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On the Range of Arithmetic Means of the Fractional Parts of Harmonic Numbers

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Abstract: In this paper, the limit points of the sequence of arithmetic means $\frac{1}{n}\sum_{m=1}^{n}\{H_m\}^{\sigma}$ for $n = 1, 2, 3, \ldots$ are studied, where H_m is the mth harmonic number with fractional part $\{H_m\}$ and σ is a fixed positive constant. In particular, for $\sigma = 1$, it is shown that the largest limit point of the above sequence is $1/(e-1) = 0.581976\ldots$, its smallest limit point is $1 - \log(e-1) = 0.458675\ldots$, and all limit points form a closed interval between these two constants. A similar result holds for the sequence $\frac{1}{n}\sum_{m=1}^{n} f(\{H_m\})$, $n = 1, 2, 3, \ldots$, where $f(x) = x^{\sigma}$ is replaced by an arbitrary absolutely continuous function f in $[0, 1]$.

Keywords: harmonic number; fractional part; arithmetic mean; Euler's constant

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1. Introduction

Recall that the mth harmonic number is the sum of the reciprocals of the first m positive integers:

$$H_m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}.$$

Harmonic numbers appear frequently in many different areas, such as combinatorial problems, expressions involving special functions in analytic number theory, probability and statistics, analysis of algorithms, etc. Sometimes they appear unexpectedly [1], but they mainly can be found in many beautiful identities. For instance, in 1775, Euler proved the following identity:

$$\sum_{m=1}^{\infty} \frac{H_m}{m^2} = 2\zeta(3),$$

where ζ is the Riemann zeta function. See, e.g., [2,3], for a short proof of this and similar identities involving zeta functions, logarithms and polylogarithms. In [4], there are many identities of a different type, such as the following:

$$\sum_{m=1}^{n} (n-2m)\binom{n}{m} H_m = 1 - 2^n$$

for $n \in \mathbb{N}$. Other more complicated identities have been proven with the help of computers. See also [5–7].

It is well known that $H_1 = 1$ is the only integer among all the harmonic numbers H_m (see, e.g., Section 1.2.7 in [8]). Thus, the fractional parts of other harmonic numbers $\{H_m\}$, where $m \geq 2$, all belong to the open interval $(0, 1)$. Note that the mth harmonic number can be written in the form $H_m = u_m/D_m$, where D_m is the least common multiple of the integers $1, 2, \ldots, m$ and $u_m \in \mathbb{N}$. Here, u_m and D_m are not necessarily coprime. In [9], Wu and Chen conjectured that $\gcd(u_m, D_m) = 1$ for infinitely many $m \in \mathbb{N}$. This conjecture is still open despite some progress in [10], showing that $\gcd(u_m, D_m)$ cannot be too large for all m. In the opposite direction, the set of $m \in \mathbb{N}$ for which $\gcd(u_m, D_m) > 1$ has been recently studied by Yan and Wu [11].

In particular, from the representation $H_m = u_m/D_m$, even in the worst case, when there is no cancellation by the factor $\gcd(u_m, D_m) > 1$, it follows that $\{H_m\} = H_m - \lfloor H_m \rfloor \geq 1/D_m$ for each $m \geq 2$. By the prime number theorem, it is well known that $\log D_m \to m$ as $m \to \infty$. This gives the exponential bound $\{H_m\} > \kappa^m$ for each positive constant $\kappa < e^{-1} = 0.367879\ldots$ and each sufficiently large integer m. Calculations show that this bound is far from optimal. However, the question of whether this bound can be replaced by the bound $\{H_m\} \gg 1/m^2$ is completely open (see, e.g., Question 258097 at MathOverflow). One should also mention recent progress on the question of Erdős and Graham [12], who were interested in the question of how close the difference $H_\ell - H_m$ can be to 1. In [13], it was shown that for any $\varepsilon > 0$, there are infinitely many pairs of positive integers $\ell > m$ such that $|H_\ell - H_m - 1| < 1/m^2(\log m)^{5/4-\varepsilon}$.

Since $H_{m+1} - H_m = \frac{1}{m+1} \to 0$ as $m \to \infty$, the sequence of the fractional parts $\{H_m\}$, $m = 1, 2, 3, \ldots$, is everywhere dense in $[0, 1]$. However, as $H_m - \log m$ tends to a finite limit $\gamma = 0.577215\ldots$, which is called *Euler's constant*, and the sequence $\log m$, $m = 1, 2, 3, \ldots$, is not uniformly distributed modulo 1, the sequence of the fractional parts $\{H_m\}$, $m = 1, 2, 3, \ldots$, is not uniformly distributed in $[0, 1]$. For a sequence $a_m \in [0, 1)$, $m = 1, 2, 3, \ldots$, which is uniformly distributed in $[0, 1]$, one has the following:

$$\frac{1}{n}\sum_{m=1}^{n} a_m \to \frac{1}{2} \quad \text{as} \quad n \to \infty.$$

We do not have this property for $a_m = \{H_m\}$, so it seems a natural problem to investigate the limit points of the sequence of arithmetic means $\frac{1}{n}\sum_{m=1}^{n}\{H_m\}$, $n = 1, 2, 3, \ldots$. In this paper, we determine the upper and lower limits of this sequence and show that all its possible limit points consist of the closed interval between them.

Theorem 1. *We have*

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n}\{H_m\} = \frac{1}{e-1} = 0.581976\ldots$$

and

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n}\{H_m\} = 1 - \log(e-1) = 0.458675\ldots.$$

Theorem 1 follows from the following more general result:

Theorem 2. *For each $\sigma > 0$, we have*

$$\frac{1}{n}\sum_{m=1}^{n}\{H_m\}^{\sigma} \sim \Phi(\sigma, \{H_n\}) \quad \text{as} \quad n \to \infty, \tag{1}$$

where

$$\Phi(\sigma, t) = e^{-t}\left(\frac{\int_0^1 x^\sigma e^x dx}{e-1} + \int_0^t x^\sigma e^x dx\right) \tag{2}$$

for each $t \in [0, 1)$.

Indeed, since the sequence of the fractional parts $\{H_m\}$, $m = 1, 2, 3, \ldots$, is everywhere dense in the closed interval $[0, 1]$, by (1) and (2), the set of limit points of the sequence of arithmetic means, $\frac{1}{n}\sum_{m=1}^{n}\{H_m\}^{\sigma}$, $n = 1, 2, 3, \ldots$, is actually the set of all values attained by the function $\Phi(\sigma, t)$ for $t \in [0, 1]$. Since $\Phi(\sigma, t)$ is continuous in $t \in [0, 1]$, the latter set is obviously the following closed interval:

$$[\min_{t\in[0,1]} \Phi(\sigma,t), \max_{t\in[0,1]} \Phi(\sigma,t)].$$

In particular, for $\sigma = 1$, the function $\Phi(\sigma,t)$ defined in (2) equals

$$\Phi(1,t) = e^{-t}\left(\frac{\int_0^1 xe^x dx}{e-1} + \int_0^t xe^x dx\right) = e^{-t}\left(\frac{1}{e-1} + (t-1)e^t + 1\right) = t - 1 + \frac{e^{1-t}}{e-1}.$$

In the closed interval $t \in [0,1]$, the maximum of $\Phi(1,t)$ is attained at $t = 0$ and at $t = 1$ and equals $1/(e-1) = 0.581976\ldots$. The minimum of $\Phi(1,t)$ is attained at the point $t_1 = 1 - \log(e-1) = 0.458675\ldots$ and is equal to the same value $t_1 = 1 - \log(e-1)$. Consequently, all limit points of the sequence $\frac{1}{n}\sum_{m=1}^{n}\{H_m\}$, $n = 1,2,3,\ldots$, form the closed interval $[1-\log(e-1), 1/(e-1)]$, which implies Theorem 1.

Observe that, for any fixed $\sigma > 0$, the derivative of the function $e^t\Phi'(\sigma,t)$ in $t \in (0,1)$ equals $\sigma t^{\sigma-1}e^t$, such that $e^t\Phi'(\sigma,t)$ is increasing in $[0,1]$ from a negative value at $t = 0$ to a positive value at $t = 1$. (The inequality $\Phi'(\sigma,0) < 0$ is immediate, while the inequality $\Phi'(\sigma,1) > 0$ follows from $\int_0^1 x^\sigma e^x dx < e - 1$.) By continuity, this implies that there is a unique t_σ in $(0,1)$ satisfying $\Phi'(\sigma,t_\sigma) = 0$ such that $\Phi'(\sigma,t) < 0$ for $t < t_\sigma$ and $\Phi'(\sigma,t) > 0$ for $t > t_\sigma$. Therefore, the function $\Phi(\sigma,t)$ is decreasing in $[0,t_\sigma]$ and increasing in $[t_\sigma,1]$. Consequently, the maximum of $\Phi(\sigma,t)$ in $[0,1]$ is attained at $t = 0$ or at $t = 1$. Since $\Phi(\sigma,0) = \Phi(\sigma,1)$, the maximum of the function $\Phi(\sigma,t)$ in the interval $t \in [0,1]$ equals $\Phi(\sigma,0) = \Phi(\sigma,1) = \frac{\int_0^1 x^\sigma e^x dx}{e-1}$, while its minimum is $\Phi(\sigma,t_\sigma)$. Hence,

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n}\{H_m\}^\sigma = \frac{\int_0^1 x^\sigma e^x dx}{e-1}.$$

However, unlike in the case where $\sigma = 1$, the smallest limit point $\Phi(\sigma,t_\sigma)$ cannot be determined by an explicit expression as before. For example, for $\sigma = 2$, we have the following:

$$\Phi(2,t) = t^2 - 2t + 2 - \frac{e^{1-t}}{e-1}.$$

Here, $\Phi(2,0) = \Phi(2,1) = (e-2)/(e-1)$, and hence, we obtain

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n}\{H_m\}^2 = \frac{e-2}{e-1} = 0.418023\ldots.$$

The minimum of the function $\Phi(2,t)$ in $[0,1]$ is attained at point $t_2 = 0.538241\ldots$ satisfying the following:

$$\Phi'(2,t_2) = 2t_2 - 2 + \frac{e^{1-t_2}}{e-1} = 0,$$

where $\Phi(2,t_2) = t_2^2 = 0.289704\ldots$. Therefore,

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n}\{H_m\}^2 = t_2^2 = 0.289704\ldots.$$

In fact, we will prove a result more precise than that stated in Theorem 2, which not only gives the asymptotical Formula (1) but also an estimate for the error term:

Theorem 3. *For each $n \geq 2$, with the notation of Theorem 2, we have*

$$\left|\frac{1}{n}\sum_{m=1}^{n}\{H_m\}^\sigma - \Phi(\sigma,\{H_n\})\right| < c\left(\frac{\log n}{n}\right)^{1/3}$$

for some constant $c > 0$ independent of n.

In the next section, we prove several auxiliary results. The proof of Theorem 3 is given in Section 3. Finally, in Section 4, we will show that $\{H_n\}^\sigma$ in (1) can be replaced by a more general function $f(\{H_m\})$ with an appropriate change in the definition of Φ in (2); see (29). Some examples of f giving explicit upper and lower limits for the sequence $\frac{1}{n}\sum_{m=1}^{n} f(\{H_m\})$, $n = 1, 2, 3, \ldots$, will be presented there as well.

2. Auxiliary Results

Throughout this paper, we will use the following notation. For any real numbers A and B satisfying $1 < A < B$, by $S(A, B)$, we will denote the set of $m \in \mathbb{N}$ such that $A < H_m \le B$. The cardinality of this set will be denoted by $\#S(A, B)$. By γ, we will denote Euler's constant:

$$\gamma = \lim_{n\to\infty}(H_n - \log n) = 0.577215\ldots.$$

We begin with the following lemma.

Lemma 1. *Let $y \ge 1$ be a real number and let $m \in \mathbb{N}$ be the largest integer for which $H_m \le y$. Then,*

$$-2 < m - e^{y-\gamma} < -\frac{1}{2}. \tag{3}$$

Proof. By the definition of m, it is clear that

$$H_m \le y < H_{m+1}.$$

A well-known approximation formula from [14] asserts that for each $n \in \mathbb{N}$, we have

$$\frac{1}{24(n+1)^2} < H_n - \log\left(n + \frac{1}{2}\right) - \gamma < \frac{1}{24n^2}.$$

Hence,

$$y - \gamma \ge H_m - \gamma > \log\left(m + \frac{1}{2}\right) + \frac{1}{24(m+1)^2} > \log\left(m + \frac{1}{2}\right),$$

which implies the upper bound in (3) by taking the exponents of both sides.

Similarly, from

$$y - \gamma < H_{m+1} - \gamma < \log\left(m + \frac{3}{2}\right) + \frac{1}{24(m+1)^2} < \log(m+2),$$

we deduce the lower bound in (3). Here, the last inequality follows from

$$e^{\frac{1}{24(m+1)^2}} < 1 + \frac{1}{12(m+1)^2} < 1 + \frac{1}{2m+3} == \frac{2m+4}{2m+3} = \frac{m+2}{m+3/2},$$

which is true due to $e^x < 1 + 2x$ for $0 < x < 1$. $\square$

An exact evaluation of m defined in Lemma 1 in terms of the integral part $\lfloor e^{y-\gamma} \rfloor$ is a problem studied by Hardy in 1924; see [15,16].

Now, we will estimate the number of indices m for which $K + u < H_m \le K + v$:

Lemma 2. *Let u, v be real numbers satisfying $0 < u < v \le 1$ and $K \in \mathbb{N}$. Then,*

$$-\frac{3}{2} < \#S(K+u, K+v) - (e^v - e^u)e^{K-\gamma} < \frac{3}{2}. \tag{4}$$

Proof. Let U and V be the largest positive integers for which $H_U \le K + u$ and $H_V \le K + v$. Then,

$$\#S(K+u, K+v) = V - U.$$

By Lemma 1, we have

$$-2 < V - e^{K+v-\gamma} < -\frac{1}{2}$$

and

$$-2 < U - e^{K+u-\gamma} < -\frac{1}{2}.$$

It follows that the difference

$$V - U - e^{K+v-\gamma} + e^{K+u-\gamma} = \#S(K+u, K+v) - (e^v - e^u)e^{K-\gamma}$$

is in the interval $(-3/2, 3/2)$, which implies (4). □

Now, we are ready to state our main auxiliary lemma:

Lemma 3. *Let $\sigma > 0$ be a real number. Then, for each sufficiently large $K \in \mathbb{N}$ and each real t satisfying $18e^{-K} \le t \le 1$, the set $S(K, K+t)$ is nonempty and*

$$\left| \sum_{m \in S(K,K+t)} \{H_m\}^\sigma - \#S(K, K+t) \cdot \frac{\int_0^t x^\sigma e^x dx}{e^t - 1} \right| < 3\big(\#S(K, K+t)\big)^{2/3}. \tag{5}$$

Proof. Take an integer L satisfying

$$6 \le L \le \frac{te^K}{3}. \tag{6}$$

Note that $S(K, K+t)$ is the union of L disjoint sets $S(K + jt/L, K + (j+1)t/L)$, where $j = 0, 1, \dots, L-1$. By Lemma 2, we have

$$\#S(K + jt/L, K + (j+1)t/L) = (e^{t/L} - 1)e^{K+jt/L-\gamma} + \delta(K, j, L, t), \tag{7}$$

with $\delta(K, j, L, t) \in (-3/2, 3/2)$. From

$$(e^{t/L} - 1)e^{K+jt/L-\gamma} > \frac{t}{L}e^{K-\gamma} \ge \frac{3Le^{-K}}{L}e^{K-\gamma} = 3e^{-\gamma} > \frac{3}{2},$$

we see that the set $S(K + jt/L, K + (j+1)t/L)$ is nonempty for each $L \in \mathbb{N}$ satisfying the upper bound in (6) and for each sufficiently large K. In particular, this implies that the set $S(K, K+t)$ is nonempty.

For each $m \in S(K + jt/L, K + (j+1)t/L)$, we have

$$\left(\frac{jt}{L}\right)^\sigma < \{H_m\}^\sigma \le \left(\frac{(j+1)t}{L}\right)^\sigma.$$

Thus, the sum

$$\sum_{m \in S(K,K+t)} \{H_m\}^\sigma = \sum_{j=0}^{L-1} \sum_{m \in S(K+jt/L, K+(j+1)t/L)} \{H_m\}^\sigma \tag{8}$$

is greater than

$$B_1 := \sum_{j=0}^{L-1} \#S(K + jt/L, K + (j+1)t/L)\left(\frac{jt}{L}\right)^\sigma$$

and smaller than or equal to

$$B_2 := \sum_{j=0}^{L-1} \#S(K + jt/L, K + (j+1)t/L)\left(\frac{(j+1)t}{L}\right)^\sigma$$

Note that, by Lemma 2, we also have

$$\#S(K, K+t) = (e^t - 1)e^{K-\gamma} + \delta(K, t), \tag{9}$$

with $\delta(K, t) \in (-3/2, 3/2)$. Combining (7) with (9), we deduce

$$\#S(K + jt/L, K + (j+1)t/L) - \delta(K, j, L, t) = \Big(\#S(K, K+t) - \delta(K, t)\Big)\frac{(e^{t/L} - 1)e^{jt/L}}{e^t - 1}.$$

This implies

$$\#S(K + jt/L, K + (j+1)t/L) = \frac{(e^{t/L} - 1)e^{jt/L}}{e^t - 1}\#S(K, K+t) + \mu(K, j, L, t), \tag{10}$$

where

$$\mu(K, j, L, t) = \delta(K, j, L, t) - \delta(K, t)\frac{(e^{t/L} - 1)e^{jt/L}}{e^t - 1}.$$

Here, $\delta(K, j, L, t)$ and $\delta(K, t)$ are both at most $3/2$ in absolute value. By $0 < t \leq 1$ and $L \geq 6$ (see the lower bound in (6)), we have $0 < e^{t/L} - 1 < 1.1t/L$. Hence, as $j \leq L - 1$ and $1.1te^t/(e^t - 1) < 1.75$, we deduce

$$0 < \frac{(e^{t/L} - 1)e^{jt/L}}{e^t - 1} < \frac{1.1te^t}{L(e^t - 1)} < \frac{1.75}{L} \leq \frac{1.75}{6} < \frac{1}{3}.$$

Thus,

$$|\mu(K, j, L, t)| < 2.$$

Now, by (9), (10) and $e^{t/L} - 1 > t/L$, we deduce

$$\begin{aligned}
B_1 &> \sum_{j=0}^{L-1}\left(\#S(K, K+t)\frac{(e^{t/L} - 1)e^{jt/L}}{e^t - 1} - 2\right)\left(\frac{jt}{L}\right)^{\sigma} \\
&> \sum_{j=0}^{L-1}\left(\#S(K, K+t)\frac{(e^{t/L} - 1)\left(\frac{jt}{L}\right)^{\sigma}e^{jt/L}}{e^t - 1} - 2\right) \\
&= \#S(K, K+t)\left(\frac{e^{t/L} - 1}{e^t - 1}\right)\sum_{j=0}^{L-1}\left(\frac{jt}{L}\right)^{\sigma}e^{jt/L} - L(L-1) \\
&> \frac{\#S(K, K+t)}{e^t - 1}\cdot\frac{t}{L}\sum_{j=0}^{L-1}\left(\frac{jt}{L}\right)^{\sigma}e^{jt/L} - L^2
\end{aligned}$$

and, similarly,

$$\begin{aligned}
B_2 &< \frac{\#S(K, K+t)}{e^t - 1}\sum_{j=0}^{L-1}(e^{t/L} - 1)\left(\frac{(j+1)t}{L}\right)^{\sigma}e^{jt/L} + L(L-1) \\
&= \#S(K, K+t)\left(\frac{1 - e^{-t/L}}{e^t - 1}\right)\sum_{j=0}^{L-1}\left(\frac{(j+1)t}{L}\right)^{\sigma}e^{(j+1)t/L} + L(L-1) \\
&= \#S(K, K+t)\left(\frac{1 - e^{-t/L}}{e^t - 1}\right)\sum_{j=1}^{L}\left(\frac{jt}{L}\right)^{\sigma}e^{jt/L} + L(L-1) \\
&< \frac{\#S(K, K+t)}{e^t - 1}\cdot\frac{t}{L}\sum_{j=1}^{L}\left(\frac{jt}{L}\right)^{\sigma}e^{jt/L} + L^2.
\end{aligned}$$

Therefore, by (8) and the definitions of B_1, B_2, we see that the quantity

$$\frac{\sum_{m\in S(K,K+t)}\{H_m\}^\sigma}{\#S(K,K+t)}$$

is greater than

$$\frac{t}{(e^t-1)L}\sum_{j=0}^{L-1}\left(\frac{jt}{L}\right)^\sigma e^{jt/L}-\frac{L^2}{\#S(K,K+t)} \tag{11}$$

and smaller than

$$\frac{t}{(e^t-1)L}\sum_{j=1}^{L}\left(\frac{jt}{L}\right)^\sigma e^{jt/L}+\frac{L^2}{\#S(K,K+t)}. \tag{12}$$

Furthermore, since the function $x^\sigma e^x$ is increasing in x for $x\in[0,t]$, we have

$$\frac{t}{L}\sum_{j=0}^{L-1}(\frac{jt}{L})^\sigma e^{jt/L}<\int_0^t x^\sigma e^x dx<\frac{t}{L}\sum_{j=1}^{L}(\frac{jt}{L})^\sigma e^{jt/L}=\frac{t}{L}\sum_{j=0}^{L-1}(\frac{jt}{L})^\sigma e^{jt/L}+\frac{t}{L}t^\sigma e^t.$$

Thus, (11) is greater than

$$\frac{\int_0^t x^\sigma e^x dx}{e^t-1}-\frac{t^{1+\sigma}e^t}{L(e^t-1)}-\frac{L^2}{\#S(K,K+t)}$$

and (12) is smaller than

$$\frac{\int_0^t x^\sigma e^x dx}{e^t-1}+\frac{t^{1+\sigma}e^t}{L(e^t-1)}+\frac{L^2}{\#S(K,K+t)}.$$

Therefore,

$$\left|\frac{\sum_{m\in S(K,K+t)}\{H_m\}^\sigma}{\#S(K,K+t)}-\frac{\int_0^t x^\sigma e^x dx}{e^t-1}\right|<\frac{t^{1+\sigma}e^t}{L(e^t-1)}+\frac{L^2}{\#S(K,K+t)}<\frac{1.6}{L}+\frac{L^2}{\#S(K,K+t)}.$$

Now, selecting, for instance, $L=\lfloor(\#S(K,K+t)\cdot\frac{4}{5})^{1/3}\rfloor$, and multiplying both sides of the last inequality by $\#S(K,K+t)$, we derive the desired inequality (5). By (9), it is clear that this choice of L satisfies (6) for a sufficiently large K. □

In particular, from Lemma 3, we will derive the following:

Lemma 4. *Let $\sigma>0$ be a real number. Then, there is $K_0\in\mathbb{N}$ such that for each integer $M>K_0$, we have*

$$\left|\sum_{m\in S(K_0,M)}\{H_m\}^\sigma-\#S(K_0,M)\cdot\frac{\int_0^1 x^\sigma e^x dx}{e-1}\right|<3M^{1/3}\big(\#S(K_0,M)\big)^{2/3}. \tag{13}$$

Proof. Fix $\sigma>0$. Assume that K_0 is the integer as claimed in Lemma 3. Applying Lemma 3 to $t=1$ and to $K\in\{K_0,K_0+1,\dots,M-1\}$, we deduce

$$\left|\sum_{m\in S(K,K+1)}\{H_m\}^\sigma-\#S(K,K+1)\cdot\frac{\int_0^1 x^\sigma e^x dx}{e-1}\right|<3\big(\#S(K,K+1)\big)^{2/3}.$$

Adding those inequalities for $K=K_0,K_0+1,\dots,M-1$, we obtain

$$\left|\sum_{m\in S(K_0,M)}\{H_m\}^\sigma-\#S(K_0,M)\cdot\frac{\int_0^1 x^\sigma e^x dx}{e-1}\right|<3\sum_{K=K_0}^{M-1}\big(\#S(K,K+1)\big)^{2/3}. \tag{14}$$

By Hölder's inequality, for any $\ell \in \mathbb{N}$ and any non-negative real number x_j, we have

$$\sum_{j=1}^{\ell} x_j^{2/3} \leq \ell^{1/3} \Big(\sum_{j=1}^{\ell} x_j \Big)^{2/3}.$$

Thus, because of

$$\sum_{K=K_0}^{M-1} \#S(K, K+1) = \#S(K_0, M),$$

the right-hand side of inequality (14) does not exceed $3(M-K_0)^{1/3}\big(\#S(K_0, M)\big)^{2/3}$. This completes the proof of (13). □

3. Proof of Theorem 3

Proof. Let $n \geq 2$ be an integer. Set $M = \lfloor H_n \rfloor$ and $t = \{H_n\}$. Here, $0 < t < 1$ because for $n \geq 2$, the number H_n is not an integer. Assume that the inequality (5) of Lemma 3 holds for $K \geq K_0$, where K_0 depends on σ and t. There is nothing to prove if $M \leq K_0$, since then $H_n = M + t < K_0 + 1$ and n is bounded by an absolute constant; so, assume that $M > K_0$. Observe that

$$n = 1 + \#S(1, K_0) + \#S(K_0, M) + \#S(M, M+t).$$

Applying Lemma 1 to $y = M + t$, we find that

$$n = e^{M+t-\gamma} - \eta_0, \tag{15}$$

where $1/2 < \eta_0 < 2$. Similarly, applying the same lemma to $y = M$, we deduce

$$1 + \#S(1, K_0) + \#S(K_0, M) = e^{M-\gamma} - \eta_1,$$

where $1/2 < \eta_1 < 2$, and hence,

$$\#S(K_0, M) = e^{M-\gamma} - \eta_2 \tag{16}$$

for some positive constant η_2. Also, by Lemma 2, we obtain

$$\#S(M, M+t) = (e^t - 1)e^{M-\gamma} + \eta_3, \tag{17}$$

where $|\eta_3| < 3/2$.

By (15), we have $e^{M-\gamma} = (n + \eta_0)e^{-t}$. Inserting this into (16) and (17), we derive

$$\#S(K_0, M) = ne^{-t} + \eta_4 \tag{18}$$

and

$$\#S(M, M+t) = n(1 - e^{-t}) + \eta_5, \tag{19}$$

respectively. Here, η_4, η_5 are bounded constants.

Consider the sum

$$\sum_{m=1}^{n} \{H_m\}^{\sigma} = \sum_{m=2}^{n} \{H_m\}^{\sigma} = \sum_{m \in S(1,K_0)} \{H_m\}^{\sigma} + \sum_{m \in S(K_0,M)} \{H_m\}^{\sigma} + \sum_{m \in S(M,M+t)} \{H_m\}^{\sigma}. \tag{20}$$

Here, the first sum, $\sum_{m \in S(1,K_0)} \{H_m\}^{\sigma}$, is a non-negative constant that depends on K_0 and σ say, $\theta_0 = \theta_0(K_0, \sigma)$, namely,

$$\sum_{m \in S(1,K_0)} \{H_m\}^{\sigma} = \theta_0. \tag{21}$$

By Lemma 4, the second sum is

$$\sum_{m\in S(K_0,M)}\{H_m\}^\sigma = \#S(K_0,M)\cdot\frac{\int_0^1 x^\sigma e^x dx}{e-1} + \theta_1 M^{1/3}\big(\#S(K_0,M)\big)^{2/3}, \tag{22}$$

where $|\theta_1| < 3$. Note that for sufficiently large n, we have $M \le 2\log n$ by (15). So, inserting into (22) the value of $\#S(K_0,M)$ from (18), we obtain

$$\sum_{m\in S(K_0,M)}\{H_m\}^\sigma = ne^{-t}\cdot\frac{\int_0^1 x^\sigma e^x dx}{e-1} + \theta_2 n^{2/3}(\log n)^{1/3}, \tag{23}$$

where θ_2 depends on n and σ but is bounded.

To evaluate the third sum, we will consider two cases: firstly, $18e^{-M} \le t < 1$, and, secondly, $0 < t < 18e^{-M}$. In the first case, $18e^{-M} \le t < 1$, applying Lemma 3, we deduce

$$\sum_{m\in S(M,M+t)}\{H_m\}^\sigma = \#S(M,M+t)\cdot\frac{\int_0^t x^\sigma e^x dx}{e^t-1} + \theta_3\big(\#S(M,M+t)\big)^{2/3}, \tag{24}$$

where $|\theta_3| < 3$. Now, inserting into (24) the value of $\#S(K_0,M)$ from (19), we obtain

$$\sum_{m\in S(M,M+t)}\{H_m\}^\sigma = ne^{-t}\cdot\int_0^t x^\sigma e^x dx + \frac{\eta_5\int_0^t x^\sigma e^x dx}{e^t-1} + \theta_4 n^{2/3},$$

and hence,

$$\sum_{m\in S(M,M+t)}\{H_m\}^\sigma = ne^{-t}\cdot\int_0^t x^\sigma e^x dx + \theta_5 n^{2/3}, \tag{25}$$

where θ_4 and θ_5 depend on n and σ but are bounded. From (20), (21), (23) and (25), we deduce

$$\sum_{m=1}^n\{H_m\}^\sigma = ne^{-t}\left(\frac{\int_0^1 x^\sigma e^x dx}{e-1} + \int_0^t x^\sigma e^x dx\right) + \theta_6 n^{2/3}(\log n)^{1/3}, \tag{26}$$

with θ_6 bounded. Dividing (26) by n, we obtain

$$\left|\frac{1}{n}\sum_{m=1}^n\{H_m\}^\sigma - \Phi(\sigma,\{H_n\})\right| < c\left(\frac{\log n}{n}\right)^{1/3} \tag{27}$$

for some $c > 0$ independent of n, which is the required estimate.

We now turn to the case when $0 < t < 18e^{-M}$. Then, by (17), $\#S(M,M+t)$ is bounded from above by an absolute constant. So, instead of (25), we have

$$\sum_{m\in S(M,M+t)}\{H_m\}^\sigma = \theta_7,$$

where θ_7 is bounded. Combining this with (20), (21), and (23), we obtain

$$\sum_{m=1}^n\{H_m\}^\sigma = ne^{-t}\cdot\frac{\int_0^1 x^\sigma e^x dx}{e-1} + \theta_8 n^{2/3}(\log n)^{1/3},$$

with θ_8 bounded. Now, to derive Formula (26) from this, we need only show that the integral $ne^{-t}\int_0^t x^\sigma e^x dx$ is small for small t. We will show that, under our assumption on t, this integral is bounded. Indeed, as $0 < t < 18e^{-M}$, using (15), we obtain

$$0 < ne^{-t}\int_0^t x^\sigma e^x dx < 2ne^{-t}\frac{t^{\sigma+1}}{\sigma+1} < 2nt < 36ne^{-M} = 36(e^{M+t-\gamma}-\eta_0)e^{-M} < 36e^{t-\gamma} < 36.$$

Hence, (26) holds with an appropriate θ_6 (depending on n and σ but bounded). As stated above, we see that (26) implies (27), which completes the proof of the theorem. □

4. Concluding Remarks

In the proof of Theorem 3 and Lemma 3, we mainly used the fact that the function x^σ is continuous, non-negative and non-decreasing in $[0,1]$, implying that the function $x^\sigma e^x$ is as well. By exactly the same argument, one can show that, for every continuous, non-negative and non-decreasing function f in $[0,1]$, we have

$$\frac{1}{n}\sum_{m=1}^{n} f(\{H_m\}) \sim \Phi_f(\{H_n\}) \quad \text{as} \quad n \to \infty, \tag{28}$$

where

$$\Phi_f(t) = e^{-t}\left(\frac{\int_0^1 f(x)e^x dx}{e-1} + \int_0^t f(x)e^x dx\right) \tag{29}$$

for each $t \in [0,1)$. (The specific form of f, namely $f(x) = x^\sigma$, has been used only in the estimate of the error term as in Theorem 3, which we will not do for a general f.)

Thus, Theorem 2 can be generalized as follows:

Theorem 4. *Let $f(x)$ be an absolutely continuous function on $[0,1]$. Then,*

$$\frac{1}{n}\sum_{m=1}^{n} f(\{H_m\}) \sim \Phi_f(\{H_n\})$$

as $n \to \infty$, where $\Phi_f(t)$ is defined in (29).

Indeed, since f is absolutely continuous, it is a function of bounded variation. (The definition and basic properties of functions of bounded variation can be found in the following monographs [17,18]). Next, every function of bounded variation is the difference between two monotonically non-decreasing functions. Adding an appropriate positive constant to both of them, we conclude that f is expressible in the form

$$f = f_1 - f_2,$$

where the functions f_1 and f_2 are both continuous, positive, and non-decreasing in $[0,1]$. In view of (29), we clearly have

$$\Phi_f(t) = \Phi_{f_1-f_2}(t) = \Phi_{f_1}(t) - \Phi_{f_2}(t).$$

Thus, applying the asymptotic Formula (28) to f_1 and f_2 and then subtracting one formula from another, we derive Theorem 4.

Selecting in (29), for instance, $f(x) = e^x$, we find that

$$\Phi_f(t) = \frac{e^{1-t} + e^t}{2}.$$

The maximum of this function for $t \in [0,1]$ is attained at $t = 0$ and $t = 1$ and equals $(e+1)/2$, while its minimum is attained at $t = 1/2$ and equals $e^{1/2}$. Hence, by Theorem 4, it follows that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} e^{\{H_m\}} = \frac{e+1}{2} = 1.859140\ldots$$

and

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} e^{\{H_m\}} = e^{1/2} = 1.648721\ldots.$$

Likewise, selecting in (29), for instance, $f(x) = e^{-x}$, we obtain

$$\Phi_f(t) = e^{-t}\Big(\frac{1}{e-1} + t\Big).$$

This time, unlike in all previous examples, not the maximum but the minimum of the function $\Phi_f(t)$ is attained at $t = 0$ and $t = 1$, and it equals $1/(e-1)$. Its maximum is attained at $t = (e-2)/(e-1)$ and equals $e^{-\frac{e-2}{e-1}}$. Therefore, by Theorem 4,

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} e^{-\{H_m\}} = \frac{1}{e-1} = 0.581976\ldots$$

and

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{m=1}^{n} e^{-\{H_m\}} = e^{-\frac{e-2}{e-1}} = 0.658346\ldots.$$

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