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Extremal *k*-Connected Graphs with Maximum Closeness

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Abstract: Closeness is a measure that quantifies how quickly information can spread from a given node to all other nodes in the network, reflecting the efficiency of communication within the network by indicating how close a node is to all other nodes. For a graph G, the subset S of vertices of $V(G)$ is called vertex cut of G if the graph $G - S$ becomes disconnected. The minimum cardinality of S for which $G - S$ is either disconnected or contains precisely one vertex is called connectivity of G. A graph is called k-connected if it stays connected even when any set of fewer than k vertices is removed. In communication networks, a k-connected graph improves network reliability; even if up to $k - 1$ nodes fail, the network remains operational, maintaining connectivity between devices. This paper aims to study the concept of closeness within n-vertex graphs with fixed connectivity. First, we identify the graphs that maximize the closeness among all graphs of order n with fixed connectivity k. Then, we determine the graphs that achieve the maximum closeness within all k-connected graphs of order n, given specific fixed parameters such as diameter, independence number, and minimum degree.

Keywords: closeness; connectivity; k-connected graph; diameter; independence number; minimum degree

MSC: 05C12; 05C35; 68M15

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1. Introduction

Network science has undergone substantial advancements in the last ten years and has emerged as the foremost scientific discipline in the analysis of complex networks. Consequently, the study of complex networks represents a crucial domain within the broader field of complexity science. A network is commonly represented as an undirected simple graph, where the vertices are denoted by nodes and the edges by the connections between the nodes. A key focus of network analysis is to determine which nodes occupy critical positions within the network. Graph theory has proven to be a powerful mathematical framework for conducting network analysis, providing a variety of techniques and methodologies. Within this context, several graph-theoretic parameters contribute to the analysis of networks. One important parameter is closeness, which helps identify nodes that can effectively spread information across the network. In essence, a node with high closeness can reach other nodes swiftly and efficiently. This indicates that the node is well-connected to the broader network and has the potential to influence or be influenced by other nodes more quickly than those with lower closeness values.

Nodes characterized by high closeness are vital in numerous practical applications. They significantly improve the efficiency of communication networks by enabling swift data transfers, which is critical for real-time communication systems. In the context of social networks, these nodes act as pivotal influencers, promoting the rapid spread of information and encouraging community involvement, thus influencing marketing strategies and public awareness initiatives. In transportation networks, high closeness nodes can optimize routing and scheduling, ensuring timely responses in emergencies and improving overall traffic management.

The first closeness concept was given by Bavelas [1]. Then, Freeman also presented the concept of closeness [2], but his methodology was ineffective for disconnected graphs and revealed constraints during graph operations. To overcome the initial limitation, Latora and Marchiori developed a new measure of closeness applicable to disconnected graphs [3], although it still faces challenges related to the second limitation. In response, Danglachev offered an alternative definition [4], which successfully tackles the issues associated with disconnected graphs and allows for the derivation of useful formulas for graph operations. Building on this definition of closeness, several vulnerability measures have been created to evaluate the resilience of a network. Notably, the vertex (or edge) residual closeness parameters have been introduced to measure the closeness of a graph after the removal of vertices (or edges) [4]. Additionally, the concept of additional closeness has been defined to determine the maximum potential of a graph's closeness through the addition of an edge [5,6]. For a more comprehensive discussion of these new sensitive parameters, we suggest consulting [7–13].

The analysis of closeness across different categories of graphs has attracted considerable interest [4,14–16]. For example, Danglachev explored the closeness of splitting networks [17]. In a subsequent study [18], the same researcher assessed the closeness of line graphs for a specific family of graphs. Golpek [19] derived the closeness for some classes of graphs. Additionally, Poklukar and Žerovnik [20] determined the graphs with maximum or minimum closeness in the class of all graphs and trees, and determined the graphs with maximum closeness over cacti with given order and cycles. Hayat and Xu [21] identified the graph having minimum closeness among cacti with predetermined numbers of vertices and cycles. Furthermore, Zheng and Zhou [22] recently integrated the concept of closeness within spectral graph theory. They discovered the closeness matrix and established a relationship between the closeness eigenvalues and the structure of a graph. Recently, Hayat and Otera [23] explored the graphs having maximum closeness in the class of bipartite graphs with some given parameters such as diameter, dissociation number, connectivity, and cut edges.

The k-connected graphs, where the removal of any $k-1$ vertices does not disconnect the graph, have numerous practical applications across various fields. For instance, k-connected graphs are essential in designing robust communication networks, ensuring that the network remains connected even if several nodes fail. This is particularly important for telecommunications and data networks, where reliability is crucial [24]. In transportation networks, k-connectivity ensures that multiple routes exist between destinations, enhancing resilience against disruptions. This is important for urban planning and optimizing public transit systems [25]. For more details about k-connected graphs and its applications, we recommend referring to [26–28].

To investigate the connection between closeness and the structural characteristics of a graph, we explore extremal problems aimed at maximizing closeness in graphs with n vertices and specified connectivity. In particular, we obtain the unique graph that attains the maximum closeness among all graphs of order n with connectivity k. Then, we identify those graphs that maximize closeness within k-connected graphs of order n and one of the fixed parameters such as diameter, independence number, and minimum degree.

2. Basic Definitions

Let G represent a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. For any two vertices u and v within $V(G)$, the distance between u and v in the graph G is defined as the length of the shortest path that links them, represented as $d_G(u,v)$.

In [4], for a vertex u of G, the closeness of u in G is defined as

$$C_G(u) = \sum_{v \in V(G) \setminus \{u\}} 2^{-d_G(u,v)}.$$

The closeness of G is defined as

$$C(G) = \sum_{u \in V(G)} C_G(u) = \sum_{u \in V(G)} \sum_{v \in V(G) \setminus \{u\}} 2^{-d_G(u,v)}.$$

A subset H of $V(G)$ is referred to as a vertex cut of G if the graph $G - H$ becomes disconnected. The smallest cardinality of H for which $G - H$ is either disconnected or contains precisely one vertex is called connectivity of G, and denoted by $k(G)$. A graph is called k-connected if it stays connected even when any set of fewer than k vertices is removed. The diameter of graph G is defined as the greatest distance between any two vertices. A subset M of the vertex set $V(G)$ is referred to as an independent set of G if the vertices within M are mutually non-adjacent. The independence number of G represents the largest size of independent sets within the graph. The degree of a vertex v in G is quantified by the number of edges that are incident to v, which is denoted as $deg_G(v)$.

For any $e \in E(G)$, the expression $G - e$ denotes the subgraph obtained by removing the edge e from G, and $G + uv$ indicates a graph formed by adding an edge between vertices u and v, where $u, v \in V(G)$. The operation of removing a vertex $x \in V(G)$, along with all edges incident to it, is denoted as $G - x$. The symbols P_n and K_n represent the path graph and complete graph of order n, respectively.

The union of two graphs G_1 and G_2, denoted $G_1 \cup G_2$, is defined as the graph where $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. The join of two graphs G_1 and G_2, represented as $G_1 \vee G_2$, is created by connecting every vertex in G_1 to each vertex in G_2. For disjoint graphs $G_1, G_2, \ldots, G_s$ where $s \geq 3$, the sequential join $G_1 \vee G_2 \vee \cdots \vee G_s$ is the graph obtained from $G_1, G_2, \ldots, G_s$ is formed by first joining each vertex of G_1 to all vertices of G_2, then connecting each vertex of G_2 to all vertices of G_3, and continuing this process until each vertex of G_{s-1} is connected to all vertices of G_s. For convenience, sG (and $[s]G$) is used to denote the union (and sequential join) of s disjoint copies of G. For instance, $sK_1 = \overline{K_s}$ represents s isolated vertices and $[a]G_1 \vee G_2 \vee [b]G_3$ is the sequential join $\underbrace{G_1 \vee G_1 \vee \cdots \vee G_1}_{a} \vee G_2 \vee \underbrace{G_3 \vee G_3 \vee \cdots \vee G_3}_{b}$.

The following lemma will be utilized in the proof of the main results.

Lemma 1 ([4,13]). *Let G be a graph with vertices u and v in $V(G)$. If the edge uv in not present in $E(G)$, then $C(G) < C(G + uv)$. If uv is an edge in $E(G)$, then $C(G) > C(G - uv)$.*

3. Main Results

In [20], it was proved that K_n uniquely maximizes the closeness among all connected graphs of order n. Note that the connectivity of K_n is $(n - 1)$. Now the question arises: which graph maximizes the closeness when the connectivity is fixed and less than $n - 1$? To answer this question, in the following result, we determine the graph that maximizes the closeness among all graphs of order n with connectivity k, where $k \leq n - 2$.

Let G_k^n be the graph obtained from K_{n-1} by adding edges between a single vertex and k vertices of K_{n-1}. Clearly, G_k^n is a graphs of order n with connectivity k, where $k \leq n - 2$.

Theorem 1. *Let G be a graph of order n with connectivity k that maximizes closeness, where $k \leq n - 2$. Then, $G \cong G_k^n$.*

Proof. Let G be a graph of order n with connectivity k such that G has the largest closeness. Denote by X the set of cut vertices of G with $|X| = k$. Let $H_1, H_2, \ldots, H_t$ represent the components of $G - X$ with $t \geq 2$. We assert that $t = 2$. Assume, for the sake of contradiction, that $t \geq 3$, and consider H_1, H_2, H_3 the three components of G. Let $G' = G + \{uv : u \in V(H_2), v \in V(H_3)\}$. Obviously, G' is a graph of order n with connectivity k. According to Lemma 1, we obtain $C(G) < C(G')$, a contradiction. So, $t = 2$. Also by Lemma 1, $G[V(H_1) \cup X]$ and $G[V(H_2) \cup X]$ are complete graphs. Hence, we obtain $G \cong K_k \vee (K_{q_1} \cup K_{q_2})$, where $q_1 = |V(H_1)|, q_2 = |V(H_2)|$, and $q_1 + q_2 = n - k$. We suppose that $q_1 \geq q_2$.

To finalize the proof it suffices to prove that $q_2 = 1$. Assume that $q_2 \geq 2$. For $u \in V(H_2)$, let $G'' = G - \{uv : v \in V(H_2) \setminus \{u\}\} + \{uv : v \in V(H_1)\}$. Clearly, G'' is k-connected graphs of order n.

From G to G'', we have $C_G(x) = C_{G''}(x)$ for each $x \in V(G) \setminus \{u\}$; $C_G(u) = 2^{-2}q_1 + 2^{-1}(q_2 - 1)$; $C_{G''}(u) = 2^{-1}q_1 + 2^{-4}(q_2 - 1)$.

This gives

$$\begin{aligned} C(G) - C(G'') &= C_G(u) - C_{G''}(u) \\ &= 2^{-2}q_1 + 2^{-1}(q_2 - 1) - 2^{-1}q_1 - 2^{-4}(q_2 - 1) \\ &= 2^{-2}(q_2 - q_1 - 1) \\ &< 0, \end{aligned}$$

a contradiction. So, $q_2 = 1$. Thus, $G \cong G_k^n$. This completes the proof. □

In the following result, we identify the graph that maximizes the closeness within k-connected graphs of order n with a fixed diameter α. Note that K_n uniquely maximized the closeness among k-connected graphs of order n having diameter 1. Therefore, in the following, we take $\alpha \geq 2$. For even α, we define

$$E_k(n, \alpha) = K_1 \vee [(\alpha - 2)/2]K_k \vee K_{n-k\alpha+2k-2} \vee [(\alpha - 2)/2]K_k \vee K_1,$$

and

$$\mathcal{O}_k(n, \alpha) = K_1 \vee [(\alpha - 3)/2]K_k \vee K_{a+1} \vee K_{b+1} \vee [(\alpha - 3)/2]K_k \vee K_1$$

when α is odd, $a, b \geq k - 1$ and $a + b = n - k\alpha + 3k - 4$.

Clearly, $E_k(n,)\alpha$ is a k-connected graph of order n with diameter α, and $\mathcal{O}_k(n, \alpha)$ is a set of n-vertex k-connected graphs with diameter α.

Theorem 2. *Let G be a k-connected graph of order n with a diameter $\alpha \geq 2$ that maximizes closeness. Then, $G \cong E_k(n, \alpha)$ if α is even, and $G \in \mathcal{O}_k(n, \alpha)$ otherwise.*

Proof. Let G be a k-connected graph of order n with a diameter α such that $C(G)$ is as large as possible. Consider a diametral path $P := u_0u_1 \cdots u_\alpha$ within G. Let $J_i = \{x \in V(G) : d_G(x, u_0) = i\}$. Then, $|J_0| = 1$ and $J_0 \cup J_1 \cup \cdots \cup J_\alpha$ is a partition of $V(G)$.

Given that G is a k-connected, we observe that $|J_i| \geq k$ for each $i \in \{1, 2, \ldots, \alpha - 1\}$. According to Lemma 1, the subgraphs $G[J_i]$ and $G[J_{i-1} \cup J_i]$ are complete graphs for $1 \leq i \leq \alpha$. We claim that $|J_\alpha| = 1$. Assume that $|J_\alpha| \geq 2$, and we choose a vertex $x \in J_\alpha \setminus \{u_\alpha\}$ and let $G' = G + \{xw : w \in J_{\alpha-2}\}$. Clearly, G' is k-connected graph having diameter α. By Lemma 1, $C(G) < C(G')$, leading to a contradiction. Therefore, we conclude that $|J_\alpha| = 1$. Thus, we obtain $|J_0| = |J_\alpha| = 1$, and $|J_i| \geq k$ for each $2 \leq i \leq \alpha - 1$. We considering the following three possible cases.

Case 1. If $\alpha = 2$, then $|J_0| = |J_2| = 1$ one has $|J_1| = n - 2$ and the result holds.

Case 2. $\alpha \geq 3$ is even. We claim that $|J_1| = |J_2| = \cdots = |J_{\frac{\alpha}{2}-1}| = |J_{\frac{\alpha}{2}+1}| = \cdots = |J_{\alpha-1}| = k$ and $|J_{\frac{\alpha}{2}}| = n - k\alpha + 2k - 2$.

First, we prove that $|J_1| = |J_{\alpha-1}| = k$. Assume that $|J_1| \geq k + 1$; then, we select $w \in J_1 \setminus \{u_1\}$ and let $G' = G - wu_0 + \{wx : x \in J_3\}$. Clearly, $J_0 \cup (J_1 \setminus \{w\}) \cup (J_2 \cup \{w\}) \cup J_3 \cup \cdots \cup J_\alpha$ is a vertex partition of $V(G')$. From the structure of G', one has $C_G(v) = C_{G'}(v)$

for each $v \in (J_1 \setminus \{w\}) \cup J_2$; $C_G(v) = C_{G'}(v) - \sum_{i=3}^{\alpha} 2^{-(i-1)}$ for each $v \in J_3 \cup \cdots \cup J_\alpha$; $C_G(u_0) = C_{G'}(u_0) + 2^{-2}$: $C_G(w) = C_{G'}(w) + 2^{-2} - \sum_{i=3}^{\alpha} 2^{-(i-1)}$. This gives

$$\begin{aligned} C(G) - C(G') &= \sum_{u \in V(G)} C_G(u) - \sum_{u \in V(G')} C_{G'}(u) \\ &= [C_G(u_0) - C_{G'}(u_0)] + [C_G(w) - C_{G'}(w)] \\ &+ \sum_{v \in J_3 \cup \cdots \cup J_\alpha} (C_G(v) - C_{G'}(v)) \\ &= 2^{-2} + 2^{-2} - \sum_{i=3}^{\alpha} 2^{-(i-1)} - \sum_{i=3}^{\alpha} 2^{-(i-1)} \\ &< 0. \end{aligned}$$

Thus, $C(G) < C(G')$, a contradiction. Thus, $|J_1| = k$. Similarly, $|J_{\alpha-1}| = k$, as claimed. We can similarly demonstrate that $|J_2| = |J_{\alpha-2}| = k, \ldots, |J_{\frac{\alpha}{2}-1}| = |J_{\frac{\alpha}{2}+1}| = k$. As $|J_0| = |J_\alpha| = 1$ and $|J_1| = |J_2| = \cdots = |J_{\frac{\alpha}{2}-1}| = |J_{\frac{\alpha}{2}+1}| = \cdots = |J_{\alpha-1}| = k$, one has $|J_{\frac{\alpha}{2}}| = n - k\alpha + 2k - 2$. Thus, $G \cong E_k(n, d)$.

Case 3. $\alpha \geq 3$ is odd. By the similar way as in Case 2, we have $|J_1| = |J_2| = \cdots = |J_{\frac{\alpha-3}{2}}| = |J_{\frac{\alpha+3}{2}}| = \cdots = |J_{\alpha-1}| = k$. This gives $|J_{\frac{\alpha-1}{2}}| + |J_{\frac{\alpha-1}{2}}| = n - k\alpha + 3k - 2$. Hence, $G \in \mathcal{O}_k(n, \alpha)$.

To finalize the proof, it suffices to prove that all graphs within $\mathcal{O}_k(n, \alpha)$ have equal closeness. Let $H_1 = K_1 \vee [(\alpha-3)/2]K_k \vee K_{r+1} \vee [(\alpha-3)/2]K_k \vee K_1$, where $r = n - k\alpha + 2k - 3$. Clearly, $H_1 \in \mathcal{O}_k(n, \alpha)$. For a graph $H_2 = K_1 \vee [(\alpha-3)/2]K_k \vee K_{s+1} \vee K_{t+1} \vee [(\alpha-3)/2]K_k \vee K_1$, we suppose its vertex partition $J_0 \cup J_1 \cup \cdots \cup J_\alpha$ defined as above. We show that $C(H_1) = C(H_2)$. If one of s, t is $k-1$, then $C(H_1) \cong C(H_2)$. Suppose that $s, t \geq k$. Let $Q \subseteq J_{\frac{\alpha+1}{2}} \setminus \{u_{\frac{\alpha+1}{2}}\}$ and $|Q| = t - k + 1$. We derive H_1 from H_2 through the subsequent graph transformation:

$$H_1 = H_2 - \{xy : x \in Q, y \in J_{\frac{\alpha+3}{2}}\} + \{xy : x \in Q, y \in J_{\frac{\alpha-3}{2}}\}.$$

It is straightforward to verify that $J_0 \cup J_1 \cup \cdots \cup J_{\frac{\alpha-3}{2}} \cup (J_{\frac{\alpha-1}{2}} \cup Q) \cup (J_{\frac{\alpha+1}{2}} \setminus Q) \cup J_{\frac{\alpha+3}{2}} \cup \cdots \cup J_\alpha$ is a partition of $V(H_1)$. From the structure of H_1 and H_2, it is straightforward to verify that

$$\begin{aligned} C_{H_2}(v) &= C_{H_1}(v) \text{ for each } v \in J_{\frac{\alpha-1}{2}} \cup J_{\frac{\alpha+1}{2}}, \\ C_{H_2}(v) &= C_{H_1}(v) - 2^{-2}|Q| \text{ for each } v \in J_0 \cup \cdots \cup J_{\frac{\alpha-3}{2}}, \\ C_{H_2}(v) &= C_{H_1}(v) + 2^{-2}|Q| \text{ for each } v \in J_{\frac{\alpha+3}{2}} \cup \cdots \cup J_\alpha. \end{aligned}$$

This gives

$$\begin{aligned} C(H_2) - C(H_1) &= \sum_{v \in J_{\frac{\alpha-1}{2}} \cup J_{\frac{\alpha+1}{2}}} (C_{H_2}(v) - C_{H_1}(v)) + \sum_{v \in J_0 \cup \cdots \cup J_{\frac{\alpha-3}{2}}} (C_{H_2}(v) - C_{H_1}(v)) \\ &+ \sum_{v \in J_{\frac{\alpha+3}{2}} \cup \cdots \cup J_\alpha} (C_{H_2}(v) - C_{H_1}(v)) \\ &= \sum_{v \in J_0 \cup \cdots \cup J_{\frac{\alpha-3}{2}}} (C_{H_1}(v) - 2^{-2}|Q| - C_{H_1}(v)) \\ &+ \sum_{v \in J_{\frac{\alpha+3}{2}} \cup \cdots \cup J_\alpha} (C_{H_1}(v) + 2^{-2}|Q| - C_{H_1}(v)) \\ &= 0. \end{aligned}$$

Thus, $C(H_2) = C(H_1)$. This completes the proof. □

In the following, we obtain the graph that maximizes the closeness within k-connected graphs of order n and a fixed independence number λ.

If $\lambda = 1$, then K_n is a unique k-connected graph of independence number of 1 that achieve maximum closeness. Consequently, we will focus on cases $\lambda \geq 2$ in the following.

Let $A_k(n, \lambda) = K_k \vee [K_1 \cup (K_{n-k-\lambda} \vee (\lambda - 1)K_1)]$. Evidently, $A_k(n, \lambda)$ is a k-connected graph of order n and independence number λ.

Theorem 3. *Let G be a k-connected graph of order n with independence number $\lambda \geq 2$ that maximizes closeness. Then, $G \cong A_k(n, \lambda)$.*

Proof. Note that G is k-connected graph of order n with independence number λ. It follows that the inequality $k + \lambda \leq n$ holds. In the specific scenario where $k + \lambda = n$, it can be established that $G \cong K_k \vee \lambda K_1$, and the result is valid in this case. Consequently, we will focus on the case where $k + \lambda + 1 \leq n$ in the subsequent.

Assume G be a k-connected graph of order n with independence number λ, and let $C(G)$ be maximized. Denote M as the maximum independent set and Z as the vertex cut of G, where $|M| = \lambda$ and $|Z| = k$. Let $H_1, H_2, \ldots, H_t$ represent the components of $G - Z$ with $t \geq 2$. We will assume that the sizes of these components satisfy $|H_1| \geq |H_2| \geq \cdots \geq |H_t|$. It is claimed that H_1 must be non-trivial; if it were trivial, then each H_i for $i \in \{1, 2, \ldots, t\}$ would also be trivial, leading to an independence number for G of at least $n - k$ $(\geq \lambda + 1)$, resulting in a contradiction. Therefore, we conclude that H_1 is indeed non-trivial. Let us define $|Z \cap M| = a, |Z \setminus M| = b$ and $|V(H_i) \cap M| = s_i, |V(H_i) \setminus M| = m_i$ for $i \in \{1, 2, \ldots, t\}$. It is evident that $k = a + b$ and $V(H_i) = s_i + m_i$ for $i \in \{1, 2, \ldots, t\}$. We proceed with the subsequent claims.

Claim 1. The structure $G - Z$ comprises precisely two components, i.e., $t = 2$.

Proof of Claim 1. Assume that $t \geq 3$. Given that H_1 is non-trivial, it follows that $V(H_1) \setminus M \neq \emptyset$. We can select $x \in V(H_1) \setminus M$ and $y \in V(H_2)$. Define the graph $Q = G + xy$. It is evident that Q is k-connected graph of order n with an independence number λ. By Lemma 1, it holds that $C(G) < C(Q)$, which leads to a contradiction. Therefore, we conclude that $t = 2$.

Claim 2. $G[Z] \cong K_b \vee aK_1, H_i \cong K_{m_i} \vee s_i K_1$ and $G[V(H_i) \cup Z] \cong K_{b+m_i} \vee (a + s_i)K_1$ for $i = 1, 2$.

Proof of Claim 2. Initially, we establish that $G[Z] \cong K_b \vee aK_1$. Assume, for the sake of contradiction, that $G[Z] \ncong K_b \vee aK_1$. This assumption implies the existence of $x, y \in Z \setminus M$ or $x \in Z \setminus M, y \in Z \cap M$. Consider the graph $Q' = G + uv$. It is evident that Q' is k-connected graph of order n with an independence number λ. According to Lemma 1, it follows that $C(G) < C(Q')$, leading to a contradiction. Therefore, we conclude that $G[Z] \cong K_b \vee aK_1$. Similarly, we can prove that $H_i \cong K_{m_i} \vee s_i K_1$ and $G[V(H_i) \cup Z] \cong K_{b+m_i} \vee (a + s_i)K_1$ for $i = 1, 2$.

Claim 3. H_2 is trivial.

Proof of Claim 3. Assuming that H_2 is non-trivial. Then, we consider the following two possible cases.

Case 1. $s_2 = 0$.

If $a = 0$, then $M = V(H_1) \cap M$. We can choose $w \in V(H_2) \setminus M$, we arrive at the conclusion that the set $M \cup \{w\}$ constitutes an independent set, with the cardinality $|M \cup \{w\}| = \lambda + 1$, which presents a contradiction. Therefore, it follows that $a \geq 1$.

Let $G' = G - \{wu : u \in V(H_2) \setminus \{w\}\} + \{uv : u \in V(H_2) \setminus \{w\}, v \in V(H_1)\}$. It is evident that G' is k-connected graph of order n and an independence number λ. From the construction of G', it is straight forward to verify that $C_G(v) = C_{G'}(v)$ for $v \in Z$. Moreover, $C_G(w) = C_{G'}(w) + 2^{-2}(m_2 - 1)$; $C_G(v) = C_{G'}(v) - 2^{-2}(m_2) + 2^{-2}$ for each $v \in V(H_1)$; $C_G(v) = C_{G'}(v) - 2^{-2}(s_1) - 2^{-2}(m_1) + 2^{-2}$ for each $v \in V(H_2) \setminus \{w\}$.

This gives

$$\begin{aligned} C(G) - C(G') &= [C_G(w) - C_{G'}(w)] + \sum_{v \in V(H_1)} [C_G(v) - C_{G'}(v)] \\ &+ \sum_{v \in V(H_2)\setminus\{w\}} [C_G(v) - C_{G'}(v)] \\ &= [C_{G'}(w) + 2^{-2}(m_2 - 1) - C_{G'}(w)] \\ &+ \sum_{v \in V(H_1)} [C_{G'}(v) - 2^{-2}(m_2) + 2^{-2} - C_{G'}(v)] \\ &+ \sum_{v \in V(H_2)\setminus\{w\}} [C_{G'}(v) - 2^{-2}(s_1) - 2^{-2}(m_1) + 2^{-2} - C_{G'}(v)] \\ &< 2^{-2}(-s_1 - m_1 + 1) \\ &< 0. \end{aligned}$$

The last inequality follows from the fact that $V(H_1) = s_1 + m_1 \geq 2$. Consequently, this leads to the conclusion that $C(G) < C(G')$, a contradiction.

Case 2. $s_2 \neq 0$.

Choose $w \in V(H_2) \cap M$. Let $G'' = G - \{wu : u \in V(H_2)\} + \{uv : u \in V(H_1) \cap M, v \in V(H_2) \setminus M\} + \{uv : u \in V(H_1) \setminus M, v \in V(H_2) \setminus \{w\}\}$. Clearly, G'' is k-connected graph of order n with an independence number λ. From the structure of G'', it is not difficult to verify that $C_G(v) = C_{G''}(v)$ for each $v \in Z$. Moreover, $C_G(w) = C_{G''}(w) + 2^{-2}(m_2)$; $C_G(v) = C_{G''}(v) - 2^{-2}(m_2)$ for each $v \in V(H_1) \cap M$; $C_G(v) = C_{G''}(v) - 2^{-2}(s_2) - 2^{-2}(m_2) + 2^{-2}$ for each $v \in V(H_1) \setminus M$; $C_G(v) = C_{G''}(v) - 2^{-2}(m_1)$ for each $v \in (V(H_2) \cap M) \setminus \{w\}$; $C_G(v) = C_{G''}(v) - 2^{-2}(s_1) - 2^{-2}(m_1) + 2^{-2}$ for each $v \in V(H_2) \setminus M$.

This provides

$$\begin{aligned} C(G) - C(G'') &= [C_G(w) - C_{G''}(w)] + \sum_{v \in V(H_1)\cap M} [C_G(v) - C_{G''}(v)] \\ &+ \sum_{v \in V(H_1)\setminus M} [C_G(v) - C_{G''}(v)] \\ &+ \sum_{v \in (V(H_2)\cap M)\setminus\{w\}} [C_G(v) - C_{G''}(v)] + \sum_{v \in V(H_2)\setminus M} [C_G(v) - C_{G''}(v)] \\ &= [C_{G''}(w) + 2^{-2}(m_2) - C_{G''}(w)] + \\ &+ \sum_{v \in V(H_1)\cap M} [C_{G''}(v) - 2^{-2}(m_2) - C_{G''}(v)] \\ &+ \sum_{v \in V(H_1)\setminus M} [C_{G''}(v) - 2^{-2}(s_2) - 2^{-2}(m_2) + 2^{-2} - C_{G''}(v)] \\ &+ \sum_{v \in (V(H_2)\cap M)\setminus\{w\}} [C_{G''}(v) - 2^{-2}(m_1) - C_{G''}(v)] \\ &+ \sum_{v \in V(H_2)\setminus M} [C_{G''}(v) - 2^{-2}(s_1) - 2^{-2}(m_1) + 2^{-2} - C_{G''}(v)] \\ &< 2^{-2}(-s_1 - m_1 + 1) \\ &< 0, \end{aligned}$$

a contradiction. Thus, H_2 is trivial.

Claim 4. $V(H_2) \subseteq M$.

Proof of Claim 4. Assume that $V(H_2) \nsubseteq M$; then, $a \geq 2$. Suppose that $a \leq 1$. If $a = 1$, i.e., $Z \cap M = \{u\}$. Let $G^* = G + \{ux : x \in V(H_1) \cap M\}$. Clearly, G^* is k-connected graph of order n with an independence number λ. According to Lemma 1, we obtain $C(G) < C(G^*)$, which leads to a contradiction. If $a = 0$, then $M \cup V(H_2)$ is an independent set of G such that $|M \cup V(H_2)| = \lambda + 1$, a contradiction. So, $a \geq 2$. Given that H_1 is non-trivial, then $V(H_1) \setminus M \neq \emptyset$. We select $u' \in V(H_1) \setminus M$. Define a graph $G^{**} = G - \{uv : \{v\} = V(G_2)\} + \{u'v : \{v\} = V(G_2)\}$. Evidently, G^{**} is k-connected graph of order n with an independence number λ.

From G to G^{**}, it is easy to verify that $C_G(x) = C_{G^{**}}(x)$ for each $x \in V(G) \setminus \{u, u'\}$. Furthermore, $C_G(u) = C_{G^{**}}(u) + 2^{-2}$; $C_G(u') = C_{G^{**}}(u') - 2^{-1}$.

We obtain

$$\begin{aligned} C(G) - C(G^{**}) &= [C_G(u) - C_{G^{**}}(u)] + [C_G(u') - C_{G^{**}}(u')] \\ &= [C_{G^{**}}(u) + 2^{-2} - C_{G^{**}}(u)] + [C_{G^{**}}(u') - 2^{-1} - C_{G^{**}}(u')] \\ &= -\frac{1}{4} \\ &< 0, \end{aligned}$$

a contradiction. Thus, $V(H_2) \subseteq M$. From claims 1 to 4, it follows that $G \cong A_k(n, \lambda)$. □

In the following, we determine the graph that maximizes the closeness within k-connected graphs of order n with a fixed minimum degree β.

Let $F_k(n, \beta) = K_k \vee (K_{\beta-k+1} \cup K_{n-\beta-1})$. Obviously, $F_k(n, \beta)$ is a k-connected graphs of order n with a minimum degree β.

Theorem 4. *Let G be a k-connected graph of order n with a minimum degree β that maximizes closeness. Then, $G \cong F_k(n, \beta)$.*

Proof. Let us consider a graph G that is k-connected of order n with a minimum degree β. It is important to note that $k + 1 \leq n$. In the case where $k + 1 = n$, it follows that $G \cong F_k(n, \beta)$, and the result is valid in this case. Consequently, we will focus on the case where $k + 2 \leq n$.

Let G be a k-connected graph of order n with a minimum degree β, with the largest possible value of $C(G)$. Let Z denote the vertex cut of G such that $|Z| = k$. The components of of $G - Z$ are represented as $H_1, H_2, \ldots, H_t$ where $t \geq 2$. We assert that $t = 2$. Suppose that $t \geq 3$; then, let H_1, H_2, H_3 represent at least three components of G. We can construct a new graph $G' = G + \{uv : u \in V(H_2), v \in V(H_3)\}$. It is evident that G' remains a k-connected graph of order n with a minimum degree β. According to Lemma 1, it follows that $C(G) < C(G')$, leading to a contradiction. Therefore, we conclude that $t = 2$. Furthermore, by applying Lemma 1, we find that both $G[V(H_1) \cup Z]$ and $G[V(H_2) \cup Z]$ are complete graphs. This leads us to the conclusion that $G \cong K_k \vee (K_{q_1} \cup K_{q_2})$, where $q_1 = |V(H_1)|, q_2 = |V(H_2)|$, and $q_1 + q_2 = n - k$. For the sake of simplicity, we will assume $q_1 \leq q_2$.

To finalize the proof, it is sufficient to prove that either $q_1 = \beta - k + 1$ or $q_2 = \beta - k + 1$. Assume, for the sake of argument, that $q_1 > \beta - k + 1$. Consider a vertex $u_0 \in V(H_1)$. We can define a new graph $G' = G - \{u_0v : v \in V(H_1) \setminus \{u_0\}\} + \{u_0v : v \in V(H_2)\}$. It is evident that G' remains a k-connected graph of order n with a minimum degree β.

From G to G', it is easy to verify that $C_G(v) = C_{G'}(v)$ for each $v \in Z$; $C_G(u_0) = C_{G'}(u_0) - 2^{-2}$; $C_G(v) = C_{G'}(v) + 2^{-2}$ for each $v \in V(H_1) \setminus \{u_0\}$; $C_G(v) = C_{G'}(v) - 2^{-2}$ for each $v \in V(H_2)$.

This provides

$$\begin{aligned} C(G) - C(G') &= [C_G(u_0) - C_{G'}(u_0)] + \sum_{v \in V(H_1) \setminus \{u_0\}} [C_G(v) - C_{G'}(v)] \\ &+ \sum_{v \in V(H_2)} [C_G(v) - C_{G'}(v)] \\ &= -2^{-2} + 2^{-2} - 2^{-2} \\ &< 0, \end{aligned}$$

a contradiction. Therefore, we have $q_1 = \beta - k + 1$ and consequently, $q_2 = n - \beta - 1$. Hence, it follows that $G \cong F_k(n, \beta)$. This concludes the proof. □

4. Concluding Remarks

Nodes with high closeness are vital in numerous network applications, including communication networks, social networks, and transportation networks, as they enable rapid information exchange, impact decision-making, and improve overall network resilience. Therefore, comprehending the closeness of nodes offers significant insights into the structural and functional attributes of complex networks. In this paper, we have thoroughly investigated the concept of closeness within n-vertex graphs with fixed connectivity k. We identified the graphs that maximize closeness among all graphs of order n with connectivity k. We also determined those graphs that achieved the maximum closeness over the k-connected graphs with fixed order and one of the additional fixed parameters such as diameter, independence number, and minimum degree. However, finding the graphs that minimize closeness in this same class is still an open problem. This leads to an interesting direction for future research.

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