_{ck} - k\mathbf{v}_{sk}, \\
-\nabla \cdot \mathbf{v}_{ck} &= 0, \\
\mathbf{v}_{ck}|_{\partial\Omega} &= \mathbf{0}
\end{cases}$$

and

$$\begin{cases}
-\nu \Delta \mathbf{v}_{sk} + \nabla p_{sk} &= \mathbf{f}_{sk} + k\mathbf{v}_{ck}, \\
-\nabla \cdot \mathbf{v}_{sk} &= 0, \\
\mathbf{v}_{sk}|_{\partial\Omega} &= \mathbf{0}.
\end{cases}$$

From inequality (1.2.7) it follows that

$$\begin{aligned}
&\|(\rho_\beta)^{1/2}(\mathbf{f}_{ck} - k\mathbf{v}_{sk})\|_{L^2(\Omega)} + \|(\rho_\beta)^{1/2}(\mathbf{f}_{sk} + k\mathbf{v}_{ck})\|_{L^2(\Omega)} \\
&\qquad \leq c\|(\rho_\beta)^{1/2}\mathbf{f}_{ck}\|_{L^2(\Omega)} + \|(\rho_\beta)^{1/2}\mathbf{f}_{sk}\|_{L^2(\Omega)},
\end{aligned}$$

and, therefore, (see Theorem 3.2 in [80]), the estimate (1.2.4) holds.

Since the gradients of the pressures p_{ck} and p_{sk} decay exponentially, we get that

$$\lim_{x\in\Omega^j_+,\, x^j_3\to\infty} p_{ck}(x) = p^j_{ck}, \qquad \lim_{x\in\Omega^j_+,\, x^j_3\to\infty} p_{sk}(x) = p^j_{sk}, \quad j=1,\dots,J.$$

The considerations below are valid for both functions p_{ck} and p_{sk}, therefore we write index k instead of indexes ck and sk for the pressure functions. Let $0<\beta'<\beta<\beta^*$. Then from the equality

$$p_k(x) - p^j_k = -\int_{x^j_3}^{\infty} \frac{\partial p_k}{\partial y^j_3}(y^j, z^j)dz^j$$

we get the estimate

$$e^{\beta' x^j_3}\int_{\omega^j} |p_k(y^j, x^j_3) - p^j_k| dy^j \le \int_{x^j_3}^{\infty}\int_{\omega^j} e^{(\beta'-\beta)z^j} e^{\beta z^j} |\nabla p_k(y^j, z^j)| dy^j dz^j$$

$$\le \left(\int_{x^j_3}^{\infty}\int_{\omega^j} e^{2(\beta'-\beta)z^j} dy^j dz^j\right)^{1/2} \left(\int_{x^j_3}^{\infty}\int_{\omega^j} e^{2\beta z^j}|\nabla p_k(y^j, z^j)|^2 dy^j dz^j\right)^{1/2}$$

$$\le c e^{(\beta'-\beta)x^j_3}\left(\|\rho_\beta \mathbf{f}_{ck}\|_{L^2(\Omega)} + \|\rho_\beta \mathbf{f}_{sk}\|_{L^2(\Omega)}\right).$$

Since $\beta'-\beta<0$, we can integrate both parts of the above inequality with respect to x^j_3 over $(0,\infty)$. As a result we obtain

$$\|\rho^{1/2}_{\beta'}(p_k - p^j_k)\|_{L^1(\Omega^j_+)} \le c\left(\|\rho^{1/2}_\beta \mathbf{f}_{ck}\|_{L^2(\Omega)} + \|\rho^{1/2}_\beta \mathbf{f}_{sk}\|_{L^2(\Omega)}\right). \tag{1.2.8}$$

Denote $G^j_l = \Omega^j_{l+1}\setminus\Omega^j_l$. Then the following interpolation inequality (see Chapter 2 in [43])

$$\|p_k - p^j_k\|^2_{L^2(G^j_l)} \le c\left(\|\nabla p_k\|^2_{L^2(G^j_l)} + \|p_k - p^j_k\|^2_{L^1(G^j_l)}\right), \tag{1.2.9}$$

holds. Here constant c is independent of l. Multiplying (1.2.9) by $e^{2\beta' l}$ we get

$$e^{2\beta' l}\int_{G^j_l} |p_k(x) - p^j_k|^2 dx$$

$$\le c\left(\left(e^{\beta' l}\int_{G^j_l} |\nabla p_k(x)|^2 dx\right)^2 + \left(e^{\beta' l}\int_{G^j_l} |p_k(x) - p^j_k| dx\right)^2\right).$$

Since $e^{2\beta' l} \leq e^{2\beta' x_3^j} \leq e e^{2\beta' l}$ for $x_3^j \in [l, l+1]$, we have

$$\int_{G_l^j} e^{2\beta' x_3^j} |p_k(x) - p_k^j|^2 dx$$
$$\leq c \left(e^{2(\beta'-\beta)l} \int_{G_l^j} e^{2\beta x_3^j} |\nabla p_k(x)|^2 dx + \left(\int_{G_l^j} e^{\beta' x_3^j} |p_k(x) - p_k^j| dx \right)^2 \right).$$

Summing the above inequalities by l from 0 to ∞ yields

$$\begin{aligned} \|e^{\beta' x_3^j}(p_k - p_k^j)\|^2_{L^2(\Omega_+^j)} &\leq c \left(\|e^{\beta x_3^j} \nabla p_k\|^2_{L^2(\Omega_+^j)} \sum_{l=0}^{\infty} e^{2(\beta'-\beta)l} \right. \\ + \sum_{l=0}^{\infty} \left(\left(\int_{G_l^j} e^{\beta' x_3^j} |p_k(x) - p_k^j| dx \right)^2 \right) &\leq c \left(\|e^{\beta x_3^j} \nabla p_k\|^2_{L^2(\Omega_+^j)} \sum_{l=0}^{\infty} e^{2(\beta'-\beta)l} \right. \\ &\quad \left. + \|e^{\beta' x_3^j}(p_k(x) - p_k^j\|_{L^1(\Omega_+^j)} \sum_{l=0}^{\infty} \int_{G_l^j} e^{\beta' x_3^j} |p_k(x) - p_k^j| dx \right) \\ &\leq c \left(\|e^{\beta x_3^j} \nabla p_k\|^2_{L^2(\Omega_+^j)} + \|e^{\beta' x_3^j}(p_k - p_k^j)\|^2_{L^1(\Omega_+^j)} \right). \end{aligned} \tag{1.2.10}$$

Estimate (1.2.5) follows from estimates (1.2.4), (1.2.8) and (1.2.10). □

Remark 1.2.3. According to Theorem 1.2.2 the exponents β and β' do not depend on k. Therefore in the rest of the Thesis we fix the same exponents $0 < \beta' < \beta < \beta^*$ for all $k = 0, 1, \dots$ (unless otherwise stated).

1.3 Generalized Green's formula for Stokes-type problems

In the following we consider solutions from the class $\mathcal{D}^l_{-\beta}H(\Omega)$ (non-decaying at infinity) to problem (1.1.2) with exponentially vanishing data, i.e., with $\mathbf{f}_k \in \mathcal{R}^l_\beta H(\Omega)$. Since $\mathcal{R}^l_\beta H(\Omega) \subset \mathcal{R}^l_{-\beta} H(\Omega)$ the solution from this class exists for every $\mathbf{f}_k \in \mathcal{R}^l_\beta H(\Omega)$, however it is not unique (see (1.1.21)). Let $\mathbb{D}^l_{\pm\beta} H(\Omega) \subset \mathcal{D}^l_{-\beta} H(\Omega)$ denotes the pre-image of the set $\mathcal{R}^l_\beta H(\Omega)$. Then the corresponding operator[vii]

$$\mathbf{A}^l_{-\beta\to\beta,k} : \mathbb{D}^l_{\pm\beta} H(\Omega) \to \mathcal{R}^l_\beta H(\Omega)$$

[vii]This operator maps an element of $\mathcal{D}^l_{-\beta}H(\Omega)$, a function that may grow, to an element form $\mathcal{R}^l_\beta H(\Omega)$, an exponentially decaying function. This fact is emphasized by using subscript $-\beta \to \beta$ in the notation of the operator.

inherits the Fredholm property from $\mathbf{A}^l_{-\beta,k}$ and the following equalities

$$\dim\ker\mathbf{A}^l_{-\beta\to\beta,k}=2J,\quad \dim\operatorname{coker}\mathbf{A}^l_{-\beta\to\beta,k}=0 \tag{1.3.1}$$

hold. Moreover, according to Theorem 1.1.5 elements of $\mathbb{D}^l_{\pm\beta}H(\Omega)$ admit representations (1.1.22).

Analogously we denote by $\mathbb{D}^l_{\pm\beta}H(\Omega)^*$ the subset of $\mathcal{D}^l_{-\beta}H(\Omega)$ consisting from solutions to the formally adjoint problem (1.1.16) with the exponentially decaying data. This set consists from the vector-fields having form (1.1.25). The corresponding operator

$$\left(\mathbf{A}^l_{-\beta\to\beta,k}\right)^*:\mathbb{D}^l_{\pm\beta}H(\Omega)^*\to\mathcal{R}^l_\beta H(\Omega)$$

possesses the same properties as $\mathbf{A}^l_{-\beta\to\beta,k}$:

$$\dim\ker\left(\mathbf{A}^l_{-\beta\to\beta,k}\right)^*=2J,\quad \dim\operatorname{coker}\left(\mathbf{A}^l_{-\beta\to\beta,k}\right)^*=0. \tag{1.3.2}$$

In order to get a unique growing solution $\mathbf{u}_k\in\mathbb{D}^l_{\pm\beta}H(\Omega)$ to problem (1.1.2), one should fix constants $\{a^j_{ck},a^j_{sk},b^j_{ck},b^j_{sk}\}_{j=1}^J$ in the main part of the asymptotic expression (1.1.22). However the first equation in (1.3.1) indicates that only the half of them may be selected independently. Furthermore, examples in [54], where the analogous problems for the steady Stokes system were considered, show that not every collection of $2J$ constants from the set $\{a^j_{ck},a^j_{sk},b^j_{ck},b^j_{sk}\}_{j=1}^J$ is admissible. Proper selection of these constants may be carried out with the help of the generalized Green formula.

Recall that in Section 1.1 we derived the classical Green formula (1.1.15) which is valid in the case of smooth or sufficiently fast decaying functions (see Remark 1.1.3). Below we derive the Green formula which holds for functions $\mathbf{u}_k\in\mathbb{D}^l_{\pm\beta}H(\Omega)$ and $\mathbf{U}_k\in\mathbb{D}^l_{\pm\beta}H(\Omega)^*$ (notice that in this case (1.1.15) is not valid since the elements of $\mathbb{D}^l_{\pm\beta}H(\Omega)$ and $\mathbb{D}^l_{\pm\beta}H(\Omega)^*$, in general, do not vanish at infinity).

Let us apply the operator $\mathbf{S}_k$ to the function $\mathbf{u}_k$ having asymptotic representation (1.1.22). We get the expression

$$\mathbf{S}_k\left(\sum_{j=1}^J\chi^j\left\{a^j_{ck}\mathbf{u}^{j0}_{ck}+a^j_{sk}\mathbf{u}^{j0}_{sk}+b^j_{ck}\mathbf{u}^{j1}_{ck}+b^j_{sk}\mathbf{u}^{j1}_{sk}\right\}+\tilde{\mathbf{u}}_k\right)$$

with terms which either have compact supports, e.g., $\mathbf{S}_k(\chi^j a^j_{ck}\mathbf{u}^{j0}_{ck})$, or decay at infinity exponentially, e.g., $\mathbf{S}_k(\tilde{\mathbf{u}}_k)$. Therefore we can multiply this expression by

$\mathbf{U}_k \in \mathbb{D}^l_{\pm\beta} H(\Omega)^*$ and integrate over the truncated domain

$$\Omega_L = \left\{ x \in \Omega : \text{ if } x \in \Omega^j_+ \text{ then } x^j_3 < L, j = 1, \dots, J \right\}.$$

The boundary $\partial\Omega_L$, besides the part which belongs to $\partial\Omega$, contains the sets ω^j, $j = 1, \dots, J$. Consequently, integration by parts results in Green's formula[viii]

$$(\mathbf{S}_k \mathbf{u}_k, \mathbf{U}_k)_{\Omega_L} - (\mathbf{u}_k, \mathbf{S}^*_k \mathbf{U}_k)_{\Omega_L} = q_L(\mathbf{u}_k, \mathbf{U}_k), \tag{1.3.3}$$

with

$$\begin{aligned} q_L(\mathbf{u}_k, \mathbf{U}_k) = \sum_{j=1}^{J} \int_{\omega^j} \Big(\mathbf{v}_{ck} \cdot (\mathbf{n} P_{ck} - \nu \partial_3 \mathbf{V}_{ck}) + \mathbf{v}_{sk} \cdot (\mathbf{n} P_{sk} - \nu \partial_3 \mathbf{V}_{sk}) \Big)\Big|_{x^j_3 = L} dy^j \\ - \int_{\omega^j} \Big((\mathbf{n} p_{ck} - \nu \partial_3 \mathbf{v}_{ck}) \cdot \mathbf{V}_{ck} - (\mathbf{n} p_{sk} - \nu \partial_3 \mathbf{v}_{sk}) \cdot \mathbf{V}_{sk} \Big)\Big|_{x^j_3 = L} dy^j. \end{aligned}$$

In the last formula we denoted by ∂_3 the partial derivative $\partial / \partial x^j_3$ and by $\mathbf{n} = (0, 0, 1)$ the outward normal vector to ω^j. We can evaluate boundary integrals $q_L(\mathbf{u}_k, \mathbf{U}_k)$ using expressions (1.1.22) and (1.1.25). Indeed the terms

$$q_L(\tilde{\mathbf{u}}_k, \widetilde{\mathbf{U}}_k), \quad q_L(\mathbf{u}_k, \widetilde{\mathbf{U}}_k), \quad q_L(\tilde{\mathbf{u}}_k, \mathbf{U}_k)$$

vanish as $L \to \infty$ due to the fact that functions $\tilde{\mathbf{u}}_k, \widetilde{\mathbf{U}}_k \in \mathcal{D}^l_\beta H(\Omega)$ exponentially decay at infinity. Since the cut-off functions χ^j and χ^l have disjoint supports for $j \neq l$, we also obtain

$$\begin{aligned} q_L(\chi^j \mathbf{u}^{jh}_{ck}, \chi^l \mathbf{U}^{lm}_{ck}) = 0, \quad q_L(\chi^j \mathbf{u}^{jh}_{ck}, \chi^l \mathbf{U}^{lm}_{sk}) = 0, \\ q_L(\chi^j \mathbf{u}^{jh}_{sk}, \chi^l \mathbf{U}^{lm}_{ck}) = 0, \quad q_L(\chi^j \mathbf{u}^{jh}_{sk}, \chi^l \mathbf{U}^{lm}_{sk}) = 0, \end{aligned} \tag{1.3.4}$$

for $h, m \in \{0, 1\}$. Let us compute the rest terms of $q_L(\mathbf{u}_k, \mathbf{U}_k)$. Consider, for example, the term $q_L(\chi^j \mathbf{u}^{j0}_{ck},\ \chi^j \mathbf{U}^{j1}_{ck})$. Since the function χ^j is supported in the outlet Ω^j_+ and $\chi^j(x^j_3) = 1$ if $x^j_3 \geq 1$, we get from formulas (1.1.11), (1.1.17) the following relation

$$\begin{aligned} q_L(\chi^j \mathbf{u}^{j0}_{ck}, \chi^j \mathbf{U}^{j1}_{ck}) = (\mathbf{v}^{j0}_{ck}, \mathbf{n} P^{j1}_{ck} - \nu \partial_3 \mathbf{V}^{j1}_{ck})_{\omega^j} + (\mathbf{v}^{j0}_{sk}, \mathbf{n} P^{j1}_{sk} - \nu \partial_3 \mathbf{V}^{j1}_{sk})_{\omega^j} \\ - (\mathbf{n} p^{j0}_{ck} - \nu \partial_3 \mathbf{v}^{j0}_{ck}, \mathbf{V}^{j1}_{ck})_{\omega^j} - (\mathbf{n} p^{j0}_{sk} - \nu \partial_3 \mathbf{v}^{j0}_{sk}, \mathbf{V}^{j1}_{sk})_{\omega^j} = - \int_{\omega^j} \varphi^j_k \, dy^j, \quad L > 1. \end{aligned}$$

[viii]Notice that the integrals over the lateral surface $\partial\Omega_L \setminus \cup^J_{j=1} \omega^j$ vanish.

In the same way we derive the equalities

$$q_L(\chi^j \mathbf{u}_{sk}^{j0}, \chi^j \mathbf{U}_{sk}^{j1}) = q_L(\chi^j \mathbf{u}_{ck}^{j1}, \chi^j \mathbf{U}_{ck}^{j0}) = q_L(\chi^j \mathbf{u}_{sk}^{j1}, \chi^j \mathbf{U}_{sk}^{j0}) = -\int_{\omega^j} \varphi_k^j \, dy^j,$$

$$\begin{aligned} q_L(\chi^j \mathbf{u}_{ck}^{j0}, \chi^j \mathbf{U}_{sk}^{j1}) = q_L(\chi^j \mathbf{u}_{sk}^{j0}, \chi^j \mathbf{U}_{ck}^{j1}) = q_L(\chi^j \mathbf{u}_{ck}^{j1}, \chi^j \mathbf{U}_{sk}^{j1}) &= -q_L(\chi^j \mathbf{u}_{sk}^{j1}, \chi^j \mathbf{U}_{ck}^{j0}) \\ &= \int_{\omega^j} \psi_k^j \, dy^j. \end{aligned}$$

Analogous computations yield

$$q_L(\chi^j \mathbf{u}_{ck}^{jh}, \chi^j \mathbf{U}_{ck}^{jh}) = 0, \quad q_L(\chi^j \mathbf{u}_{ck}^{jh}, \chi^j \mathbf{U}_{sk}^{jh}) = 0,$$
$$q_L(\chi^j \mathbf{u}_{sk}^{jh}, \chi^j \mathbf{U}_{ck}^{jh}) = 0, \quad q_L(\chi^j \mathbf{u}_{sk}^{jh}, \chi^j \mathbf{U}_{sk}^{jh}) = 0$$

for $h = 0, 1$ and all $j = 1, \ldots, J$.

Now we pass to the limit as $L \to \infty$ in relation (1.3.3) and obtain the following Green formula

$$(\mathbf{S}_k \mathbf{u}_k, \mathbf{U}_k)_\Omega - (\mathbf{u}_k, \mathbf{S}_k^* \mathbf{U}_k)_\Omega = q_\infty(\mathbf{u}_k, \mathbf{U}_k). \tag{1.3.5}$$

Here

$$\begin{aligned} q_\infty(\mathbf{u}_k, \mathbf{U}_k) = \sum_{j=1}^{J} \Big\{ & \Big(a_{ck}^j \, B_{ck}^j - a_{sk}^j \, B_{sk}^j - b_{ck}^j \, A_{ck}^j - b_{sk}^j \, A_{sk}^j \Big) \, (\varphi_k^j, 1)_{\omega^j} \\ & + \Big(a_{ck}^j \, B_{sk}^j + a_{sk}^j \, B_{ck}^j + b_{ck}^j \, A_{sk}^j - b_{sk}^j \, A_{ck}^j \Big) (\psi_k^j, 1)_{\omega^j} \Big\}. \end{aligned} \tag{1.3.6}$$

In the same way as in [54] we define the projectors of $\mathbf{u}_k \in \mathbb{D}_{\pm\beta}^l H(\Omega)$ onto $\mathbb{R}^J$:

$$\begin{aligned} \pi_c^0 \mathbf{u}_k = (a_{ck}^1, a_{ck}^2, \ldots, a_{ck}^J), \quad \pi_s^0 \mathbf{u}_k = (a_{sk}^1, a_{sk}^2, \ldots, a_{sk}^J), \\ \pi_c^1 \mathbf{u}_k = (b_{ck}^1, b_{ck}^2, \ldots, b_{ck}^J), \quad \pi_s^1 \mathbf{u}_k = (b_{sk}^1, b_{sk}^2, \ldots, b_{sk}^J), \end{aligned} \tag{1.3.7}$$

where the constants $\{a_{ck}^j, a_{sk}^j, b_{ck}^j, b_{sk}^j\}_{j=1}^J$ are taken form the asymptotic representation (1.1.22) of the function $\mathbf{u}_k$. The projectors from $\mathbb{D}_{\pm\beta}^l H(\Omega)^*$ to $\mathbb{R}^J$ are defined analogously. Let us define $J \times J$ diagonal matrices

$$\mathcal{C}_k = \operatorname{diag}\Big(c_k^1, c_k^2, \ldots, c_k^J\Big), \quad \mathcal{D}_k = \operatorname{diag}\Big(d_k^1, d_k^2, \ldots, d_k^J\Big) \tag{1.3.8}$$

with the entries

$$c_k^j = \int_{\omega^j} \varphi_k^j dy^j, \quad d_k^j = -\int_{\omega^j} \psi_k^j dy^j. \tag{1.3.9}$$

Then relation (1.3.6) may be rewritten[ix] as follows

$$q_\infty(\mathbf{u}_k, \mathbf{U}_k) = \langle \pi_c^0 \mathbf{u}_k, \mathcal{C}_k \pi_c^1 \mathbf{U}_k - \mathcal{D}_k \pi_s^1 \mathbf{U}_k \rangle_J + \langle \pi_s^0 \mathbf{u}_k, -\mathcal{D}_k \pi_c^1 \mathbf{U}_k - \mathcal{C}_k \pi_s^1 \mathbf{U}_k \rangle_J \\ - \langle \mathcal{C}_k \pi_c^1 \mathbf{u}_k - \mathcal{D}_k \pi_s^1 \mathbf{u}_k, \pi_c^0 \mathbf{U}_k \rangle_J - \langle \mathcal{D}_k \pi_c^1 \mathbf{u}_k + \mathcal{C}_k \pi_s^1 \mathbf{u}_k, \pi_s^0 \mathbf{U}_k \rangle_J. \tag{1.3.10}$$

Here $\langle\, , \,\rangle_N$ denotes the inner product in $\mathbb{R}^N$. Setting

$$\pi^1 = \begin{pmatrix} \pi_c^1 \\ \pi_s^1 \end{pmatrix}, \quad \pi^0 = \begin{pmatrix} \pi_c^0 \\ \pi_s^0 \end{pmatrix}, \quad \pi = \begin{pmatrix} \pi^1 \\ \pi^0 \end{pmatrix} \tag{1.3.11}$$

and

$$\mathbb{F}_k = \begin{pmatrix} \mathcal{C}_k & -\mathcal{D}_k \\ \mathcal{D}_k & \mathcal{C}_k \end{pmatrix}, \quad \mathbb{G}_k = \begin{pmatrix} \mathcal{C}_k & -\mathcal{D}_k \\ -\mathcal{D}_k & -\mathcal{C}_k \end{pmatrix}, \quad \mathbb{J}_k = \begin{pmatrix} \mathbb{O} & \mathbb{G}_k \\ -\mathbb{F}_k & \mathbb{O} \end{pmatrix} \tag{1.3.12}$$

we rewrite the right-hand side of (1.3.10) in the compact form

$$\begin{aligned} &\Big\langle \begin{pmatrix} \mathbb{O} & \mathbb{I} \end{pmatrix} \pi \mathbf{u}_k, \begin{pmatrix} \mathbb{G}_k & \mathbb{O} \end{pmatrix} \pi \mathbf{U}_k \Big\rangle_{2J} - \Big\langle \begin{pmatrix} \mathbb{F}_k & \mathbb{O} \end{pmatrix} \pi \mathbf{u}_k, \begin{pmatrix} \mathbb{O} & \mathbb{I} \end{pmatrix} \pi \mathbf{U}_k \Big\rangle_{2J} \\ = &\Big\langle \begin{pmatrix} \mathbb{O} & \mathbb{G}_k \end{pmatrix} \pi \mathbf{u}_k, \begin{pmatrix} \mathbb{I} & \mathbb{O} \end{pmatrix} \pi \mathbf{U}_k \Big\rangle_{2J} - \Big\langle \begin{pmatrix} \mathbb{F}_k & \mathbb{O} \end{pmatrix} \pi \mathbf{u}_k, \begin{pmatrix} \mathbb{O} & \mathbb{I} \end{pmatrix} \pi \mathbf{U}_k \Big\rangle_{2J} \\ &= \langle \mathbb{J}_k \pi \mathbf{u}_k, \pi \mathbf{U}_k \rangle_{4J}, \end{aligned}$$

where $\mathbb{I}$ and $\mathbb{O}$ denote $2J \times 2J$ identity and zero matrices.

Assume that $2J \times 4J$ real matrices $\mathbb{B}_k$, $\mathbb{T}_k$, $\mathbb{S}_k$ and $\mathbb{Q}_k$ are such that the $4J \times 4J$ matrices $\mathbb{X}_k = \begin{pmatrix} \mathbb{B}_k \\ \mathbb{S}_k \end{pmatrix}$ and $\mathbb{Y}_k = \begin{pmatrix} -\mathbb{T}_k \\ \mathbb{Q}_k \end{pmatrix}$ satisfy the relation

$$(\mathbb{Y}_k)^T \mathbb{X}_k = \mathbb{J}_k. \tag{1.3.13}$$

Then we get the equalities

$$\begin{aligned} \langle \mathbb{J}_k \pi \mathbf{u}_k, \pi \mathbf{U}_k \rangle_{2J} &= \langle \mathbb{X}_k \pi \mathbf{u}_k, \mathbb{Y}_k \pi \mathbf{U}_k \rangle_{2J} \\ = \langle \mathbb{S}_k \pi \mathbf{u}_k, \mathbb{Q}_k \pi \mathbf{U}_k \rangle_{2J} &- \langle \mathbb{B}_k \pi \mathbf{u}_k, \mathbb{T}_k \pi \mathbf{U}_k \rangle_{2J}. \end{aligned}$$

This relation and formulas (1.3.5), (1.3.6), (1.3.10) lead to the following statement.

Theorem 1.3.1. *Assume that matrices $\mathbb{B}_k$, $\mathbb{T}_k$, $\mathbb{S}_k$, $\mathbb{Q}_k$ satisfy condition* (1.3.13).

[ix] We treat the projections (1.3.7) as the column-vectors.

Then functions $\mathbf{u}_k \in \mathbb{D}^l_{\pm\beta}H(\Omega)$ *and* $\mathbf{U}_k \in \mathbb{D}^l_{\pm\beta}H(\Omega)^*$ *satisfy the following relation*

$$(\mathbf{S}_k\mathbf{u}_k, \mathbf{U}_k)_\Omega + \langle \mathbb{B}_k\pi\mathbf{u}_k, \mathbb{T}_k\pi\mathbf{U}_k\rangle_{2J} = (\mathbf{u}_k, \mathbf{S}_k^*\mathbf{U}_k)_\Omega + \langle \mathbb{S}_k\pi\mathbf{u}_k, \mathbb{Q}_k\pi\mathbf{U}_k\rangle_{2J}. \qquad (1.3.14)$$

Formula (1.3.14) is called *the generalized Green formula.* Assume that $\mathbf{h}_k \in \mathbb{R}^{2J}$ is a given vector. The relation

$$\mathbb{B}_k\pi\mathbf{u}_k = \mathbf{h}_k \qquad (1.3.15)$$

is called *the asymptotic conditions at infinity* (see Sections 4, 6 in [54]). We emphasize that (1.3.15) determines $2J$ relations between coefficients from the asymptotic representation (1.1.22) of the function $\mathbf{u}_k \in \mathbb{D}^l_{\pm\beta}H(\Omega)$. According to formula (1.6.6), $2J$ restrictions to the constants $\{a^j_{ck}, a^j_{sk},\ b^j_{ck}, b^j_{sk},\}_{j=1}^J$ shall be imposed in order to ensure the uniqueness of the solution from the class $\mathbb{D}^l_{\pm\beta}H(\Omega)$ for problem (1.1.2).

Remark 1.3.2. Straightforward computations show that

$$\det\mathbb{F}_k = \det\left(\mathcal{C}_k^2 + \mathcal{D}_k^2\right) = \prod_{j=1}^J \left((c_k^j)^2 + (d_k^j)^2\right),$$

$$\det\mathbb{G}_k = \det\left(-\mathcal{C}_k^2 - \mathcal{D}_k^2\right) = -\prod_{j=1}^J \left((c_k^j)^2 + (d_k^j)^2\right).$$

Then, according to formula (1.3.13),

$$\det\mathbb{J}_k = \det\left(\mathbb{F}_k\mathbb{G}_k\right) = -\prod_{j=1}^J \left((c_k^j)^2 + (d_k^j)^2\right)^2.$$

Taking into account Part (2) in Lemma A.0.1 (see Appendix A), we conclude that $(c_k^j)^2 + (d_k^j)^2 > 0$ for all $j = 1, \ldots, J$ and $k = 0, 1, \ldots$. Therefore the rank of $\mathbb{J}_k$ is equal to $4J$. Applying Silvester's inequality (see, e.g., [29])

$$\text{rank}AB \le \min\{\text{rank}A, \text{rank}B\}$$

to the product $(\mathbb{Y}_k)^T\mathbb{X}_k = \mathbb{J}_k$, we conclude

$$\text{rank}\begin{pmatrix}\mathbb{B}_k\\ \mathbb{S}_k\end{pmatrix} = 4J, \quad \text{rank}\begin{pmatrix}-\mathbb{T}_k\\ \mathbb{Q}_k\end{pmatrix} = 4J.$$

1.4 Basis for the homogeneous Stokes-type problem

Let us define the set

$$\mathfrak{D}_k = \left\{ \mathbf{u}_k \in \mathbb{D}^l_{\pm\beta} H(\Omega) : \pi^1 \mathbf{u}_k = \mathbf{0} \right\}. \tag{1.4.1}$$

According to definitions (1.3.7), (1.3.11) we see that $\mathfrak{D}_k$ consists of the elements having $b^j_{ck} = 0$, $b^j_{sk} = 0$, $j = 1, \dots, J$, in the asymptotic representation (1.1.22). Taking into account the fact that the rest of the terms in (1.1.22) have exponentially decaying velocity-fields, we conclude that functions in (1.4.1) possess finite energy dissipation, i.e., have bounded Dirichlet's integrals

$$\int_\Omega |\nabla \mathbf{v}_{ck}(x)|^2 + |\nabla \mathbf{v}_{sk}(x)|^2 dx < \infty.$$

It is obvious that the codimension[x] of the subspace $\mathfrak{D}_k \subset \mathbb{D}^l_{\pm\beta} H(\Omega)$ is $2J$. Consider the restriction of the operator $\mathbf{A}^l_{-\beta\to\beta,k}$ on the subspace $\mathfrak{D}_k$:

$$\mathbf{A}_{\mathfrak{D}_k,k} : \mathfrak{D}_k \to \mathcal{R}^l_\beta H(\Omega).$$

Let us recall that $\mathrm{ind}\mathbf{A}^l_{-\beta\to\beta,k} = \dim\ker\mathbf{A}^l_{-\beta\to\beta,k} - \dim\operatorname{coker}\mathbf{A}^l_{-\beta\to\beta,k} = 2J$ (see (1.3.1)). Therefore the operator $\mathbf{A}_{\mathfrak{D}_k,k}$ (the restriction of $\mathbf{A}^l_{-\beta\to\beta,k}$ on the subspace of the codimension $2J$) has the index equal to zero. Below we formulate the statement concerning existence and uniqueness of the solution from $\mathfrak{D}_k$ to the problem (1.1.2) (see Section 5 in [54] for analogous results for the steady Stokes problem).

Theorem 1.4.1. *(a) The solutions* $\mathbf{u}_k = (\mathbf{v}_{ck}, p_{ck}, \mathbf{v}_{sk}, p_{sk}) \in \mathfrak{D}_k$ *to the homogeneous Stokes-type problem* (1.1.2) *have zero velocity coefficients* $\mathbf{v}_{ck}$, $\mathbf{v}_{sk}$ *and constant pressure coefficients* p_{ck}, p_{sk}. *The basis in* $\ker\mathbf{A}_{\mathfrak{D}_k,k}$ *consists of two vector-fields*

$$\mathbf{u}^c_k = (0,0,0,\ 1,\ 0,0,0,\ 0), \quad \mathbf{u}^s_k = (0,0,0,\ 0,\ 0,0,0,\ 1). \tag{1.4.2}$$

(b) Problem (1.1.2) *is solvable in the class* $\mathfrak{D}_k$ *for every right-hand side* $\mathbf{f}_k = (\mathbf{f}_{ck}, 0, \mathbf{f}_{sk}, 0) \in \mathcal{R}^l_\beta H(\Omega)$.

[x] We recall that the codimension of the subspace S in the vector space V is the dimension of the quotient space V/S.

Proof. (a) Assume that $\mathbf{u}_k = (\mathbf{v}_{ck}, p_{ck}, \mathbf{v}_{sk}, p_{sk}) \in \ker \mathbf{A}_{\mathfrak{D},k}$, i.e.,

$$\left\{ \begin{array}{rcll} -\nu\Delta\mathbf{v}_{ck} + \nabla p_{ck} + k\mathbf{v}_{sk} & = & \mathbf{0}, & x \in \Omega, \\ -\nabla \cdot \mathbf{v}_{ck} & = & 0, & x \in \Omega, \\ -\nu\Delta\mathbf{v}_{sk} + \nabla p_{sk} - k\mathbf{v}_{ck} & = & \mathbf{0}, & x \in \Omega, \\ -\nabla \cdot \mathbf{v}_{sk} & = & 0, & x \in \Omega, \\ \mathbf{v}_{ck} = \mathbf{0},\ \mathbf{v}_{sk} & = & \mathbf{0}, & x \in \partial\Omega. \end{array} \right. \tag{1.4.3}$$

Multiplying equations (1.4.3$_1$) and (1.4.3$_3$) by $\mathbf{v}_{ck}$ and $\mathbf{v}_{sk}$, respectively, applying integration by parts and summing the obtained expressions, we derive

$$\nu(\nabla\mathbf{v}_{ck}, \nabla\mathbf{v}_{ck})_\Omega + \nu(\nabla\mathbf{v}_{sk}, \nabla\mathbf{v}_{sk})_\Omega - (p_{ck}, \nabla \cdot \mathbf{v}_{ck})_\Omega - (p_{sk}, \nabla \cdot \mathbf{v}_{sk})_\Omega$$
$$+(\mathbf{n}p_{ck} - \nu\frac{\partial\mathbf{v}_{ck}}{\partial\mathbf{n}}, \mathbf{v}_{ck})_{\partial\Omega} + (\mathbf{n}p_{sk} - \nu\frac{\partial\mathbf{v}_{sk}}{\partial\mathbf{n}}, \mathbf{v}_{sk})_{\partial\Omega} = 0.$$

From this identity we get, taking into account equations (1.4.3$_2$), (1.4.3$_4$) and boundary conditions (1.4.3$_5$), the relation

$$\int_\Omega |\nabla\mathbf{v}_{ck}(x)|^2 + |\nabla\mathbf{v}_{sk}(x)|^2 dx = 0.$$

Therefore $\mathbf{v}_{ck} = \text{const}$ and $\mathbf{v}_{sk} = \text{const}$. Since these two vector-fields vanish on the boundary $\partial\Omega$, we get that $\mathbf{v}_{ck} \equiv \mathbf{0}$ and $\mathbf{v}_{sk} \equiv \mathbf{0}$. Substituting zero velocity coefficients into (1.4.3$_1$) and (1.4.3$_3$) we conclude that $p_{ck} = \text{const}$ and $p_{sk} = \text{const}$.

(b) Denote by $\mathfrak{D}_k^*$ the subspace of $\mathbb{D}^l_{\pm\beta}H(\Omega)^*$ consisting of functions $\mathbf{U}_k$ with $B^j_{ck} = 0$, $B^j_{sk} = 0$, $j = 1, \dots, J$, in the asymptotic representation (1.1.25). Then the velocity fields of $\mathbf{u}_k \in \mathfrak{D}_k$ and $\mathbf{U}_k \in \mathfrak{D}_k^*$ decay exponentially. Therefore for elements from the subspaces $\mathfrak{D}_k$ and $\mathfrak{D}_k^*$ the classical Green formula (1.1.15) holds. Consequently, the problem (1.1.2) is solvable if and only if the right-hand side $\mathbf{f}_k$ satisfies the following compatibility condition

$$\int_\Omega \mathbf{f}_k \cdot \boldsymbol{\mathcal{U}}_k dx = 0$$

for every solution $\boldsymbol{\mathcal{U}}_k$ of the homogeneous formally adjoint problem (1.1.16). In the same way as in the Part (a) we show, for every $k = 0, 1, \dots,$ that the homogeneous formally adjoint problem (1.1.16) in the class $\mathfrak{D}_k^*$ has only two linearly independent solutions:

$$\mathbf{U}_k^c = (0, 0, 0,\ 1,\ 0, 0, 0,\ 0), \quad \mathbf{U}_k^s = (0, 0, 0,\ 0,\ 0, 0, 0,\ 1).$$

Hence, the function $\mathbf{f}_k$ shall satisfy conditions

$$\int_\Omega \mathbf{f}_k \cdot \mathbf{U}_k^c dx = 0, \quad \int_\Omega \mathbf{f}_k \cdot \mathbf{U}_k^s dx = 0. \tag{1.4.4}$$

Recall that the index of the operator of $\mathbf{A}_{\mathfrak{D}_k,k}$ is zero. This means that the dimensions of it's kernel and co-kernel are equal. In other words, the number of compatibility conditions for the data $\mathbf{f}_k$ of the non-homogeneous problem (1.1.2) is the same as the number of linearly independent solutions of the corresponding homogeneous problem. Since the number of conditions (1.4.4) coincides with the dimension of the set $\ker \mathbf{A}_{\mathfrak{D}_k,k}$, conditions (1.4.4) become sufficient ones. Obviously, conditions (1.4.4) automatically hold for any function $\mathbf{f}_k \in \mathcal{R}^l_\beta H(\Omega)$ admitting the representation $\mathbf{f}_k = (\mathbf{f}_{ck}, 0, \mathbf{f}_{sk}, 0)$[xi]. $\square$

Below we present a basis in $\ker \mathbf{A}^l_{-\beta\to\beta}$, i.e., in the set of solutions to the homogeneous Stokes-type problem (1.1.2). According to formula (1.3.1), the set $\ker \mathbf{A}^l_{-\beta\to\beta}$ is spanned by $2J$ linearly independent elements. Let us denote these elements by

$$\boldsymbol{u}_k^1, \dots \boldsymbol{u}_k^{2J} \tag{1.4.5}$$

Notice that for the solutions of the homogeneous problem Theorem 1.1.5 may be applied. As a consequence we get that every element in basis (1.4.5) has the following asymptotic representation

$$\boldsymbol{u}_k^i = \sum_{j=1}^{J} \chi^j \left\{ a_{ck}^{i,j} \mathbf{u}_{ck}^{j0} + a_{sk}^{i,j} \mathbf{u}_{sk}^{j0} + b_{ck}^{i,j} \mathbf{u}_{ck}^{j1} + b_{sk}^{i,j} \mathbf{u}_{sk}^{j1} \right\} + \widetilde{\boldsymbol{u}}_k^i. \tag{1.4.6}$$

Construction of the element $\boldsymbol{u}_k^i$, is divided into three steps. Let us briefly describe the procedure. First we define the flux carrier $\boldsymbol{u}_k^{p,i} = \sum_{j=1}^J \chi^j \boldsymbol{u}_k^{p,i,j}$, which is represented as a sum of the Poiseuille flows $\boldsymbol{u}_k^{p,i,1}, \dots, \boldsymbol{u}_k^{p,i,J}$. Notice that due to multiplication of the exact solutions $\boldsymbol{u}_k^{p,i,j}$ to the homogeneous problem (1.1.2) by the cut-off functions χ^j, the vector-field $\boldsymbol{u}_k^{p,i}$ has non-zero divergence. In order to obtain a solenoidal flux carrier, we construct a special vector field $\boldsymbol{u}_k^{d,i}$ with the divergence equal to $-\nabla \cdot \boldsymbol{u}_k^{p,i}$. The sum $\boldsymbol{u}_k^{p,i} + \boldsymbol{u}_k^{d,i}$ has zero divergence, however it does not satisfy the homogeneous equations $(1.1.2_1)$ and $(1.1.2_3)$. Therefore in the last step we solve in the class $\mathfrak{D}_k$ the Stokes-type problem $\mathbf{S}_k \widehat{\boldsymbol{u}}_k^i = \widehat{\mathbf{f}}_k^i$ with the right-hand side $\widehat{\mathbf{f}}_k^i = -\mathbf{S}_k(\boldsymbol{u}_k^p + \boldsymbol{u}_k^d)$. Then the element in the basis of $\ker \mathbf{A}^l_{-\beta\to\beta}$ is defined as the sum $\boldsymbol{u}_k^i = \boldsymbol{u}_k^{p,i} + \boldsymbol{u}_k^{d,i} + \widehat{\boldsymbol{u}}_k^i$.

[xi]Let us notice, that this structure of the right-hand $\mathbf{f}_k$ in (1.1.2) implies that the divergence equations $(1.1.2_2)$ and $(1.1.2_4)$ are necessary homogeneous.

Step (1): Consider the linear combinations of the vector-fields $\mathbf{u}^{j1}_{ck}, \mathbf{u}^{j1}_{sk}$ (see (1.1.11), (1.1.12)):

$$\boldsymbol{u}^{p,i,j}_k = b^{i,j}_{ck}\mathbf{u}^{j1}_{ck} + b^{i,j}_{sk}\mathbf{u}^{j1}_{sk}.$$

For every $i = 1, \dots, 2J$ and $j = 1, \dots, J$ the vector-field $\boldsymbol{u}^{p,i,j}_k$ has the velocity coefficients

$$\boldsymbol{v}^{p,i,j}_{ck} = (0, 0, b^{i,j}_{ck}\phi^j_k + b^{i,j}_{sk}\psi^j_k), \quad \boldsymbol{v}^{p,i,j}_{sk} = (0, 0, b^{i,j}_{sk}\phi^j_k - b^{i,j}_{ck}\psi^j_k) \tag{1.4.7}$$

and the linear pressure terms

$$p^{p,i,j}_{ck} = -b^{i,j}_{ck}x^j_3, \quad p^{p,i,j}_{sk} = -b^{i,j}_{sk}x^j_3.$$

Moreover $\boldsymbol{u}^{p,i,j}_k$ satisfy the homogeneous Stokes-type system set in the cylinder $\Omega^j = \omega^j \times \mathbb{R}$ (see (1.1.7), (1.1.11), (1.1.12)):

$$\begin{cases} -\nu\Delta\boldsymbol{v}^{p,i,j}_{ck} + \nabla p^{p,i,j}_{ck} + k\boldsymbol{v}^{p,i,j}_{sk} &= \mathbf{0}, \qquad x \in \Omega^j, \\ -\nabla \cdot \boldsymbol{v}^{p,i,j}_{ck} &= 0, \qquad x \in \Omega^j, \\ -\nu\Delta\boldsymbol{v}^{p,i,j}_{sk} + \nabla p^{i,j}_{sk} - k\boldsymbol{v}^{p,i,j}_{ck} &= \mathbf{0}, \qquad x \in \Omega^j, \\ -\nabla \cdot \boldsymbol{v}^{p,i,j}_{sk} &= 0, \qquad x \in \Omega^j, \\ \boldsymbol{v}^{p,i,j}_{ck} = \mathbf{0}, \ \boldsymbol{v}^{p,i,j}_{sk} &= \mathbf{0}, \qquad x \in \partial\Omega^j. \end{cases} \tag{1.4.8}$$

Let us define the flux carrier

$$\boldsymbol{u}^{p,i}_k = \sum_{j=1}^J \chi^j \boldsymbol{u}^{p,i,j}_k. \tag{1.4.9}$$

The velocity components $\boldsymbol{v}^{p,i}_{ck} = \sum_{j=1}^J \chi^j \boldsymbol{v}^{p,i,j}_{ck}$ and $\boldsymbol{v}^{p,i}_{sk} = \sum_{j=1}^J \chi^j \mathbf{v}^{p,i,j}_{sk}$ of the vector-field $\boldsymbol{u}^{p,i,j}_k$ generate in every outlet the flow-rates

$$\begin{aligned} \phi^{i,j}_{ck} &= \int_{\omega^j} b^{i,j}_{ck}\phi^j_k(y^j) + b^{i,j}_{sk}\psi^j_k(y^j)\, dy^j = b^{i,j}_{ck}c^j_k - b^{i,j}_{sk}d^j_k, \\ \phi^{i,j}_{sk} &= \int_{\omega^j} -b^{i,j}_{ck}\psi^j_k(y^j) + b^{i,j}_{sk}\phi^j_k(y^j)\, dy^j = b^{i,j}_{ck}d^j_k + b^{i,j}_{sk}c^j_k. \end{aligned} \tag{1.4.10}$$

For every $i = 1, \dots, 2J$ we select the constants $\{b^{i,j}_{ck}, b^{i,j}_{sk}\}^J_{j=1}$ in (1.4.10) in such a way that the element $\boldsymbol{u}^{p,i}_k$ has the following distribution of flow-rates:

$$
\begin{aligned}
(\phi_{ck}^{i,1},\ldots,\phi_{ck}^{i,J},\ \phi_{sk}^{i,1},\ldots,\phi_{sk}^{i,J}) &= (\delta_i^1-\frac{1}{J},\ldots,\delta_i^J-\frac{1}{J},\ 0,\ldots,0) \\
&\qquad \text{for } i=1,\ldots,J, \\
(\phi_{ck}^{i,1},\ldots,\phi_{ck}^{i,J},\ \phi_{sk}^{i,1},\ldots,\phi_{sk}^{i,J}) &= (0,\ldots,0,\ \delta_i^{J+1}-\frac{1}{J},\ldots,\delta_i^{2J}-\frac{1}{J}) \\
&\qquad \text{for } i=J+1,\ldots,2J,
\end{aligned}
\tag{1.4.11}
$$

here δ_i^j denotes Kroneker's delta. To realize (1.4.11) we solve the following systems of linear equations:

$$
\begin{cases}
b_{ck}^{i,j}c_k^j - b_{sk}^{i,j}d_k^j = \delta_i^j - 1/J, \\
b_{ck}^{i,j}d_k^j + b_{sk}^{i,j}c_k^j = 0,
\end{cases}
\qquad i=1,\ldots,J,
$$

$$
\begin{cases}
b_{ck}^{i,j}c_k^j - b_{sk}^{i,j}d_k^j = 0, \\
b_{ck}^{i,j}d_k^j + b_{sk}^{i,j}c_k^j = \delta_i^{J+j} - 1/J,
\end{cases}
\qquad i=J+1,\ldots,2J.
$$

According to Lemma A.0.1 the quantity $(c_k^j)^2+(d_k^j)^2>0$ for every $j=1,\ldots,J$ and all $k=0,1,\ldots$. Therefore we get that

$$
\begin{aligned}
b_{ck}^{i,j} &= \frac{c_k^j(\delta_i^j-1/J)}{(c_k^j)^2+(d_k^j)^2}, \qquad b_{sk}^{i,j} = -\frac{d_k^j(\delta_i^j-1/J)}{(c_k^j)^2+(d_k^j)^2}, \qquad i=1,\ldots,J, \\
b_{ck}^{i,j} &= \frac{d_k^j(\delta_i^{J+j}-1/J)}{(c_k^j)^2+(d_k^j)^2}, \quad b_{sk}^{i,j} = \frac{c_k^j(\delta_i^{J+j}-1/J)}{(c_k^j)^2+(d_k^j)^2}, \quad i=J+1,\ldots,2J.
\end{aligned}
\tag{1.4.12}
$$

We notice that in $\boldsymbol{u}_k^{p,i}=(\boldsymbol{v}_{ck}^{p,i},p_{ck}^{p,i},\boldsymbol{v}_{sk}^{p,i},p_{sk}^{p,i})$ the flux is carried by the components $\boldsymbol{v}_{ck}^{p,i}$ for $i=1,\ldots,J$, and by the components $\boldsymbol{v}_{sk}^{p,i}$ for $i=J+1,\ldots,2J$. In both cases one of the cylinders is a source with the flux equal to $1-1/J$, while the other cylinders are drains with outflows equal to $-1/J$.

Step (2): It is obvious form the definitions (1.4.7) that $\nabla\cdot\boldsymbol{v}_{ck}^{p,i,j}=0$ and $\nabla\cdot\boldsymbol{v}_{sk}^{p,i,j}=0$ for every $i=1,\ldots,2J$ and $j=1,\ldots,J$. However due to multiplication by the cut-off functions χ^j, the velocity-fields $\boldsymbol{v}_{ck}^{p,i}=\sum_{j=1}^J\chi^j\boldsymbol{v}_{ck}^{p,i,j}$ and $\boldsymbol{v}_{sk}^{p,i}=\sum_{j=1}^J\chi^j\boldsymbol{v}_{sk}^{p,i,j}$ of the flux carrier (1.4.9) are no longer divergence-free. In order to restore the incompressibility of the flow generated by $\boldsymbol{u}_k^{p,i}$, we use the following result of [11]:

Lemma 1.4.2. *Let $G\subset\mathbb{R}^n$ be a bounded domain with the Lipschitz boundary ∂G*

and let the function $g \in \mathring{H}^1(G)$ *satisfies the condition* $\int_G g\,dx = 0$. *Then the problem*

$$\begin{cases} -\nabla\cdot \mathbf{w} &= g \\ \mathbf{w}|_{\partial G} &= 0, \end{cases} \tag{1.4.13}$$

admits a solution $\mathbf{w} \in \mathring{H}^1(G)\cap H^2(G)$ *satisfying the estimates*

$$\|\nabla \mathbf{w}\|_{L_2(G)} \le c\|g\|_{L_2(G)}, \quad \|\mathbf{w}\|_{H^2(G)} \le c\|g\|_{H^1(G)}$$

with a constant c *independent of* g.

Taking into account the definition of the cut-off function χ^j (we recall that $\chi^j(x_3^j) = 0$ for $x_3^j \le 0$ and $\chi^j(x_3^j) = 1$ for $x_3^j \ge 1$) we see that the functions $\nabla\cdot \boldsymbol{v}_{ck}^{p,i} = \sum_{j=1}^J \frac{d\chi^j}{dx_3^j}\boldsymbol{v}_{ck}^{p,i,j}$ and $\nabla\cdot \boldsymbol{v}_{sk}^{p,i,j} = \sum_{j=1}^J \frac{d\chi^j}{dx_3^j}\boldsymbol{v}_{sk}^{p,i,j}$ have compact supports and belong to the space $\mathring{H}^1(\Omega_1)$[xii]. Moreover, integrating by parts we get

$$\int_{\Omega_1} \nabla\cdot \boldsymbol{v}_{ck}^{p,i} dx = -\sum_{j=1}^J \int_{\omega^j} \boldsymbol{v}_{ck}^{p,i}\cdot \mathbf{n} dy^j.$$

Here we have used the fact that the velocity field $\boldsymbol{v}_{ck}^{p,i}$ vanish on the lateral boundary $\partial\Omega_1 \setminus \cup_{j=1}^J \omega^j$. The integral $\int_{\omega^j} \boldsymbol{v}_{ck}^{p,i}\cdot \mathbf{n} dy^j$ is equal to the flow-rate $\phi^{i,j}$ generated by $\boldsymbol{v}_{ck}^{p,i}$ over the section ω^j. Therefore, taking into account the flow-rate distributions (1.4.11) we conclude, that the sum in the last identity is equal to zero:

$$\sum_{j=1}^J \int_{\omega^j} \boldsymbol{v}_{ck}^{p,i}\cdot \mathbf{n} dy^j = \sum_{j=1}^J \phi_{ck}^{i,j} = \sum_{j=1}^J \left(\delta_i^j - 1/J\right) = 0.$$

Consequently,

$$\int_{\Omega_1} \nabla\cdot \boldsymbol{v}_{ck}^{p,i} dx = 0.$$

In the same way we derive the equalities

$$\int_{\Omega_1} \nabla\cdot \boldsymbol{v}_{ck}^{p,i} dx = -\sum_{j=1}^J \int_{\omega^j} \boldsymbol{v}_{sk}^{p,i}\cdot \mathbf{n} dy^j = -\sum_{j=1}^J \phi_{sk}^{i,j} = -\sum_{j=1}^J \left(\delta_i^j - 1/J\right) = 0.$$

Therefore Lemma 1.4.2 may be applied in the case when $g = \nabla\cdot \boldsymbol{v}_{ck}^{p,i}$ and $g = \nabla\cdot \boldsymbol{v}_{sk}^{p,i}$, and we conclude that there exist vector-fields $\boldsymbol{v}_{ck}^{d,i}$, $\boldsymbol{v}_{sk}^{d,i} \in \left(\mathring{H}^1(\Omega_1)\cap H^2(\Omega_1)\right)^3$,

[xii] We recall that $\Omega_1 = \left\{x \in \Omega : x_3^j < 1, j = 1, \dots, J\right\}$, see (0.1.3).

satisfying the following problems set in the domain Ω_1

$$\left\{\begin{array}{rcl} -\nabla \cdot \boldsymbol{v}_{ck}^{d,i} & = & \nabla \cdot \boldsymbol{v}_{ck}^{p,i}, \\ \boldsymbol{v}_{ck}^{d,i}|_{\partial\Omega_1} & = & 0, \end{array}\right. \qquad \left\{\begin{array}{rcl} -\nabla \cdot \boldsymbol{v}_{sk}^{d,i} & = & \nabla \cdot \boldsymbol{v}_{sk}^{p,i}, \\ \boldsymbol{v}_{sk}^{d,i}|_{\Omega_1} & = & 0, \end{array}\right.$$

Moreover the following estimates

$$\begin{gathered} \|\boldsymbol{v}_{ck}^{d,i}\|_{H^1(\Omega_1)} \leq c\|\nabla \cdot \boldsymbol{v}_{ck}^{p,i}\|_{L^2(\Omega_1)}, \quad \|\boldsymbol{v}_{ck}^{d,i}\|_{H^2(\Omega_1)} \leq c\|\nabla \cdot \boldsymbol{v}_{ck}^{p,i}\|_{H^1(\Omega_1)}, \\ \|\boldsymbol{v}_{sk}^{d,i}\|_{H^1(\Omega_1)} \leq c\|\nabla \cdot \boldsymbol{v}_{sk}^{p,i}\|_{L^2(\Omega_1)}, \quad \|\boldsymbol{v}_{sk}^{d,i}\|_{H^2(\Omega_1)} \leq c\|\nabla \cdot \boldsymbol{v}_{sk}^{p,i}\|_{H^1(\Omega_1)} \end{gathered} \tag{1.4.14}$$

hold. We extend functions $\boldsymbol{v}_{ck}^{d,i}$ and $\boldsymbol{v}_{sk}^{d,i}$ by zero to the whole domain Ω and define the vector-field $\boldsymbol{u}_k^{d,i} = (\boldsymbol{v}_{ck}^{d,i}, 0, \boldsymbol{v}_{sk}^{d,i}, 0)$. Then the sum $\boldsymbol{u}_k^{p,i} + \boldsymbol{u}_k^{d,i}$ is a divergence-free flux carrier.

Step (3): We look for the elements in (1.4.5) in the form $\boldsymbol{u}_k^i = \boldsymbol{u}_k^{p,i} + \boldsymbol{u}_k^{d,i} + \widehat{\boldsymbol{u}}_k^i$, where $\widehat{\boldsymbol{u}}_k^i \in \mathfrak{D}_k$. Substituting $\boldsymbol{u}_k^i$ into the homogeneous system (1.1.2) we get for the function $\widehat{\boldsymbol{u}}_k^i$ the Stokes-type problem $\mathbf{S}_k \widehat{\boldsymbol{u}}_k^i = \widehat{\mathbf{f}}_k^i$ with $\widehat{\mathbf{f}}_k^i = -\mathbf{S}_k(\boldsymbol{u}_k^{p,i} + \boldsymbol{u}_k^{d,i})$, i.e., the function $\widehat{\boldsymbol{u}}_k^i$ shall satisfy in the domain Ω the following problem

$$\left\{\begin{array}{rcl} -\nu\Delta\widehat{\boldsymbol{v}}_{ck}^i + \nabla\widehat{p}_{ck}^i + k\widehat{\boldsymbol{v}}_{sk}^i & = & \widehat{\mathbf{f}}_{ck}^i, \\ -\nabla \cdot \widehat{\boldsymbol{v}}_{ck}^i & = & 0, \\ -\nu\Delta\widehat{\boldsymbol{v}}_{sk}^i + \nabla\widehat{p}_{sk}^i - k\widehat{\boldsymbol{v}}_{ck}^i & = & \widehat{\mathbf{f}}_{sk}^i, \\ -\nabla \cdot \widehat{\boldsymbol{v}}_{sk}^i & = & 0, \\ \widehat{\boldsymbol{v}}_{ck}^i|_{\partial\Omega} = \mathbf{0}, \quad \widehat{\boldsymbol{v}}_{sk}^i|_{\partial\Omega} & = & \mathbf{0}, \end{array}\right. \tag{1.4.15}$$

with

$$\begin{aligned} \widehat{\mathbf{f}}_{ck}^i &= \mathbf{f}_{ck}^{p,i} + \mathbf{f}_{ck}^{d,i} = \nu\Delta\boldsymbol{v}_{ck}^{p,i} - \nabla p_{ck}^{p,i} - k\boldsymbol{v}_{sk}^{p,i} + \nu\Delta\boldsymbol{v}_{ck}^{d,i} - k\boldsymbol{v}_{sk}^{d,i}, \\ \widehat{\mathbf{f}}_{sk}^i &= \mathbf{f}_{sk}^{p,i} + \mathbf{f}_{sk}^{d,i} = \nu\Delta\boldsymbol{v}_{sk}^{p,i} - \nabla p_{sk}^{p,i} + k\boldsymbol{v}_{ck}^{p,i} + \nu\Delta\boldsymbol{v}_{sk}^{d,i} + k\boldsymbol{v}_{ck}^{d,i}. \end{aligned} \tag{1.4.16}$$

Let us notice that the right-hand in (1.4.15) has a compact support. Indeed, the supports of the vector-fields $\mathbf{f}_{ck}^{d,i}$ and $\mathbf{f}_{sk}^{d,i}$ are finite due to the fact that outside the domain Ω_1 the functions $\boldsymbol{v}_{ck}^{d,i}$ and $\boldsymbol{v}_{sk}^{d,i}$ vanish. The functions $\mathbf{f}_{ck}^{p,i}$ and $\mathbf{f}_{sk}^{p,i}$ are obtained when the Stokes operator $\mathbf{S}_k$ is applied to the flux carrier $\boldsymbol{v}_k^{p,i}$. Since the vector-field $\boldsymbol{v}_k^{p,i}$ is a combination of the Poiseuille flows (exact solutions the to the homogeneous problem) multiplied by the corresponding cut-off functions χ^j, the functions $\mathbf{f}_{ck}^{p,i}$ and $\mathbf{f}_{sk}^{p,i}$ are equal to zero whenever $\chi^1, \dots, \chi^J$ are constant, i.e., outside the domain $\cup_{j=1}^J G_0^j$, where $G_0^j = \omega^j \times (0,1)$. Moreover, the regularity of $\boldsymbol{u}_k^{p,i}$ $\boldsymbol{u}_k^{d,i}$ is enough for the vector-field $\widehat{\mathbf{f}}_k^i = (\widehat{\mathbf{f}}_{ck}^i, 0, \widehat{\mathbf{f}}_{ck}^i, 0)$ to be an element of the space $\mathcal{R}_\beta^2 H(\Omega)$. Hence we may apply Theorem 1.4.1 to problem (1.4.15) and conclude that it has a solution in the class $\mathfrak{D}_k$. Theorem 1.1.5 shows that this solution admits the asymptotic representation (1.1.22) which, due to the boundedness of the Dirichlet integral of elements in $\mathfrak{D}_k$, turns into

$$\widehat{\boldsymbol{u}}_k^i = \sum_{j=1}^{J} \chi^j \{a_{ck}^{i,j} \mathbf{u}_{ck}^{j0} + a_{sk}^{i,j} \mathbf{u}_{sk}^{j0}\} + \widetilde{\boldsymbol{u}}^i.$$

In the same way as in Theorem 1.2.2 we obtain the estimate

$$\begin{aligned} &\|\widehat{\boldsymbol{v}}_{ck}^i\|_{H_\beta^2(\Omega)} + \|\widehat{\boldsymbol{v}}_{sk}^i\|_{H_\beta^2(\Omega)} + \|\nabla \widehat{p}_{ck}^i\|_{L_\beta^2(\Omega)} + \|\nabla \widehat{p}_{sk}^i\|_{L_\beta^2(\Omega)} \\ &\qquad \leq c \left(\|\widehat{\mathbf{f}}_{ck}^i\|_{L_\beta^2(\Omega)} + \|\widehat{\mathbf{f}}_{sk}^i\|_{L_\beta^2(\Omega)} \right). \end{aligned} \tag{1.4.17}$$

Remark 1.4.3. For any constants a_{ck}^i and a_{sk}^i the vector-field $\widehat{\boldsymbol{u}}_k^i + a_{ck}^i \mathbf{u}_k^c + a_{sk}^i \mathbf{u}_k^s$, where

$$\mathbf{u}_k^c = (0,0,0,\ 1,\ 0,0,0,\ 0), \quad \mathbf{u}_k^s = (0,0,0,\ 0,\ 0,0,0,\ 1),$$

is also the solution of problem (1.4.15).

According to Remark 1.4.3, we can add arbitrary constants to the pressure terms p_{ck} and p_{sk} of the solution to the homogeneous problem (1.1.2). This means that in the asymptotic representation of the function $\widehat{\boldsymbol{u}}_k^i$ (see (1.4.6)) we can substitute the constants $\{a_{ck}^{i,j}\}_{j=1}^J$ and $\{a_{sk}^{i,j}\}_{j=1}^J$ by the constants $\{a_{ck}^{i,j} + a_{ck}^i\}_{j=1}^J$ and $\{a_{ck}^{i,j} + a_{sk}^i\}_{j=1}^J$, with an arbitrary a_{ck}^i and a_{sk}^i. Therefore taking suitable constants a_{ck}^i and a_{sk}^i we may attribute to the sums $a_{ck}^{i,J} + a_{ck}^i$ and $a_{sk}^{i,J} + a_{sk}^i$ any values[xiii]. If not said otherwise, in the following we fix

$$\begin{aligned} a_{ck}^{i,J} = 1, \quad a_{sk}^{i,J} = 0 \quad \text{for} \quad i = 1, \dots, J, \\ a_{ck}^{i,J} = 0, \quad a_{sk}^{i,J} = 1 \quad \text{for} \quad i = J+1, \dots, 2J. \end{aligned} \tag{1.4.18}$$

The rest of the constants in (1.4.6) is defined by the problem itself. In the next subsection we will derive formulas to compute $\{a_{ck}^{i,j}, a_{ck}^{i,j}\}_{j=1}^{J-1}$ (they will be expressed in terms of the data of the problem and the geometry of the domain Ω).

We recall that the projectors $\pi^0, \pi^1 : \mathbb{D}_{\pm\beta}^l H(\Omega) \to \mathbb{R}^{2J}$ defined by (1.3.7) and (1.3.11) generate $2J$-dimensional column-vectors composed from the constants in the asymptotic representation (1.1.22) of a function $\mathbf{u}_k \in \mathbb{D}_{\pm\beta}^l H(\Omega)$. Thus for every

[xiii]Notice that $a_{ck}^{i,J} + a_{ck}^i$ and $a_{sk}^{i,J} + a_{sk}^i$ stands for the pressure constants in the outlet Ω_+^J. Obviously, we may chose any other outlet. That is, one constant in the set $\{a_{ck}^{i,j}\}_{j=1}^J$ and one constant in the set $\{a_{sk}^{i,j}\}_{j=1}^J$ may be chosen arbitrarily.

$i = 1, \dots, 2J$ we have

$$\pi^0 \boldsymbol{u}_k^i = (a_{ck}^{i,1}, \dots, a_{ck}^{i,J}, a_{sk}^{i,1}, \dots, a_{sk}^{i,J}),$$
$$\pi^1 \boldsymbol{u}_k^i = (b_{ck}^{i,1}, \dots, b_{ck}^{i,J}, b_{sk}^{i,1}, \dots, b_{sk}^{i,J}),$$

with the constants $b_{ck}^{i,j}$, $b_{sk}^{i,j}$ and $a_{ck}^{i,J}$, $a_{sk}^{i,J}$ defined by (1.4.12) and (1.4.18). It is not difficult to verify that the flow-rate coefficients (1.4.11) generated by the vector-field $\boldsymbol{u}_k^i$ can be expressed as the product $\mathbb{F}_k \pi^1 \boldsymbol{u}_k^i$, where $2J \times 2J$ matrix $\mathbb{F}_k$ is given by (1.3.12) (see also (1.3.8)). According to Remark 1.3.2, the matrix $\mathbb{F}_k$ is non-singular for every k. Moreover, one can straightforwardly verify that in the set $\mathbb{F}_k \pi^1 \boldsymbol{u}_k^1, \dots, \mathbb{F}_k \pi^1 \boldsymbol{u}_k^J$ any $J-1$ vectors are linearly independent, while the sum $\mathbb{F}_k \pi^1 \boldsymbol{u}_k^1 + \dots + \mathbb{F}_k \pi^1 \boldsymbol{u}_k^J = \mathbf{0}$[xiv]. Since $\mathbb{F}_k$ is non-singular, any $J-1$ vectors among $\pi^1 \boldsymbol{u}_k^1, \dots, \pi^1 \boldsymbol{u}_k^J$ are linearly independent, whereas the vector $\pi^1 \boldsymbol{u}_k^1 + \dots + \pi^1 \boldsymbol{u}_k^J$ is equal to zero one. Taking into account the definition (1.4.1) we conclude that the sum $\boldsymbol{u}_k^1 + \dots + \boldsymbol{u}_k^J$, denote it by $\boldsymbol{u}_k^\sharp$, belong to the subspace $\mathfrak{D}_k$. Using Theorem 1.4.1 and formulas (1.4.18) we see that $\boldsymbol{u}_k^\sharp$ (as the solution to the homogeneous problem) must be proportional to the vector $\mathbf{u}_k^c$ defined in (1.4.2). For the vector $\mathbf{u}_k^c$ we have

$$\pi^1 \mathbf{u}_k^c = (0, \dots, 0), \quad \pi^0 \mathbf{u}_k^c = (1, \dots, 1, 0, \dots, 0).$$

Hence $\mathbf{u}_k^c$ and the vectors $\boldsymbol{u}_k^1, \dots, \boldsymbol{u}_k^{J-1}$ form the system of J linearly independent vector-fields in $\ker \mathbf{A}^l_{-\beta\to\beta}$. In the same way we can show linear independence of the system composed from $\boldsymbol{u}_k^{J+1}, \dots, \boldsymbol{u}_k^{2J-1}$ and the vector $\mathbf{u}_k^s$, having projections

$$\pi^1 \mathbf{u}_k^s = (0, \dots, 0), \quad \pi^0 \mathbf{u}_k^s = (0, \dots, 0, 1, \dots, 1).$$

As a consequence we have $2J$ linearly independent solutions

$$\boldsymbol{u}_k^1, \dots, \boldsymbol{u}_k^{J-1}, \mathbf{u}_k^c, \boldsymbol{u}_k^{J+1}, \dots, \boldsymbol{u}_k^{2J-1}, \mathbf{u}_k^s \tag{1.4.19}$$

to the homogeneous problem (1.1.2). Since the dimension of the subspace $\ker \mathbf{A}^l_{-\beta\to\beta}$ is equal to $2J$, the vector-fields (1.4.19) form the basis in the set of solutions to the

[xiv] For example, in the case $J = 3$ we have

$$\mathbb{F}_k \pi^1 \boldsymbol{u}_k^1 = (2/3, -1/3, -1/3, 0, 0, 0), \qquad \mathbb{F}_k \pi^1 \boldsymbol{u}_k^4 = (0, 0, 0, 2/3, -1/3, -1/3),$$
$$\mathbb{F}_k \pi^1 \boldsymbol{u}_k^2 = (-1/3, 2/3, -1/3, 0, 0, 0), \qquad \mathbb{F}_k \pi^1 \boldsymbol{u}_k^5 = (0, 0, 0, -1/3, 2/3, -1/3),$$
$$\mathbb{F}_k \pi^1 \boldsymbol{u}_k^3 = (-1/3, -1/3, 2/3, 0, 0, 0), \qquad \mathbb{F}_k \pi^1 \boldsymbol{u}_k^6 = (0, 0, 0, -1/3, -1/3, 2/3).$$

homogeneous Stokes-type problem.

Remark 1.4.4. The sums $\boldsymbol{u}_k^1 + \ldots + \boldsymbol{u}_k^J$ and $\boldsymbol{u}_k^{J+1} + \ldots + \boldsymbol{u}_k^{2J}$ are proportional to the vectors $\mathbf{u}_k^c$ and $\mathbf{u}_k^s$, respectively. Therefore the system of vector-fields $\boldsymbol{u}_k^1, \ldots, \boldsymbol{u}_k^{2J}$ also forms the basis in the set $\ker \mathbf{A}^l_{-\beta\to\beta}$.

Let us describe certain quantities characterising the basis presented in this Remark. Consider the following $2J \times 2J$ matrices

$$\mathcal{A}_k = \left(\pi^0 \boldsymbol{u}_k^1 \cdots \pi^0 \boldsymbol{u}_k^{2J}\right), \quad \mathcal{B}_k = \left(\pi^1 \boldsymbol{u}_k^1 \cdots \pi^1 \boldsymbol{u}_k^{2J}\right). \tag{1.4.20}$$

Taking into account formulas (1.4.10), the flow-rate distributions (1.4.11) and the definition of the matrices $\mathcal{B}_k$ and $\mathbb{F}_k$ (see (1.3.12)) we deduce that

$$\mathbb{F}_k \mathcal{B}_k = \begin{pmatrix} \mathcal{C}_k & -\mathcal{D}_k \\ \mathcal{D}_k & \mathcal{C}_k \end{pmatrix} \mathcal{B}_k = \begin{pmatrix} \mathcal{F} & \mathbb{O} \\ \mathbb{O} & \mathcal{F} \end{pmatrix}, \tag{1.4.21}$$

here $\mathcal{F}$ is the $J \times J$ matrix with the entries equal to $\delta_i^j - 1/J$. The matrix on the right-hand side of (1.4.21) describes the flow-rates of the basis $\boldsymbol{u}_k^1, \ldots, \boldsymbol{u}_k^{2J}$ and is called *the flux distribution matrix.* According to formulas (1.1.24) the pressure coefficients p_{ck} and p_{sk} in the outlet Ω^j admit the representation

$$p_{ck}(x) = a_{ck}^j - b_{ck}^j x_3^j + \widetilde{p}_{ck}^j(x), \quad p_{sk}(x) = a_{sk}^j - b_{sk}^j x_3^j + \widetilde{p}_{sk}^j(x),$$

where $\widetilde{p}_{ck}^j = o(e^{-\beta x_3^j})$ and $\widetilde{p}_{sk}^j = o(e^{-\beta x_3^j})$ as $x_3^j \to \infty$. The matrix $\mathcal{A}_k$ is called *the pressure distribution matrix.* The flux and the pressure distribution matrices for the steady-state Stokes problem were presented in Section 5 of [54].

1.4.1 Estimates for the elements $\boldsymbol{u}_k^1, \ldots, \boldsymbol{u}_k^{2J}$

The estimates (1.4.14), (1.4.17) allow us to evaluate the L^2-norm of the velocity coefficients $\boldsymbol{v}_{ck}^i$, $\boldsymbol{v}_{ck}^i$ of the element $\boldsymbol{u}_k^i = \boldsymbol{u}_k^{p,i} + \boldsymbol{u}_k^{d,i} + \widehat{\boldsymbol{u}}_k^i$ in terms of the flux carriers $\boldsymbol{v}_{ck}^{p,i}$, $\boldsymbol{v}_{sk}^{p,i}$ and, consequently, in terms of the coefficients $\{b_{ck}^{i,j}, b_{sk}^{i,j}\}$.

Let us first denote

$$\alpha_k^j = \int_{\omega^j} \left(|\varphi_k^j|^2 + |\psi_k^j|^2\right) dy^j, \quad \gamma_k^j = \int_{\omega^j} \left(|\Delta\varphi_k^j|^2 + |\Delta\psi_k^j|^2\right) dy^j. \tag{1.4.22}$$

Taking into account estimates (1.4.14), the definition of the flux carrier $\boldsymbol{u}_k^{p,i}$ and the fact that $(\chi^j)' = (\chi^j(x_3^j))'$ is supported on the interval $(0,1)$, we get

$$\|\boldsymbol{v}_{ck}^{d,i}\|^2_{H^1(\Omega_1)} \leq c\|\nabla\cdot\boldsymbol{v}_{ck}^{p,i}\|^2_{L^2(\Omega_1)} \leq c\sum_{j=1}^{J}\left\|(\chi^j)'\left(b_{ck}^{i,j}\varphi_k^j + b_{sk}^{i,j}\psi_k^j\right)\right\|^2_{L^2(\Omega_1)}$$
$$= c\sum_{j=1}^{J}\int_0^1\int_{\omega^j}|(\chi^j)'\left(b_{ck}^{i,j}\varphi_k^j + b_{sk}^{i,j}\psi_k^j\right)|^2 dy^j dx_3^j \quad (1.4.23)$$
$$\leq c\sum_{j=1}^{J}\left((b_{ck}^{i,j})^2 + (b_{sk}^{i,j})^2\right)\int_{\omega^j}|\varphi_k^j|^2 + |\psi_k^j|^2 dy^j = c\sum_{j=1}^{J}\alpha_k^j\left((b_{ck}^{i,j})^2 + (b_{sk}^{i,j})^2\right).$$

Analogously we derive the inequality

$$\|\boldsymbol{v}_{sk}^{d,i}\|^2_{H^1(\Omega_1)} \leq c\sum_{j=1}^{J}\alpha_k^j\left((b_{ck}^{i,j})^2 + (b_{sk}^{i,j})^2\right). \quad (1.4.24)$$

Reasoning in the same way as in Theorem 1.2.2 we derive the estimate for the velocity fields $\widehat{\boldsymbol{v}}_{ck}^i$ and $\widehat{\boldsymbol{v}}_{sk}^i$ of the function $\widehat{\boldsymbol{u}}_k^i$ satisfying equations (1.4.15):

$$\frac{\nu k}{2}\int_\Omega \rho_\beta\left(|\nabla\widehat{\boldsymbol{v}}_{ck}^i|^2 + |\nabla\widehat{\boldsymbol{v}}_{sk}^i|^2\right)dx + \frac{k^2}{2}\int_\Omega\rho_\beta\left(|\widehat{\boldsymbol{v}}_{ck}^i|^2 + |\widehat{\boldsymbol{v}}_{sk}^i|^2\right)dx$$
$$\leq c\int_\Omega\rho_\beta\left(|\widehat{\mathbf{f}}_{ck}^i|^2 + |\widehat{\mathbf{f}}_{sk}^i|^2\right)dx.$$

Since the functions $\widehat{\mathbf{f}}_{ck}^i$ and $\widehat{\mathbf{f}}_{sk}^i$ (see (1.4.16)) on the right-hand of the last estimate are supported on Ω_1 and the weight function ρ_β is bounded on Ω_1, we get

$$\|\widehat{\boldsymbol{v}}_{ck}^i\|^2_{L^2(\Omega)} + \|\widehat{\boldsymbol{v}}_{sk}^i\|^2_{L^2(\Omega)} \leq \|\widehat{\boldsymbol{v}}_{ck}^i\|^2_{L^2_\beta(\Omega)} + \|\widehat{\boldsymbol{v}}_{sk}^i\|^2_{L^2_\beta(\Omega)}$$
$$\leq \frac{c}{k^2}\left(\|\widehat{\mathbf{f}}_{ck}^i\|^2_{L^2(\Omega_1)} + \|\widehat{\mathbf{f}}_{sk}^i\|^2_{L^2(\Omega_1)}\right). \quad (1.4.25)$$

Recall that in (1.4.25) the right-hand side is expressed as $\widehat{\mathbf{f}}_{ck}^i = \mathbf{f}_{ck}^{p,i} + \mathbf{f}_{ck}^{d,i}$ and $\widehat{\mathbf{f}}_{sk}^i = \mathbf{f}_{sk}^{p,i} + \mathbf{f}_{sk}^{d,i}$ with

$$\mathbf{f}_{ck}^{p,i} = \nu\Delta\boldsymbol{v}_{ck}^{p,i} - \nabla p_{ck}^{p,i} - k\boldsymbol{v}_{sk}^{p,i}, \quad \mathbf{f}_{ck}^{d,i} = \nu\Delta\boldsymbol{v}_{ck}^{d,i} - k\boldsymbol{v}_{sk}^{d,i},$$
$$\mathbf{f}_{sk}^{p,i} = \nu\Delta\boldsymbol{v}_{sk}^{p,i} - \nabla p_{sk}^{p,i} + k\boldsymbol{v}_{ck}^{p,i}, \quad \mathbf{f}_{sk}^{d,i} = \nu\Delta\boldsymbol{v}_{sk}^{d,i} + k\boldsymbol{v}_{ck}^{d,i}.$$

Consider the terms $\mathbf{f}_{ck}^{p,i}$ and $\mathbf{f}_{sk}^{p,i}$. According to formulas (1.1.11), (1.1.12) and (1.4.6), the first two coordinates of the vector-fields $\mathbf{f}_{ck}^{p,i}$ and $\mathbf{f}_{sk}^{p,i}$ are equal to zero, while the third ones are equal to

$$\sum_{j=1}^{J}\left\{\nu(\chi^j)''(b^{i,j}_{ck}\varphi^j_k + b^{i,j}_{sk}\psi^j_k) + (\chi^j)'b^{i,j}_{ck}x^j_3\right\},$$
$$\sum_{j=1}^{J}\left\{\nu(\chi^j)''(b^{i,j}_{sk}\varphi^j_k - b^{i,j}_{ck}\psi^j_k) + (\chi^j)'b^{i,j}_{sk}x^j_3\right\},$$

respectively. It is easy to see that

$$\|\mathbf{f}^{p,i}_{ck}\|^2_{L^2(\Omega_1)} + \|\mathbf{f}^{p,i}_{sk}\|^2_{L^2(\Omega_1)} \le c\sum_{j=1}^{J}(1+\alpha^j_k)\left((b^{ij}_{ck})^2 + (b^{ij}_{sk})^2\right). \tag{1.4.26}$$

To estimate the L^2-norm of the term $\mathbf{f}^{d,i}_{ck}$ we use estimates (1.4.14) and the interpolation inequality $\|\nabla u\|_{L_2(G)} \le c\left(\|u\|_{L_2(G)} + \|\Delta u\|_{L_2(G)}\right)$, which is valid when the boundary ∂G is of class C^2 and u vanishes at ∂G (see Section 3.8 in [43]). Since for every $j = 1,\dots,J$, the boundary $\partial\omega^j \in C^2$ and functions φ^j_k, ψ^j_k belong to $\mathring{H}^1(\omega^j)\cap H^2(\omega^j)$, we have

$$\begin{aligned}
\int_{\Omega_1}|\mathbf{f}^{d,i}_{ck}|^2dx &\le c\int_{\Omega_1}\left(|\Delta\boldsymbol{v}^{d,i}_{ck}|^2 + k^2|\boldsymbol{v}^{d,i}_{sk}|^2\right)dx \le c\|\nabla\cdot\boldsymbol{v}^{p,i}_{ck}\|^2_{H^1(\Omega_1)} \\
&+ck^2\|\nabla\cdot\boldsymbol{v}^{p,i}_{sk}\|^2_{L^2(\Omega_1)} = c\|\sum_{j=1}^{J}\chi'_j\boldsymbol{v}^{p,i,j}_{ck}\|^2_{H^1(\Omega_1)} + ck^2\|\sum_{j=1}^{J}\chi'_j\boldsymbol{v}^{p,i,j}_{sk}\|^2_{L^2(\Omega_1)} \\
&\le c\sum_{j=1}^{J}\left(\|\boldsymbol{v}^{p,i,j}_{ck}\|^2_{L^2(\omega^j)} + \|\Delta\boldsymbol{v}^{p,i,j}_{ck}\|^2_{L^2(\omega^j)} + k^2\|\boldsymbol{v}^{p,i,j}_{sk}\|^2_{L^2(\omega^j)}\right) \\
&\le c\sum_{j=1}^{J}\int_{\omega^j}\left(|b^{i,j}_{ck}\varphi^j_k + b^{i,j}_{sk}\psi^j_k|^2 + |b^{i,j}_{ck}\Delta\varphi^j_k + b^{i,j}_{sk}\Delta\psi^j_k|^2 + k^2|b^{i,j}_{sk}\varphi^j_k - b^{i,j}_{ck}\psi^j_k|^2\right)dS.
\end{aligned}$$

Combining this estimate with the analogous estimate for the function $\mathbf{f}^{d,i}_{sk}$ we get that

$$\|\mathbf{f}^{d,i}_{ck}\|^2_{L^2(\Omega_1)} + \|\mathbf{f}^{d,i}_{sk}\|^2_{L^2(\Omega_1)} \le c\sum_{j=1}^{J}(\alpha^j_k + k^2\alpha^j_k + \gamma^j_k)\left((b^{i,j}_{ck})^2 + (b^{i,j}_{sk})^2\right). \tag{1.4.27}$$

Collecting estimates (1.4.23)–(1.4.27) we obtain the inequality

$$\|\boldsymbol{v}^i_{ck}\|^2_{L^2(\Omega_1)} + \|\boldsymbol{v}^i_{sk}\|^2_{L^2(\Omega_1)} \le c\sum_{j=1}^{J}\left(\alpha^j_k + \frac{1}{k^2}(1+\alpha^j_k+\gamma^j_k)\right)\left((b^{i,j}_{ck})^2 + (b^{i,j}_{sk})^2\right). \tag{1.4.28}$$

Remark 1.4.5. The coefficients $b^{i,j}_{ck}$, $b^{i,j}_{sk}$ grow unboundedly as $k \to \infty$. Indeed, from formulas (1.4.12) and Lemma A.0.1 we see that the sequence $\{c_k^j/d_k^j\}_{k=1}^\infty$ is bounded:

$$0 < \frac{c_k^j}{d_k^j} \le \frac{|\omega^j|}{k d_k^j} \to 1, \quad k \to \infty.$$

Then from the definition of $b_{sk}^{i,j}$ (see (1.4.12)) we have

$$|b_{sk}^{i,j}| = \frac{d_k^j(\delta_i^j - 1/J)}{(c_k^j)^2 + (d_k^j)^2} = \frac{1}{d_k^j} \frac{(\delta_i^j - 1/J)}{\left(c_k^j/d_k^j\right)^2 + 1} \sim \frac{1}{d_k^j}, \quad k \to \infty.$$

Since $d_k^j = O(1/k)$ as $k \to \infty$, the quantity $|b_{sk}^{i,j}|$ is $O(k)$ as $k \to \infty$.

In spite of this fact, the norms $\|\boldsymbol{v}_{ck}^i\|_{L^2(\Omega_1)}$ and $\|\boldsymbol{v}_{sk}^i\|_{L^2(\Omega_1)}$ (estimated by (1.4.28)) remain bounded as $k \to \infty$. The functions φ_k^j and ψ_k^j decay sufficiently fast and ensure that the right hand-side of inequality (1.4.28) is bounded (uniformly with respect to k). Indeed, from (1.4.12) we get, for every $i = 1, \ldots, 2J$ and every $j = 1, \ldots, J$, that

$$(b_{ck}^{i,j})^2 + (b_{sk}^{i,j})^2 = \frac{\delta_i^j - 1/J}{(c_k^j)^2 + (d_k^j)^2}.$$

Then definitions (1.4.22) and Lemma A.0.1 yield the estimates

$$\Big(\alpha_k^j + \frac{1}{k^2}(1 + \alpha_k^j + \gamma_k^j)\Big)\Big((b_{ck}^{i,j})^2 + (b_{sk}^{i,j})^2\Big) \le \left(\frac{d_k^j}{k} + \frac{d_k^j}{k^3} + \frac{1 + |\omega^j|}{k^2}\right) \frac{1}{(c_k^j)^2 + (d_k^j)^2}$$

$$\le \left(\frac{d_k^j}{k} + \frac{d_k^j}{k^3} + \frac{1 + |\omega^j|}{k^2}\right) \frac{1}{(d_k^j)^2} = \frac{1}{k d_k^j} + \frac{1}{k^3 d_k^j} + \frac{1 + |\omega^j|}{(k d_k^j)^2} \to \frac{1}{|\omega^j|} + \frac{1 + |\omega^j|}{|\omega^j|^2},$$

$$\text{as} \quad k \to \infty, \quad \text{for all} \quad j = 1, \ldots, J.$$

1.4.2 Basis for the homogeneous formally adjoint Stokes-type problem

Applying the three step procedure, presented in the beginning of this Section, we can construct the basis $\boldsymbol{\mathcal{U}}_k^1, \ldots, \boldsymbol{\mathcal{U}}_k^{2J}$ in the set of solutions to the homogeneous adjoint Stokes-type problem (1.1.16). Namely, for every $i = 1, \ldots, 2J$ we define the flux carrier $\boldsymbol{\mathcal{U}}_k^{p,i} = (\boldsymbol{\mathcal{V}}_{ck}^{p,i}, \mathcal{P}_{ck}^{p,i}, \boldsymbol{\mathcal{V}}_{sk}^{p,i}, \mathcal{P}_{sk}^{p,i})$ with the specific flow-rates (see below). Then we construct the vector-field $\boldsymbol{\mathcal{U}}_k^{d,i} = (\boldsymbol{\mathcal{V}}_{ck}^{d,i}, 0, \boldsymbol{\mathcal{V}}_{sk}^{p,i}, 0)$ which annuls the divergence of $\boldsymbol{\mathcal{V}}_{ck}^{p,i}$ and $\boldsymbol{\mathcal{V}}_{sk}^{p,i}$. Finally, in the class of functions with the finite Dirichlet integral we solve the adjoint Stokes-type problem $\mathbf{S}^*\widehat{\boldsymbol{\mathcal{U}}}_k^i = \widehat{\mathbf{F}}_k^i$ with the right-hand side $\widehat{\mathbf{F}}_k^i = -\mathbf{S}^*(\boldsymbol{\mathcal{U}}_k^{p,i} + \boldsymbol{\mathcal{U}}_k^{d,i})$.

For every $i = 1, \ldots, 2J$ the vector-fields $\boldsymbol{\mathcal{U}}_k^i$ admit the asymptotic representation (see (1.1.25))

$$\boldsymbol{\mathcal{U}}_k^i = \sum_{j=1}^{J} \chi^j \left\{ A_{ck}^{i,j} \mathbf{U}_{ck}^{j0} + A_{sk}^{i,j} \mathbf{U}_{sk}^{j0} + B_{ck}^{i,j} \mathbf{U}_{ck}^{j1} + B_{sk}^{i,j} \mathbf{U}_{sk}^{j1} \right\} + \widetilde{\boldsymbol{\mathcal{U}}}_k^i.$$

In order to prescribe the flow-rates and the pressure constants of the elements in the basis, we substitute in the last formula

$$\begin{aligned} B_{ck}^{i,j} &= \frac{c_k^j(\delta_i^j - 1/J)}{(c_k^j)^2 + (d_k^j)^2}, \quad B_{sk}^{i,j} = -\frac{d_k^j(\delta_i^j - 1/J)}{(c_k^j)^2 + (d_k^j)^2}, \\ A_{ck}^{i,J} &= 1, \quad A_{sk}^{i,J} = 0, \quad \text{for} \quad i = 1, \ldots, J, \\ B_{ck}^{i,j} &= -\frac{d_k^j(\delta_i^{J+j} - 1/J)}{(c_k^j)^2 + (d_k^j)^2}, \quad B_{sk}^{i,j} = -\frac{c_k^j(\delta_i^{J+j} - 1/J)}{(c_k^j)^2 + (d_k^j)^2}, \\ A_{ck}^{i,J} &= 0, \quad A_{sk}^{i,J} = 1, \quad \text{for} \quad i = J+1, \ldots, 2J. \end{aligned} \tag{1.4.29}$$

With the coefficients defined by (1.4.29), the basis $\boldsymbol{\mathcal{U}}_k^1, \ldots, \boldsymbol{\mathcal{U}}_k^{2J}$ in $\ker \left(\mathbf{A}^l_{-\beta\to\beta}\right)^*$ has the same flow-rate distributions as the basis $\boldsymbol{u}_k^1, \ldots, \boldsymbol{u}_k^{2J}$ in $\ker \mathbf{A}^l_{-\beta\to\beta}$ (see (1.4.11)) while the pressure distribution matrix

$$\mathcal{A}_k^* = \left(\pi^0 \boldsymbol{\mathcal{U}}_k^1 \cdots \pi^0 \boldsymbol{\mathcal{U}}_k^{2J}\right)$$

in general may be different from the matrix $\mathcal{A}_k$.

In the same way as in Subsection 1.4.1 we derive the estimate

$$\begin{aligned} &\|\boldsymbol{\mathcal{V}}_{ck}^i\|^2_{L^2(\Omega_1)} + \|\boldsymbol{\mathcal{V}}_{sk}^i\|^2_{L^2(\Omega_1)} \\ &\le c \sum_{j=1}^{J} \left(\alpha_k^j + \frac{1}{k^2}(1 + \alpha_k^j + \gamma_k^j)\right) \left((B_{ck}^{i,j})^2 + (B_{sk}^{i,j})^2\right). \end{aligned} \tag{1.4.30}$$

Comparing formulas (1.4.12) and (1.4.29) we see that $(b_{ck}^{i,j})^2 + (b_{sk}^{i,j})^2 = (B_{ck}^{i,j})^2 + (B_{sk}^{i,j})^2$. Therefore, repeating the same arguments as in the end of the previous Subsection, we show that the norms $\|\boldsymbol{\mathcal{V}}_{ck}^i\|_{L^2(\Omega_1)}$, $\|\boldsymbol{\mathcal{V}}_{sk}^i\|_{L^2(\Omega_1)}$ remain bounded for all k, i.e., the estimate

$$\|\boldsymbol{\mathcal{V}}_{ck}^i\|_{L^2(\Omega_1)} + \|\boldsymbol{\mathcal{V}}_{sk}^i\|_{L^2(\Omega_1)} \le c, \tag{1.4.31}$$

holds with a constant c independent of k.

1.5 Representation and estimates of the pressure constants $a_{ck}^{i,j}$ and $a_{sk}^{i,j}$

In order to derive certain relations between the coefficients of the basis (1.4.5), we need special solutions of the homogeneous adjoint Stokes-type problem. Let us take the basis $\boldsymbol{\mathcal{U}}_k^1, \dots, \boldsymbol{\mathcal{U}}_k^{2J}$ of $\ker \left(\mathbf{A}_{-\beta\to\beta}^l\right)^*$ constructed in Subsection 1.4.2 and for $l = 1, \dots, J-1, J+1, \dots, 2J-1$ define the functions

$$\overline{\boldsymbol{\mathcal{U}}}_k^l = \boldsymbol{\mathcal{U}}_k^l - \boldsymbol{\mathcal{U}}_k^{l+1}.$$

We recall that the flow-rate coefficients generated by the element $\boldsymbol{\mathcal{U}}_k^l$ are

$$\begin{aligned}(\delta_i^1 - \frac{1}{J}, \dots, \delta_i^J - \frac{1}{J},\ 0, \dots, 0), &\quad \text{if} \quad i = 1, \dots, J,\\ (0, \dots, 0,\ \delta_i^{J+1} - \frac{1}{J}, \dots, \delta_i^{2J} - \frac{1}{J}), &\quad \text{if} \quad i = J+1, \dots, 2J.\end{aligned}$$

Therefore the elements $\overline{\boldsymbol{\mathcal{U}}}_k^l, \overline{\boldsymbol{\mathcal{U}}}_k^{J+l}$ have unit inflows through the cylinder Ω_+^l and unit outflow through the cylinder Ω_+^{l+1}, while the flow-rates in the rest of the cylinders are equal to zero. Note that for the element $\overline{\boldsymbol{\mathcal{U}}}_k^l = (\overline{\boldsymbol{\mathcal{V}}}_{ck}^l, \overline{\mathcal{P}}_{ck}^l, \overline{\boldsymbol{\mathcal{V}}}_{sk}^l, \overline{\mathcal{P}}_{sk}^l)$ the flow is carried by the term $\overline{\boldsymbol{\mathcal{V}}}_{ck}^i$ when $l = 1, \dots, J-1$ and by the term $\overline{\boldsymbol{\mathcal{V}}}_{sk}^l$ when $l = J+1, \dots, 2J-1$. Namely, for $l = 1, \dots, J-1$ the relations

$$\begin{gathered}\int_{\omega^l} \overline{\boldsymbol{\mathcal{V}}}_{ck}^l \cdot \mathbf{n}\, dy^l = 1, \quad \int_{\omega^{l+1}} \overline{\boldsymbol{\mathcal{V}}}_{ck}^l \cdot \mathbf{n}\, dy^{l+1} = -1,\\ \int_{\omega^j} \overline{\boldsymbol{\mathcal{V}}}_{ck}^j \cdot \mathbf{n}\, dy^j = 0, \quad j \neq l,\, j \neq l+1,\\ \int_{\omega^j} \overline{\boldsymbol{\mathcal{V}}}_{sk}^j \cdot \mathbf{n}\, dy^j = 0, \quad j = 1, \dots, J\end{gathered} \tag{1.5.1}$$

hold, while for the elements $\boldsymbol{\mathcal{U}}_k^l$ with $l = J+1, \dots, 2J-1$ we have the following flow-rate distributions

$$\begin{gathered}\int_{\omega^j} \overline{\boldsymbol{\mathcal{V}}}_{ck}^j \cdot \mathbf{n}\, dy^j = 0, \quad j = 1, \dots, J\\ \int_{\omega^l} \overline{\boldsymbol{\mathcal{V}}}_{sk}^l \cdot \mathbf{n}\, dy^l = 1, \quad \int_{\omega^{l+1}} \overline{\boldsymbol{\mathcal{V}}}_{sk}^l \cdot \mathbf{n}\, dy^{l+1} = -1,\\ \int_{\omega^j} \overline{\boldsymbol{\mathcal{V}}}_{sk}^j \cdot \mathbf{n}\, dy^j = 0, \quad j \neq l,\, j \neq l+1.\end{gathered} \tag{1.5.2}$$

Let us take some $l \in \{1, 2, \ldots, J-1\}$. Multiplying the equations in the system (1.4.15) by the vector-field $(\overline{\boldsymbol{\mathcal{V}}}^l_{ck}, \overline{\mathcal{P}}^l_{ck}, \overline{\boldsymbol{\mathcal{V}}}^l_{sk}, \overline{\mathcal{P}}^l_{sk})$, integrating over the truncated domain $\Omega_L = \left\{x \in \Omega : x^j_3 < L, j = 1, \ldots, J\right\}$ and then applying integration by parts to the left-hand side of the obtained expression we derive the relation

$$\int_{\Omega_L} \left(\widehat{\boldsymbol{v}}^i_{ck} \cdot (-\nu\Delta\overline{\boldsymbol{\mathcal{V}}}^l_{ck} + \nabla\overline{\mathcal{P}}^l_{ck} - k\overline{\boldsymbol{\mathcal{V}}}^l_{sk}) + \widehat{\boldsymbol{v}}^i_{sk} \cdot (-\nu\Delta\overline{\boldsymbol{\mathcal{V}}}^l_{sk} + \nabla\overline{\mathcal{P}}^l_{sk} + k\overline{\boldsymbol{\mathcal{V}}}^l_{ck})\right) dx$$
$$+ \int_{\partial\Omega_L} \sum_{j=1}^{J} \left(\chi_j \widehat{p}^i_{ck} \mathbf{n} \cdot \overline{\boldsymbol{\mathcal{V}}}^l_{ck} + \chi_j \widehat{p}^i_{sk} \mathbf{n} \cdot \overline{\boldsymbol{\mathcal{V}}}^l_{sk}\right) dS = \int_{\Omega_L} \left(\widehat{\mathbf{f}}^i_{ck} \cdot \overline{\boldsymbol{\mathcal{V}}}^l_{ck} + \widehat{\mathbf{f}}^i_{sk} \cdot \overline{\boldsymbol{\mathcal{V}}}^l_{sk}\right) dx.$$

Since $\overline{\boldsymbol{\mathcal{U}}}^l_k$ is a solution of the homogeneous problem, the first integral on the left-hand side vanishes. The velocity-fields $\overline{\boldsymbol{\mathcal{V}}}^l_{ck}$ and $\overline{\boldsymbol{\mathcal{V}}}^l_{sk}$ are equal to zero on the boundary $\partial\Omega$, therefore the integrals over the boundary $\partial\Omega_L$ turns into the integrals over the cross-sections ω^j. Moreover, the terms $\widetilde{p}^i_{ck}$ and $\widetilde{p}^i_{sk}$ in the pressure functions $\widehat{p}^i_{ck} = \sum_{j=1}^J \chi_j a^{i,j}_{ck} + \widetilde{p}^i_{ck}$ and $\widehat{p}^i_{sk} = \sum_{j=1}^J \chi_j a^{ij}_{s,k} + \widetilde{p}^i_{sk}$ belong to the space $L^2_\beta(\Omega)$, i.e., they decay exponentially as $x^j_3 \to \infty$. Consequently, we rewrite the last identity as follows

$$\sum_{j=1}^{J} \left(a^{i,j}_{ck} \int_{\omega^j} \overline{\boldsymbol{\mathcal{V}}}^l_{ck} \cdot \mathbf{n} dy^l + a^{i,j}_{sk} \int_{\omega^j} \overline{\boldsymbol{\mathcal{V}}}^l_{sk} \cdot \mathbf{n} dy^l\right) + o(e^{\beta' L}) = \int_{\Omega_L} \left(\widehat{\mathbf{f}}^i_{ck} \cdot \overline{\boldsymbol{\mathcal{V}}}^l_{ck} + \widehat{\mathbf{f}}^i_{sk} \cdot \overline{\boldsymbol{\mathcal{V}}}^l_{sk}\right) dx.$$

Functions $\widehat{\mathbf{f}}^i_{ck}$ and $\widehat{\mathbf{f}}^i_{sk}$ have supports in Ω_1. Thus after passing to the limit as $L \to \infty$, we obtain, taking into account (1.5.1), the relation

$$a^{i,l}_{ck} - a^{i,l+1}_{ck} = \int_{\Omega_1} \left(\widehat{\mathbf{f}}^i_{ck} \cdot \overline{\boldsymbol{\mathcal{V}}}^l_{ck} + \widehat{\mathbf{f}}^i_{sk} \cdot \overline{\boldsymbol{\mathcal{V}}}^l_{sk}\right) dx. \tag{1.5.3}$$

Application of Hölder's inequality to the last formula yields the estimate

$$|a^{i,l}_{ck} - a^{i,l+1}_{ck}|^2 \leq \left(\int_{\Omega_1} |\widehat{\mathbf{f}}^i_{ck}|^2 + |\widehat{\mathbf{f}}^i_{sk}|^2 dx\right) \left(\int_{\Omega_1} |\overline{\boldsymbol{\mathcal{V}}}^l_{ck}|^2 + |\overline{\boldsymbol{\mathcal{V}}}^l_{sk}|^2 dx\right). \tag{1.5.4}$$

Both terms on the right-hand side of this inequality were estimated in the Section 2.5. Combining estimate (1.4.26) for the functions $\mathbf{f}^{p,i}_{ck}$, $\mathbf{f}^{p,i}_{sk}$ with estimate (1.4.27) for the functions $\mathbf{f}^{d,i}_{ck}$, $\mathbf{f}^{d,i}_{sk}$, we get the estimate for the elements $\widehat{\mathbf{f}}^i_{ck} = \mathbf{f}^{p,i}_{ck} + \mathbf{f}^{d,i}_{ck}$ and $\widehat{\mathbf{f}}^i_{sk} = \mathbf{f}^{p,i}_{sk} + \mathbf{f}^{d,i}_{sk}$:

$$\|\widehat{\mathbf{f}}^i_{ck}\|^2_{L^2(\Omega_1)} + \|\widehat{\mathbf{f}}^i_{sk}\|^2_{L^2(\Omega_1)} \leq c \sum_{j=1}^{J} (1 + 2\alpha^j_k + k^2\alpha^j_k + \gamma^j_k) \left((b^{i,j}_{ck})^2 + (b^{i,j}_{sk})^2\right).$$

Since $\overline{\boldsymbol{\mathcal{V}}}^l_{ck} = \boldsymbol{\mathcal{V}}^l_{ck} - \boldsymbol{\mathcal{V}}^{l+1}_{ck}$ and $\overline{\boldsymbol{\mathcal{V}}}^l_{sk} = \boldsymbol{\mathcal{V}}^l_{sk} - \boldsymbol{\mathcal{V}}^{l+1}_{sk}$ we get from (1.4.31) that

$$\|\overline{\boldsymbol{\mathcal{V}}}^l_{ck}\|_{L^2(\Omega_1)} + \|\overline{\boldsymbol{\mathcal{V}}}^l_{sk}\|_{L^2(\Omega_1)} \leq c,$$

with the constant c independent of k. Substituting the two last estimates into (1.5.4) yields

$$|a^{i,l}_{ck} - a^{i,l+1}_{ck}|^2 \leq c \sum_{j=1}^{J} (1 + 2\alpha^j_k + k^2\alpha^j_k + \gamma^j_k) \left((b^{i,j}_{ck})^2 + (b^{i,j}_{sk})^2 \right).$$

Using Lemma A.0.1 and taking into account definitions (1.4.22), we get that

$$0 < 1 + 2\alpha^j_k + k^2\alpha^j_k + \gamma^j_k \leq 1 + 2\frac{d^j_k}{k} + kd^j_k + |\omega^j|^2.$$

Since $\lim_{k\to\infty}(1 + 2\frac{d^j_k}{k} + kd^j_k + |\omega^j|^2) = 1 + |\omega^j| + |\omega^j|^2$, the sequence $\{1 + 2\alpha^j_k + k^2\alpha^j_k + \gamma^j_k\}_{k=0}^{\infty}$ is bounded. This gives the estimate

$$|a^{i,l}_{ck} - a^{i,l+1}_{ck}|^2 \leq c \sum_{j=1}^{J} \left((b^{i,j}_{ck})^2 + (b^{i,j}_{sk})^2 \right).$$

Multiplying equations (1.4.15) by $\overline{\boldsymbol{\mathcal{U}}}^l_k$ with $l \in \{J+1, \ldots, 2J\}$, using relations (1.5.2) and arguing in the same way as above, we get the expression for the differences of constants a^l_{sk}, a^{l+1}_{sk} and their estimates in terms of coefficients $b^{i,j}_{ck}, b^{i,j}_{sk}$:

$$a^{i,l}_{sk} - a^{i,l+1}_{sk} = \int_{\Omega_1} \left(\widehat{\mathbf{f}}^i_{ck} \cdot \overline{\boldsymbol{\mathcal{V}}}^l_{ck} + \widehat{\mathbf{f}}^i_{sk} \cdot \overline{\boldsymbol{\mathcal{V}}}^l_{sk} \right) dx, \tag{1.5.5}$$

$$|a^{i,l}_{sk} - a^{i,l+1}_{sk}|^2 \leq c \sum_{j=1}^{J} \left((b^{i,j}_{ck})^2 + (b^{i,j}_{sk})^2 \right), \tag{1.5.6}$$

where the constant c is independent of k.

Remark 1.5.1. Formulas (1.5.3), (1.5.5) again emphasize the fact that for the elements in the basis $\boldsymbol{u}^1_k, \ldots, \boldsymbol{u}^{2J}_k$ we may choose in (1.4.6) one arbitrary constant in the set $\{a^{i,j}_{ck}\}_{j=1}^J$ and one arbitrary constant in the set $\{a^{i,j}_{sk}\}_{j=1}^J$. The rest of the constants in these sets are defined by the given flow-rate coefficients and the geometry of the domain Ω.

1.6 Stokes-type problems with asymptotic conditions at infinity

Consider the Stokes-type problem (1.1.2) supplemented with the asymptotic conditions at infinity (1.3.15):

$$\mathbf{S}_k\mathbf{u}_k = \mathbf{f}_k,\ \ x \in \Omega, \quad \mathbf{v}_k = 0,\ \ x \in \partial\Omega, \quad \mathbb{B}_k\pi\mathbf{u}_k = \mathbf{h}_k. \tag{1.6.1}$$

Let us denote by $\mathbb{A}_k^l$ the operator corresponding to problem (1.6.1), i.e.,

$$\mathbb{A}_k^l : \mathbb{D}_{\pm\beta}^l H(\Omega) \ni \mathbf{u}_k \mapsto (\mathbf{S}_k\mathbf{u}_k, \mathbb{B}_k\pi\mathbf{u}) \in \mathcal{R}_\beta^l H(\Omega) \times \mathbb{R}^{2J}.$$

Since the operator $\mathbb{B}_k$ is finite dimensional, the operator $\mathbb{A}_k^l$ inherits Fredholm property from the operator $\mathbf{A}^l_{-\beta\to\beta,k}$. The generalized Green formula (1.3.14) determines for (1.6.1) the formally adjoint problem with the asymptotic conditions at infinity. Namely, instead of problem (1.1.16), which is formally adjoint to (1.1.2), we consider the problem

$$\mathbf{S}_k^*\mathbf{U}_k = \mathbf{F}_k,\ \ x \in \Omega, \quad \mathbf{V}_k = 0,\ \ x \in \partial\Omega, \quad \mathbb{Q}_k\pi\mathbf{U}_k = \mathbf{H}_k, \tag{1.6.2}$$

with $\mathbf{H}_k \in \mathbb{R}^{2J}$ and $\mathbf{F}_k \in \mathcal{R}_\beta^l H(\Omega)$ given. As usual, the formally adjoint problem plays crucial role in for investigation of solvability properties of the "direct" problem (1.6.1).

Theorem 1.6.1. *Assume that matrices $\mathbb{B}_k$, $\mathbb{S}_k$, $\mathbb{T}_k$, $\mathbb{Q}_k$ satisfy the relation* (1.3.13)*. Then*

(1) $\ker \mathbb{A}_k^l = \{\mathbf{u}_k : \mathbf{S}_k\mathbf{u}_k = \mathbf{0},\ \mathbf{v}_k|_{\partial\Omega} = \mathbf{0},\ \mathbb{B}_k\pi\mathbf{u}_k = \mathbf{0}\}$;

(2) $\operatorname{coker} \mathbb{A}_k^l = \{(\mathbf{U}_k, \mathbb{T}_k\pi\mathbf{U}_k) : \mathbf{S}_k^*\mathbf{U}_k = \mathbf{0},\ \mathbf{V}_k|_{\partial\Omega} = \mathbf{0},\ \mathbb{Q}_k\pi\mathbf{U}_k = \mathbf{0}\}$.

Remark 1.6.2. Part (2) in Theorem 1.6.1 states that problem (1.6.1) has a solution if and only if the data $\mathbf{f}_k$, $\mathbf{h}_k$ satisfy the following compatibility condition

$$\int\limits_\Omega \mathbf{f}_k \cdot \mathbf{U}_k dx + \langle \mathbf{h}_k, \mathbb{T}_k\pi\mathbf{U}_k\rangle_{2J} = 0 \tag{1.6.3}$$

for every function $\mathbf{U}_k$ which solves the homogeneous problem (1.6.2).

Theorem 1.6.1 can be proved similarly as Theorem 6.2 in [54]. We present the complete proof for the reader convenience.

Proof. (1) Validity of the first statement follows from the inclusion $\ker \mathbb{A}^l_k \subset \ker \mathbf{A}^l_{-\beta\to\beta,k}$.
(2). Assume that $\mathbf{U}^{B,1}_k, \dots, \mathbf{U}^{B,2J}_k$ is a basis in $\ker(\mathbf{A}^l_{-\beta\to\beta,k})^*$, i.e., $\mathbf{S}^*_k\mathbf{U}^{B,i}_k = \mathbf{0}$, $\mathbf{V}^{B,i}_k|_{\partial\Omega} = \mathbf{0}$. Suppose that this basis is such that

$$\mathbb{Q}_k\pi\mathbf{U}^{B,i}_k = \mathbf{0}, \quad j = 1, \dots, K, \quad 0 \le K \le 2J,$$

where $K = \dim\left\{\mathbf{U} \in \ker(\mathbf{A}^l_{-\beta\to\beta,k})^* : \mathbb{Q}_k\pi\mathbf{U} = \mathbf{0}\right\}$, i.e., vectors $\mathbf{U}^{B,1}_k, \dots, \mathbf{U}^{B,K}_k$ satisfy the homogeneous conditions at infinity. Since $\operatorname{rank}\mathbb{Q}_k = 2J$ (see Remark 1.3.2), the vectors $\mathbb{Q}_k\pi\mathbf{U}^{B,K+1}_k, \dots, \mathbb{Q}_k\pi\mathbf{U}^{B,2J}_k$ are linearly independent. For each function $\mathbf{u}_k \in \mathbb{D}^l_{\pm\beta}$ we get from the generalized Green's formula the equalities

$$(\mathbf{S}_k\mathbf{u}_k, \mathbf{U}^{B,i}_k)_\Omega + \langle \mathbb{B}_k\pi\mathbf{u}_k, \mathbb{T}_k\pi\mathbf{U}^{B,i}_k\rangle_{2J} = (\mathbf{u}_k, \mathbf{S}^*_k\mathbf{U}^{B,i}_k)_\Omega + \langle \mathbb{S}_k\pi\mathbf{u}_k, \mathbb{Q}_k\pi\mathbf{U}^{B,i}_k\rangle_{2J},$$
$$i = 1, \dots, 2J.$$

If function $\mathbf{u}_k$ is a solution of problem (1.6.1), then the following necessary compatibility condition has to be satisfied:

$$(\mathbf{f}_k, \mathbf{U}^{B,i}_k)_\Omega + \langle \mathbf{h}_k, \mathbb{T}_k\pi\mathbf{U}^{B,i}_k\rangle_{2J} = 0, \quad \text{for} \quad i = 1, \dots, K. \tag{1.6.4}$$

Let us show that the above conditions are also sufficient. Since $\dim\operatorname{coker}\mathbf{A}^l_{-\beta\to\beta,k} = 0$ (see (1.3.1)), there always exists a solution $\mathbf{u}^0_k \in \mathbb{D}^l_{\pm\beta}$ to the non-homogeneous Stokes-type problem (1.1.2) (to the problem without asymptotic conditions at infinity). Using the substitution $\mathbf{u}_k = \mathbf{w}_k + \mathbf{u}^0_k$ we reduce problem (1.6.1) to the homogeneous Stokes-type problem with non-homogeneous conditions at infinity:

$$\mathbf{S}_k\mathbf{w}_k = \mathbf{0},\ x \in \Omega, \quad \mathbf{w}'_k = \mathbf{0},\ x \in \partial\Omega, \quad \mathbb{B}_k\pi\mathbf{w}_k = \mathbf{h}^0_k \equiv \mathbf{h}_k - \mathbb{B}_k\pi\mathbf{u}^0_k,$$

here $\mathbf{w}'_k$ denotes the velocity components of $\mathbf{w}_k$. The compatibility conditions (1.6.4) now turns into

$$\langle \mathbf{h}^0_k, \mathbb{T}_k\pi\mathbf{U}^{B,i}_k\rangle_{2J} = 0, \quad i = 1, \dots, K.$$

We look for the solution $\mathbf{w}_k$ in the form

$$\mathbf{w}_k = \sum_{i=1}^{2J} a_i \boldsymbol{u}^i,$$

where $\boldsymbol{u}^1, \dots, \boldsymbol{u}^{2J}$ is basis in $\ker \mathbf{A}^l_{-\beta\to\beta,k}$. The vector-field $\mathbf{w}_k$ automatically satisfies the homogeneous Stokes-type equations and zero boundary conditions. Substituting

$\mathbf{w}_k$ into asymptotic conditions $\mathbb{B}_k\pi\mathbf{w}_k = \mathbf{h}_k^0$ we get for the $4J$ dimensional vector $\mathbf{b}_k = a_1\pi\boldsymbol{u}^1 + \cdots + a_{2J}\pi\boldsymbol{u}^{2J}$ the system of $2J$ linear equations. Let us supply this system with another $2J$ linear equations $\mathbb{S}_k\mathbf{b}_k = \mathbf{h}_k^1$, where $\mathbf{h}_k^1$ is at the moment unknown vector, and consider the system of $4J$ equations

$$\mathbb{X}_k\mathbf{b}_k = \begin{pmatrix}\mathbf{h}_k^0\\ \mathbf{h}_k^1\end{pmatrix},$$

where $\mathbb{X}_k$ denotes the $4J \times 4J$ matrix $\begin{pmatrix}\mathbb{B}_k\\ \mathbb{S}_k\end{pmatrix}$. Since this matrix is non-singular (see Remark 1.3.2), for every right-hand side $\mathbf{h}_k^0$, $\mathbf{h}_k^1$ there exists a unique solution $\mathbf{b}_k$ of the system above. In order to find constants $a_1, \ldots, a_{2J}$ we shall select $\mathbf{h}_k^1$ in such a way, that the vector $\mathbf{b}_k$ belongs to the linear hull $\mathcal{L}\left\{\pi\boldsymbol{u}_k^i, \ldots, \pi\boldsymbol{u}_k^{2J}\right\}$.

Theorem 1.3.1 states that the generalized Greens formula

$$(\mathbf{S}_k\mathbf{u}_k, \mathbf{U}_k)_\Omega - (\mathbf{u}_k, \mathbf{S}_k^*\mathbf{U}_k)_\Omega = \langle \mathbb{J}_k\pi\mathbf{u}_k, \pi\mathbf{U}_k\rangle_{4J},$$

holds for all $\mathbf{u}_k \in \mathbb{D}_{\pm\beta}^l H(\Omega), \mathbf{U}_k \in \mathbb{D}_{\pm\beta}^l H(\Omega)^*$. Taking $\mathbf{u}_k = \mathbf{w}_k$, $\mathbf{U}_k = \mathbf{U}_k^{B,i}$ in the last formula, we get that

$$\langle \mathbf{b}_k, \mathbb{J}^*\pi\mathbf{U}_k^{B,i}\rangle_{4J} = 0, \quad i = 1, \ldots, 2J.$$

Therefore the condition $\mathbf{b}_k \in \mathcal{L}\left\{\pi\mathbf{u}_k^{B,1}, \ldots, \pi\mathbf{u}_k^{B,2J}\right\}$ is equivalent to the condition $\mathbf{b}_k \perp \mathbb{J}_k^*\mathcal{L}(\pi\mathbf{U}_k^{B,1}, \ldots, \pi\mathbf{U}_k^{B,2J})$. Let us rewrite the last equality using the relation $\mathbb{J}_k = \mathbb{Y}_k^*\mathbb{X}_k$:

$$\begin{aligned} 0 = \langle \mathbb{J}_k\mathbf{b}_k, \pi\mathbf{U}_k^{B,i}\rangle_{4J} &= \langle \mathbb{Y}_k^*\mathbb{X}_k\mathbf{b}_k, \pi\mathbf{U}_k^{B,i}\rangle_{4J} = \langle \mathbb{X}_k\mathbf{b}_k, \mathbb{Y}_k\pi\mathbf{U}_k^{B,i}\rangle_{4J} \\ &= \langle \mathbf{h}_k^0, \mathbb{T}_k\pi\mathbf{U}_k^{B,i}\rangle_{2J} - \langle \mathbf{h}_k^1, \mathbb{Q}_k\pi\mathbf{U}_k^{B,i}\rangle_{2J}, \quad i = 1, \ldots, 2J. \end{aligned}$$

If $i = 1, \ldots, K$, then both terms on the right-hand side of the last equality vanish (due to compatibility conditions and the homogeneous conditions at infinity). Taking $i = K+1, \ldots, 2J$ in the last equality we get the system of $2J - K$ linear equations with respect to the unknown $\mathbf{h}_k^1$:

$$\langle \mathbf{h}_k^1, \mathbb{Q}_k\pi\mathbf{U}_k^{B,i}\rangle_{2J} = \langle \mathbf{h}_k^0, \mathbb{T}_k\pi\mathbf{U}_k^{B,i}\rangle_{2J}, \quad i = K+1, \ldots, 2J. \tag{1.6.5}$$

Since $\mathbb{Q}_k\pi\mathbf{U}_k^{B,K+1}, \ldots, \mathbb{Q}_k\pi\mathbf{U}_k^{B,2J}$ are linearly independent vectors, the rank of the system's matrix in (1.6.5) is equal to $2J - K$. Therefore, this system has a solution

$\mathbf{h}_k^1$ for every right-hand side. If $K = 0$, the number of unknowns coincide with the rank of the system, and the solution $\mathbf{h}_k^1$ is unique. $\square$

Remark 1.6.3. Recall that the $4J \times 2J$ matrix $\left(\pi \boldsymbol{u}^1 \cdots \pi \boldsymbol{u}^{2J}\right)$ is composed from the $2J \times 2J$ matrices $\mathcal{B}_k$ and $\mathcal{A}_k$ (see (1.4.20)). Using Theorem 1.6.1 and the same arguments as in Corollary 6.6, [54] we can show that

$$\dim \ker \mathbb{A}_k^l = 2J - \operatorname{rank} \left\{ \mathbb{B}_k \begin{pmatrix} \mathcal{B}_k \\ \mathcal{A}_k \end{pmatrix} \right\}. \tag{1.6.6}$$

1.7 Examples of matrices $\mathbb{B}_k$ modelling certain class of pressure-related asymptotic conditions

Particular physical phenomena can be modelled by choosing corresponding matrices $\mathbb{B}_k$ in the asymptotic conditions (1.3.15). For example, a sequence of matrices $\mathbb{B}_k = \begin{pmatrix}\mathbb{F}_k & \mathbb{O}\end{pmatrix}$, $k = 0, 1, \ldots$, with $\mathbb{F}_k$ defined by (1.3.12) corresponds to the situation when flow-rates are prescribed in every outlet[xv]. Indeed, the product $\mathbb{B}_k \pi \mathbf{u}_k$ is equal to the vector $(\phi_{ck}^1, \ldots, \phi_{ck}^J, \phi_{sk}^1, \ldots, \phi_{sk}^J)$ consisting the Fourier coefficients (1.4.10) of the time-periodic flow-rates $\phi^j = \phi^j(t)$, $j = 1, \ldots, J$. Let us describe a class of matrices $\mathbb{B}_k$, $\mathbb{S}_k$, $\mathbb{T}_k$ and $\mathbb{Q}_k$ that satisfy condition (1.3.13) and allows to impose the asymptotic conditions at infinity which are different from prescribing only the flow-rates.

Assume that $\mathcal{I} \subset \{1, \ldots, J\}$ denotes the set of indexes of the outlets where we are going to change the flow-rate conditions by conditions of other type. Construct the $2J \times 2J$ matrix $\widehat{\mathbb{F}}_k$ by taking the lines with numbers i and $J + i$, $i \in \mathcal{I}$ the same as in the matrix $\mathbb{F}_k$, while setting the rest lines in $\widehat{\mathbb{F}}_k$ equal to the zero ones. The asymptotic conditions (1.3.15) with the matrix $\mathbb{B}_k = \begin{pmatrix}\mathbb{F}_k - \widehat{\mathbb{F}}_k & \mathbb{O}\end{pmatrix}$ still determine the flow rates in the outlets Ω_+^j with numbers $j \in \{1, \ldots, J\} \setminus \mathcal{I}$, while the rest outlets are "free" of conditions at infinity. Let $\mathbb{L}_k = \operatorname{diag}\left\{L_k^1, \ldots, L_k^{2J}\right\}$ and $\mathbb{I}$ be, respectively, a real $2J \times 2J$ diagonal and identity matrices. We construct the matrices $\widehat{\mathbb{L}}_k$ and $\widehat{\mathbb{I}}$ in the similar way as $\widehat{\mathbb{F}}_k$, i.e., the lines with indexes i and $J + i$, $i \in \mathcal{I}$, are the same as in the matrices $\mathbb{L}_k$ and $\mathbb{I}$, while the rest are zero lines. Define the matrix $\mathbb{B}_k = \begin{pmatrix}\mathbb{F}_k - \widehat{\mathbb{F}}_k - \widehat{\mathbb{L}}_k & \widehat{\mathbb{I}}\end{pmatrix}$ and consider conditions at infinity (1.3.15). One may straightforward verify that for $i \in \mathcal{I}$ the i^{th} and the $J + i^{\text{th}}$ components in the

[xv] For $\mathbb{B}_k = \begin{pmatrix}\mathbb{F}_k & \mathbb{O}\end{pmatrix}$ the condition (1.3.13) is satisfied with matrices $\mathbb{T}_k = \begin{pmatrix}\mathbb{O} & \mathbb{I}\end{pmatrix}$, $\mathbb{S}_k = \begin{pmatrix}\mathbb{O} & \mathbb{I}\end{pmatrix}$, $\mathbb{Q}_k = \begin{pmatrix}\mathbb{G}_k & \mathbb{O}\end{pmatrix}$, where $\mathbb{G}_k$ is defined by (1.3.12).

product $\mathbb{B}_k \pi \mathbf{u}_k$ are equal to

$$a^i_{ck} - L^i b^i_{ck}, \qquad a^i_{sk} - L^{J+i} b^i_{sk}. \tag{1.7.1}$$

These two quantities determine the main part of the Fourier coefficients p^i_{ck} and p^i_{sk}. Indeed, the pressure coefficients in the outlet Ω^j_+ admit the representation (see (1.1.24))

$$p_{ck}(x) = a^j_{ck} - b^j_{ck} x^j_3 + \widetilde{p}_{ck}(x), \quad p_{sk}(x) = a^j_{sk} - b^j_{sk} x^j_3 + \widetilde{p}_{sk}(x),$$

where $\widetilde{p}_{ck}, \widetilde{p}_{sk}$ decay exponentially as $x^j_3 \to \infty$, i.e., for large L they differ from (1.7.1) by the terms of order $o(e^{-\beta L})$ only.

Lemma 1.7.1. *Let $\widehat{\mathbb{G}}_k$ and $\widehat{\mathbb{M}}_k$ be the matrices obtained from the matrix $\mathbb{G}_k = \begin{pmatrix} \mathcal{C}_k & -\mathcal{D}_k \\ -\mathcal{D}_k & -\mathcal{C}_k \end{pmatrix}$, (see (1.3.8)), and from $2J \times 2J$ diagonal real matrix $\mathbb{M}_k$ by using the same procedure and the same set $\mathcal{I}$ as for projectors $\widehat{\mathbb{F}}_k$, $\widehat{\mathbb{L}}_k$, $\widehat{\mathbb{I}}$ defined above. Assume that*

$$\widehat{\mathbb{M}}_k \widehat{\mathbb{F}}_k - \widehat{\mathbb{G}}_k \widehat{\mathbb{L}}_k = \mathbb{O}. \tag{1.7.2}$$

Then the matrices

$$\begin{aligned} \mathbb{B}_k &= \begin{pmatrix} \mathbb{F}_k - \widehat{\mathbb{F}}_k - \widehat{\mathbb{L}}_k & \widehat{\mathbb{I}} \end{pmatrix}, \quad \mathbb{S}_k = \begin{pmatrix} -\widehat{\mathbb{F}}_k & \mathbb{I} - \widehat{\mathbb{I}} \end{pmatrix}, \\ \mathbb{Q}_k &= \begin{pmatrix} \mathbb{G}_k - \widehat{\mathbb{G}}_k - \widehat{\mathbb{M}}_k & \widehat{\mathbb{I}} \end{pmatrix}, \quad \mathbb{T}_k = \begin{pmatrix} -\widehat{\mathbb{G}}_k & \mathbb{I} - \widehat{\mathbb{I}} \end{pmatrix} \end{aligned} \tag{1.7.3}$$

satisfy the condition $(\mathbb{Y}_k)^T \mathbb{X}_k = \begin{pmatrix} -\mathbb{T}_k \\ \mathbb{Q}_k \end{pmatrix}^T \begin{pmatrix} \mathbb{B}_k \\ \mathbb{S}_k \end{pmatrix} = \begin{pmatrix} \mathbb{O} & \mathbb{G}_k \\ -\mathbb{F}_k & \mathbb{O} \end{pmatrix}$ *(see (1.3.13)).*

Remark 1.7.2. Consider the matrices

$$\mathbb{L}_k = \begin{pmatrix} \mathbb{L}^1_k & \mathbb{O} \\ \mathbb{O} & \mathbb{L}^2_k \end{pmatrix}, \quad \mathbb{M}_k = \begin{pmatrix} \mathbb{M}^1_k & \mathbb{O} \\ \mathbb{O} & \mathbb{M}^2_k \end{pmatrix},$$

where $\mathbb{L}^1_k$, $\mathbb{L}^2_k$, $\mathbb{M}^1_k$ and $\mathbb{M}^2_k$ are the $J \times J$ diagonal matrices. Straightforward computations show that condition (1.7.2) holds if and only if

$$\mathbb{L}^1_k = \mathbb{L}^2_k = \mathbb{M}^1_k = -\mathbb{M}^2_k. \tag{1.7.4}$$

It turns out that condition (1.7.4) (and (1.7.2)) is a natural requirement. In the following section we will see that the sequence $\{\mathbb{Q}_k\}_{k=0}^{\infty}$ defines the asymptotic conditions for the adjoint time-periodic problem which is "backward in time". The change of a time direction is already reflected in matrices $\mathbb{F}_k$ and $\mathbb{G}_k$ – their lower parts, related to Fourier coefficients $\mathbf{v}_{sk}$ and $\mathbf{V}_{sk}$, has opposite signs, while the upper parts, related to coefficients $\mathbf{v}_{ck}$ and $\mathbf{V}_{ck}$, coincide see (1.3.12). The similar situation is reflected in condition (1.7.4) for matrices $\mathbb{L}_k$ and $\mathbb{M}_k$, which are responsible for the pressure coefficients of "direct" and "backward" solutions, respectively.

Proof of Lemma 3.1. The matrices $\mathbb{G}_k$, $\mathbb{M}_k$ are symmetric, therefore we get that

$$(\mathbb{Y}_k)^T \mathbb{X}_k = \begin{pmatrix} -\mathbb{T}_k \\ \mathbb{Q}_k \end{pmatrix}^T \begin{pmatrix} \mathbb{B}_k \\ \mathbb{S}_k \end{pmatrix} = \begin{pmatrix} \widehat{\mathbb{G}}_k & \widehat{\mathbb{I}} - \mathbb{I} \\ \mathbb{G}_k - \widehat{\mathbb{G}}_k - \widehat{\mathbb{M}}_k & \widehat{\mathbb{I}} \end{pmatrix}^T \begin{pmatrix} \mathbb{F}_k - \widehat{\mathbb{F}}_k - \widehat{\mathbb{L}}_k & \widehat{\mathbb{I}}_k \\ -\widehat{\mathbb{F}} & \mathbb{I} - \widehat{\mathbb{I}} \end{pmatrix}$$

$$= \begin{pmatrix} \widehat{\mathbb{G}}_k & \mathbb{G}_k - \widehat{\mathbb{G}}_k - \widehat{\mathbb{M}}_k \\ \widehat{\mathbb{I}} - \mathbb{I} & \widehat{\mathbb{I}} \end{pmatrix} \begin{pmatrix} \mathbb{F}_k - \widehat{\mathbb{F}}_k - \widehat{\mathbb{L}}_k & \widehat{\mathbb{I}}_k \\ -\widehat{\mathbb{F}} & \mathbb{I} - \widehat{\mathbb{I}} \end{pmatrix}$$

$$= \begin{pmatrix} \widehat{\mathbb{G}}_k(\mathbb{F}_k - \widehat{\mathbb{F}}_k - \widehat{\mathbb{L}}_k) - (\mathbb{G}_k - \widehat{\mathbb{G}}_k - \widehat{\mathbb{M}}_k)\widehat{\mathbb{F}}_k & \widehat{\mathbb{G}}_k\widehat{\mathbb{I}}_k + (\mathbb{G}_k - \widehat{\mathbb{G}}_k - \widehat{\mathbb{M}}_k)(\mathbb{I} - \widehat{\mathbb{I}}) \\ (\widehat{\mathbb{I}} - \mathbb{I})(\mathbb{F}_k - \widehat{\mathbb{F}}_k - \widehat{\mathbb{L}}_k) - \widehat{\mathbb{I}}\widehat{\mathbb{F}}_k & (\widehat{\mathbb{I}} - \mathbb{I})\widehat{\mathbb{I}} + \widehat{\mathbb{I}}(\mathbb{I} - \widehat{\mathbb{I}}) \end{pmatrix}.$$

Since the matrices $\mathbb{F}_k - \widehat{\mathbb{F}}_k$, $\mathbb{G}_k - \widehat{\mathbb{G}}_k$, $\mathbb{I} - \widehat{\mathbb{I}}$ and the matrices $\widehat{\mathbb{F}}_k$, $\widehat{\mathbb{G}}_k$, $\widehat{\mathbb{L}}_k$, $\widehat{\mathbb{M}}_k$, $\widehat{\mathbb{I}}$ project $\mathbb{R}^{2J}$ onto orthogonal subspaces, the last identity is reduced to

$$(\mathbb{Y}_k)^T \mathbb{X}_k = \begin{pmatrix} \widehat{\mathbb{M}}_k\widehat{\mathbb{F}}_k - \widehat{\mathbb{G}}_k\widehat{\mathbb{L}}_k & \mathbb{G}_k \\ -\mathbb{F}_k & \mathbb{O} \end{pmatrix}.$$

Under assumption (1.7.2) the matrix on the right-hand side is equal to the matrix $\mathbb{J}$. □

Chapter 2

Time-periodic Stokes problem

This chapter is devoted to the investigation of the time-periodic problem

$$\begin{cases} \partial_t \mathbf{v} - \nu\Delta\mathbf{v} + \nabla p &= \mathbf{f}, \qquad (x,t) \in \Omega \times (0, 2\pi), \\ -\nabla \cdot \mathbf{v} &= \mathbf{0}, \qquad (x,t) \in \Omega \times (0, 2\pi), \\ \mathbf{v} &= \mathbf{0}, \qquad (x,t) \in \partial\Omega \times (0, 2\pi), \\ \mathbf{v}(x,0) &= \mathbf{v}(x, 2\pi), \quad x \in \Omega. \end{cases} \tag{2.0.1}$$

We will derive the asymptotic conditions at infinity that ensure existence of the unique time-periodic solution having unbounded Dirichlet's integral.

2.1 Structure of a time-periodic solution

Assume that the time-periodic function $\mathbf{f} = \mathbf{f}(x,t)$ in (2.0.1) satisfies the condition

$$\mathbf{f} \in L^2(0, 2\pi; L^2_\beta(\Omega)), \quad \beta > 0, \tag{2.1.1}$$

i.e., $\mathbf{f}$ decays exponentially as $x_3^j \to \infty$ for all $j = 1, \ldots, J$. Then the Fourier coefficients $\mathbf{f}_{ck}$, $\mathbf{f}_{sk}$, $k = 0, 1, \ldots$, belong to the space $\mathcal{R}^2_\beta H(\Omega)$. If β is sufficiently small, we may use results presented in Chapter 1, and conclude that for every $k = 0, 1, \ldots$ there exists $2J$ linearly independent solutions $\mathbf{u}_k = (\mathbf{v}_{ck}, p_{ck}, \mathbf{v}_{sk}, p_{sk}) \in \mathbb{D}^2_{\pm\beta} H(\Omega)$ to the Stokes-type problem (1.1.2). According to Theorem 1.1.1, the function $\mathbf{u}_k$ is defined by formulas (1.1.11), (1.1.12) and (1.1.22), namely, it has the velocity-fields $\mathbf{v}_{ck}, \mathbf{v}_{sk}$ described by (1.1.23) and the pressure functions p_{ck}, p_{sk} having the form (1.1.24). Multiplying the corresponding coefficients of the function $\mathbf{u}_k$ by $\cos kt$ and

$\sin kt$, and summing by k we get the series

$$\begin{aligned} \mathbf{v}(x,t) &= \sum_{k=0}^{\infty} \left\{ \mathbf{v}_{ck}(x)\cos kt + \mathbf{v}_{sk}(x)\sin kt \right\}, \\ p(x,t) &= \sum_{k=0}^{\infty} \left\{ p_{ck}(x)\cos kt + p_{sk}(x)\sin kt \right\}, \end{aligned} \tag{2.1.2}$$

which formally satisfy system (2.0.1)[i]. Due to the special structure of functions $\mathbf{u}_k$, $k = 0, 1, \ldots$, (see (1.1.22)) we can split series (2.1.2) as follows:

$$\mathbf{v} = \mathbf{v}_p + \widetilde{\mathbf{v}}, \quad p = p_p + p_0 + \widetilde{p}, \tag{2.1.3}$$

with the summands listed below.

1. *The Poiseuille part*

$$(\mathbf{v}_p, p_p) = \sum_{j=1}^{J} \chi^j(x_3^j) \left(\mathbf{v}_p^j(x,t), p_p^j(x,t) \right)$$

which is generated in each outlet Ω_+^j by the terms $b_{ck}^j \mathbf{u}_{ck}^{j1} + b_{sk}^j \mathbf{u}_{sk}^{j1}$, $k = 0, 1, \ldots$. Note that due to the structure of the vector-fields $\mathbf{u}_{ck}^{j1}$ and $\mathbf{u}_{sk}^{j1}$ (see (1.1.11), (1.1.12)) every pair $(\mathbf{v}_p^j, p_p^j)$ may be represented in local coordinates as

$$\mathbf{v}_p^j(y^j,t) = \left(0, 0, v^j(y^j,t)\right), \quad p_p^j(x_3^j,t) = -q^j(t)x_3^j, \tag{2.1.4}$$

where

$$v^j(y^j,t) = \sum_{k=0}^{\infty} \left\{ \left(b_{ck}^j \varphi_k^j(y^j) + b_{sk}^j \psi_k^j(y^j) \right) \cos kt + \left(b_{sk}^j \varphi_k^j(y^j) - b_{ck}^j \psi_k^j(y^j) \right) \sin kt \right\}, \tag{2.1.5}$$

$$q^j(t) = \sum_{k=0}^{\infty} \{ b_{ck}^j \cos kt + b_{sk}^j \sin kt \}. \tag{2.1.6}$$

2. *The pressure part*

$$p_0(x,t) = \sum_{j=1}^{J} \chi^j(x_3^j) p_0^j(t), \tag{2.1.7}$$

where the functions $p_0^j = p_0^j(t)$ are generated by the vector-fields $a_{ck}^j \mathbf{u}_{ck}^{j0} + a_{sk}^j \mathbf{u}_{sk}^{j0}$, $k = 0, 1, \ldots$. Taking into account the definitions of $\mathbf{u}_{ck}^{j0}$ and $\mathbf{u}_{sk}^{j0}$ (see (1.1.11), (1.1.12))

[i] We say that series (2.1.2) formally satisfy the time-periodic problem (2.0.1) if they are defined by the coefficients $(\mathbf{v}_{ck}, p_{ck}, \mathbf{v}_{sk}, p_{sk})$, $k = 0, 1, \ldots$, which are solutions to the corresponding Stokes-type problems (1.1.2).

we express every p_0^j as the series

$$p_0^j(t) = \sum_{k=0}^{\infty} \{a_{ck}^j \cos kt + a_{sk}^j \sin kt\}. \tag{2.1.8}$$

3. *The decaying (at infinity) part*

$$\begin{aligned} \widetilde{v}(x,t) &= \sum_{k=0}^{\infty} \{\widetilde{\mathbf{v}}_{ck}(x) \cos kt + \widetilde{\mathbf{v}}_{sk}(x) \sin kt\}, \\ \widetilde{p}(x,t) &= \sum_{k=0}^{\infty} \{\widetilde{p}_{ck}(x) \cos kt + \widetilde{p}_{sk}(x) \sin kt\}, \end{aligned} \tag{2.1.9}$$

generated by the terms $\widetilde{\mathbf{u}}_k = (\widetilde{\mathbf{v}}_{ck}, \widetilde{p}_{ck}, \widetilde{\mathbf{v}}_{sk}, \widetilde{p}_{sk}) \in \mathcal{D}^2_\beta H(\Omega)$, $k = 0, 1, \ldots$, which exponentially decay in every outlet Ω_+^j as $x_3^j \to \infty$.

For the reader's convenience we provide below the known facts, concerning properties of the time-periodic solution (2.1.2) to problem (2.0.1), which are the most important for our research.

2.2 Time-periodic problem in domains with cylindrical outlets

2.2.1 Stokes problem in a single pipe. Poiseuille flow

We recall that the Fourier coefficients $b_{ck}^j \mathbf{u}_{ck}^{j1} + b_{sk}^j \mathbf{u}_{sk}^{j1}$, $k = 0, 1, \ldots$, satisfy the homogeneous Stokes-type problem (1.4.8) set in the infinite cylinder $\Omega^j = \omega^j \times \mathbb{R}$. Therefore, the pair $(\mathbf{v}_p^j, p_p^j)$, defined by (2.1.4)-(2.1.6), formally satisfies the time-periodic homogeneous Stokes problem:

$$\left\{ \begin{array}{rcll} \partial_t \mathbf{v}_p^j - \nu \Delta \mathbf{v}_p^j + \nabla p_p^j &=& \mathbf{0}, & (x,t) \in \Omega^j \times (0, 2\pi), \\ -\nabla \cdot \mathbf{v}_p^j &=& \mathbf{0}, & (x,t) \in \Omega^j \times (0, 2\pi), \\ \mathbf{v}_p^j &=& \mathbf{0}, & (x,t) \in \partial\Omega^j \times (0, 2\pi), \\ \mathbf{v}_p^j(x, 0) &=& \mathbf{v}_p^j(x, 2\pi), & x \in \Omega^j. \end{array} \right. \tag{2.2.1}$$

Usually looking for the Poiseuille flow (2.1.4) one may prescribe the pressure drop $q^j = q^j(t)$ or, alternatively, the flow-rate

$$\int_{\omega^j} v^j(y,t) dy = \phi^j(t). \tag{2.2.2}$$

In the first case system (2.2.1) is reduced to the time-periodic heat equation set in the cross-section ω^j:

$$\begin{aligned} \frac{\partial v^j}{\partial t}(y^j,t) - \nu\Delta_{y^j} v^j(y^j,t) = q^j(t), \quad (y^j,t) \in \omega^j \times (0,2\pi), \\ v^j(y^j,t)|_{\partial\omega^j} = 0, \quad t \in (0,2\pi) \end{aligned} \tag{2.2.3}$$

with the given q^j. The flow-rate $\phi^j = \phi^j(t)$ can be immediately computed by formula (2.2.2).

In the second case the Poiseuille flow is determined by the solution $v^j = v^j(y^j,t)$ of problem (2.2.3), (2.2.2) with a given time-periodic function $\phi^j = \phi^j(t)$. Now the function q^j in the heat equation is not known a priori. Therefore one shall solve an inverse problem – to select in (2.2.3) the right-hand side q^j in such a way that the solution v^j satisfies condition (2.2.2). Existence of a solution to this inverse problem was proved in [7]. In [27] the relation between the flow-rate ϕ^j and the pressure drop q^j was derived. More precisely, it was shown in [27] (see Proposition 2.1 in [27]) that the Fourier coefficients of the series

$$\phi^j(t) = \sum_{k=0}^{\infty}\{\phi^j_{ck}\cos kt + \phi^j_{sk}\sin kt\} \quad \text{and} \quad q^j(t) = \sum_{k=0}^{\infty}\{q^j_{ck}\cos kt + q^j_{sk}\sin kt\},$$

satisfy the equalities

$$\phi^j_{ck} = c^j_k q^j_{ck} - d^j_k q^j_{sk}, \quad \phi^j_{sk} = d^j_k q^j_{ck} + c^j_k q^j_{sk}, \quad k = 0,1,\ldots, \tag{2.2.4}$$

where constants c^j_k and d^j_k are defined by (1.3.9). Since $(c^j_k)^2 + (d^j_k)^2 > 0$ for all $k = 0,1,\ldots,$ (see Lemma 2.1 in [27]) the inverse relations

$$q^j_{ck} = \frac{c^j_k\phi^j_{ck} + d^j_k\phi^j_{sk}}{(c^j_k)^2 + (d^j_k)^2}, \quad q^j_{sk} = \frac{c^j_k\phi^j_{sk} - d^j_k\phi^j_{ck}}{(c^j_k)^2 + (d^j_k)^2}, \tag{2.2.5}$$

are valid. Relations (2.2.4), (2.2.5) and decay properties of constants $\{c^j_k, d^j_k\}_{k=0}^{\infty}$ (see Lemma A.0.1) allow to formulate the following conclusion.

Corollary 2.2.1. (see Proposition 2.2 in [27]) If $q^j \in L^2(0,2\pi)$, then $\phi^j \in H^1(0,2\pi)$. Conversely, the assumption that $\phi^j \in H^1(0,2\pi)$ yield the inclusion $q^j \in L^2(0,2\pi)$.

This conclusion was essential to prove the following statement.

Theorem 2.2.2. (see Theorem 2.2 in [27]) Let $\omega^j \subset \mathbb{R}^2$ be a bounded domain with the boundary $\partial\omega^j \in C^2$ and let $\phi^j \in H^1(0,2\pi)$ be a time-periodic function. Then problem (2.2.2), (2.2.3) has a unique time-periodic solution $v^j = v^j(x,t), q^j = q^j(x,t)$ such that

$$\begin{aligned} &v^j \in C(0,2\pi; H^1(\omega^j)) \cap L^2(0,2\pi; H^2(\omega^j)), \\ &\partial_t v^j \in L^2(0,2\pi; L^2(\omega^j)), \quad q^j \in L^2(0,2\pi). \end{aligned} \tag{2.2.6}$$

Moreover, the following estimate

$$\begin{aligned} \max_{t\in[0,2\pi]} \|v^j(t)\|^2_{H^1(\omega^j)} + \int_0^{2\pi} \left(\left\|\partial_t v^j\right\|^2_{L^2(\omega^j)} + \|v^j(t)\|^2_{H^2(\omega^j)} + |q^j(t)|^2 \right) dt \\ \leq c \int_0^{2\pi} \left(|\phi^j(t)|^2 + \left|\frac{d\phi^j}{dt}\right|^2 \right) dt \end{aligned} \tag{2.2.7}$$

holds with a constant $c = c(\omega^j) > 0$.

2.2.2 Stokes problem in a system of pipes

Consider the time-periodic Stokes problem (2.0.1) set in the domain Ω with J outlets to infinity. Assume that the flow-rates $\phi^j = \phi^j(t)$, $j = 1, \ldots, J-1$, are given[ii]:

$$\int_{\omega^j} \mathbf{v}^j(x,t) \cdot \mathbf{n}^j dy^j = \phi^j(t), \quad j = 1, \ldots, J-1. \tag{2.2.8}$$

Let $\partial\Omega \in C^2$ and

$$\phi^1, \ldots, \phi^{J-1} \in H^1(0,2\pi). \tag{2.2.9}$$

The following result concerns the solvability of problem (2.0.1) with conditions (2.2.8) (see Theorem 5.1 in [33]).

Theorem 2.2.3. *Assume that in* (2.0.1) *the time-periodic function* $\mathbf{f}$ *belongs to* $L^2(0,2\pi; L^2_\beta(\Omega))$ *with the sufficiently small* β. *Moreover, assume that the time-periodic flow-rates* $\phi^j = \phi^j(t)$, $j = 1, \ldots, J-1$, *satisfy conditions* (2.2.9). *If the number* β *in* (2.1.1) *is sufficiently small, then problem* (2.0.1), (2.2.8) *has a time-periodic solution* $\mathbf{v} = \mathbf{v}(x,t)$, $p = p(x,t)$. *The solution admits the asymptotic representation*

$$\mathbf{v}(x,t) = \sum_{j=1}^{J} \chi^j(x_3^j)\mathbf{v}_p^j(y^j,t) + \widetilde{\mathbf{v}}(x,t), \quad p(x,t) = \sum_{j=1}^{J} \chi^j(x_3^j)p_p^j(x_3^j,t) + \widehat{p}(x,t).$$

[ii] due to incompressibility of the fluid, the flow rate ϕ^J in the last outlet is also known, it is equal to $-(\phi^1 + \cdots + \phi^{J-1})$.

Here the Poiseuille parts $\{(\mathbf{v}_p^j, p_p^j)\}_{j=1}^J$*, corresponding to the given flow-rates, are defined by* (2.1.4) *and satisfy inclusions* (2.2.6)*, while the term* $(\widetilde{\mathbf{v}}, \widehat{p})$ *is such that*

$$\widetilde{\mathbf{v}} \in L^2(0,2\pi; H^2_\beta(\Omega)), \quad \partial_t \widetilde{\mathbf{v}} \in L^2(0,2\pi; L^2_\beta(\Omega)), \quad \nabla \widehat{p} \in L^2(0,2\pi; L^2_\beta(\Omega)).$$

2.3 Structure and estimates of the pressure function

Since the gradient $\nabla \widehat{p}$ decays at infinity, the pressure function $\widehat{p} = \widehat{p}(x,t)$ in every outlet Ω_+^j, $j = 1, \dots, J$, tends to the time-dependent function $p_0^j = p_0^j(t)$, as $x_3^j \to \infty$. Namely, $\widehat{p}$ may be represented as

$$\widehat{p}(x,t) = \sum_{j=1}^J \chi^j(x_3^j) p_0^j(t) + \widetilde{p}(x,t), \tag{2.3.1}$$

where the term $\widetilde{p}$ and it's first order derivatives decay exponentially in every outlet, i.e., assume that β' is any number satisfying the condition $0 < \beta' < \beta$, then the following inclusions

$$\nabla \widetilde{p} \in L^2(0,2\pi; L^2_\beta(\Omega)), \quad \widetilde{p} \in L^2(0,2\pi; L^2_{\beta'}(\Omega))$$

hold. The first inclusion is a consequence of Theorem 2.2.3, while the second one, or, equivalently, the estimate of $\widetilde{p}$ in $L^2_{\beta'}(\Omega)$ norm, can be proved using the same arguments as in the proof of estimates (1.2.5) for the Fourier coefficients p_{ck} and p_{sk} in Section 1.2.

In order to determine the function $\widehat{p}$ completely, it is enough to prescribe only one of functions $p_0^1, \dots, p_0^J$, while the rest of the functions are determined by data of the problem. Indeed, comparing the asymptotic representation of $\widehat{p}$ with the structure of the time-periodic formal solution (2.1.3), we see that the function (2.3.1) corresponds to the part $\sum_{j=1}^J \chi^j p_0^j + \widetilde{p}$ of the time-periodic solution (2.1.3) (see also (2.1.7)). Recall that these two functions are generated by series (2.1.8) and (2.1.9). Assume that

$$p_0^J \in L^2(0,2\pi) \tag{2.3.2}$$

is a given time-periodic function with the Fourier series

$$p_0^J(t) = \sum_{k=0}^{\infty}\{a_{ck}^J \cos kt + a_{sk}^J \sin kt\}.$$

Then we look for $p_0^i = p_0^i(t)$, $i = 1, \ldots, J-1$, in the form of series (2.1.8). The Fourier coefficients of these functions are defined by the following relations

$$a_{ck}^i - a_{ck}^J = \int_\Omega (\mathbf{f}_{ck}\boldsymbol{\mathcal{V}}_{ck}^{ci} + \mathbf{f}_{sk}\boldsymbol{\mathcal{V}}_{sk}^{ci})dx - \int_\Omega (\mathbf{S}_k\mathbf{u}_k^p)\boldsymbol{\mathcal{U}}_k^{ci}dx, \tag{2.3.3}$$

$$a_{sk}^i - a_{sk}^J = \int_\Omega (\mathbf{f}_{ck}\boldsymbol{\mathcal{V}}_{ck}^{si} + \mathbf{f}_{sk}\boldsymbol{\mathcal{V}}_{sk}^{si})dx - \int_\Omega (\mathbf{S}_k\mathbf{u}_k^p)\boldsymbol{\mathcal{U}}_k^{si}dx. \tag{2.3.4}$$

To derive these relations we use the procedure similar to one used for derivation of relations (1.5.3) and (1.5.5) in the case of the homogeneous Stokes-type problem. Namely, for every $k = 0, 1, \ldots,$ we split the Fourier coefficient $\mathbf{u}_k = \mathbf{u}_k(x)$ of the function $\mathbf{u} = \mathbf{u}(x,t)$ into two parts:

$$\mathbf{u}_k^p = (\mathbf{v}_{ck}^p, p_{ck}^p, \mathbf{v}_{sk}^p, p_{sk}^p) = \sum_{j=0}^{J}\chi^j(b_{ck}^j\mathbf{u}_{ck}^{j1} + b_{sk}^j\mathbf{u}_{sk}^{j1})$$

and

$$\widehat{\mathbf{u}}_k = (\widetilde{\mathbf{v}}_{ck}, \widehat{p}_{ck}, \widetilde{\mathbf{v}}_{ck}, \widehat{p}_{sk}), \quad \widehat{p}_{ck} = \sum_{j=1}^{J}\chi^j a_{ck}^j + \widetilde{p}_{ck}, \quad \widehat{p}_{sk} = \sum_{j=1}^{J}\chi^j a_{sk}^j + \widetilde{p}_{sk}.$$

Recall that the prescribtion of flow-rates (2.2.8) determines the Poiseuille part of the solution, i.e., the Fourier coefficients $\mathbf{u}_k^p$, $k = 0, 1, \ldots$. Therefore, for every $k = 0, 1, \ldots$, we may consider the term $\widehat{\mathbf{u}}_k$ as the solution to the following Stokes-type problem

$$\begin{cases} -\nu\Delta\widetilde{\mathbf{v}}_{ck} + \nabla\widehat{p}_{ck} + k\widetilde{\mathbf{v}}_{sk} &= \mathbf{f}_{ck} + \nu\Delta\mathbf{v}_{ck}^p - \nabla p_{ck}^p - k\mathbf{v}_{sk}^p, \qquad x \in \Omega^j, \\ -\nabla\cdot\widetilde{\mathbf{v}}_{ck} &= \nabla\cdot\mathbf{v}_{ck}^p, \qquad x \in \Omega^j, \\ -\nu\Delta\widetilde{\mathbf{v}}_{sk} + \nabla\widehat{p}_{sk} - k\widetilde{\mathbf{v}}_{ck} &= \mathbf{f}_{sk} + \nu\Delta\mathbf{v}_{sk}^p - \nabla\mathbf{v}_{sk}^p + k\mathbf{v}_{ck}^p, \qquad x \in \Omega^j, \\ -\nabla\cdot\widetilde{\mathbf{v}}_{sk} &= \nabla\cdot\mathbf{v}_{sk}^p, \qquad x \in \Omega^j, \\ \widetilde{\mathbf{v}}_{ck} = \mathbf{0},\ \widetilde{\mathbf{v}}_{sk} &= \mathbf{0}, \qquad x \in \partial\Omega^j. \end{cases} \tag{2.3.5}$$

The functions $\{\boldsymbol{\mathcal{U}}_k^{ci}, \boldsymbol{\mathcal{U}}_k^{si}\}_{i=1}^{J-1} \in \mathcal{D}_{-\beta}^2 H(\Omega)$ in (2.3.3), (2.3.4) are solutions to the homogeneous adjoint Stokes-type problem (1.1.16). We suppose that:

(a) the vector-fields $\boldsymbol{\mathcal{U}}_k^{ci}, \boldsymbol{\mathcal{U}}_k^{si}$ generate the flow-rate equal to $+1$ in Ω_+^i, the flow-rate equal to -1 in Ω_+^J and zero flow-rates in the rest of the outlets;

(b) the flux is carried by the component $\boldsymbol{\mathcal{V}}^{ci}_{ck}$ of the function $\boldsymbol{\mathcal{U}}^{ci}_k = (\boldsymbol{\mathcal{V}}^{ci}_{ck}, \mathcal{P}^{ci}_{ck}, \boldsymbol{\mathcal{V}}^{ci}_{sk}, \mathcal{P}^{ci}_{sk})$ and by the component $\boldsymbol{\mathcal{V}}^{si}_{sk}$ of the function $\boldsymbol{\mathcal{U}}^{si}_k = (\boldsymbol{\mathcal{V}}^{si}_{ck}, \mathcal{P}^{si}_{ck}, \boldsymbol{\mathcal{V}}^{si}_{sk}, \mathcal{P}^{si}_{sk})$;

(c) there exists a constant C (independent of k), such that weighted[iii] estimates

$$\int_\Omega \rho_{-\beta}(|\boldsymbol{\mathcal{V}}^{ci}_{ck}|^2 + |\boldsymbol{\mathcal{V}}^{ci}_{sk}|^2)dx \le C, \quad \int_\Omega \rho_{-\beta}(|\boldsymbol{\mathcal{V}}^{si}_{ck}|^2 + |\boldsymbol{\mathcal{V}}^{si}_{sk}|^2)dx \le C \qquad (2.3.6)$$

hold for all $i = 1, \dots, J-1$.

Construction of solutions to the homogeneous problem (1.1.16) possessing properties (a) and (b) was described in Subsection 1.4.2. In fact, we may take

$$\boldsymbol{\mathcal{U}}^{ci}_k = \boldsymbol{\mathcal{U}}^{i}_k - \boldsymbol{\mathcal{U}}^{J}_k, \quad \boldsymbol{\mathcal{U}}^{si}_k = \boldsymbol{\mathcal{U}}^{J+i}_k - \boldsymbol{\mathcal{U}}^{2J}_k, \qquad (2.3.7)$$

where $\{\boldsymbol{\mathcal{U}}^i_k\}_{i=1}^{2J}$ is a basis presented in Subsection 1.4.2. It was shown there, besides other properties, that for every $i = 1, \dots, 2J$ the velocity coefficients of $\boldsymbol{\mathcal{U}}^i_k = (\boldsymbol{\mathcal{V}}^i_{ck}, \mathcal{P}^i_{ck}, \boldsymbol{\mathcal{V}}^i_{sk}, \mathcal{P}^i_{sk})$ satisfy the estimates

$$\int_{\Omega_1} (|\boldsymbol{\mathcal{V}}^i_{ck}|^2 + |\boldsymbol{\mathcal{V}}^i_{sk}|^2)dx \le C$$

in the finite domain Ω_1, with a constant C independent of k. In order to extend the last estimate to the whole domain Ω, we consider the structure of $\boldsymbol{\mathcal{V}}^i_{ck}$ and $\boldsymbol{\mathcal{V}}^i_{sk}$. Recall that the velocity field $\boldsymbol{\mathcal{V}}^i_{ck}$ was constructed as the sum $\boldsymbol{\mathcal{V}}^{p,i}_{ck} + \boldsymbol{\mathcal{V}}^{d,i}_{ck} + \widehat{\boldsymbol{\mathcal{V}}}^i_{ck}$. Here the term $\boldsymbol{\mathcal{V}}^{d,i}_{ck}$ has the compact support, while the term $\widehat{\boldsymbol{\mathcal{V}}}^i_{ck}$ decays exponentially in every outlet Ω^j_+ as $x^j_3 \to \infty$, i.e., $\widehat{\boldsymbol{\mathcal{V}}}^i_{ck} \in \mathcal{D}^2_\beta H(\Omega)$. Therefore for both of these terms the estimate of type (2.3.6) holds[iv]. Let us consider the term $\boldsymbol{\mathcal{V}}^{p,i}_{ck} = \sum_{j=1}^{j} \chi^j \boldsymbol{\mathcal{V}}^{p,i,j}_{ck}$. In every outlet Ω^j_+ the velocity-field $\boldsymbol{\mathcal{V}}^{p,i,j}_{ck}$ admits the representation

$$\Big(0, 0, B^{i,j}_{ck}\varphi^j_k(y^j) + B^{i,j}_{sk}\psi^j_k(y^j)\Big)\ ^{\mathrm{v}}.$$

Taking into account the definitions of $B^{i,j}_{ck}$, $B^{i,j}_{sk}$ (see (1.4.29)) and using Lemma A.0.1, we show that

$$\int_{\omega^j} |B^{i,j}_{ck}\varphi^j_k(y^j) + B^{i,j}_{sk}\psi^j_k(y^j)|^2 dy^j \le c\left((B^{i,j}_{ck})^2 + (B^{i,j}_{sk})^2\right)\int_{\omega^j} |\varphi^j_k|^2 + |\psi^j_k|^2 dy^j$$
$$\le c\frac{|\delta^j_i - 1/J|^2}{(c^j_k)^2 + (d^j_k)^2}\frac{d^j_k}{k} \le \frac{1}{kd^j_k} \to \frac{1}{|\omega|^j}, \quad k \to \infty.$$

[iii] The weight function $\rho_{-\beta}$ coincide in every outlet Ω^j, $j = 1, \dots, J$, with the exponent $e^{-2\beta x^j_3}$.

[iv] Analogous considerations are valid for the two last terms of the function $\boldsymbol{\mathcal{V}}^i_{sk} = \boldsymbol{\mathcal{V}}^{p,i}_{sk} + \boldsymbol{\mathcal{V}}^{d,i}_{sk} + \widehat{\boldsymbol{\mathcal{V}}}^i_{sk}$.

[v] We recall that the function cut-off function $\chi^j = \chi^j(x^j_3)$ is smooth and equal to 1 for $x^j_3 \ge 1$.

Consequently, the integrals $\int_{\omega^j} |\boldsymbol{\mathcal{V}}_{ck}^{p,i,j}|^2 dy^j$ may be bounded by the same constant for all $k = 0, 1, \ldots$, every $i = 1, \ldots, 2J$ and $j = 1, \ldots, J$. Multiplying these integrals (notice that they do not depend on x_3^j) by $\rho_{-\beta} = \rho_{-\beta}(x_3^j)$ and integrating the obtained expression by x_3^j from 0 to ∞ we get for every $i = 1, \ldots, 2J$, $j = 1, \ldots, J$ and $k = 0, 1, \ldots$ the estimate

$$\int_{\Omega_+^j} \rho_{-\beta} |\boldsymbol{\mathcal{V}}_{ck}^{p,i,j}|^2 dx \leq \frac{C}{\beta}.$$

Analogous estimate, hold for the velocity coefficient $\boldsymbol{\mathcal{V}}_{sk}^{p,i,j}$, i.e., we have that

$$\int_{\Omega_+^j} \rho_{-\beta} |\boldsymbol{\mathcal{V}}_{sk}^{p,i,j}|^2 dx \leq \frac{C}{\beta}.$$

Since β and C are fixed, the estimate (2.3.6) follows from the last two inequalities.

Multiplying the equations in (2.3.5) first by $\boldsymbol{\mathcal{U}}_k^{ci}$ and then by $\boldsymbol{\mathcal{U}}_k^{si}$, and integrating by parts in the left-hand side, we obtain, respectively, relations (2.3.3) and (2.3.4). Taking into account the definition of $\mathbf{u}_k^p$ (see also (1.1.11), (1.1.12)) and the fact that the derivatives of the cut-off functions $\chi^j = \chi^j(x_3^j)$, $j = 1, \ldots, J$, are supported on the interval $(0, 1)$, we may express the integral $\int_\Omega (\mathbf{S}_k \mathbf{u}_k^p) \boldsymbol{\mathcal{U}}_k^{ci} dx$ in (2.3.3) as follows:

$$\sum_{j=1}^J b_{ck}^j \int_0^1 \int_{\omega^j} \nu \left(\chi^j\right)'' \left(\varphi_k^j \mathcal{V}_{ck}^{ci} - \psi_k^j \mathcal{V}_{sk}^{ci}\right) + (\chi^j)' \left(x_3^j \mathcal{V}_{ck}^{ci} + \varphi_k^j \mathcal{P}_{ck}^{ci} + \psi_k^j \mathcal{P}_{sk}^{ci}\right) dy^j dx_3^j$$
$$+ \sum_{j=1}^J b_{sk}^j \int_0^1 \int_{\omega^j} \nu \left(\chi^j\right)'' \left(\psi_k^j \mathcal{V}_{ck}^{ci} + \varphi_k^j \mathcal{V}_{sk}^{ci}\right) + (\chi^j)' \left(x_3^j \mathcal{V}_{sk}^{ci} + \varphi_k^j \mathcal{P}_{sk}^{ci} - \psi_k^j \mathcal{P}_{ck}^{ci}\right) dy^j dx_3^j$$

$$=: \sum_{j=1}^J b_{ck}^j \alpha_{ck}^{c,i,j} + \sum_{j=1}^J b_{sk}^j \alpha_{sk}^{c,i,j}.$$

Here $\mathcal{V}_{ck}^{ci}$ and $\mathcal{V}_{sk}^{ci}$ denote the third components of the vectors $\boldsymbol{\mathcal{V}}_{ck}^{ci}$ and $\boldsymbol{\mathcal{V}}_{sk}^{ci}$, respectively. In the same way we obtain the equality

$$\int_\Omega (\mathbf{S}_k \mathbf{u}_k^p) \boldsymbol{\mathcal{U}}_k^{si} dx = \sum_{j=1}^J b_{ck}^j \alpha_{ck}^{s,i,j} + \sum_{j=1}^J b_{sk}^j \alpha_{sk}^{s,i,j}.$$

Now (2.3.3), (2.3.4) may be rewritten as follows

$$a_{ck}^i - a_{ck}^J = \int_\Omega (\mathbf{f}_{ck} \boldsymbol{\mathcal{V}}_{ck}^{ci} + \mathbf{f}_{sk} \boldsymbol{\mathcal{V}}_{sk}^{ci}) dx - \sum_{j=1}^J b_{ck}^j \alpha_{ck}^{c,i,j} - \sum_{j=1}^J b_{sk}^j \alpha_{sk}^{c,i,j}, \tag{2.3.8}$$

$$a^i_{sk} - a^J_{sk} = \int_\Omega (\mathbf{f}_{ck}\boldsymbol{\mathcal{V}}^{si}_{ck} + \mathbf{f}_{sk}\boldsymbol{\mathcal{V}}^{si}_{sk})dx - \sum_{j=1}^J b^j_{ck}\alpha^{s,i,j}_{ck} - \sum_{j=1}^J b^j_{sk}\alpha^{s,i,j}_{sk}. \tag{2.3.9}$$

Relations (2.3.8), (2.3.9) and Hölder's inequality provide the estimates

$$\begin{aligned}\left(a^i_{ck} - a^J_{ck}\right)^2 \le \left(\|\mathbf{f}_{ck}\|^2_{L^2_\beta(\Omega)} + \|\mathbf{f}_{sk}\|^2_{L^2_\beta(\Omega)}\right)\left(\|\boldsymbol{\mathcal{V}}^{ci}_{ck}\|^2_{L^2_{-\beta}(\Omega)} + \|\boldsymbol{\mathcal{V}}^{ci}_{sk}\|^2_{L^2_{-\beta}(\Omega)}\right) \\ + \sum_{j=1}^J \left(b^j_{ck}\alpha^{c,i,j}_{ck}\right)^2 + \sum_{j=1}^J \left(b^j_{sk}\alpha^{c,i,j}_{sk}\right)^2.\end{aligned} \tag{2.3.10}$$

$$\begin{aligned}\left(a^i_{sk} - a^J_{sk}\right)^2 \le \left(\|\mathbf{f}_{ck}\|^2_{L^2_\beta(\Omega)} + \|\mathbf{f}_{sk}\|^2_{L^2_\beta(\Omega)}\right)\left(\|\boldsymbol{\mathcal{V}}^{si}_{ck}\|^2_{L^2_{-\beta}(\Omega)} + \|\boldsymbol{\mathcal{V}}^{si}_{sk}\|^2_{L^2_{-\beta}(\Omega)}\right) \\ + \sum_{j=1}^J \left(b^j_{ck}\alpha^{s,i,j}_{ck}\right)^2 + \sum_{j=1}^J \left(b^j_{sk}\alpha^{s,i,j}_{sk}\right)^2.\end{aligned} \tag{2.3.11}$$

At the end of the Section we will prove the following

Lemma 2.3.1. *The sequences* $\{|\alpha^{c,i,j}_{ck}|\}_{k=0}^\infty$, $\{|\alpha^{c,i,j}_{sk}|\}_{k=0}^\infty$, $\{|\alpha^{s,i,j}_{ck}|\}_{k=0}^\infty$, $\{|\alpha^{s,i,j}_{sk}|\}_{k=0}^\infty$ *are bounded for every* $j = 1, \dots, J$ *and* $i = 1, \dots, J-1$.

As a consequence of this Lemma we have

Remark 2.3.2. Suppose that sequences $\{b_{ck}\}_{k=0}^\infty$ and $\{b_{sk}\}_{k=0}^\infty$ belong to the space l^2. Then for all $i = 1, \dots, J-1$ and $j = 1, \dots, J$ the sequences

$$\{\alpha^{c,i,j}_{ck} b_{ck}\}_{k=0}^\infty, \quad \{\alpha^{c,i,j}_{sk} b_{sk}\}_{k=0}^\infty, \quad \{\alpha^{s,i,j}_{ck} b_{ck}\}_{k=0}^\infty, \quad \{\alpha^{s,i,j}_{sk} b_{ck}\}_{k=0}^\infty \tag{2.3.12}$$

also belong to l^2.

Consider problem (2.0.1) with prescribed flow-rates (2.2.8) in $J-1$ outlet and the function $p^J_0 = p^J_0(t)$ given in the outlet Ω^J_+. Suppose that conditions (2.1.1), (2.2.9) and (2.3.2) hold. Inclusions $\mathbf{f} \in L^2(0, 2\pi; L^2_\beta(\Omega))$ and $p^J_0 \in L^2(0, 2\pi)$ are equivalent to the conditions

$$\sum_{k=0}^\infty \int_\Omega \rho_\beta(x)\left(|\mathbf{f}_{ck}(x)|^2 + |\mathbf{f}_{sk}(x)|^2\right)dx < \infty, \tag{2.3.13}$$

$$\sum_{k=0}^\infty (a^J_{ck})^2 + (a^J_{sk})^2 < \infty, \tag{2.3.14}$$

respectively. The assumption that the flow-rates $\phi^1, \dots, \phi^J$ belong to $H(0, 2\pi)$ ensures that the corresponding pressure drops $q^1, \dots, q^J$ belong to $L^2(0, 2\pi)$ (see Corollary 2.2.1). These inclusions yield convergence of the series (2.1.6) or, equivalently, the conditions

$$\sum_{k=0}^{\infty}(b^j_{ck})^2+(b^j_{sk})^2<\infty, \quad j=1,\dots,J. \tag{2.3.15}$$

Now, taking into account (2.3.6), (2.3.13) and conditions (2.3.15) together with Remark 2.3.2, we get from estimates (2.3.10) and (2.3.11) that the series

$$\sum_k (a^j_{ck}-a^J_{ck})^2, \quad \sum_k (a^j_{sk}-a^J_{sk})^2$$

converge for every $j=1,\dots,J-1$. By assumptions (2.3.2) and (2.3.14), we immediately obtain the convergence of the series

$$\sum_{k=0}^{\infty}(a^j_{ck})^2+(a^j_{sk})^2<\infty, \quad j=1,\dots,J-1,$$

or, equivalently, the inclusions $p^1_0,\dots,p^{J-1}_0\in L^2(0,2\pi)$.

The results presented in the current Section and Sections 2.1-2.3 can be summarized as follows.

Corollary 2.3.3. *Consider the time-periodic Stokes problem* (2.0.1) *with the prescribed time-periodic flow-rates $\phi^1,\dots,\phi^J\in H^1(0,2\pi)$ (see* (2.2.8)*) and the given time-periodic pressure term $p^J_0\in L^2(0,2\pi)$. Assume that the external force $\mathbf{f}=\mathbf{f}(x,t)$ in* (2.0.1) *belongs to $L^2(0,2\pi;L^2_\beta(\Omega))$. Then problem* (2.0.1) *has a unique solution $\mathbf{v}=\mathbf{v}(x,t)$, $p=p(x,t)$. The solution admits the following representation*

$$\begin{gathered}\mathbf{v}(x,t)=\sum_{j=1}^{J}\chi^j(x^j_3)\mathbf{v}^j_p(y^j,t)+\widetilde{\mathbf{v}}(x,t),\\ p(x,t)=\sum_{j=1}^{J}\chi^j(x^j_3)p^j_p(x^j_3,t)+\sum_{j=1}^{J}\chi^j(x^j_3)p^j_0(t)+\widetilde{p}(x,t).\end{gathered} \tag{2.3.16}$$

Here functions $(\mathbf{v}^j_p,p^j_p)$, $j=1,\dots,J$, defined by (2.1.4)*, correspond to the given flow rates $\phi^1,\dots,\phi^J$ and satisfy inclusions* (2.2.6)*. Functions $p^1_0,\dots,p^{J-1}_0$, defined by* (2.1.8), (2.3.8) *and* (2.3.9)*, belong to the space $L^2(0,2\pi)$. The exponentially decaying part $(\widetilde{\mathbf{v}},\widetilde{p})$ satisfies the inclusions (with any β' such that $0<\beta'<\beta$):*

$$\begin{gathered}\widetilde{\mathbf{v}}\in L^2(0,2\pi;H^2_\beta(\Omega)), \quad \partial_t\widetilde{\mathbf{v}}\in L^2(0,2\pi;L^2_\beta(\Omega)),\\ \nabla\widetilde{p}\in L^2(0,2\pi;L^2_\beta(\Omega)) \quad \widetilde{p}\in L^2(0,2\pi;L^2_{\beta'}(\Omega)).\end{gathered} \tag{2.3.17}$$

Finally, the following estimate

$$\int_0^{2\pi}\left(\sum_{j=1}^{J}\|\widetilde{\mathbf{v}}(t)\|^2_{H^2_\beta(\Omega)}+\|\partial_t\widetilde{\mathbf{v}}(t)\|^2_{L^2_\beta(\Omega)}+\right.$$

$$\left.|p_0^j(t)|^2+\|\nabla\widetilde{p}(t)\|^2_{L^2_\beta(\Omega)}+\|\widetilde{p}(t)\|^2_{L^2_{\beta'}(\Omega)}\right)dt \tag{2.3.18}$$

$$\leq c\int_0^{2\pi}\left(\|\mathbf{f}(t)\|^2_{L^2_\beta(\Omega)}+\sum_{j=1}^{J-1}\left\{|\phi^j(t)|^2+\left|\frac{d\phi^j}{dt}(t)\right|^2\right\}+|\psi^J(t)|^2\right)dt$$

holds.

We finish the Section by proving Lemma 2.3.1 formulated above.

Proof. Let us prove the boundedness of $\{|\alpha^{c,i,j}_{ck}|\}_{k=0}^{\infty}$. Consider the quantity

$$\alpha^{c,i,j}_{ck}=\int_0^1\int_{\omega^j}\nu\left(\chi^j\right)''\left(\varphi^j_k\mathcal{V}^{ci}_{ck}-\psi^j_k\mathcal{V}^{ci}_{sk}\right)+(\chi^j)'\left(x^j_3\mathcal{V}^{ci}_{ck}+\varphi^j_k\mathcal{P}^{ci}_{ck}+\psi^j_k\mathcal{P}^{ci}_{sk}\right)dy^j dx^j_3.$$

Since the function $\chi^j=\chi^j(x^j_3)$ is smooth, one easily gets the following estimate

$$|\alpha^{c,i,j}_{ck}|\leq c\left(\|\varphi^j_k\|_{L^2(\omega^j)}\|\mathcal{V}^{ci}_{ck}\|_{L^2(G^j_1)}+\|\psi^j_k\|_{L^2(\omega^j)}\|\mathcal{V}^{ci}_{sk}\|_{L^2(G^j_1)}+\|\mathcal{V}^{ci}_{sk}\|_{L^2(G^j_1)}\right.$$
$$\left.+\|\varphi^j_k\|_{L^2(\omega^j)}\|\mathcal{P}^{ci}_{ck}\|_{L^2(G^j_1)}+\|\psi^j_k\|_{L^2(\omega^j)}\|\mathcal{P}^{ci}_{sk}\|_{L^2(G^j_1)}\right), \tag{2.3.19}$$

here $G^j_1=\omega^j\times(0,1)$. Boundedness (uniform with respect to k) of the terms

$$\|\varphi^j_k\|_{L^2(\omega^j)}\|\mathcal{V}^{ci}_{ck}\|_{L^2(G^j_1)}+\|\psi^j_k\|_{L^2(\omega^j)}\|\mathcal{V}^{ci}_{sk}\|_{L^2(G^j_1)}+\|\mathcal{V}^{ci}_{sk}\|_{L^2(G^j_1)},\quad j=1,\dots,J$$

immediately follows from formula (2.3.7), estimate (2.3.6) and properties of the functions φ^j_k, ψ^j_k presented in Lemma A.0.1.

Let us consider the pressure functions $\mathcal{P}^{ci}_{ck}$ and $\mathcal{P}^{ci}_{sk}$. Structure of the elements in the basis $\boldsymbol{\mathcal{U}}^1_k,\dots,\boldsymbol{\mathcal{U}}^{2J}_k$ of the subspace $\ker\left(\mathbf{A}^l_{-\beta\to\beta,k}\right)^*$ allows to represent these functions as $\mathcal{P}^{ci}_{ck}=\mathcal{P}^{ci,p}_{ck}+\widehat{\mathcal{P}}^{ci}_{ck}$ and $\mathcal{P}^{ci}_{sk}=\mathcal{P}^{ci,p}_{sk}+\widehat{\mathcal{P}}^{ci}_{sk}$. Here

$$\mathcal{P}^{ci,p}_{ck}=-\sum_{j=1}^{J}\chi^j B^j_{ck}x^j_3,\quad \mathcal{P}^{ci,p}_{sk}=\sum_{j=1}^{J}\chi^j B^j_{sk}x^j_3 \tag{2.3.20}$$

and

$$\widehat{\mathcal{P}}^{ci}_{ck}(x,t)=\sum_{j=1}^{J}\chi^j(x^j_3)A^j_{ck}+\widetilde{\mathcal{P}}_{ck}(x,t),\quad \widehat{\mathcal{P}}^{ci}_{sk}(x,t)=\sum_{j=1}^{J}\chi^j(x^j_3)A^j_{sk}+\widetilde{\mathcal{P}}_{sk}(x,t).$$

Using the same scheme as was used for the elements $\boldsymbol{u}_k^1, \ldots, \boldsymbol{u}_k^{2J} \in \ker \mathbf{A}^l_{-\beta\to\beta,k}$ in Section 1.5, we derive the relations (analogous to (1.5.3))

$$A^l_{ck} - A^J_{ck} = \int_\Omega \mathbf{S}^*_k(\boldsymbol{\mathcal{U}}^{ci,p}_k + \boldsymbol{\mathcal{U}}^{ci,d}_k) \cdot \boldsymbol{u}^l_k dx,$$

$$A^l_{sk} - A^J_{sk} = \int_\Omega \mathbf{S}^*_k(\boldsymbol{\mathcal{U}}^{ci,p}_k + \boldsymbol{\mathcal{U}}^{ci,d}_k) \cdot \boldsymbol{u}^{l+J}_k dx.$$

Therefore taking $A^J_{ck} = 0, A^J_{sk} = 0$ we may describe the constants $\{A^j_{ck}, A^j_{sk}\}_{j=1}^{J-1}$ in terms of B^j_{ck}, B^j_{sk} and derive the following estimates

$$(A^j_{ck})^2 + (A^j_{sk})^2 \le c\sum_{j=1}^J \left((B^j_{ck})^2 + (B^j_{sk})^2\right), \quad j = 1, \ldots, J-1. \tag{2.3.21}$$

Finally, in the same way as in Theorem 1.2.2, we derive the estimate (analogous to (1.2.5))

$$\int_{\Omega^j_+} e^{2\beta' x^j_3} |\widehat{\mathcal{P}}^{ci}_{ck}(y^j, x^j_3) - A^j_{ck}|^2 dy^j dx^j_3 \le c \int_\Omega \rho_\beta |\mathbf{S}^*_k(\boldsymbol{\mathcal{U}}^{ci,p}_k + \boldsymbol{\mathcal{U}}^{ci,d}_k)|^2 dx.$$

The two last estimates and the following expression

$$\int_{G^j_1} |\widetilde{\mathcal{P}}^{ci}_{ck}|^2 dx = \int_{G^j_1} |\widehat{\mathcal{P}}^{ci}_{ck}(y^j, x^j_3) - \chi^j A^j_{ck} \pm A^j_{ck}|^2 dx$$

$$\le \int_{\Omega^j_+} e^{2\beta' x^j_3} |\widehat{\mathcal{P}}^{ci}_{ck}(y^j, x^j_3) - A^j_{ck}|^2 dy^j dx^j_3 + |A^j_{ck}|^2 \int_{G^j_1} |1 - \chi^j(x^j_3)|^2 dx$$

leads to the conclusion that the function $\widetilde{\mathcal{P}}^{ci}_{ck}$ and, consequently, the function $\mathcal{P}^{ci}_{ck}$ can be estimated in terms of the coefficients $\{B^j_{ck}, B^j_{sk}\}_{j=1}^J$. Namely there holds the inequality

$$\int_{G^j_1} |\mathcal{P}_{ck}|^2 dx \le c\sum_{j=1}^J \left((B^j_{ck})^2 + (B^j_{sk})^2\right) \tag{2.3.22}$$

with some constant c independent of k. In the same way we may derive the estimate

$$\int_{G^j_1} |\mathcal{P}_{sk}|^2 dx \le c\sum_{j=1}^J \left((B^j_{ck})^2 + (B^j_{sk})^2\right). \tag{2.3.23}$$

It was assumed that the vector-fields $\boldsymbol{\mathcal{U}}^{ci}_k$ and $\boldsymbol{\mathcal{U}}^{si}_k$ generate the flow-rate equal to $+1$ in Ω^i_+, the flow-rate equal to -1 in Ω^J_+ and zero flow-rates in the rest of the outlets (see the assumptions below formula (2.3.5)). In order to achieve such flow-rate distribution, we shall use in the construction of $\boldsymbol{\mathcal{U}}_{ck}^{ci}$ and $\boldsymbol{\mathcal{U}}_{sk}^{ci}$ the constants

$$B_{ck}^{i} = \frac{c_k^i}{(c_k^i)^2 + (d_k^i)^2}, \quad B_{sk}^{i} = -\frac{d_k^i}{(c_k^i)^2 + (d_k^i)^2},$$
$$B_{sk}^{i+1} = -\frac{c_k^{i+1}}{(c_k^{i+1})^2 + (d_k^{i+1})^2}, \quad B_{sk}^{i+1} = \frac{d_k^{i+1}}{(c_k^{i+1})^2 + (d_k^{i+1})^2},$$
$$B_{ck}^{j} = 0, \quad B_{sk}^{j} = 0 \quad \text{for} \quad j \neq i, i+1.$$

Let us notice that the coefficients B_{ck}^i and B_{sk}^i may grow as k becomes large, for example, $B_{sk}^i = O(\frac{1}{d_k^j}) \sim \frac{k}{|\omega^j|}$, as $k \to \infty$. However, the terms

$$\|\varphi_k^j\|_{L^2(\omega^j)} \|\mathcal{P}_{ck}^{ci}\|_{L^2(G_1^j)} + \|\psi_k^j\|_{L^2(\omega^j)} \|\mathcal{P}_{sk}^{ci}\|_{L^2(G_1^j)}$$

in (2.3.19) are bounded by the same constant for all $k = 0, 1, \ldots$, due to the "good" behaviour of the functions φ_k^j and ψ_k^j. Using Lemma A.0.1, estimates (2.3.22), (2.3.23) and definitions of the coefficients B_{ck}^j, B_{sk}^j we obtain the following estimate:

$$\|\varphi_k^j\|_{L^2(\omega^j)} \|\mathcal{P}_{ck}^{ci}\|_{L^2(G_1^j)} + \|\psi_k^j\|_{L^2(\omega^j)} \|\mathcal{P}_{sk}^{ci}\|_{L^2(G_1^j)} \leq \Big(\|\varphi_k^j\|_{L^2(\omega^j)}^2 + \|\psi_k^j\|_{L^2(\omega^j)}^2 \Big)$$
$$\times \sum_{j=1}^{J} \Big((B_{ck}^j)^2 + (B_{sk}^j)^2 \Big) \leq \frac{d_k^j}{k} \sum_{j=1}^{J} \frac{1}{(c_k^j) + (d_k^j)^2} \leq \frac{|\omega^j|}{k^2} \sum_{j=1}^{J} \frac{1}{(d_k^j)^2} \equiv N_k.$$

Since $d_k^j k \to |\omega^j|$ as $k \to \infty$, the sequence N_k tends to $|\omega^j| \sum_{j=1}^{J} \frac{1}{|\omega|^j}$ and, consequently, is bounded by a constant independent of k. Using this fact we conclude from (2.3.19) that the sequence $\{|\alpha_{ck}^{c,i,j}|\}_{k=0}^{\infty}$ is bounded for each $j = 1, \ldots, J$ and $i = 1, \ldots, J-1$. In the same way one can prove boundedness of the sequences $\{|\alpha_{sk}^{c,i,j}|\}_{k=0}^{\infty}$, $\{|\alpha_{ck}^{s,i,j}|\}_{k=0}^{\infty}$ and $\{|\alpha_{sk}^{s,i,j}|\}_{k=0}^{\infty}$. □

2.4 Time-periodic problem with general asymptotic conditions at infinity

2.4.1 Function spaces for the time-periodic problem

Let $\mathbb{D}^2_{\pm\beta}H(\Omega\times(0,2\pi))$ denotes the set of time-periodic functions $\mathbf{u}=(\mathbf{v},p)$ having the form and regularity described below. Namely, we assume that

$$\begin{gathered}\mathbf{v}(x,t)=\sum_{j=1}^{J}\chi^j(x_3^j)\mathbf{v}_p^j(y^j,t)+\widetilde{\mathbf{v}}(x,t),\\ p(x,t)=\sum_{j=1}^{J}\chi^j(x_3^j)p_p^j(x_3^j,t)+\sum_{j=1}^{J}\chi^j(x_3^j)p_0^j(t)+\widetilde{p}(x,t),\\ \mathbf{v}_p^j(y^j,t)=\Big(0,0,v^j(y^j,t)\Big),\quad p_p^j(x_3^j,t)=-q^j(t)x_3^j,\end{gathered}\tag{2.4.1}$$

where

$$v^j(y^j,t)=\sum_{k=0}^{\infty}\Big\{\Big(b_{ck}^j\varphi_k^j(y^j)+b_{sk}^j\psi_k^j(y^j)\Big)\cos kt+\Big(b_{sk}^j\varphi_k^j(y^j)-b_{ck}^j\psi_k^j(y^j)\Big)\sin kt\Big\},$$

$$q^j(t)=\sum_{k=0}^{\infty}\{b_{ck}^j\cos kt+b_{sk}^j\sin kt\},$$

and that the following inclusions (with any β' satisfying the condition $0<\beta'<\beta$):

$$\begin{gathered}v^j\in C(0,2\pi;\mathring{H}^1(\omega^j))\cap L^2(0,2\pi;H^2(\omega^j)),\\ \partial_t v^j\in L^2(0,2\pi;L^2(\omega^j)),\quad q^j\in L^2(0,2\pi),\\ p_0^j\in L^2(0,2\pi),\quad \text{for all}\quad j=1,\dots,J,\\ \widetilde{\mathbf{v}}\in L^2(0,2\pi;H^2_\beta(\Omega)),\quad \partial_t\widetilde{\mathbf{v}}\in L^2(0,2\pi;L^2_\beta(\Omega)),\\ \nabla\widetilde{p}\in L^2(0,2\pi;L^2_\beta(\Omega))\quad \widetilde{p}\in L^2(0,2\pi;L^2_{\beta'}(\Omega))\end{gathered}\tag{2.4.2}$$

hold.

Analogously, consider time-periodic functions

$$\begin{gathered}\mathbf{V}(x,t)=\sum_{j=1}^{J}\chi^j(x_3^j)\mathbf{V}_p^j(x,t)+\widetilde{\mathbf{V}}(x,t),\\ P(x,t)=\sum_{j=1}^{J}\chi^j(x_3^j)P_p^j(x,t)+\sum_{j=1}^{J}\chi^j(x_3^j)P_0^j(t)+\widetilde{P}(x,t),\end{gathered}\tag{2.4.3}$$

such that the terms $\mathbf{V}_p^j(x,t)=\Big(0,0,V_p^j(x,t)\Big)$, $P_p^j(x,t)=Q_p^j(t)x_3^j$, $j=1,\dots,J$, are generated by the Fourier coefficients

$$\mathbf{V}_{ck}^{p,j}=\Big(0,0,B_{ck}^j\varphi_k^j+B_{sk}^j\psi_k^j\Big),\quad P_{ck}^{p,j}=-B_{ck}^jx_3^j,$$
$$\mathbf{V}_{sk}^{p,j}=\Big(0,0,B_{sk}^j\psi_k^j-B_{ck}^j\varphi_k^j\Big),\quad P_{sk}^{p,j}=B_{sk}^jx_3^j.$$

Assume that for $j=1,\dots,J$

$$\begin{aligned}&V_p^j\in C(0,2\pi;\mathring{H}^1(\omega^j))\cap L^2(0,2\pi;H^2(\omega^j)),\\&\partial_tV_p^j\in L^2(0,2\pi;L^2(\omega^j)),\quad Q_p^j\in L^2(0,2\pi),\end{aligned}\tag{2.4.4}$$

$$P_0^1,\dots,P_0^J\in L^2(0,2\pi).\tag{2.4.5}$$

and

$$\begin{aligned}&\widetilde{\mathbf{V}}\in L^2(0,2\pi;H^2_\beta(\Omega)),\quad \partial_t\widetilde{\mathbf{V}}\in L^2(0,2\pi;L^2_\beta(\Omega)),\\&\nabla\widetilde{P}\in L^2(0,2\pi;L^2_\beta(\Omega)),\quad \widetilde{P}\in L^2(0,2\pi;L^2_{\beta'}(\Omega)).\end{aligned}\tag{2.4.6}$$

Note that $\{(\mathbf{V}_{ck}^{p,j},P_{ck}^{p,j},\mathbf{V}_{sk}^{p,j},P_{sk}^{p,j})\}_{j=1}^J$ and $P_0^1,\dots,P_0^J$ form the main part in the asymptotic representation (1.1.25) of functions $\mathbf{U}_k=(\mathbf{V}_{ck},P_{ck},\mathbf{V}_{sk},P_{sk})\in\mathbb{D}^2_{\pm\beta}H(\Omega)^*$. Therefore we denote the set of functions (2.4.3) with the regularity (2.4.4)–(2.4.6) by $\mathbb{D}^2_{\pm\beta}H(\Omega\times(0,2\pi))^*$.

2.4.2 Conditions at infinity

Below we study the solvability of problem (2.0.1) in the class $\mathbb{D}^2_{\pm\beta}H(\Omega\times(0,2\pi))$ when general conditions at infinity are imposed. Assume that the right-hand side $\mathbf{f}$ in (2.0.1) belongs to $L^2(0,2\pi;L^2_\beta(\Omega))$. Suppose that the time-periodic functions $h^j=h^j(t)$, $j=1,\dots,J$, are given, and

$$h^1,\dots,h^J\in L^2(0,2\pi).\tag{2.4.7}$$

Let us represent these functions as the Fourier series

$$h^j(t)=\sum_{k=0}^\infty\{h_{ck}^j\cos kt+h_{sk}^j\sin kt\},\quad j=1,\dots,J,\tag{2.4.8}$$

and construct, for every $k = 0, 1, \ldots$, the $2J$-dimensional vector

$$\mathbf{h}_k = (h^1_{ck}, \ldots, h^J_{ck}, h^1_{sk}, \ldots, h^J_{sk}). \tag{2.4.9}$$

Now we may consider the sequence of Stokes-type problems (1.1.2) supplied with the asymptotic conditions at infinity:

$$\mathbf{S}_k \mathbf{u}_k = \mathbf{f}_k, \quad x \in \Omega, \qquad \mathbf{v}_k = \mathbf{0}, \quad x \in \partial\Omega, \quad \mathbb{B}_k \pi \mathbf{u}_k = \mathbf{h}_k, \quad k = 0, 1, \ldots \tag{2.4.10}$$

Results presented in Chapter 1 guarantee that problem (1.1.2) has at least one solution $\mathbf{u}_k \in \mathbb{D}^2_{\pm\beta} H(\Omega)$. If the matrix $\mathbb{B}_k$ in (2.4.10) is selected "properly", the solution $\mathbf{u}_k$ becomes unique (see Theorem 1.6.1). Then the sequence

$$\{\mathbf{u}_k = (\mathbf{v}_{ck}, p_{ck}, \mathbf{v}_{sk}, p_{sk})\}_{k=0}^{\infty}.$$

generates series (2.1.2), which may be treated as a unique formal solution to problem (2.0.1) supplied with the following conditions at infinity

$$\mathbb{B}\Pi\mathbf{u} = \mathbf{h}. \tag{2.4.11}$$

Here $\mathbf{h} = (\mathbf{h}_0, \mathbf{h}_1, \ldots)$ is a sequence composed from the vectors (2.4.9), while the projector Π and the operator (an infinite matrix) $\mathbb{B} : \mathbb{R}^\infty \to \mathbb{R}^\infty$ are defined as follows

$$\Pi\mathbf{u} = (\pi\mathbf{u}_0, \pi\mathbf{u}_1, \ldots), \quad \mathbb{B} = \mathrm{diag}(\mathbb{B}_0, \mathbb{B}_1, \ldots).$$

If functions $h^1, \ldots, h^J$ are "sufficiently" regular, series (2.1.2) converge to the solution $\mathbf{u} = (\mathbf{v}, p) \in \mathbb{D}^2_{\pm\beta} H(\Omega \times (0, 2\pi))$. We notice that the regularity required for these functions depends on the choice of the operator $\mathbb{B}$.

Example 1. Consider the time-periodic problem (2.0.1) with the flow-rates $\phi^j = \phi^j(t)$ prescribed in $J-1$ outlets and with the given time-periodic pressure function $p^J_0 = p^J_0(t)$. In the case $J = 3$ these conditions are described by the sequence of systems of linear algebraic equations

$$\mathbb{B}_k \pi \mathbf{u}_k = \mathbf{h}_k, \quad k = 0, 1, \ldots$$

with the matrices

$$\mathbb{B}_k = \begin{pmatrix} c_k^1 & 0 & 0 & d_k^1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_k^2 & 0 & 0 & d_k^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -d_k^1 & 0 & 0 & c_k^1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -d_k^2 & 0 & 0 & c_k^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.4.12}$$

here c_k^j and d_k^j are defined by (1.3.9), while the column-vectors $\mathbf{h}_k$ are given by

$$\mathbf{h}_k = (\phi_{ck}^1, \phi_{ck}^2, p_0{}_{ck}^3, \phi_{sk}^1, \phi_{sk}^2, p_0{}_{sk}^3).$$

One can verify, using relations (1.4.10), (1.4.11) and (1.4.18) that in this case

$$\mathbb{B}_k \begin{pmatrix} \mathcal{B}_k \\ \mathcal{A}_k \end{pmatrix} = \begin{pmatrix} 2/3 & -1/3 & -1/3 & 0 & 0 & 0 \\ -1/3 & 2/3 & -1/3 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2/3 & -1/3 & -1/3 \\ 0 & 0 & 0 & -1/3 & 2/3 & -1/3 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

The determinant of the matrix above is equal to 1. Therefore, for every $k = 0, 1, \ldots$, the rank of the product $\mathbb{B}_k \begin{pmatrix} \mathcal{B}_k \\ \mathcal{A}_k \end{pmatrix}$ is equal to 6. This fact, together with formula (1.6.6), confirm the uniqueness of the coefficients $\mathbf{u}_k$, $k = 0, 1, \ldots$, determining the formal solution

$$\mathbf{u}(x,t) = \sum_{k=0}^{\infty} \mathbf{u}_{ck}(x) \cos kt + \mathbf{u}_{sk}(x) \sin kt \tag{2.4.13}$$

to problem (2.0.1), (2.2.8), (2.3.2).

Let us assume that the components of the vectors $\mathbf{h}_k$, $k = 0, 1, \ldots$, generate the convergent series

$$\sum_{k=0}^{\infty} \left\{ \left(\phi_{ck}^j\right)^2 + \left(\phi_{sk}^j\right)^2 \right\}, \quad \sum_{k=0}^{\infty} \left\{ \left(k\phi_{ck}^j\right)^2 + \left(k\phi_{sk}^j\right)^2 \right\}, \quad j = 1, 2,$$

$$\sum_{k=0}^{\infty} \left\{ \left(p_0{}_{ck}^3\right)^2 + \left(p_0{}_{sk}^3\right)^2 \right\}.$$

Convergence of these series yields the inclusions ϕ^1, $\phi^2 \in H^1(0, 2\pi)$ and $p_0^3 \in L^2(0, 2\pi)$. Therefore we conclude, taking into account Corollary 2.3.3, that series (2.4.13) converge to the solution $\mathbf{u} \in \mathbb{D}^2_{\pm\beta}H(\Omega \times (0, 2\pi))$ of problem (2.0.1), (2.2.8), (2.3.2). $\square$

2.4.3 Generalized Green's formula. Solvability of the time-periodic problem

Let us derive for the time-periodic Stokes problem (2.0.1) the generalized Green formula which holds for the functions $\mathbf{u} \in \mathbb{D}^2_{\pm\beta}H(\Omega \times (0, 2\pi))$ and $\mathbf{U} \in \mathbb{D}^2_{\pm\beta}H(\Omega \times (0, 2\pi))^*$. Denote by $\mathbf{Su}$ the left-hand side of the equations in (2.0.1). Substitute in $\mathbf{Su}$ the function $\mathbf{u} = (\mathbf{v}, p) \in \mathbb{D}^2_{\pm\beta}H(\Omega \times (0, 2\pi))$ and multiply the obtained expression by the function $\mathbf{U} \in \mathbb{D}^2_{\pm\beta}H(\Omega \times (0, 2\pi))^*$. Taking into account the periodicity in time of $\mathbf{u}$ and $\mathbf{U}$ and the orthogonality in $L_2(0, 2\pi)$ of functions $\{\cos kt, \sin kt\}_{k=0}^{\infty}$, we get the relation

$$\int_0^{2\pi}\int_\Omega \mathbf{Su}(x,t)\cdot\mathbf{U}(x,t)dxdt = \sum_{k=0}^{\infty}\int_\Omega \mathbf{S}_k\mathbf{u}_k(x)\cdot\mathbf{U}_k(x)dx.$$

Here $\mathbf{u}_k \in \mathbb{D}^2_{\pm\beta}H(\Omega)$ and $\mathbf{U}_k \in \mathbb{D}^2_{\pm\beta}H(\Omega)^*$ are the Fourier coefficients of functions $\mathbf{u}$ and $\mathbf{U}$, respectively. Using formula (1.3.14) we rewrite the right-hand side of the last identity as follows:

$$\begin{aligned}\int_0^{2\pi}\int_\Omega \mathbf{Su}(x,t)\cdot\mathbf{U}(x,t)dxdt &= \sum_{k=0}^{\infty}\int_\Omega \mathbf{u}_k(x)\cdot\mathbf{S}_k^*\mathbf{U}_k(x)dx \\ &+ \sum_{k=0}^{\infty}\left\{\langle\mathbb{S}_k\pi\mathbf{u}_k, \mathbb{Q}_k\pi\mathbf{U}_k\rangle_{2J} - \langle\mathbb{B}_k\pi\mathbf{u}_k, \mathbb{T}_k\pi\mathbf{U}_k\rangle_{2J}\right\}.\end{aligned} \tag{2.4.14}$$

The sequence $\{\mathbf{S}_k^*\mathbf{U}_k = \mathbf{F}_k\}_{k=0}^{\infty}$ of formally adjoint Stokes-type problems (1.1.16) generates the following time-periodic problem

$$\begin{cases} -\partial_t\mathbf{V} - \nu\Delta\mathbf{V} + \nabla P = \mathbf{F}, & (x,t) \in \Omega \times (0, 2\pi), \\ -\nabla\cdot\mathbf{V} = \mathbf{0}, & (x,t) \in \Omega \times (0, 2\pi), \\ \mathbf{V} = \mathbf{0}, & (x,t) \in \partial\Omega \times (0, 2\pi), \\ \mathbf{V}(x,0) = \mathbf{V}(x,2\pi), & x \in \Omega. \end{cases} \tag{2.4.15}$$

Namely, looking for the solution of this problem in the form

$$\mathbf{V}(x,t) = \sum_{k=0}^{\infty} \mathbf{V}_{ck}(x) \cos kt + \mathbf{V}_{sk}(x) \sin kt$$

$$P(x,t) = \sum_{k=0}^{\infty} P_{ck}(x) \cos kt + P_{sk}(x) \sin kt,$$

and substituting these series into (2.4.15), we get for the Fourier coefficients $\{(\mathbf{V}_{ck}, P_{ck}, \mathbf{V}_{sk}, P_{sk})\}_{k=0}^{\infty}$ the sequence of the formally adjoint Stokes-type problems $\{\mathbf{S}_k^* \mathbf{U}_k = \mathbf{F}_k\}_{k=0}^{\infty}$.

Remark 2.4.1. We would like to emphasize that problem (2.4.15) is a "backward time" problem – it has a negative sign in front of the time derivative. In general, backward time problems may be ill-posed. However assumption about the time-periodicity allows to avoid any peculiarities related to the "change of the time direction" and we may consider problem (2.4.15) in the same way as problem (2.0.1).

Let $\mathbf{S}^*\mathbf{U}$ denotes the left-hand side of equations (2.4.15$_1$)-(2.4.15$_2$). Assume that the operators (infinite matrices) $\mathbb{B}, \mathbb{S}, \mathbb{T}, \mathbb{Q} : \mathbb{R}^\infty \to \mathbb{R}^\infty$ are defined as follows:

$$\begin{aligned} \mathbb{B} &= \operatorname{diag}(\mathbb{B}_0, \mathbb{B}_1, \ldots), \quad \mathbb{S} = \operatorname{diag}(\mathbb{S}_0, \mathbb{S}_1, \ldots), \\ \mathbb{T} &= \operatorname{diag}(\mathbb{T}_0, \mathbb{T}_1, \ldots), \quad \mathbb{Q} = \operatorname{diag}(\mathbb{Q}_0, \mathbb{Q}_1, \ldots), \end{aligned} \tag{2.4.16}$$

where $\mathbb{B}_k, \mathbb{S}_k, \mathbb{T}_k, \mathbb{Q}_k$, $k = 0, 1, \ldots$, are $2J \times 4J$ matrices satisfying condition (1.3.13). Then (2.4.14) yields the Green formula

$$\begin{aligned} &\int_0^{2\pi} \int_\Omega \mathbf{S}\mathbf{u}(x,t) \cdot \mathbf{U}(x,t) dx dt + \langle \mathbb{B}\Pi\mathbf{u},\ \mathbb{T}\Pi\mathbf{U} \rangle_\infty \\ = &\int_0^{2\pi} \int_\Omega \mathbf{u}(x,t) \cdot \mathbf{S}^*\mathbf{U}(x,t) dx dt + \langle \mathbb{S}\Pi\mathbf{u},\ \mathbb{Q}\Pi\mathbf{U} \rangle_\infty. \end{aligned} \tag{2.4.17}$$

Here $\langle\, ,\, \rangle_\infty$ denotes the scalar product of two sequences. For the time-periodic Stokes problem (2.0.1) supplied with the asymptotic conditions at infinity $\mathbb{B}\Pi\mathbf{u} = \mathbf{h}$, Green's formula (2.4.17) determines the formally adjoint problem (2.4.15) with the following conditions at infinity

$$\mathbb{Q}\Pi\mathbf{U} = \mathbf{H}. \tag{2.4.18}$$

Here $\mathbf{H}$ is a given real number sequence.

We recall that the projector Π maps elements $\mathbf{u} \in \mathbb{D}^2_{\pm\beta} H(\Omega \times (0, 2\pi))$ and $\mathbf{U} \in$

$\mathbb{D}^2_{\pm\beta}H(\Omega\times(0,2\pi))^*$ to the sequences

$$(\pi\mathbf{u}_0,\pi\mathbf{u}_1,\dots,),\quad (\pi\mathbf{U}_0,\pi\mathbf{U}_1,\dots,),$$

where

$$\pi\mathbf{u}_k=(b^1_{ck},\dots,b^J_{ck},b^1_{sk},\dots,b^J_{sk},a^1_{ck},\dots,a^J_{ck},a^1_{sk},\dots,a^J_{sk}),$$
$$\pi\mathbf{U}_k=(B^1_{ck},\dots,B^J_{ck},B^1_{sk},\dots,B^J_{sk},A^1_{ck},\dots,A^J_{ck},A^1_{sk},\dots,A^J_{sk}).$$

These sequences belong to the space l^2. If we assume, for example, that the elements of the matrices $\mathbb{B}_k$ and $\mathbb{Q}_k$ in (2.4.16) are bounded[vi] with respect to k, then conditions (2.4.11) and (2.4.18) define the operators $\mathbb{B}\Pi:\mathbb{D}^2_{\pm\beta}H(\Omega\times(0,2\pi))\to l^2$ and $\mathbb{Q}\Pi:\mathbb{D}^2_{\pm\beta}H(\Omega\times(0,2\pi))^*\to l^2$, respectively. These operators play the essential role when the questions of solvability of the time-periodic Stokes problem (2.0.1) and the question of uniqueness of the solution are considered.

Theorem 2.4.2. *Assume that the time-periodic functions* $\mathbf{f}=\mathbf{f}(x,t)$ *and* $h^j=h^j(t), j=1,\dots,J$, *are sufficiently smooth (see details in the proof). Assume also that Green's formula* (2.4.17) *is valid.*

(i) If the homogeneous formally adjoint problem (2.4.15) *with the homogeneous asymptotic conditions at infinity*

$$\mathbb{Q}\Pi\mathbf{U}=\mathbf{0} \tag{2.4.19}$$

has the trivial solution only, then there exists a unique solution $\mathbf{v}=\mathbf{v}(x,t)$, $p=p(x,t)$ *from* $\mathbb{D}^2_{\pm\beta}H(\Omega\times(0,2\pi))$ *of problem* (2.0.1), (2.4.11).

(ii) Assume that the homogeneous problem (2.4.15), (2.4.19) *has non-trivial solutions* $\mathbf{U}(x,t)=(\mathbf{V}(x,t),\ P(x,t))$. *Then the problem* (2.0.1), (2.4.11) *has a solution* $\mathbf{v}=\mathbf{v}(x,t)$, $p=p(x,t)$ *in the space* $\mathbb{D}^2_{\pm\beta}H(\Omega\times(0,2\pi))$ *if and only if the compatibility condition*

$$\int_0^{2\pi}\int_\Omega \mathbf{f}(x,t)\cdot\mathbf{V}(x,t)dxdt+\langle\mathbf{h},\mathbb{T}\Pi\mathbf{U}\rangle_\infty=0 \tag{2.4.20}$$

is satisfied for all solutions $\mathbf{U}=\mathbf{U}(x,t)$ *of the homogeneous problem* (2.4.15), (2.4.19).

[vi]This is the case when the flow-rate or various pressure-type conditions are imposed, for example.

The couple $(\mathbf{v}, p)$ *in this case is not unique*[vii].

Proof. Part (i). Let us consider first the time-periodic Stokes problem $\mathbf{Su} = \mathbf{f}$ with zero flow-rates $\phi^1, \ldots, \phi^J$ and zero pressure part p_0^J, i.e., the problem (2.0.1) with the homogeneous conditions (2.2.8). According to Corollary 2.3.3, this problem has a unique solution $\widehat{\mathbf{u}} = \widehat{\mathbf{u}}(x,t)$ defined by formula (2.3.16). Notice that due to conditions $\phi^1 = \ldots = \phi^J = 0$ and $p_0^J = 0$, we have

$$\widehat{b}^j_{ck} = 0, \quad \widehat{b}^j_{sk} = 0, \quad \widehat{a}^J_{ck} = 0, \quad \widehat{a}^J_{sk} = 0, \tag{2.4.21}$$

for all $k = 0, 1, \ldots$ and for all $j = 1, \ldots, J$, while the coefficients $\{\widehat{a}^j_{ck}, \widehat{a}^j_{sk}\}_{k=0}^{\infty}$ of the pressure functions $p_0^j = p_0^j(t)$, $j = 1, \ldots, J-1$, are defined by formulas (2.3.8), (2.3.9):

$$\widehat{a}^j_{ck} = \int_\Omega (\mathbf{f}_{ck}\boldsymbol{\mathcal{V}}^{cj}_{ck} + \mathbf{f}_{sk}\boldsymbol{\mathcal{V}}^{cj}_{sk})dx, \quad \widehat{a}^j_{sk} = \int_\Omega (\mathbf{f}_{ck}\boldsymbol{\mathcal{V}}^{sj}_{ck} + \mathbf{f}_{sk}\boldsymbol{\mathcal{V}}^{sj}_{sk})dx. \tag{2.4.22}$$

Let us return now to problem (2.0.1), (2.4.11), i.e., to the Stokes system $\mathbf{Su} = \mathbf{f}$ with the conditions at infinity $\mathbb{B}\Pi\mathbf{u} = \mathbf{h}$. Using the substitution $\mathbf{u} = \widehat{\mathbf{u}} + \bar{\mathbf{u}}$ we reduce this problem to the homogeneous one:

$$\mathbf{S}\bar{\mathbf{u}} = \mathbf{0}, \quad \text{in} \quad \Omega \times (0, 2\pi), \qquad \bar{\mathbf{v}} = \mathbf{0}, \quad \text{on} \quad \partial\Omega \times (0, 2\pi),$$

with the following conditions at infinity

$$\mathbb{B}\Pi\bar{\mathbf{u}} = \bar{\mathbf{h}} \equiv \mathbf{h} - \mathbb{B}\Pi\widehat{\mathbf{u}}.$$

The last time-periodic problem is formally equivalent to the sequence of the Stokes-

[vii] We can illustrate this situation with the standard example. Assume that the flow-rates $\phi^j(t)$, $j = 1, \ldots, J$, of the solution $\mathbf{u} = \mathbf{u}(x,t)$ to problem (2.0.1) are given. The corresponding adjoint problem consists of the system (2.4.15) and the conditions at infinity $\mathbb{Q}\Pi\mathbf{U} = \mathbf{H}$, which are determined by the sequence of matrices $\mathbb{Q}_k = \begin{pmatrix}\mathbb{G}_k & \mathbb{O}\end{pmatrix}$, $k = 0, 1, \ldots$. Here $\mathbb{G}_k$ is defined by (1.3.12) (see footnote xv on the page 61). Taking into account the definition of $\mathbb{G}_k$, we notice that conditions $\mathbb{Q}\Pi\mathbf{U} = \mathbf{H}$ prescribe in every outlet the flow-rate generated by $\mathbf{U} = (\mathbf{V}, P)$. In this case the homogeneous adjoint problem has a family of non-trivial solutions admitting representation $(\mathbf{V}, P_0) = ((0,0,0), P_0(t))$, where $P_0 \in L^2(0, 2\pi)$ is any time-periodic function. Taking into account the structure of the operator $\mathbb{T}$ (see (2.4.16) and footnote xv) we conclude the well-known compatibility condition

$$\phi^1 + \cdots + \phi^J = 0.$$

Note that in this case the solution $(\mathbf{v}(x,t), p(x,t))$ to problem (2.0.1) is defined up to an additive time-dependent function in the pressure term, i.e., for an arbitrary $p_0 \in L^2(0, 2\pi)$ the couple $(\mathbf{v}(x,t), p(x,t) + p_0(t))$ is also a solution.

type problems

$$\mathbf{S}_k\bar{\mathbf{u}}_k = \mathbf{0}, \quad \text{in} \quad \Omega, \quad \bar{\mathbf{v}}_k = \mathbf{0}, \quad \text{on} \quad \partial\Omega, \quad \mathbb{B}_k\pi\bar{\mathbf{u}}_k = \bar{\mathbf{h}}_k, \quad k = 0, 1, \ldots. \tag{2.4.23}$$

We have assumed that the homogeneous adjoint Stokes problem (2.4.15) with conditions (2.4.19) has the solution $\mathbf{U}(x,t) \equiv 0$ only. Therefore each of the following homogeneous adjoint problems

$$\mathbf{S}_k^*\mathbf{U}_k = \mathbf{0}, \quad \text{in} \quad \Omega, \qquad \mathbf{V}_k = \mathbf{0}, \quad \text{on} \quad \partial\Omega, \qquad \mathbb{Q}_k\pi\mathbf{U}_k = \mathbf{0}, \quad k = 0, 1, \ldots,$$

also has only the trivial solution. Then, according to the Part (2) in Theorem 1.6.1, problem $(2.4.23)_k$ has a unique solution $\mathbf{u}_k$ for every right-hand side and all $k = 0, 1, \ldots$. The vector-field $\mathbf{u}_k$ is a linear combination

$$\bar{\mathbf{u}}_k = \xi_k^1\boldsymbol{u}_k^1 + \cdots + \xi_k^{2J}\boldsymbol{u}_k^{2J}$$

of the elements (1.4.5) forming the basis in the set of solutions to the homogeneous Stokes like problem (1.1.2). This linear combination satisfies the homogeneous equations and boundary conditions in (2.4.23). Substituting the above representation of $\bar{\mathbf{u}}_k$ into $\mathbb{B}_k\pi\bar{\mathbf{u}}_k = \widehat{\mathbf{h}}_k$ and taking into account the matrices $\mathcal{A}_k$ and $\mathcal{B}_k$ (see (1.4.20)), generated by the basis elements $\boldsymbol{u}_k^1, \ldots, \boldsymbol{u}_k^{2J}$, we get the following system of $2J$ linear equations for the column-vector $\boldsymbol{\xi}_k = (\xi_k^1, \ldots, \xi_k^{2J})$:

$$\mathbb{B}_k \begin{pmatrix} \mathcal{B}_k \\ \mathcal{A}_k \end{pmatrix} \boldsymbol{\xi}_k = \bar{\mathbf{h}}_k.$$

Since the solution $\bar{\mathbf{u}}_k$ is unique, this system also has a unique solution $\boldsymbol{\xi}_k$, which is expressed as

$$\boldsymbol{\xi}_k = \left(\mathbb{B}_k \begin{pmatrix} \mathcal{B}_k \\ \mathcal{A}_k \end{pmatrix}\right)^{-1} (\mathbf{h}_k - \mathbb{B}_k\pi\widehat{\mathbf{u}}_k). \tag{2.4.24}$$

The vector-fields $\mathbf{u}_k = \widehat{\mathbf{u}}_k + \bar{\mathbf{u}}_k$, $k = 0, 1, \ldots$, define series (2.1.2), which formally satisfy the system of equations (2.0.1) and conditions (2.4.11). As it was shown in the beginning of this Section, the convergence of series (2.1.2) to the solution $\mathbf{v} = \mathbf{v}(x,t)$, $p = p(x,t)$ from $\mathbb{D}^2_{\pm\beta}H(\Omega \times (0, 2\pi))$ depends on the behaviour of the coefficients $b^j_{ck}, b^j_{sk}, a^j_{ck}, a^j_{sk}$ in the asymptotic representation (1.1.22) of $\mathbf{u}_k$. Namely, if the sequences

$$a^j_{c0}, a^j_{s0}, a^j_{c1}, a^j_{s1}, \ldots, \quad b^j_{c0}, b^j_{s0}, b^j_{c1}, b^j_{s1}, \ldots \tag{2.4.25}$$

for every $j = 1, \dots, J$, belong to the space l^2, then the functions defined by (2.1.6), (2.1.8) satisfy the inclusions

$$p_0^1, \dots, p_0^J,\ q^1, \dots, q^J \in L^2(0, 2\pi). \tag{2.4.26}$$

This condition guarantees that the corresponding flow-rate functions $\phi^1, \dots, \phi^J$ belong to the space $H^1(0, 2\pi)$ (see Corollary 2.2.1) and, according to Corollary 2.3.3, ensure existence of the solution $\mathbf{u} \in \mathbb{D}^2_{\pm\beta} H(\Omega \times (0, 2\pi))$.

Let us show that the sequences (2.4.25) belong to the space l^2. Since $\mathbf{u}_k = \hat{\mathbf{u}}_k + \bar{\mathbf{u}}_k$, the Fourier coefficients (2.4.25) of functions in (2.4.26) are equal to

$$b^j_{ck} = \widehat{b}^j_{ck} + \bar{b}^j_{ck}, \quad b^j_{sk} = \widehat{b}^j_{sk} + \bar{b}^j_{sk}, \quad a^j_{ck} = \widehat{a}^j_{ck} + \bar{a}^j_{ck}, \quad a^j_{sk} = \widehat{a}^j_{sk} + \bar{a}^j_{sk}. \tag{2.4.27}$$

The constants on the right-hand sides in (2.4.27) depend on the data of problem (2.0.1), (2.4.11). Indeed, the coefficients $\widehat{b}^j_{ck}, \widehat{b}^j_{sk}, \widehat{a}^j_{ck}, \widehat{a}^j_{sk}$ are either equal to zero (see (2.4.21)) or are expressed in terms of the Fourier coefficients $\mathbf{f}_{ck}$, $\mathbf{f}_{sk}$ and the special solutions $\boldsymbol{\mathcal{U}}^{cj}_k, \boldsymbol{\mathcal{U}}^{sj}_k \in \mathcal{D}^2_{-\beta} H(\Omega)^*$ for the homogeneous formally adjoint Stokes-type problem (see (2.4.22)). As it was shown in Section 3.3, the assumption $\mathbf{f} \in L^2(0, 2\pi; L^2_\beta(\Omega))$ yields the inclusions $\{\widehat{a}^j_{ck}\}_{k=0}^\infty$, $\{\widehat{a}^j_{sk}\}_{k=0}^\infty \in l^2$ for all $j = 1, \dots, J-1$.

Let us recall that $\bar{\mathbf{u}}_k = \xi^1_k \boldsymbol{u}^1_k + \cdots + \xi^{2J}_k \boldsymbol{u}^{2J}_k$ and that constants $\{\bar{b}^j_{ck}, \bar{a}^j_{ck}, \bar{a}^j_{ck}, \bar{a}^j_{sk}\}_{j=1}^J$ are defined by the relation

$$\pi \bar{\mathbf{u}}_k = \begin{pmatrix} \mathcal{B}_k \\ \mathcal{A}_k \end{pmatrix} \boldsymbol{\xi}_k.$$

According to formulas (2.4.22) and (2.4.24), the vector $\boldsymbol{\xi}_k$ depends on the matrices $\mathbb{B}_k$, $\mathcal{A}_k$, $\mathcal{B}_k$ and on the Fourier coefficients $\mathbf{h}_k$ and $\mathbf{f}_{ck}$, $\mathbf{f}_{sk}$. The matrices $\mathbb{B}_k$, $\mathcal{A}_k$, $\mathcal{B}_k$ depend on k, therefore elements of the inverse matrix $\left(\mathbb{B}_k \begin{pmatrix} \mathcal{B}_k \\ \mathcal{A}_k \end{pmatrix} \right)^{-1}$ may grow[viii] as $k \to \infty$. However, assuming that the sequence $\mathbf{h}$ decays fast enough and, if necessary, assuming the higher regularity of the function $\mathbf{f}$, we may achieve the inclusions

$$\{\bar{b}^j_{ck}\}_{k=0}^\infty, \quad \{\bar{b}^j_{sk}\}_{k=0}^\infty, \quad \{\bar{a}^j_{ck}\}_{k=0}^\infty, \quad \{\bar{a}^j_{sk}\}_{k=0}^\infty \in l^2, \quad j = 1, \dots, J.$$

We notice that the regularity of $\mathbf{f}$ and $\mathbf{h}$, necessary for the last inclusions, depend on

[viii] We restrict ourself to the polynomial growth of the elements in this inverse matrix. Various physically sensible conditions at infinity possess this property. For example, in the case of prescribed flow-rates and pressures, the elements of the corresponding inverse matrices $\left(\mathbb{B}_k \begin{pmatrix} \mathcal{B}_k \\ \mathcal{A}_k \end{pmatrix} \right)^{-1}$, $k = 0, 1, \dots$, are either bounded or are quantities of order $O(k)$, as $k \to \infty$.

the structure of the operator $\mathbb{B}$.

Part (ii). Assume that problem (2.0.1), (2.4.11) has a time-periodic solution $\mathbf{u} \in \mathbb{D}^2_{\pm\beta} H(\Omega \times (0, 2\pi))$. Then the Fourier coefficients $\mathbf{u}_0, \mathbf{u}_1, \ldots$ satisfy the Stokes-type problems (2.4.10). Consequently (see the Part (2) in Theorem 3.2), the compatibility conditions

$$\int\limits_\Omega \mathbf{f}_k \cdot \mathbf{U}_k dx + \langle \mathbf{h}_k, \mathbb{T}_k \pi \mathbf{U}_k \rangle_{2J} = 0, \quad k = 0, 1, \ldots \tag{2.4.28}$$

are satisfied for all solutions $\mathbf{U}_k = (\mathbf{V}_{ck}, P_{ck}, \mathbf{V}_{sk}, P_{sk})$ of the homogeneous adjoint problem

$$\mathbf{S}^*_k \mathbf{U}_k = \mathbf{0}, \quad x \in \Omega, \quad \mathbf{V}_k = \mathbf{0}, \quad x \in \partial\Omega, \quad \mathbb{Q}_k \pi \mathbf{U}_k = \mathbf{0}, \quad k = 0, 1, \ldots. \tag{2.4.29}$$

Any time-periodic function $\mathbf{V} = \mathbf{V}(x,t)$, $P = P(x,t)$ satisfies the homogeneous problem (2.4.15), (2.4.19) if and only if it's Fourier's coefficients are solutions to problems (2.4.29). Therefore conditions (2.4.28) imply condition (2.4.20).

Let us show that (2.4.20) is a sufficient condition. Consider a time-periodic solution $\mathbf{U}(x,t) = (\mathbf{V}(x,t), P(x,t))$ of the homogeneous adjoint problem (2.4.15), (2.4.19) with the Fourier coefficients $\{\mathbf{V}_{ck}, P_{ck}, \mathbf{V}_{sk}, P_{sk}\}_{k=0}^\infty$. For every $k = 0, 1, \ldots$, the pair of functions

$$\mathbf{V}_k(x,t) = \mathbf{V}_{ck}(x) \cos kt + \mathbf{V}_{sk}(x) \sin kt, \quad P_k(x,t) = P_{ck}(x) \cos kt + P_{sk}(x) \sin kt,$$

is also a solution to the homogeneous problem (2.4.15), (2.4.19). It is obvious that the pair $(\mathbf{V}_k(x,t), P_k(x,t))$ solves the homogeneous problem (2.4.15), (2.4.19) if and only if the Fourier coefficients $(\mathbf{V}_{ck}(x), P_{ck}(x), \mathbf{V}_{sk}(x), P_{sk}(x))$ solve the adjoint Stokes-type problem (2.4.29). Consequently the condition (2.4.20) implies conditions (2.4.28). Applying Part (2) in Theorem 1.6.1, we conclude that, for every $k = 0, 1, \ldots$, there exist at least one solution $\mathbf{u}_k \in \mathbb{D}^2_{\pm\beta} H(\Omega)$ to the Stokes-type problem (2.4.10). Then the sequence $\mathbf{u}_0, \mathbf{u}_1, \ldots$ defines at least one "formal" solution (2.1.2) of problem (2.0.1), (2.4.11). If the data of this problem is sufficiently smooth, we may repeat arguments of the Part (i) and show that the series (2.1.2) converge in the space $\mathbb{D}^2_{\pm\beta} H(\Omega \times (0, 2\pi))$ to the solution described in Corollary 2.3.3. $\square$

2.5 Other versions of Green's formula and corresponding conditions at infinity.

Relation (2.4.11) combines conditions corresponding to different cylinders into a single equation $\mathbb{B}\Pi\mathbf{u} = \mathbf{h}$. However in some cases it is more convenient to use conditions separately. For this purpose, we define the projectors $\pi^j : \mathbb{D}^2_{\pm\beta}H(\Omega) \to \mathbb{R}^4$ and $\Pi^j : \mathbb{D}^2_{\pm\beta}H(\Omega \times (0, 2\pi)) \to l^2$, $j = 1, \dots J$, as follows:

$$\pi^j \mathbf{u}_k = (b^j_{ck}, b^j_{sk}, a^j_{ck}, a^j_{sk}), \quad \Pi^j \mathbf{u} = (\pi^j \mathbf{u}_0, \pi^j \mathbf{u}_1, \dots).$$

Assume that, for all $j = 1, \dots, J$ and $k = 0, 1, \dots$, the 2×4 matrices $\mathbb{B}^j_k$, $\mathbb{T}^j_k$, $\mathbb{S}^j_k$ and $\mathbb{Q}^j_k$ satisfy the condition (analogous to (1.3.13))

$$\begin{pmatrix} -\mathbb{T}^j_k \\ \mathbb{Q}^j_k \end{pmatrix}^T \begin{pmatrix} \mathbb{B}^j_k \\ \mathbb{S}^j_k \end{pmatrix} = \begin{pmatrix} \mathbb{O} & \mathbb{G}^j_k \\ -\mathbb{F}^j_k & \mathbb{O} \end{pmatrix},$$

where

$$\mathbb{F}^j_k = \begin{pmatrix} c^j_k & -d^j_k \\ d^j_k & c^j_k \end{pmatrix}, \quad \mathbb{G}^j_k = \begin{pmatrix} c^j_k & -d^j_k \\ -d^j_k & -c^j_k \end{pmatrix}.$$

Let $\mathbb{B}^j$, $\mathbb{T}^j$, $\mathbb{S}^j$, $\mathbb{Q}^j$, $j = 1, \dots, J$, denote the operators

$$\mathbb{B}^j = \mathrm{diag}\,(\mathbb{B}^j_0, \mathbb{B}^j_1, \dots), \quad \mathbb{S}^j = \mathrm{diag}\,(\mathbb{S}^j_0, \mathbb{S}^j_1, \dots),$$
$$\mathbb{T}^j = \mathrm{diag}\,(\mathbb{T}^j_0, \mathbb{T}^j_1, \dots), \quad \mathbb{Q}^j = \mathrm{diag}\,(\mathbb{Q}^j_0, \mathbb{Q}^j_1, \dots).$$

Then the Green formula (2.4.17) may be rewritten as follows:

$$\begin{aligned} &\int_0^{2\pi} \int_\Omega \mathbf{S}\mathbf{u}(x,t) \cdot \mathbf{U}(x,t)dxdt + \sum_{j=1}^J \langle \mathbb{B}^j \Pi^j \mathbf{u},\, \mathbb{T}^j \Pi^j \mathbf{U} \rangle_\infty \\ = &\int_0^{2\pi} \int_\Omega \mathbf{u}(x,t) \cdot \mathbf{S}^* \mathbf{U}(x,t)dxdt + \sum_{j=1}^J \langle \mathbb{S}^j \Pi^j \mathbf{u},\, \mathbb{Q}^j \Pi^j \mathbf{U} \rangle_\infty. \end{aligned} \tag{2.5.1}$$

This generalized Green's formula supplies the time-periodic Stokes problem (2.0.1) with the set of conditions imposed for every outlet separately:

$$\mathbb{B}^j \Pi^j \mathbf{u} = \mathbf{h}^j, \quad j = 1, \dots, J. \tag{2.5.2}$$

The corresponding changes should be made in Theorem 2.4.2: formula (2.4.19) shall be substituted by the equations

$$\mathbb{Q}^j \Pi^j \mathbf{U} = \mathbf{0}, \quad j = 1, \ldots, J,$$

while the compatibility condition (2.4.20) takes the form

$$\int_0^{2\pi} \int_\Omega \mathbf{f}(x,t) \cdot \mathbf{V}(x,t) dx dt + \sum_{j=1}^{J} \langle \mathbf{h}^j, \mathbb{T}^j \Pi^j \mathbf{U} \rangle_\infty = 0. \tag{2.5.3}$$

Remark 2.5.1. Conditions (2.5.2) are formulated in every outlet separately. However, formula (2.5.3) provides the global compatibility condition including the given data that correspond to all outlets at infinity.

Let us express the conditions at infinity (2.5.2) in another form. In (2.5.2) the sequence $\Pi^j \mathbf{u} = (b^j_{c0}, b^j_{s0}, a^j_{c0}, a^j_{s0}, b^j_{c1}, b^j_{s1},\ a^j_{c1}, a^j_{s1}, \ldots) \in l^2$, which is composed from the coefficients of a function $\mathbf{u} \in \mathbb{D}^2_{\pm\beta} H(\Omega \times (0, 2\pi))$, is mapped to the sequence $\mathbf{h}^j = (h^j_{c0}, h^j_{s0}, h^j_{c1}, h^j_{s1}, \ldots)$. This sequence may be treated as the sequence of the Fourier coefficients of some time-periodic function $h^j = h^j(t)$. In general, relations (2.5.2) define the operators

$$B^j : \mathbb{D}^l_{\pm\beta} H(\Omega \times (0, 2\pi)) \to H^m(0, 2\pi), \quad j = 1, \ldots, J$$

(the value of the integer m depends on the choice of the operator $\mathbb{B}^j$). Therefore, instead of conditions at infinity (2.5.2), formulated in terms of Fourier coefficients, we may impose for the solution of problem (2.0.1) conditions similar to boundary conditions for non-stationary problems, i.e., the set of relations

$$[B^j \mathbf{u}](t) = h^j(t), \quad j = 1, \ldots, J. \tag{2.5.4}$$

We emphasize that these conditions are formulated for the function $\mathbf{u} = \mathbf{u}(x,t)$ itself, while (2.5.2) are formulated for the Fourier coefficients of $\mathbf{u}$. Recall that formulas (2.2.8), (2.3.2) impose the flow-rates and pressure conditions of type (2.5.4), while the same conditions in terms of Fourier's coefficients were formulated in Example 1 (see the page 81).

Assume that operators

$$\begin{aligned} S^j &: \mathbb{D}^l_{\pm\beta} H(\Omega \times (0,2\pi)) \to H^m(0,2\pi), \quad j = 1,\dots,J, \\ T^j &: \mathbb{D}^l_{\pm\beta} H(\Omega \times (0,2\pi))^* \to H^m(0,2\pi), \quad j = 1,\dots,J, \\ Q^j &: \mathbb{D}^l_{\pm\beta} H(\Omega \times (0,2\pi))^* \to H^m(0,2\pi), \quad j = 1,\dots,J \end{aligned}$$

are defined in the same way as B^j, i.e., instead of using the relations $\mathbb{S}^j \Pi^j \mathbf{u} = \mathbf{h}$, $\mathbb{T}^j \Pi^j \mathbf{U} = \mathbf{H}$, $\mathbb{Q}^j \Pi^j \mathbf{U} = \mathbf{H}$ between the Fourier coefficients, we define the corresponding operators "mapping the functions $\mathbf{u}$ and $\mathbf{U}$ to the time-periodic functions from $H^m(0,2\pi)$ with some integer m". Then we may consider the time-periodic Stokes problem (2.0.1) supplied with conditions (2.5.4) instead of (2.5.2). In this case condition (2.5.3) shall be substituted by the following compatibility condition

$$\int_0^{2\pi} \int_\Omega \mathbf{f}(x,t) \cdot \mathbf{V}(x,t) dx dt + \sum_{j=1}^J \int_0^{2\pi} h^j(t) [T^j \mathbf{U}](t) dt = 0. \tag{2.5.5}$$

2.6 Examples

According to material of this Chapter, a time-periodic solution $\mathbf{u} \in \mathbb{D}^2_{\pm\beta} H(\Omega \times (0,2\pi))$ of problem (2.0.1) may be determined uniquely if one imposes correct asymptotic conditions at infinity. In Section 2.4 these conditions were formulated for the sequence of Fourier coefficients $\{\mathbf{u}_k\}_{k=0}^\infty$ (see (2.4.11)). In this case one shall solve for each $k = 0, 1, \dots$ a system of linear equations $\mathbb{B}_k \pi \mathbf{u}_k = \mathbf{h}_k$ (see Example 1 on the page 80). However, in some cases the systems of equations may be different for each k and, for example, showing that every system is solvable or proving that the series $\sum_k (\mathbf{u}_{ck} \cos kt + \mathbf{u}_{sk} \sin kt)$ converges may be difficult. Therefore sometimes it is more convenient to impose conditions of type (2.5.4).

Let us recall that the time-periodic solution $(\mathbf{v}, p) \in \mathbb{D}^2_{\pm\beta} H(\Omega \times (0,2\pi))$ is fully determined if we know the functions $q^j = q^j(t)$ and $p_0^j = p_0^j(t)$, $j = 1,\dots,J$, in the representation (2.4.1). Unfortunately, one cannot select in (2.4.1) arbitrary functions $q^1, \dots, q^J$ and $p_0^1, \dots, p_0^J$. For example, due to incompressibility of the fluid, the sum $\phi^1 + \dots + \phi^J$ shall vanish. We know that the Fourier coefficients of the flow-rate $\phi^j = \phi^j(t)$ and the pressure drop $q^j = q^j(t)$ are related by equations (2.2.4), i.e., for every pressure drop $q^j \in L^2(0,2\pi)$ there exists the flow rate $\phi^j \in H^1(0,2\pi)$ and vice versa (see Corollary 2.2.1). In other words there exists a bounded linear operator $\mathcal{F}^j : L^2(0,2\pi) \to H^1(0,2\pi)$ with a bounded inverse $(\mathcal{F}^j)^{-1} : H^1(0,2\pi) \to L^2(0,2\pi)$ such that $\phi^j = \mathcal{F}^j q^j$ and $q^j = (\mathcal{F}^j)^{-1} \phi^j$. Having this in mind, the zero total flow-rate condition may be rewritten in terms of functions q^j, $j = 1, \dots, J$:

$$\mathcal{F}^1 q^1(t) + \cdots \mathcal{F}^J q^J(t) = 0, \quad \forall t \in (0, 2\pi), \tag{2.6.1}$$

Another type restrictions arise for the functions $p_0^1, \dots, p_0^J$. The differences of the Fourier coefficients of these functions are determined by formulas (2.3.8), (2.3.9). These relations yield the following pressure jump conditions

$$p_0^i(t) - p_0^J(t) = \eta^i(t) + \mathcal{G}^{i,1} q^1(t) + \dots \mathcal{G}^{i,J} q^J(t), \quad i = 1, \dots, J-1. \tag{2.6.2}$$

Here the functions $\eta^i = \eta^i(t)$, $i = 1, \dots, J-1$, depend on the external force $\mathbf{f}$ and the domain Ω only. In the case $\mathbf{f} \in L^2(0, 2\pi; L^2_\beta(\Omega))$, they satisfy inclusions $\eta^1, \dots, \eta^{J-1} \in L^2(0, 2\pi)$. The operators $\mathcal{G}^{i,j} : L^2(0, 2\pi) \to L^2(0, 2\pi)$, $i = 1, \dots, J-1$, $j = 1, \dots, J$, are defined by (2.3.12) and are bounded according to Remark 2.3.2.

Below we investigate a couple of situations when mixed – flow-rate and pressure type – conditions at infinity are imposed.

Example 2

Let us consider the domain Ω with three outlets to infinity. Suppose that we can measure a time-periodic flow-rate $h^1 \in H^1(0, 2\pi)$ through the cross-section of the cylinder Ω_+^1 and values h^2, $h^3 \in L^2(0, 2\pi)$ of the pressure $p = p(x, t)$ at the distances $x_3^2 = R_2$ and $x_3^3 = R_3$ in cylinders Ω_+^2 and Ω_+^3. This situation is described by the conditions

$$\begin{aligned} \mathcal{F}^1 q^1(t) &= h^1(t), \\ q^2(t) R_2 + p_0^2(t) &= h^2(t), \\ q^3(t) R_3 + p_0^3(t) &= h^3(t). \end{aligned} \tag{2.6.3}$$

We notice from (2.4.1) that for large R_j the pressure $p(x, t)$ in the outlet Ω_+^j differs from the function $q^j(t) R_j + p_0^j(t)$ by the quantity of order $o(e^{-\beta R_j})$.

Straightforward computations show that system (2.6.3) supplemented with compatibility conditions (2.6.1) and (2.6.2) (for $J = 3$) has a unique solution $\{q^j, p_0^j\}_{j=1}^3 \in L^2(0, 2\pi)$ if the distances R_2 and R_3 are sufficiently large. Indeed, from the equations (2.6.3) and the compatibility condition (2.6.1) we immediately obtain $q^1 = (\mathcal{F}^1)^{-1} h^1 \in L^2(0, 2\pi)$ and the relations

$$\begin{aligned} q^2 &= -\left(\mathcal{F}^2\right)^{-1} h^1 - \left(\mathcal{F}^2\right)^{-1} \mathcal{F}^3 q^3, \\ p_0^2 = h^2 + R_2 \left(\mathcal{F}^2\right)^{-1} h^1 &+ R_2 \left(\mathcal{F}^2\right)^{-1} \mathcal{F}^3 q^3, \\ p_0^3 &= h^3 - R_3 q^3. \end{aligned} \tag{2.6.4}$$

Substituting these relations to (2.6.2) (with $i = 2$), we get the equation for the function q^3:

$$\begin{aligned} &\left(R_2 \left(\mathcal{F}^2\right)^{-1} \mathcal{F}^3 + R_3\right) q^3 + \left(\mathcal{G}^{2,2} \left(\mathcal{F}^2\right)^{-1} \mathcal{F}^3 - \mathcal{G}^{2,3}\right) q^3 \\ &= \eta^2 + \left(\mathcal{G}^{2,1} \left(\mathcal{F}^1\right)^{-1} - \mathcal{G}^{2,2} \left(\mathcal{F}^2\right)^{-1} - R_2 \left(\mathcal{F}^2\right)^{-1}\right) h^1 - h^2 + h^3. \end{aligned} \tag{2.6.5}$$

Let us denote

$$\begin{aligned} \mathcal{H}^1 = R_2 \left(\mathcal{F}^2\right)^{-1} \mathcal{F}^3 + R_3, \quad \mathcal{H}^2 &= \mathcal{G}^{2,2} \left(\mathcal{F}^2\right)^{-1} \mathcal{F}^3 - \mathcal{G}^{2,3}, \\ \mathcal{H}^3 = \mathcal{G}^{2,1} \left(\mathcal{F}^1\right)^{-1} &- \mathcal{G}^{2,2} \left(\mathcal{F}^2\right)^{-1} - R_2 \left(\mathcal{F}^2\right)^{-1}. \end{aligned}$$

Taking into account boundedness of $\mathcal{F}^j$, $(\mathcal{F}^j)^{-1}$ and $\mathcal{G}^{i,j}$, we see that the operators $\mathcal{H}_1, \mathcal{H}_2 : L^2(0, 2\pi) \to L^2(0, 2\pi)$ and $\mathcal{H}_3 : H^1(0, 2\pi) \to L^2(0, 2\pi)$ are bounded.

We recall that for every $j = 1, \ldots, J$, the operator $\mathcal{F}^j$ and its inverse $(\mathcal{F}^j)^{-1}$ are defined as infinite matrices acting on the sequence of Fourier coefficients of functions from $L^2(0, 2\pi)$, i.e.,

$$\mathcal{F}^j = \operatorname{diag}(\mathcal{F}^j_0, \mathcal{F}^j_1, \ldots), \quad \left(\mathcal{F}^j\right)^{-1} = \operatorname{diag}\left(\left(\mathcal{F}^j_0\right)^{-1}, \left(\mathcal{F}^j_1\right)^{-1}, \ldots\right),$$

where

$$\mathcal{F}^j_k = \begin{pmatrix} c^j_k & -d^j_k \\ d^j_k & c^j_k \end{pmatrix}, \quad \left(\mathcal{F}^j_k\right)^{-1} = \frac{1}{\left(c^j_k\right)^2 + \left(d^j_k\right)^2} \begin{pmatrix} c^j_k & d^j_k \\ -d^j_k & c^j_k \end{pmatrix}.$$

Straightforward computations show that the matrices $R_2 \left(\mathcal{F}^2_k\right)^{-1} \mathcal{F}^3_k + R_3 \mathbb{I}$ are nonsingular for all $k = 0, 1, \ldots$ and all positive R_2, R_3. Moreover, it is easy to verify that the inverse of the operator $\mathcal{H}_1$ is bounded. Therefore, taking sufficiently large R_2 and R_3 we get the linear operator $\mathcal{H}_1 + \mathcal{H}_2$ with a bounded inverse (for "large" R_2 and R_3 the operator $\mathcal{H}_1 + \mathcal{H}_2$ is a "small" perturbation of $\mathcal{H}_1$). As a consequence, we obtain from equation (2.6.5) the function $q^3 \in L^2(0, 2\pi)$. Substituting this function into formulas (2.6.4) and (2.6.2) (for $i = 1$) we restore the functions $p^1_0, p^2_0, p^3_0, q^2 \in$

$L^2(0, 2\pi)$. As it was explained in the beginning of the Section, the set $\{p_0^j, q^j\}_{j=1}^3 \in L^2(0, 2\pi)$ determines the unique time-periodic solution $(\mathbf{v}, p) \in \mathbb{D}^2_{\pm\beta} H(\Omega \times (0, 2\pi))$.

Example 3

Let us consider a time-periodic flow in the domain Ω with four cylindrical outlets. Assume that we know the time periodic flow-rate $h^1 \in H^1(0, 2\pi)$ in the cylinder Ω^1_+ and the total pressure $h^2 \in L^2(0, 2\pi)$ at the distance $x_3^2 = R_2$ in the cylinder Ω^2_+. These conditions are expressed by the equations

$$\mathcal{F}^1 q^1(t) = h^1(t), \quad q^2(t) R_2 + p_0^2(t) = h^2(t), \quad \forall t \in (0, 2\pi). \tag{2.6.6}$$

Moreover, assume that the outlets Ω^3_+ and Ω^4_+ have the same length R, are parallel and connected at their end (see Figure 3.1). In this case the sum of the flow-rates in these pipes should vanish, and pressures at their end should coincide. We can express this by the equations

$$\mathcal{F}^3 q^3(t) + \mathcal{F}^4 q^4(t) = 0, \quad q^3(t) R + p_0^3(t) = q^4(t) R + p_0^4(t), \quad \forall t \in (0, 2\pi). \tag{2.6.7}$$

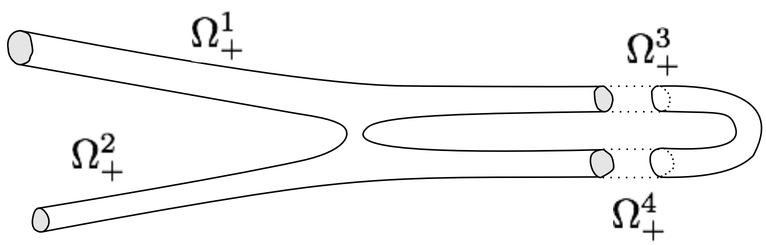

Figure 2.1: Domain Ω with two connected outlets.

Using the similar procedure as in Example 2, we get from relations (2.6.1), (2.6.2), (2.6.6) and (2.6.7) the equation for the function q^3:

$$\begin{aligned} &\left(R \left(1 + \left(\mathcal{F}^4\right)^{-1} \mathcal{F}^3 \right) + 2\mathcal{G}^{3,2} \left(\mathcal{F}^2\right)^{-1} \mathcal{F}^3 - \mathcal{G}^{3,4} \left(\mathcal{F}^4\right)^{-1} \mathcal{F}^3 - \mathcal{G}^{3,3} \right) q^3 \\ &\qquad = \eta^3 + \left(\mathcal{G}^{3,1} \left(\mathcal{F}^1\right)^{-1} - \mathcal{G}^{3,2} \left(\mathcal{F}^3\right)^{-1} \right) h^1. \end{aligned} \tag{2.6.8}$$

Let us denote

$$\mathcal{H}_1 = 1 + \left(\mathcal{F}^4\right)^{-1} \mathcal{F}^3,$$

$$\mathcal{H}_2 = 2\mathcal{G}^{3,2} \left(\mathcal{F}^2\right)^{-1} \mathcal{F}^3 - \mathcal{G}^{3,4} \left(\mathcal{F}^4\right)^{-1} \mathcal{F}^3 - \mathcal{G}^{3,3},$$

$$\mathcal{H}_3 = \mathcal{G}^{3,1} \left(\mathcal{F}^1\right)^{-1} - \mathcal{G}^{3,2} \left(\mathcal{F}^3\right)^{-1}.$$

It is obvious that the mappings $\mathcal{H}_1, \mathcal{H}_2 : L^2(0, 2\pi) \to L^2(0, 2\pi)$ and $\mathcal{H}_3 : H^1(0, 2\pi) \to L^2(0, 2\pi)$ are bounded.

One can straightforwardly verify that the matrices $\mathbb{I} + \left(\mathcal{F}_k^4\right)^{-1} \mathcal{F}_k^3$ are non-singular for all $k = 0, 1, \ldots$, and that the inverse of the operator $\mathcal{H}_1$ is bounded. Therefore, for sufficiently large R we get that the linear operator $R\mathcal{H}_1 + \mathcal{H}_2$ has a bounded inverse, i.e., from the equation (2.6.8) we obtain the function $q^3 \in L^2(0, 2\pi)$. Substituting this function to the first equation in (2.6.7), we find $q^4 \in L^2(0, 2\pi)$. Then the second relation in (2.6.7) gives the difference $p_0^3 - p_0^4 \in L^2(0, 2\pi)$, which, together with (2.6.6$_1$) and (2.6.2) allow to find $q^2 \in L^2(0, 2\pi)$ and, from (2.6.6$_2$) we find the function $p_0^2 \in L^2(0, 2\pi)$. Now we determine, from relations (2.6.2), the functions $p_0^1, p_0^3, p_0^4 \in L^2(0, 2\pi)$. As it was explained in the beginning of the Section, the set $\{p_0^j, q^j\}_{j=1}^4 \in L^2(0, 2\pi)$ determines the unique time-periodic solution $(\mathbf{v}, p) \in \mathbb{D}^2_{\pm\beta} H(\Omega \times (0, 2\pi))$.

Conclusions

The object of our investigations was the time-periodic Stokes system set in domains with cylindrical outlets to infinity. The aim of our research was to find a way how to select a unique solution having the infinite Dirichlet's integral, i.e., to find the methods of imposing the asymptotic conditions at infinity which ensure the existence and uniqueness of the solution. In order to achieve this goal we have reduced the time-periodic Stokes problem into a sequence of elliptic Stokes-type problems. Following the ideas proposed in [54] and [59], we studied these problems in the weighted Sobolev spaces $\mathbb{D}^l_{\pm\beta}H(\Omega)$, consisting the vector-fields with unbounded Dirichlet integrals. We have demonstrated that uniqueness of solutions from this class can be guaranteed by imposing the asymptotic conditions at infinity. We have shown that the correct asymptotic conditions may be formulated with the help of the generalized Green formula. In particular, we described a class of matrices which may be used to impose the flow-rate and the total pressure conditions.

Combining results obtained for the elliptic Stokes-type problems and the known results for the non-steady problems set in cylindrical domains, we have defined a set $\mathbb{D}^l_{\pm\beta}H(\Omega \times (0, 2\pi))$ consisting of time-periodic functions. These functions admit the special asymptotic representations and may have infinite Dirichlet integrals. For the time-periodic Stokes problem we have derived the generalized Green formula which is valid for functions from the class $\mathbb{D}^2_{\pm\beta}H(\Omega \times (0, 2\pi))$. It was shown that:

- the uniqueness of the time-periodic solution $(\mathbf{v}, p) \in \mathbb{D}^2_{\pm\beta}H(\Omega \times (0, 2\pi))$ can be achieved by imposing asymptotic conditions at infinity;
- general conditions at infinity may be obtained from the generalized Green formula.

Finally, we have presented several examples, when combination of the flow-rate condition with the prescribtion of the total pressure in one or several outlets yield the existence and uniqueness of the time-periodic solution in $\mathbb{D}^2_{\pm\beta}H(\Omega \times (0, 2\pi))$.

Appendix A

Consider the problem

$$\begin{cases} k\psi_k + \nu\Delta\varphi_k = -1, & y \in \omega, \\ k\varphi_k - \nu\Delta\psi_k = 0, & y \in \omega, \\ \varphi_k = 0, \;\; \psi_k = 0, & y \in \partial\omega. \end{cases} \tag{A.0.1}$$

Multiplying the homogeneous equation $(A.0.1_1)$ by φ_k and equation $(A.0.1_2)$ by ψ_k, subtracting the obtained relations and integrating by parts, we get

$$\int_\omega |\nabla\varphi_k|^2 + |\nabla\psi|^2 dy = 0.$$

This identity and the boundary conditions yield the uniqueness of the solution to problem (A.0.1). The uniqueness property and the Fredholm alternative for linear elliptic equations ensure the existence of the solution $(\varphi_k, \psi_k) \in \left(\mathring{H}^1(\omega)\right)^2$ (see [19]). Moreover, if the boundary is of class C^2, the solution (φ_k, ψ_k) belongs to $\left(H^2(\omega)\right)^2$.

Let us recall the definitions of the constants c_k^j and d_k^j (see (1.3.9)):

$$c_k^j = \int_{\omega^j} \varphi_k^j dy^j, \quad d_k^j = -\int_{\omega^j} \psi_k^j dy^j.$$

The following properties of the constants c_k^j, d_k^j and the solution of (A.0.1) were proved in Lemma 2.1 in [27].

Lemma A.0.1. *Let* $(\varphi_k^j, \psi_k^j) \in (H^1(\omega^j))^2$ *be the solution of problem* (A.0.1). *Then the following estimates*

$$\|\varphi_k^j\|^2_{L_2(\omega^j)} + \|\psi_k^j\|^2_{L_2(\omega^j)} \le \frac{d_k^j}{k}, \quad \|\Delta\varphi_k^j\|^2_{L_2(\omega^j)} + \|\Delta\psi_k^j\|^2_{L_2(\omega^j)} \le |\omega^j|^2$$

hold. Here $|\omega^j|$ *denotes the area of* ω^j. *Moreover,* *(1)* $0 < c_0^j \leq \frac{|\omega^j|^2}{2}, \quad d_0^j = 0$;

(2) $0 < c_k^j \leq \frac{|\omega^j|}{k}, \quad 0 < d_k^j \leq \frac{|\omega^j|}{k}$, *for all* $k = 1, 2, \ldots$;

(3) $\lim_{k\to\infty}(kd_k^j) = |\omega^j|$.

Appendix B

Consider the system of partial differential equations set in the domain $\Omega \subset \mathbb{R}^n$

$$\mathcal{L}\left(x, \nabla_x\right) \mathbf{u}(x) = \mathbf{f}(x). \tag{B.0.1}$$

where $\mathbf{u}$ and $\mathbf{f}$ are m-dimensional vector-fields, namely,

$$\mathbf{u}(x) = (u_1(x), \ldots, u_m(x)), \quad \mathbf{f}(x) = (f_1(x), \ldots, f_m(x)),$$

and $\mathcal{L}\left(x, \nabla_x\right)$ is an $m \times m$ matrix with elements $l_{ij}\left(x, \nabla_x\right)$, $i, j = 1, \ldots, m$, being the differential operators. System (B.0.1) is called elliptic in the sense of Agmon, Douglis, Nirenberg (see [2], [3]) if there exist integers s_i, t_i, $i = 1, \ldots, m$, such that:

(a) The degree of the operator $l_{ij}\left(x, \nabla_x\right)$ does not exceed $s_i + t_j$, and $l_{ij} = 0$ if $s_i + t_j < 0$.

(b) Let $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$ and $l^0_{ij}(x, \boldsymbol{\xi})$ be the polynomial in $\xi_1, \ldots, \xi_n$, composed from those terms of the polynomial $l_{ij}(x, \boldsymbol{\xi})$ which has the degree equal to $s_i + t_j$. Moreover, assume that $\mathcal{L}^0(x, \boldsymbol{\xi})$ is the matrix composed from the elements l^0_{ij}. Then

$$\det \mathcal{L}^0(x, \boldsymbol{\xi}) \neq 0 \quad \text{for all} \quad \boldsymbol{\xi} \neq \mathbf{0}.$$

ADN ellipticity of the steady-state Stokes system

$$\begin{cases} -\nu \Delta \mathbf{v} + \nabla p &= \mathbf{f}, \qquad x \in \Omega, \\ -\nabla \cdot \mathbf{v} &= 0, \qquad x \in \Omega, \end{cases}$$

was proved in [88]. Let us show that the Stokes-type problem (1.1.2) is also elliptic in this sense. Consider the case $\Omega \subset \mathbb{R}^2$. System (1.1.2) admits the following form

$$\left\{\begin{array}{rcll} -\nu\Delta(v_{ck})_1 + \dfrac{p_{ck}}{\partial x_1} + k(v_{sk})_1 & = & (f_{ck})_1, & x \in \Omega, \\ -\nu\Delta(v_{ck})_2 + \dfrac{p_{ck}}{\partial x_2} + k(v_{sk})_2 & = & (f_{ck})_2, & x \in \Omega, \\ -\dfrac{\partial(v_{ck})_1}{\partial x_1} - \dfrac{\partial(v_{ck})_2}{\partial x_2} & = & 0, & x \in \Omega, \\ -\nu\Delta(v_{sk})_1 + \dfrac{p_{sk}}{\partial x_1} - k(v_{ck})_1 & = & (f_{sk})_1, & x \in \Omega, \\ -\nu\Delta(v_{sk})_2 + \dfrac{p_{sk}}{\partial x_2} - k(v_{ck})_2 & = & (f_{sk})_2, & x \in \Omega, \\ -\dfrac{\partial(v_{sk})_1}{\partial x_1} - \dfrac{\partial(v_{sk})_2}{\partial x_2} & = & 0, & x \in \Omega, \end{array}\right.$$

where $\Delta = \dfrac{\partial}{\partial x_1^2} + \dfrac{\partial}{\partial x_2^2}$, $\mathbf{v}_{ck} = ((v_{ck})_1, (v_{ck})_2)$, $\mathbf{v}_{sk} = ((v_{sk})_1, (v_{sk})_2)$, $\mathbf{f}_{ck} = ((f_{ck})_1, (f_{ck})_2)$ and $\mathbf{f}_{sk} = ((f_{sk})_1, (f_{sk})_2)$. In this case $m = 6$ and $\boldsymbol{\xi} = (\xi_1, \xi_2)$. One may verify that condition (a) is satisfied by the numbers $s_1 = s_2 = 0$, $s_3 = -1$, $s_4 = s_5 = 0$, $s_6 = -1$ and $t_1 = t_2 = 2$, $t_3 = -1$, $t_4 = t_5 = 2$, $t_6 = -1$. Using these numbers and taking into account the structure of the equations, we compose the matrix:

$$\mathcal{L}^0(x, \boldsymbol{\xi}) = \begin{pmatrix} -\nu(\xi_1^2 + \xi_2^2) & 0 & -\xi_1 & 0 & 0 & 0 \\ 0 & -\nu(\xi_1^2 + \xi_2^2) & -\xi_2 & 0 & 0 & 0 \\ -\xi_1 & -\xi_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\nu(\xi_1^2 + \xi_2^2) & 0 & -\xi_1 \\ 0 & 0 & 0 & 0 & -\nu(\xi_1^2 + \xi_2^2) & -\xi_2 \\ 0 & 0 & 0 & -\xi_1 & -\xi_2 & 0 \end{pmatrix}.$$

Notice that degrees of the terms $k(v_{ck})_l$ and $k(v_{sk})_l$ are equal to 0, while the corresponding quantities $s_i + t_j = 2$, therefore the corresponding entries of the matrix $\mathcal{L}^0$ are zero. Straightforward computations yield

$$\det \mathcal{L}_0(x, \boldsymbol{\xi}) = \nu^2(\xi_1^2 + \xi_2^2)^4.$$

Obviously, the matrix $\mathcal{L}^0(x, \boldsymbol{\xi})$ is non-singular for $\boldsymbol{\xi} \neq \mathbf{0}$. In the same way it can be proved that the Stokes-type problem (1.1.2) is ADN elliptic in $3D$ (in this case $\det \mathcal{L}_0(x, \boldsymbol{\xi}) = \nu^2(\xi_1^2 + \xi_2^2 + \xi_3^2)^6$).

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