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On the distribution-tail of the product of gamma random variables

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Abstract. In this paper, we consider the product $\Pi_n := \prod_{k=1}^{n} \xi_k$ of n independent identically distributed gamma random variables $\xi_1, \xi_2, \dots, \xi_n$. We derive an asymptotic formula for the survival probability $\mathbf{P}(\Pi_n > x)$ as $x \to \infty$ with the first two remaining terms.

Keywords: gamma distribution, tail distribution function, saddle-point method, asymptotic functions, product of random variables.

1 Introduction

Let ξ be a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. A random variable (r.v.) $\xi : \Omega \to \mathbb{R}$ is said to have a gamma distribution $\Gamma(\alpha, \beta)$ if its distribution is given by the following density function:

$$f_\xi(x) = \frac{x^{\alpha-1}\mathrm{e}^{-\beta x}\beta^\alpha}{\Gamma(\alpha)}, \quad x > 0,$$

where $\alpha > 0$ and $\beta > 0$ are two parameters, and Γ denotes the standard gamma function, i.e.,

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}\mathrm{e}^{-x}\,\mathrm{d}x.$$

Thus, the distribution function (d.f.) $F_\xi(x) = \mathbf{P}(\xi \leqslant x)$ of the r.v. ξ with gamma distribution has the form

$$F_\xi(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^x y^{\alpha-1}\mathrm{e}^{-\beta y}\,\mathrm{d}y, \quad x > 0.$$

A glance at the literature shows that the theory of the product of independent r.v.s is much less studied than the theory of the sum of independent r.v.s; we mention books

[3,14,34]. Most of the literature on product theory is devoted to normal random variables (see, e.g., [2, 7, 15, 20, 23, 32, 33]), but one can also find articles on gamma random variables.

Nadarajah and Kotz [27,28] obtained exact d.f.s by multiplying two gamma and beta, and gamma and Weibull distributions, respectively. Malik [24] expressed the density function in terms of two generalized gamma distributions. Springer and Thompson [35], using the Melin transformation, obtained the exact density function of n gamma distributions

$$f_{\Pi_n}(x) = \frac{1}{n\Gamma(\alpha)} G_{0,n}^{n,0}(x \mid \alpha-1, \alpha-1, \dots, \alpha-1) = \frac{1}{2\pi \mathrm{i} n\,\Gamma(\alpha)} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} x^{-s}\,\Gamma^n(s+\alpha-1)\,\mathrm{d}s,$$

where f_{Π_n} is the density function of the product of the gamma distributions, c is any positive constant, and G is the Meijer G-function defined by the following equation in the general case:

$$G_{p,q}^{m,n}\left(z \;\middle|\; \begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix}\right) = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{L}} z^{-s} \frac{\prod_{j=1}^{m}\Gamma(b_j+s)\prod_{j=1}^{n}\Gamma(1-a_j-s)}{\prod_{j=n+1}^{p}\Gamma(a_j+s)\prod_{j=m+1}^{q}\Gamma(1-b_j-s)}\,\mathrm{d}s,$$

where the infinite contour of integration $\mathcal{L}$ separates the poles of $\Gamma(1-a_j-s)$, $j \in \{1,\dots,p\}$, from the poles of $\Gamma(b_j+s)$, $j \in \{1,\dots,q\}$; see, for instance, [4, 31] and reference therein.

In this paper, the asymptotic behaviour of the tail function of the product of gamma distributions will be investigated using the saddle-point method and the theorem developed by Arendarzcyk and Dębicki [1]. For the saddle-point method, we will use an auxiliary lemma of Wong [37], which will help us evaluate the special form of the integral.

2 Asymptotic behaviour of the product distribution tail

The asymptotic behaviour of the tail of the product of gamma r.v.s can be derived from exact formulas or using the saddle-point method described in detail by Butler [6], Fedoryuk [12], and Jensen [17], for instance. Using the latter method described in [17], Arendarczyk and Dębicki obtained the following result on the Weibull-type tail distributions; see [1, Lemma 2.1].

Theorem 1. *Let ξ_1 and ξ_2 be two independent, nonnegative r.v.s such that*

$$\overline{F}_{\xi_1}(x) \underset{x\to\infty}{\sim} C_1 x^{\gamma_1} \exp\{-\beta_1 x^{\alpha_1}\}, \qquad \overline{F}_{\xi_2}(x) \underset{x\to\infty}{\sim} C_2 x^{\gamma_2} \exp\{-\beta_2 x^{\alpha_2}\},$$

where $C_i > 0$, $\gamma_i \in \mathbb{R}$, $\beta_i > 0$, $\alpha_i > 0$, $i = 1, 2$. Then

$$\overline{F}_{\xi_1\xi_2}(x) \underset{x\to\infty}{\sim} C x^{\gamma} \exp\{-\beta x^{\alpha}\},$$

where

$$\alpha = \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2},$$
$$\beta = \beta_1^{\alpha_2/(\alpha_1+\alpha_2)}\beta_2^{\alpha_1/(\alpha_1+\alpha_2)}\biggl(\biggl(\frac{\alpha_1}{\alpha_2}\biggr)^{\alpha_2/(\alpha_1+\alpha_2)} + \biggl(\frac{\alpha_2}{\alpha_1}\biggr)^{\alpha_1/(\alpha_1+\alpha_2)}\biggr),$$
$$\gamma = \frac{\alpha_1\alpha_2+2\alpha_1\gamma_2+2\alpha_2\gamma_1}{2(\alpha_1+\alpha_2)},$$
$$C = \sqrt{2\pi}\frac{C_1C_2}{\sqrt{\alpha_1+\alpha_2}}(\alpha_1\beta_1)^{(\alpha_2-2\gamma_1+2\gamma_2)/(2(\alpha_1+\alpha_2))} \times (\alpha_2\beta_2)^{(\alpha_1-2\gamma_2+2\gamma_1)/(2(\alpha_1+\alpha_2))}.$$

In particular, this theorem implies that if ξ_1, ξ_2, ξ_3 are independent and distributed according to the gamma law $\Gamma(\alpha,\beta)$, then

$$\overline{F}_{\xi_1\xi_2}(x) \underset{x\to\infty}{\sim} \sqrt{\pi}\,\frac{\beta^{2(\alpha-1)+1/2}}{\Gamma^2(\alpha)}\,x^{\alpha-3/4}\exp\bigl\{-2\beta x^{1/2}\bigr\}, \qquad (1)$$
$$\overline{F}_{\xi_1\xi_2\xi_3}(x) \underset{x\to\infty}{\sim} \frac{2\pi}{\sqrt{3}}\,\frac{\beta^{3(\alpha-1)+1}}{\Gamma^3(\alpha)}\,x^{\alpha-2/3}\exp\bigl\{-3\beta x^{1/3}\bigr\}.$$

We observe that the product of gamma-distributed r.v.s, which are light-tailed, produces heavy-tailed or even subexponential distributions widely described in books [11, 13, 19, 21, 25, 29] and in reference therein. Recall that a distribution F on $\mathbb{R}$ is said to be heavy-tailed, denoted $F \in \mathscr{H}$, if its Laplace–Stieltjes transform satisfies the following property:

$$\int_{-\infty}^{\infty} \mathrm{e}^{\delta x}\,\mathrm{d}F(x) = \infty \quad \text{for any } \delta > 0.$$

Otherwise, F is said to be light-tailed, denoted $F \in \mathscr{H}^c$. A distribution F on $\mathbb{R}_+$ is said to be subexponential, $F \in \mathscr{S}$, if

$$\overline{F*F}(x) \underset{x\to\infty}{\sim} 2\overline{F}(x),$$

where $F*F$ denotes the convolution of d.f. F with itself. In general, it is not easy to verify this subexponentiality condition. When F is absolutely continuous with density f, Pitman [30] provided a complete characterization of subexponential distributions on $\mathbb{R}_+$ in terms of their hazard rate function $q(x) = f(x)/\overline{F}(x)$. According to [30, Thm. 2], a sufficient condition for subexponentiality is the integrability of function $\mathrm{e}^{xq(x)}f(x)$ on $\mathbb{R}_+ = [0,\infty)$. In addition, according to [30, Cor. 1], the class of d.f.s $\mathscr{S}$ is closed under strong tail-equivalence. This means the following:

$$F \in \mathscr{S}, \quad \overline{G}(x) \underset{x\to\infty}{\sim} c\overline{F}(x), \quad c > 0 \implies G \in \mathscr{S}.$$

Consequently, the asymptotic relation (1) implies that d.f. $F_{\xi_1\xi_2} \in \mathscr{S}$ because

$$\int_0^{\infty} \mathrm{e}^{xq(x)}f(x)\,\mathrm{d}x < \infty$$

with

$$f(x) = d^* \left(\left(\frac{3}{4} - \alpha \right) x^{\alpha - 7/4} + \beta x^{\alpha - 5/4} \right) \mathrm{e}^{-2\beta\sqrt{x}}, \quad x \geqslant d_*,$$
$$xq(x) = \left(\frac{3}{4} - \alpha \right) + \beta\sqrt{x}, \quad x \geqslant d_*,$$

and suitable positive constants d^*, d_* depending on parameters α, β. We get obviously that $F_{\xi_1\xi_2} \in \mathscr{H}$ for independent r.v.s distributed according to the gamma law $\Gamma(\alpha,\beta)$ because $\mathscr{S} \subset \mathscr{H}$; see, for instance, [10, Lemma 2.4]. Among other things, it follows from this that d.f. $F_{\xi_1\xi_2\ldots\xi_n}$ with independent gamma-distributed r.v.s $\{\xi_1, \xi_2, \ldots, \xi_n\}$ belongs to the class $\mathscr{H}$ for any $n \geqslant 2$ due to the obvious estimates

$$\mathbf{E}\mathrm{e}^{\lambda\xi_1\xi_2\cdots\xi_{n+1}} \geqslant \mathbf{E}\mathrm{e}^{\lambda\xi_1\xi_2\cdots\xi_{n+1}}\mathbf{I}_{\{\xi_{n+1}>1\}} \geqslant \mathbf{E}\mathrm{e}^{\lambda\xi_1\xi_2\cdots\xi_n}\mathbf{P}(\xi_{n+1} > 1),$$

provided that $n \geqslant 1$ and $\lambda > 0$.

We refer to [8, 18, 22, 26, 36] and references therein for various extensions related to the phenomenon showing how multiplying random variables change the heaviness of their tails.

Hashorva and Weng [16] derived the following general relation (see also [5, 9] for details) by using Theorem 1:

$$\overline{F}_{\Pi_n}(x) \underset{x\to\infty}{\sim} \frac{(2\pi)^{(n-1)/2}}{\sqrt{n}} \frac{\beta^{n(\alpha-1)+(n-1)/2}}{\Gamma^n(\alpha)} x^{\alpha-(n+1)/(2n)} \exp\bigl\{-n\beta x^{1/n}\bigr\}$$

valid for each natural n, where Π_n is the product of n independent r.v.s distributed according to $\Gamma(\alpha,\beta)$.

In our paper, we derive the more precise formula for the tail of the product of gamma r.v.s. The following statement is the main result of the paper.

Theorem 2. *Let $\xi_1, \xi_2, \ldots$ be i.i.d. r.v.s such that, for each k, $\xi_k \sim \Gamma(\alpha,\beta)$, and let $\Pi_n := \prod_{k=1}^n \xi_k$. Then*

$$\overline{F}_{\Pi_n}(x) = \frac{(2\pi)^{(n-1)/2}}{\sqrt{n}} \frac{\beta^{n(\alpha-1)+(n-1)/2}}{\Gamma^n(\alpha)} x^{\alpha-(n+1)/(2n)} \exp\bigl\{-n\beta x^{1/n}\bigr\} \times \bigl(1 + \beta^{-1}x^{-1/n}(\alpha - 1 - \mathcal{S}_n) + O_n\bigl(x^{-3/(2n)}\bigr)\bigr), \tag{2}$$

where $\mathcal{S}_1 = 0$,

$$\mathcal{S}_n = \sum_{k=2}^n \frac{k(k-1) - 11}{24k(k-1)}, \quad n \geqslant 2,$$

and the constant in the symbol O_n depends on α, β, and n.

It is well known that the exponential, Erlang, and χ^2 are separate cases of the gamma distribution. The exponential distribution is obtained by choosing $\alpha = 1$, the Erlang distribution is obtained by choosing $\alpha \in \mathbb{N}$, and chi-square distribution $\chi^2_\varkappa \sim \Gamma(\varkappa/2, 1/2)$, where $\varkappa$ is the number of degrees of freedom. We formulate two corollaries, obtaining the product of tail functions for exponential and chi-square distributions.

Corollary 1. *Let $\eta_1, \eta_2, \dots$ be i.i.d. r.v.s such that, for each k, $\eta_k \sim \mathrm{d}\exp(\lambda)$, and let $\Pi_n := \prod_{k=1}^n \eta_k$. Then*

$$\overline{F}_{\Pi_n}(x) = \frac{(2\pi\lambda)^{(n-1)/2}}{\sqrt{n}} x^{(n-1)/(2n)} \exp\bigl\{-n\lambda x^{1/n}\bigr\} \times \bigl(1 - \lambda^{-1} x^{-1/n} \mathcal{S}_n + O_n\bigl(x^{-3/(2n)}\bigr)\bigr),$$

where $\mathcal{S}_n$, $n \geqslant 1$, are defined in Theorem 2, and the constant in the symbol O_n depends on λ and n.

Corollary 2. *Let $\eta_1, \eta_2, \dots$ be i.i.d. r.v.s such that, for each k, $\eta_k \sim \chi^2_\varkappa$, and let $\Pi_n := \prod_{k=1}^n \eta_k$. Then*

$$\overline{F}_{\Pi_n}(x) = \frac{(2\pi)^{(n-1)/2}}{\sqrt{n}} \frac{x^{\varkappa/2-(n+1)/2}}{2^{n(\varkappa/2-1)+(n-1)/2}\,\Gamma^n(\frac{\varkappa}{2})} \exp\biggl\{-\frac{1}{2} n x^{1/n}\biggr\} \times \biggl(1 + x^{-1/n} \frac{1}{2}\biggl(\frac{\varkappa}{2} - 1 - \mathcal{S}_n\biggr) + O_n\bigl(x^{-3/(2n)}\bigr)\biggr),$$

where $\mathcal{S}_n$, $n \geqslant 1$, are defined above, and the constant in the symbol O_n depends on $\varkappa$ and n but not depends on x.

3 Proof of the main theorem

The tail function $\overline{F}_\xi = 1 - F_\xi$ of the r.v. ξ distributed according to the law $\Gamma(\alpha, \beta)$ satisfies the following inequalities for all positive x.

(i) If $0 < \alpha \leqslant 1$, then

$$\overline{F}_\xi(x) \geqslant \frac{x^{\alpha-1} \mathrm{e}^{-\beta x} \beta^{\alpha-1}}{\Gamma(\alpha)} \biggl(1 + \frac{\alpha - 1}{\beta x}\biggr), \tag{3}$$

$$\overline{F}_\xi(x) \leqslant \frac{x^{\alpha-1} \mathrm{e}^{-\beta x} \beta^{\alpha-1}}{\Gamma(\alpha)} \biggl(1 + \frac{\alpha - 1}{\beta x} + \frac{(\alpha-1)(\alpha-2)}{\beta^2 x^2}\biggr). \tag{4}$$

(ii) If $1 < \alpha \leqslant 2$, then

$$\overline{F}_\xi(x) \geqslant \frac{x^{\alpha-1} \mathrm{e}^{-\beta x} \beta^{\alpha-1}}{\Gamma(\alpha)} \biggl(1 + \frac{\alpha - 1}{\beta x} + \frac{(\alpha-1)(\alpha-2)}{\beta^2 x^2}\biggr), \tag{5}$$

$$\overline{F}_\xi(x) \leqslant \frac{x^{\alpha-1} \mathrm{e}^{-\beta x} \beta^{\alpha-1}}{\Gamma(\alpha)} \biggl(1 + \frac{\alpha - 1}{\beta x}\biggr). \tag{6}$$

(iii) If $\alpha > 2$, then

$$\overline{F}_\xi(x) \geqslant \frac{x^{\alpha-1} \mathrm{e}^{-\beta x} \beta^{\alpha-1}}{\Gamma(\alpha)} \biggl(1 + \frac{\alpha - 1}{\beta x}\biggr), \tag{7}$$

$$\overline{F}_\xi(x) \leqslant \frac{x^{\alpha-1} \mathrm{e}^{-\beta x} \beta^{\alpha-1}}{\Gamma(\alpha)} \Biggl(1 + (\alpha - 1)\beta^{-1} x^{-1} + C_3 \sum_{k=2}^{[\alpha]} x^{-k}\Biggr), \tag{8}$$

where $[\alpha]$ denotes the integer part of parameter α, and the positive quantity $C_3 = C_3(\alpha, \beta)$ depends only on α and β.

Inequalities (3)–(7) follow from representation

$$\overline{F}_\xi(x) = \frac{x^{\alpha-1}\mathrm{e}^{-\beta x}\beta^{\alpha-1}}{\Gamma(\alpha)} + \frac{x^{\alpha-2}\mathrm{e}^{-\beta x}\beta^{\alpha-2}(\alpha-1)}{\Gamma(\alpha)} + \frac{\beta^{\alpha-2}(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\int_x^\infty y^{\alpha-3}\mathrm{e}^{-\beta y}\,\mathrm{d}y$$

valid for all $x > 0$. While inequality (8) follows from the more detailed expression

$$\begin{aligned}\overline{F}_\xi(x) &= \frac{x^{\alpha-1}\mathrm{e}^{-\beta x}\beta^{\alpha-1}}{\Gamma(\alpha)} + \frac{\mathrm{e}^{-\beta x}}{\Gamma(\alpha)}\sum_{k=1}^{[\alpha]}\beta^{\alpha-k-1}x^{\alpha-k-1}(\alpha-1)(\alpha-2)\cdots(\alpha-k)\\ &\quad + \frac{\beta^{\alpha-[\alpha]-1}(\alpha-1)(\alpha-2)\cdots(\alpha-[\alpha]-1)}{\Gamma(\alpha)}\int_x^\infty y^{\alpha-[\alpha]-2}\mathrm{e}^{-\beta y}\,\mathrm{d}y\\ &\leqslant \frac{x^{\alpha-1}\mathrm{e}^{-\beta x}\beta^{\alpha-1}}{\Gamma(\alpha)} + \frac{\mathrm{e}^{-\beta x}}{\Gamma(\alpha)}\sum_{k=1}^{[\alpha]}\beta^{\alpha-k-1}x^{\alpha-k-1}(\alpha-1)(\alpha-2)\cdots(\alpha-k).\end{aligned}$$

From the obtained inequalities (3)–(8) we derive that the following asymptotic equality holds for all cases of the parameter α:

$$\overline{F}_\xi(x) = \frac{x^{\alpha-1}\mathrm{e}^{-\beta x}\beta^{\alpha-1}}{\Gamma(\alpha)}\bigl(1 + (\alpha-1)\beta^{-1}x^{-1} + O\bigl(x^{-2}\bigr)\bigr), \tag{9}$$

where the constant in the symbol O depends on α and β.

In order to prove the main theorem, we provide the reader with an auxiliary lemma, which we will use multiple times in our proof. The lemma below can be obtained by applying the saddle-point method to a real integral of a special form. The proof of the lemma can be found in [37, Chap. II, Thm. 1].

Lemma 1. *Let h and g be two real functions defined on an interval $[a, b)$ (b can be finite or infinite) such that*:

(i) *as $z \searrow a$ for all $N \geqslant 1$,*

$$h(z) = h(a) + \sum_{k=0}^{N} a_k(z-a)^{k+\mu} + o\bigl((z-a)^{N+\mu}\bigr),$$

$$g(z) = \sum_{k=0}^{N} b_k(z-a)^{k+\nu-1} + o\bigl((z-a)^{N+\nu-1}\bigr),$$

$$h'(z) = \sum_{k=0}^{N}(k+\mu)a_k(z-a)^{k+\mu-1} + o\bigl((z-a)^{N+\mu-1}\bigr),$$

where $a_0 \neq 0$, $b_0 \neq 0$, $\mu > 0$, and $\nu > 0$;

(ii) $h(z) > h(a)$, *where* $z \in (a, b)$, *and for all* $\delta > 0$,

$$\inf_{z\in[a+\delta,b)} \bigl(h(z) - h(a)\bigr) > 0;$$

(iii) h' *and* g *are continuous in a neighborhood of point* a.

If the integral $\int_a^b g(z)\mathrm{e}^{-xh(z)}\,\mathrm{d}z$ *converges absolutely for all sufficiently large* x, *then*

$$\int\limits_a^b g(z)\mathrm{e}^{-xh(z)}\,\mathrm{d}z = \mathrm{e}^{-xh(a)}\left(\sum_{k=0}^{N}\Gamma\left(\frac{k+\nu}{\mu}\right)d_k x^{-(k+\nu)/\mu} + O\bigl(x^{-(N+\nu+1)/\mu}\bigr)\right)$$

for all $N \in \mathbb{N}$, *where coefficients* d_k *are expressible in terms of* a_k *and* b_k.

In [37], Wong provided the explicit forms of the first three coefficients:

$$d_0 = \frac{b_0}{\mu a_0^{\nu/\mu}}, \quad d_1 = \left(\frac{b_1}{\mu} - \frac{(\nu+1)a_1 b_0}{\mu^2 a_0}\right)\frac{1}{a_0^{(\nu+1)/\mu}},$$
$$d_2 = \left(\frac{b_2}{\mu} - \frac{(\nu+2)a_1 b_1}{\mu^2 a_0} + \bigl((\nu+\mu+2)a_1^2 - 2\mu a_0 a_2\bigr)\frac{(\nu+2)b_0}{2\mu^3 a_0^2}\right)\frac{1}{a_0^{(\nu+2)/\mu}}.$$

3.1 The case $n = 2$

To prove Theorem 2, we use induction on n. The asymptotic equality (9) implies the assertion of Theorem 2 in the case of $n = 1$. Additionally, we note that constant in the Landau's symbol $O()$, which we use below in this subsection many times, depends on α and β.

Suppose that $n = 2$ and x is sufficiently large. We have

$$\begin{aligned}\overline{F}_{\xi_1\xi_2}(x) &= \frac{\beta^\alpha}{\Gamma(\alpha)}\int\limits_0^\infty \overline{F}_{\xi_1}\left(\frac{x}{y}\right)y^{\alpha-1}\mathrm{e}^{-\beta y}\,\mathrm{d}y\\ &= \frac{\beta^\alpha}{\Gamma(\alpha)}\left(\int\limits_0^{2x^{1/2}} + \int\limits_{2x^{1/2}}^\infty\right)\overline{F}_{\xi_1}\left(\frac{x}{y}\right)y^{\alpha-1}\mathrm{e}^{-\beta y}\,\mathrm{d}y\\ &=: \mathcal{I}_1 + \mathcal{I}_2. \end{aligned} \tag{10}$$

Applying the variable change $y = u\sqrt{x}$ and the upper bound in (9), we get that

$$\begin{aligned}\mathcal{I}_2 = \frac{\beta^{2\alpha-1}x^{\alpha-1}}{\Gamma^2(\alpha)}\int\limits_{2x^{1/2}}^{3x^{1/2}} \mathrm{e}^{-\beta(y+x/y)}\left(1 + \frac{\alpha-1}{\beta}\frac{y}{x} + O\left(\frac{y}{x}\right)^2\right)\mathrm{d}y\\ + \frac{\beta^\alpha}{\Gamma(\alpha)}\int\limits_{3x^{1/2}}^\infty y^{\alpha-1}\mathrm{e}^{-\beta y}\,\mathrm{d}y\end{aligned}$$

$$
\begin{aligned}
&= \frac{\beta^{2\alpha-1}x^{\alpha-1/2}}{\Gamma^2(\alpha)} \int_2^3 \mathrm{e}^{-\beta\sqrt{x}(u+1/u)} \left(1 + \frac{\alpha-1}{\beta}\frac{u}{\sqrt{x}} + O\left(\frac{u^2}{x}\right)\right) \mathrm{d}u \\
&\quad + \frac{\beta^\alpha}{\Gamma(\alpha)} \int_{3x^{1/2}}^{\infty} y^{\alpha-1}\mathrm{e}^{-\beta y}\,\mathrm{d}y \\
&= O\left(x^{\alpha-1/2}\exp\left\{-\frac{5}{2}\beta\sqrt{x}\right\}\right).
\end{aligned} \tag{11}
$$

For the integral $\mathcal{I}_1$, applying the same variable change $y = u\sqrt{x}$, we obtain

$$
\begin{aligned}
\mathcal{I}_1 &= \frac{\beta^{2\alpha-1}x^{\alpha-1}}{\Gamma^2(\alpha)} \int_0^{2x^{1/2}} \mathrm{e}^{-\beta(y+x/y)} \left(1 + \frac{\alpha-1}{\beta}\frac{y}{x} + O\left(\frac{y}{x}\right)^2\right) \mathrm{d}y \\
&= \frac{\beta^{2\alpha-1}x^{\alpha-1/2}}{\Gamma^2(\alpha)} \int_0^2 \mathrm{e}^{-\beta\sqrt{x}(1/u+u)} \left(1 + \frac{\alpha-1}{\beta}\frac{u}{\sqrt{x}} + O\left(\frac{u^2}{x}\right)\right) \mathrm{d}u \\
&= \frac{\beta^{2\alpha-1}x^{\alpha-1/2}}{\Gamma^2(\alpha)} \left(\int_0^2 \mathrm{e}^{-\beta\sqrt{x}(1/u+u)}\,\mathrm{d}u + \frac{\alpha-1}{\beta\sqrt{x}} \int_0^2 u\,\mathrm{e}^{-\beta\sqrt{x}(1/u+u)}\,\mathrm{d}u\right) \\
&\quad + O\left(\mathrm{e}^{-2\beta\sqrt{x}}x^{\alpha-3/2}\right) \\
&=: \frac{\beta^{2\alpha-1}x^{\alpha-1/2}}{\Gamma^2(\alpha)} \left(\mathcal{I}_{11} + \frac{\alpha-1}{\beta\sqrt{x}}\mathcal{I}_{12}\right) + O\left(\mathrm{e}^{-2\beta\sqrt{x}}x^{\alpha-3/2}\right).
\end{aligned} \tag{12}
$$

After changing of variables $v = 1/u$, $u \in (0, 1]$, in integrals $\mathcal{I}_{11}$ and $\mathcal{I}_{12}$, we derive

$$
\begin{aligned}
\mathcal{I}_{11} &= \int_1^\infty \frac{1}{v^2}\,\mathrm{e}^{-\beta\sqrt{x}(1/v+v)}\,\mathrm{d}v + \int_1^2 \mathrm{e}^{-\beta\sqrt{x}(1/u+u)}\,\mathrm{d}u \\
&=: \mathcal{I}_{111} + \mathcal{I}_{112}
\end{aligned} \tag{13}
$$

and

$$
\begin{aligned}
\mathcal{I}_{12} &= \int_1^\infty \frac{1}{v^3}\,\mathrm{e}^{-\beta\sqrt{x}(1/v+v)}\,\mathrm{d}v + \int_1^2 u\,\mathrm{e}^{-\beta\sqrt{x}(1/u+u)}\,\mathrm{d}u \\
&=: \mathcal{I}_{121} + \mathcal{I}_{122}.
\end{aligned} \tag{14}
$$

We have $v + 1/v|_{v=1} = 2$, and

$$
v + \frac{1}{v} = 2 + \sum_{k=0}^{N}(-1)^{k+2}(v-1)^{k+2} + o\left((v-1)^{N+2}\right), \quad v \to 1, \tag{15}
$$

$$\left(v+\frac{1}{v}\right)' = \sum_{k=0}^{N}(-1)^k(k+2)(v-1)^{k+1} + o\big((v-1)^{N+1}\big), \quad v\to 1, \tag{16}$$

$$\frac{1}{v^2} = \sum_{k=0}^{N}(-1)^k(k+1)(v-1)^k + o\big((v-1)^N\big), \quad v\to 1,$$

for each $N \geqslant 1$ according to the Taylor's formula. Hence, using Lemma 1 for integral $\mathcal{I}_{111}$ with suitable parameters

$$a=1, \quad \mu=2, \quad \nu=1,$$
$$a_0=b_0=1, \quad a_1=-1, \quad a_2=1, \quad b_1=-2, \quad b_2=3,$$
$$d_0=\frac{1}{2}, \qquad d_1=-\frac{1}{2}, \quad d_2=\frac{3}{16},$$

we get that

$$\mathcal{I}_{111} = \mathrm{e}^{-2\beta\sqrt{x}}\left(\frac{\sqrt{\pi}}{2}(\beta\sqrt{x})^{-1/2} - \frac{(\beta\sqrt{x})^{-1}}{2} + \frac{3\sqrt{\pi}}{32}(\beta\sqrt{x})^{-3/2} + O\left(\frac{1}{x}\right)\right). \tag{17}$$

The asymptotic relations (15), (16) imply that conditions of Lemma 1 hold for integral $\mathcal{I}_{112}$ with parameters

$$a=1, \quad \mu=2, \quad \nu=1,$$
$$a_0=b_0=1, \quad a_1=-1, \quad a_2=1, \quad b_1=0, \quad b_2=0,$$
$$d_0=\frac{1}{2}, \quad d_1=\frac{1}{2}, \quad d_2=\frac{3}{16}.$$

Therefore, for large x,

$$\mathcal{I}_{112} = \mathrm{e}^{-2\beta\sqrt{x}}\left(\frac{\sqrt{\pi}}{2}(\beta\sqrt{x})^{-1/2} + \frac{(\beta\sqrt{x})^{-1}}{2} + \frac{3\sqrt{\pi}}{32}(\beta\sqrt{x})^{-3/2} + O\left(\frac{1}{x}\right)\right).$$

This, together with (13) and (17), implies that

$$\mathcal{I}_{11} = \mathrm{e}^{-2\beta\sqrt{x}}\left(\sqrt{\pi}(\beta\sqrt{x})^{-1/2} + \frac{3\sqrt{\pi}}{16}(\beta\sqrt{x})^{-3/2} + O\left(\frac{1}{x}\right)\right) \tag{18}$$

for large x.

The asymptotic relations (15), (16) and relation

$$\frac{1}{v^3} = \sum_{k=0}^{N}(-1)^k(k+1)\left(\frac{k}{2}+1\right)(v-1)^k + o\big((v-1)^N\big), \quad v\to 1,$$

imply that Lemma 1 is applicable to $\mathcal{I}_{121}$ with parameters

$$a=1, \quad \mu=2, \quad \nu=1,$$
$$a_0=b_0=1, \quad a_1=-1, \quad b_1=-3, \quad d_0=\frac{1}{2}, \quad d_1=-1.$$

Therefore, for sufficiently large x,

$$\mathcal{I}_{121} = \mathrm{e}^{-2\beta\sqrt{x}}\biggl(\frac{\sqrt{\pi}}{2}(\beta\sqrt{x})^{-1/2} - (\beta\sqrt{x})^{-1} + O\bigl(x^{-3/4}\bigr)\biggr). \tag{19}$$

In a similar way, we derive that

$$\mathcal{I}_{122} = \mathrm{e}^{-2\beta\sqrt{x}}\biggl(\frac{\sqrt{\pi}}{2}(\beta\sqrt{x})^{-1/2} + (\beta\sqrt{x})^{-1} + O\bigl(x^{-3/4}\bigr)\biggr).$$

This estimate, representation (14), and estimate (19) imply that

$$\mathcal{I}_{12} = \mathrm{e}^{-2\beta\sqrt{x}}\bigl(\sqrt{\pi}(\beta\sqrt{x})^{-1/2} + O\bigl(x^{-3/4}\bigr)\bigr)$$

for large x. Estimates (12), (18) and the above asymptotic equality imply that

$$\begin{aligned}\mathcal{I}_1 &= \sqrt{\pi}\,\frac{\beta^{2\alpha-3/2}x^{\alpha-3/4}}{\Gamma^2(\alpha)}\,\mathrm{e}^{-2\beta\sqrt{x}}\\ &\quad\times\biggl(1+\beta^{-1}x^{-1/2}\biggl(\alpha-1+\frac{3}{16}\biggr)+O\bigl(x^{-3/4}\bigr)\biggr)\\ &\quad+O\bigl(\mathrm{e}^{-2\beta\sqrt{x}}x^{\alpha-3/2}\bigr)\\ &= \sqrt{\pi}\,\frac{\beta^{2\alpha-3/2}x^{\alpha-3/4}}{\Gamma^2(\alpha)}\,\mathrm{e}^{-2\beta\sqrt{x}}\\ &\quad\times\biggl(1+\beta^{-1}x^{-1/2}\biggl(\alpha-1+\frac{3}{16}\biggr)+O\bigl(x^{-3/4}\bigr)\biggr)\end{aligned} \tag{20}$$

for large x. Finally, substituting derived estimates (11) and (20) into (10), we obtain

$$\begin{aligned}\overline{F}_{\xi_1\xi_2}(x) &= \sqrt{\pi}\,\frac{\beta^{2\alpha-3/2}x^{\alpha-3/4}}{\Gamma^2(\alpha)}\,\mathrm{e}^{-2\beta\sqrt{x}}\\ &\quad\times\biggl(1+\beta^{-1}x^{-1/2}\biggl(\alpha-1+\frac{3}{16}\biggr)+O\bigl(x^{-3/4}\bigr)\biggr),\end{aligned}$$

which implies that the theorem statement holds in the case where $n = 2$.

3.2 The key step of induction

Suppose now that the asymptotic formula (2) is true when $n = m$, $m \geqslant 2$, i.e.,

$$\begin{aligned}\overline{F}_{\Pi_m}(x) &= \frac{(2\pi)^{(m-1)/2}}{\sqrt{m}}\,\frac{\beta^{m(\alpha-1)+(m-1)/2}}{\Gamma^m(\alpha)}\,x^{\alpha-(m+1)/(2m)}\exp\bigl\{-m\beta x^{1/m}\bigr\}\\ &\quad\times\bigl(1+\beta^{-1}x^{-1/m}(\alpha-1-\mathcal{S}_m)+O_m\bigl(x^{-3/(2m)}\bigr)\bigr),\end{aligned} \tag{21}$$

where and everywhere in this section, the constant in the symbol $O_m()$ depends on α, β, and m. Since

$$\prod_{k=1}^{m+1} \xi_k = \xi_{m+1} \prod_{k=1}^{m} \xi_k,$$

then we obtain

$$\begin{aligned}
\overline{F}_{\Pi_{m+1}}(x) &= \int_0^\infty \overline{F}_{\Pi_m}\left(\frac{x}{y}\right) f_\xi(y)\,\mathrm{d}y = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \overline{F}_{\Pi_m}\left(\frac{x}{y}\right) y^{\alpha-1}\mathrm{e}^{-\beta y}\,\mathrm{d}y \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)}\left(\int_0^{(m+1)x^{1/(m+1)}} + \int_{(m+1)x^{1/(m+1)}}^\infty\right) \overline{F}_{\Pi_m}\left(\frac{x}{y}\right) y^{\alpha-1}\mathrm{e}^{-\beta y}\,\mathrm{d}y \\
&=: \mathcal{J}_1 + \mathcal{J}_2. \qquad (22)
\end{aligned}$$

Using the variable change $y = ux^{1/(m+1)}$, similarly as in (11), we get that

$$\begin{aligned}
\mathcal{J}_2 = \mathcal{K}_x &\int_{(m+1)x^{1/(m+1)}}^{(m+2)x^{1/(m+1)}} y^{(1-m)/(2m)} \exp\left\{-\beta\left(y + m\left(\frac{x}{y}\right)^{1/m}\right)\right\} \\
&\quad \times \left(1 + \beta^{-1}\left(\frac{x}{y}\right)^{-1/m}(\alpha-1-\mathcal{S}_m) + O_m\left(\frac{x}{y}\right)^{-3/(2m)}\right)\mathrm{d}y \\
&+ \frac{\beta^\alpha}{\Gamma(\alpha)} \int_{(m+2)x^{1/(m+1)}}^\infty y^{\alpha-1}\mathrm{e}^{-\beta y}\,\mathrm{d}y \\
= \mathcal{L}_x &\int_{m+1}^{m+2} u^{(1-m)/(2m)} \exp\left\{-\beta x^{1/(m+1)}\left(u + mu^{-1/m}\right)\right\} \\
&\quad \times \left(1+\beta^{-1}u^{1/m}x^{-1/(m+1)}(\alpha-1-\mathcal{S}_m) + O_m\left(u^{3/(2m)}x^{-3/(2(m+1))}\right)\right)\mathrm{d}u \\
&+ \frac{\beta^\alpha}{\Gamma(\alpha)} \int_{(m+2)x^{1/(m+1)}}^\infty y^{\alpha-1}\mathrm{e}^{-\beta y}\,\mathrm{d}y \\
= O_m&\left(x^{\alpha-(m+1)/(2m)} \exp\left\{-\beta x^{1/(m+1)}\left(m+1+m(m+1)^{-1/m}\right)\right\}\right),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{K}_x &= \frac{(2\pi)^{(m-1)/2}}{\sqrt{m}} \frac{\beta^{\alpha(m+1)-(m+1)/2}}{\Gamma^{m+1}(\alpha)} x^{\alpha-(m+1)/(2m)}, \\
\mathcal{L}_x &= x^{1/(2m)}\mathcal{K}_x, \qquad (23)
\end{aligned}$$

and x is sufficiently large. Using the induction hypothesis (21) and the same variable change $y = ux^{1/(m+1)}$, for large x, we find that

$$\begin{aligned}\mathcal{J}_1 = \mathcal{L}_x\Biggl(&\int_0^{m+1} u^{(1-m)/(2m)} \exp\bigl\{-\beta x^{1/(m+1)}\bigl(u + mu^{-1/m}\bigr)\bigr\}\,\mathrm{d}u \\ &+ \beta^{-1}x^{-1/(m+1)}(\alpha - 1 - \mathcal{S}_m) \\ &\times \int_0^{m+1} u^{(3-m)/(2m)} \exp\bigl\{-\beta x^{1/(m+1)}\bigl(u + mu^{-1/m}\bigr)\bigr\}\,\mathrm{d}u \\ &+ O_m\Biggl(x^{-3/(2(m+1))}\int_0^{m+1} u^{(4-m)/(2m)} \exp\bigl\{-\beta x^{1/(m+1)}\bigl(u + mu^{-1/m}\bigr)\bigr\}\,\mathrm{d}u\Biggr)\Biggr) \\ =: \mathcal{L}_x\bigl(&\mathcal{J}_{11} + \beta^{-1}x^{-1/(m+1)}(\alpha - 1 - \mathcal{S}_m)\mathcal{J}_{12} + O_m\bigl(x^{-3/(2(m+1))}\mathcal{J}_{13}\bigr)\bigr), \qquad (24)\end{aligned}$$

where $\mathcal{L}_x$ is defined in (23).

Similarly, as demonstrated in (13) and (14), we obtain the following divisions:

$$\begin{aligned}\mathcal{J}_{11} &= \int_1^{\infty} v^{-(3m+1)/(2m)} \exp\biggl\{-\beta x^{1/(m+1)}\biggl(\frac{1}{v} + mv^{1/m}\biggr)\biggr\}\,\mathrm{d}v \\ &\quad + \int_1^{m+1} u^{(1-m)/(2m)} \exp\bigl\{-\beta x^{1/(m+1)}\bigl(u + mu^{-1/m}\bigr)\bigr\}\,\mathrm{d}u \\ &=: \mathcal{J}_{111} + \mathcal{J}_{112}, \qquad (25) \\ \mathcal{J}_{12} &= \int_1^{\infty} v^{-(3m+3)/(2m)} \exp\biggl\{-\beta x^{1/(m+1)}\biggl(\frac{1}{v} + mv^{1/m}\biggr)\biggr\}\,\mathrm{d}v \\ &\quad + \int_1^{m+1} u^{(3-m)/(2m)} \exp\bigl\{-\beta x^{1/(m+1)}\bigl(u + mu^{-1/m}\bigr)\bigr\}\,\mathrm{d}u \\ &=: \mathcal{J}_{121} + \mathcal{J}_{122}. \qquad (26)\end{aligned}$$

While for the last integral in (24), we get that

$$\begin{aligned}\mathcal{J}_{13} &= \int_1^{\infty} v^{-(3m+4)/(2m)} \exp\biggl\{-\beta x^{1/(m+1)}\biggl(\frac{1}{v} + mv^{1/m}\biggr)\biggr\}\,\mathrm{d}v \\ &\quad + \int_1^{m+1} u^{(4-m)/(2m)} \exp\bigl\{-\beta x^{1/(m+1)}\bigl(u + mu^{-1/m}\bigr)\bigr\}\,\mathrm{d}u \\ &=: \mathcal{J}_{131} + \mathcal{J}_{132}.\end{aligned}$$

Below we find the suitable asymptotic formulas for all the marked integrals.

• We begin with $\mathcal{J}_{111}$, $\mathcal{J}_{112}$, and $\mathcal{J}_{11}$. We have $1/v + mv^{1/m}|_{v=1} = m+1$, and

$$\frac{1}{v} + mv^{1/m} = m + 1 + \sum_{k=0}^{N}\left(\frac{\prod_{l=1}^{k+1}(1-lm)}{(k+2)!m^{k+1}} + (-1)^k\right)(v-1)^{k+2} + o\big((v-1)^{N+2}\big), \quad v \to 1, \tag{27}$$

$$\left(\frac{1}{v} + mv^{1/m}\right)' = \sum_{k=0}^{N}\left(\frac{\prod_{l=1}^{k+1}(1-lm)}{(k+2)!m^{k+1}} + (-1)^k\right)(k+2)(v-1)^{k+1} + o\big((v-1)^{N+1}\big), \quad v \to 1, \tag{28}$$

$$v^{-(3m+1)/(2m)} = 1 + \sum_{k=1}^{N}\frac{(-1)^k\prod_{l=1}^{k}((2l+1)m+1)}{k!(2m)^k}(v-1)^k + o\big((v-1)^N\big), \quad v \to 1,$$

for any $N \geqslant 1$ according to the Taylor's formula. Hence, using Lemma 1 for integral $\mathcal{J}_{111}$ with suitable parameters

$$a = 1, \quad \mu = 2, \quad \nu = 1,$$

$$a_0 = \frac{m+1}{2m}, \quad a_1 = -\frac{(4m-1)(m+1)}{6m^2}, \quad a_2 = \frac{18m^3 + 11m^2 - 6m + 1}{24m^3},$$

$$b_0 = 1, \quad b_1 = -\frac{3m+1}{2m}, \quad b_2 = \frac{(3m+1)(5m+1)}{8m^2},$$

$$d_0 = \sqrt{\frac{m}{2(m+1)}}, \quad d_1 = -\frac{m+5}{6(m+1)}, \quad d_2 = -\frac{2^{-5/2}(m^2+m-11)}{3m^{1/2}(m+1)^{3/2}},$$

we get that

$$\mathcal{J}_{111} = \exp\big\{-\beta(m+1)x^{1/(m+1)}\big\}\Bigg(\sqrt{\frac{\pi m}{2(m+1)}}\big(\beta x^{1/(m+1)}\big)^{-1/2} - \frac{m+5}{6(m+1)}\big(\beta x^{1/(m+1)}\big)^{-1} - \frac{\sqrt{\pi}(m^2+m-11)}{3(2^{7/2}m^{1/2}(m+1)^{3/2})}\big(\beta x^{1/(m+1)}\big)^{-3/2} + O_m\big(x^{-2/(m+1)}\big)\Bigg) \tag{29}$$

if x is sufficiently large.

Similarly, we have $u + mu^{-1/m}|_{u=1} = m+1$, and for any $N \geqslant 1$,

$$u + mu^{-1/m} = m + 1 + \sum_{k=0}^{N}(-1)^k\frac{\prod_{l=1}^{k+1}(1+lm)}{(k+2)!m^{k+1}}(u-1)^{k+2} + o\big((u-1)^{N+2}\big), \quad u \to 1,$$

$$\left(u+mu^{-1/m}\right)' = \sum_{k=0}^{N}(-1)^k \frac{\prod_{l=1}^{k+1}(1+lm)}{(k+1)!m^{k+1}}(k+2)(u-1)^{k+1} + o\big((u-1)^{N+1}\big), \quad u\to 1,$$

$$u^{(1-m)/(2m)} = 1+\sum_{k=1}^{N}\frac{(-1)^k\prod_{l=1}^{k}((2l-1)m-1)}{k!(2m)^k}(u-1)^k + o\big((u-1)^{N}\big), \quad u\to 1,$$

according to the Taylor's formula again. The derived equalities imply that conditions of Lemma 1 hold for integral $\mathcal{J}_{112}$ with parameters

$$a=1,\quad \mu=2,\quad \nu=1,$$

$$a_0=\frac{m+1}{2m},\quad a_1=-\frac{(m+1)(2m+1)}{6m^2},\quad a_2=\frac{6m^3+11m^2+6m+1}{24m^3},$$

$$b_0=1,\quad b_1=\frac{1-m}{2m},\quad b_2=\frac{(3m-1)(m-1)}{8m^2},$$

$$d_0=\sqrt{m/2(m+1)},\quad d_1=\frac{m+5}{6(m+1)},\quad d_2=-\frac{2^{-5/2}(m^2+m-11)}{3m^{1/2}(m+1)^{3/2}}.$$

Therefore, for large x,

$$\mathcal{J}_{112} = \exp\bigl\{-\beta(m+1)x^{1/(m+1)}\bigr\}\Biggl(\sqrt{\frac{\pi m}{2(m+1)}}\bigl(\beta x^{1/(m+1)}\bigr)^{-1/2} + \frac{m+5}{6(m+1)}\bigl(\beta x^{1/(m+1)}\bigr)^{-1} - \frac{\sqrt{\pi}(m^2+m-11)}{3(2^{7/2}m^{1/2}(m+1)^{3/2})}\bigl(\beta x^{1/(m+1)}\bigr)^{-3/2} + O_m\bigl(x^{-2/(m+1)}\bigr)\Biggr).$$

This, together with (25) and (29), implies that

$$\mathcal{J}_{11} = \exp\bigl\{-\beta(m+1)x^{1/(m+1)}\bigr\}\Biggl(\sqrt{\frac{2\pi m}{(m+1)}}\bigl(\beta x^{1/(m+1)}\bigr)^{-1/2} - \frac{\sqrt{\pi}(m^2+m-11)}{3(2^{5/2}m^{1/2}(m+1)^{3/2})}\bigl(\beta x^{1/(m+1)}\bigr)^{-3/2} + O_m\bigl(x^{-2/(m+1)}\bigr)\Biggr) \tag{30}$$

for large x.

• Now let us consider the integrals $\mathcal{J}_{121}$, $\mathcal{J}_{122}$, and $\mathcal{J}_{12}$. We can once more use asymptotic relations (27), (28) and the asymptotic equality

$$v^{-(3m+3)/(2m)} = 1+\sum_{k=1}^{N}\frac{(-1)^k\prod_{l=1}^{k}((2l+1)m+3)}{k!(2m)^k}(v-1)^k + o\big((v-1)^{N}\big), \quad v\to 1,$$

for any $N \geqslant 1$ due to the Taylor's formula. Therefore, for integral $\mathcal{J}_{121}$, the assertion of Lemma 1 holds with parameters

$$a = 1, \quad \mu = 2, \quad \nu = 1, \quad a_0 = \frac{m+1}{2m}, \quad a_1 = -\frac{(4m-1)(m+1)}{6m^2},$$
$$b_0 = 1, \quad b_1 = -\frac{3m+3}{2m}, \quad d_0 = \sqrt{\frac{m}{2(m+1)}}, \quad d_1 = -\frac{m+11}{6(m+1)}.$$

Thus, when x is large,

$$\mathcal{J}_{121} = \exp\bigl\{-\beta(m+1)x^{1/(m+1)}\bigr\}\left(\sqrt{\frac{\pi m}{2(m+1)}}\bigl(\beta x^{1/(m+1)}\bigr)^{-1/2} - \frac{m+11}{6(m+1)}\bigl(\beta x^{1/(m+1)}\bigr)^{-1} + O_m\bigl(x^{-3/(2(m+1))}\bigr)\right). \tag{31}$$

In a similar way, we derive that

$$\mathcal{J}_{122} = \exp\bigl\{-\beta(m+1)x^{1/(m+1)}\bigr\}\left(\sqrt{\frac{\pi m}{2(m+1)}}\bigl(\beta x^{1/(m+1)}\bigr)^{-1/2} + \frac{m+11}{6(m+1)}\Bigl(\beta x^{1/(m+1)}\Bigr)^{-1} + O_m\bigl(x^{-3/(2(m+1))}\bigr)\right).$$

This, together with (26) and (31), implies that

$$\mathcal{J}_{12} = \exp\bigl\{-\beta(m+1)x^{1/(m+1)}\bigr\}\left(\sqrt{\frac{2\pi m}{(m+1)}}\bigl(\beta x^{1/(m+1)}\bigr)^{-1/2} + O_m\bigl(x^{-3/(2(m+1))}\bigr)\right). \tag{32}$$

• Finally, with regard to the integrals $\mathcal{J}_{131}$, $\mathcal{J}_{132}$, and $\mathcal{J}_{13}$, we can equivalently employ Lemma 1 to find that for large x,

$$\mathcal{J}_{131} = \exp\bigl\{-\beta(m+1)x^{1/(m+1)}\bigr\}\left(\sqrt{\frac{\pi m}{2(m+1)}}\bigl(\beta x^{1/(m+1)}\bigr)^{-1/2} - \frac{m+14}{6(m+1)}\bigl(\beta x^{1/(m+1)}\bigr)^{-1} + O_m\bigl(x^{-3/(2(m+1))}\bigr)\right),$$
$$\mathcal{J}_{132} = \exp\bigl\{-\beta(m+1)x^{1/(m+1)}\bigr\}\left(\sqrt{\frac{\pi m}{2(m+1)}}\bigl(\beta x^{1/(m+1)}\bigr)^{-1/2} + \frac{m+14}{6(m+1)}\bigl(\beta x^{1/(m+1)}\bigr)^{-1} + O_m\bigl(x^{-3/(2(m+1))}\bigr)\right),$$
$$\mathcal{J}_{13} = \exp\bigl\{-\beta(m+1)x^{1/(m+1)}\bigr\}\left(\sqrt{\frac{2\pi m}{(m+1)}}\bigl(\beta x^{1/(m+1)}\bigr)^{-1/2} + O_m\bigl(x^{-3/(2(m+1))}\bigr)\right). \tag{33}$$

By substituting the derived asymptotic equalities (26), (30), (32), and (33) into (24) and (22) we get that

$$\begin{aligned}\overline{F}_{\Pi_{m+1}}(x) = \frac{(2\pi)^{m/2}}{\sqrt{m+1}}\,\frac{\beta^{(m+1)(\alpha-1)+m/2}}{\Gamma^{m+1}(\alpha)}x^{\alpha-(m+2)/(2(m+1))}\\ \times\exp\bigl\{-(m+1)\beta x^{1/(m+1)}\bigr\}\\ \times\bigl(1+\beta^{-1}x^{-1/(m+1)}(\alpha-1-\mathcal{S}_{m+1})+O_m\bigl(x^{-3/(2(m+1))}\bigr)\bigr)\end{aligned}$$

with the constant in the symbol $O_m()$ not depending on x but depending on α, β, and m.

The last estimate implies the desired formula in the case $n=m+1$. According to the induction principle, the validity of (2) follows for each $n\in\mathbb{N}$. Theorem 2 is proved.

4 Numerical illustrations

In this section, we present two examples involving the products of gamma-distributed random variables. For both cases, we compare the theoretical results with the tail probabilities estimated through the Monte Carlo method.

Example 1. Consider the product $\Pi_3=\xi_1\xi_2\xi_3$ of three independent random variables ξ_1, ξ_2, and ξ_3 distributed according to the gamma law with parameters $\alpha=3/2$ and $\beta=2$.

According to Theorem 2, there exist positive constants $\widehat{C}_1$ and $\widehat{C}_2$ such that, for $x\geqslant C_2$,

$$\begin{aligned}\overline{F}_{\Pi_3}(x) &\leqslant \frac{2^{7/2}\pi x^{5/6}}{\sqrt{3}\Gamma^3(\frac{3}{2})}\exp\bigl\{-6x^{1/3}\bigr\}\biggl(1+\frac{x^{-1/3}}{2}\biggl(\frac{1}{2}-\mathcal{S}_3\biggr)+\widehat{C}_1x^{-1/2}\biggr)\\ &= \frac{2^{7/2}\pi x^{5/6}}{\sqrt{3}\Gamma^3(\frac{3}{2})}\exp\bigl\{-6x^{1/3}\bigr\}\biggl(1+\frac{13}{36}x^{-1/3}+\widehat{C}_1x^{-1/2}\biggr),\\ \overline{F}_{\Pi_3}(x) &\geqslant \frac{2^{7/2}\pi x^{5/6}}{\sqrt{3}\Gamma^3(\frac{3}{2})}\exp\bigl\{-6x^{1/3}\bigr\}\biggl(1+\frac{13}{36}x^{-1/3}-\widehat{C}_1x^{-1/2}\biggr).\end{aligned}$$

Figure 1 below shows that we can expect that $\widehat{C}_1=0.9$ and $\widehat{C}_2=3.5$ in the case under consideration.

Example 2. Consider the product $\widetilde{\Pi}_4=\eta_1\eta_2\eta_3\eta_4$ of four independent random variables η_1, η_2, η_3, and η_4 distributed according to the exponential law with parameter $\lambda=3$.

Due to Corollary 1, there exist positive constants $\widetilde{C}_1$ and $\widetilde{C}_2$ such that, for $x\geqslant\widetilde{C}_2$,

$$\begin{aligned}\overline{F}_{\widetilde{\Pi}_4}(x) &\leqslant \frac{(6\pi)^{3/2}}{2}x^{3/8}\exp\bigl\{-12x^{1/4}\bigr\}\biggl(1+\frac{7}{96}x^{-1/4}+\widetilde{C}_1x^{-3/8}\biggr),\\ \overline{F}_{\widetilde{\Pi}_4}(x) &\geqslant \frac{(6\pi)^{3/2}}{2}x^{3/8}\exp\bigl\{-12x^{1/4}\bigr\}\biggl(1+\frac{7}{96}x^{-1/4}-\widetilde{C}_1x^{-3/8}\biggr).\end{aligned}$$

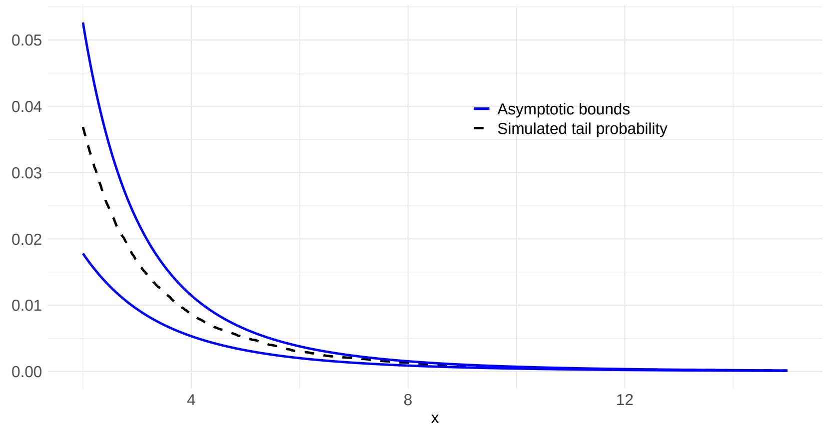

Figure 1. Simulated data compared to asymptotic bounds of tail probability for Π_3 from Example 1.

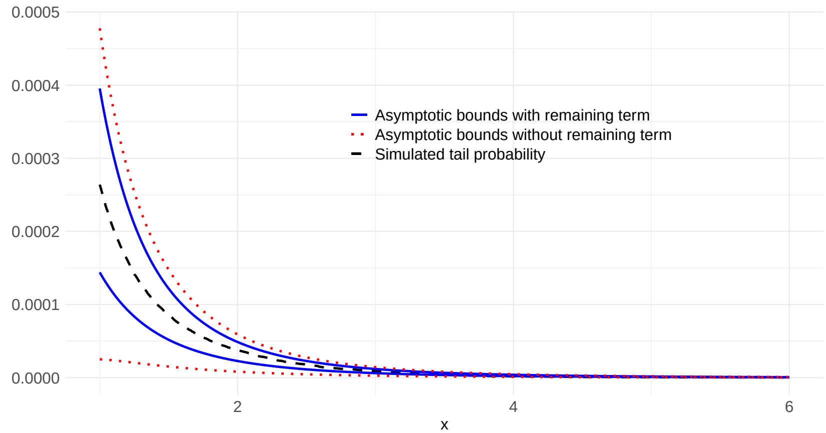

Figure 2. Simulated data compared to asymptotic bounds of tail probability for $\widetilde{\Pi}_4$ from example 2.

In addition, according to the same corollary, there exists the positive constant $\widetilde{C}_3$ such that, for $x \geqslant \widetilde{C}_2$,

$$\overline{F}_{\widetilde{\Pi}_4}(x) \leqslant \frac{(6\pi)^{3/2}}{2} x^{3/8} \exp\bigl\{-12x^{1/4}\bigr\}\bigl(1 + \widetilde{C}_3 x^{-1/4}\bigr),$$
$$\overline{F}_{\widetilde{\Pi}_4}(x) \geqslant \frac{(6\pi)^{3/2}}{2} x^{3/8} \exp\bigl\{-12x^{1/4}\bigr\}\bigl(1 - \widetilde{C}_3 x^{-1/4}\bigr).$$

Fig. 2 displays the simulated values of $\widetilde{\Pi}_4$ along with the upper and lower bounds. Specifically, Fig. 2 demonstrates that in this example, it is possible to set $\widetilde{C}_1 = 0.5$, $\widetilde{C}_3 = 0.9$, and $\widetilde{C}_2 = 2$. This figure shows that by taking more residual terms the tail function of the product $\widetilde{\Pi}_4$ can be approximated more accurately.

References

1. M. Arendarczyk, K. Dębicki, Asymptotics of supremum distribution of a Gaussian process over a Weibullian time, *Bernoulli*, **7**(1):194–210, 2011, https://doi.org/10.3150/10-BEJ266.
2. L.A. Aroian, V.S. Taneja, L.W. Cornwell, Mathematical forms of the distribution of the product of two normal variables, *Commun. Stat., Theory Methods*, **7**(2):165–172, 1978, https://doi.org/10.1080/03610927808827610.
3. G. Bareikis, J. Šiaulys, *Products of Independent Random Variables*, Vilnius Univ. Press, Vilnius, 1998 (in Lithuanian).
4. H. Bateman, *Tables of Integral Transforms*, McGraw-Hill, New York, 1954.
5. A. Bose, R.S. Hazra, K. Saha, Product of exponentials and spectral radius of random k-circulants, *Ann. Inst. H. Poincaré, Probab. Stat.*, **48**(2):424–443, 2012, https://doi.org/10.1214/10-AIHP404.
6. R. Butler, *Saddlepoint Approximations with Applications*, Cambridge Univ. Press, Cambridge, 2007, https://doi.org/10.1017/CBO9780511619083.
7. C.C. Craig, On the frequency function of xy, *Ann. Math. Stat.*, **7**(1):1–15, 1936, https://doi.org/10.1214/aoms/1177732541.
8. Z. Cui, Y. Wang, On the long tail property of product convolution, *Lith. Math. J.*, **60**(3):315–329, 2020, https://doi.org/10.1007/s10986-020-09482-w.
9. K. Dębicki, J. Farkas, E. Hashorva, Extremes of randomly scaled Gumbel risks, *J. Math. Anal. Appl.*, **458**(1):30–42, 2018, https://doi.org/10.1016/j.jmaa.2017.08.055.
10. P. Embrechts, C.M. Goldie, On convolution tails, *Stochastic Processes Appl.*, **13**:263–278, 1982, https://doi.org/10.1016/0304-4149(82)90013-.
11. P. Embrechts, C. Klüppelberg, T. Mikosch, *Modelling Extremal Events for Insurance and Finance*, Springer, Berlin, Heidelberg, 1997, https://doi.org/10.1007/978-3-642-33483-2.
12. M. Fedoryuk, *The Saddle Point Method*, Nauka, Moscow, 1977.
13. S. Foss, D. Korshunov, S. Zachary, *An Introduction to Heavy-Tailed and Subexponential Distributions,* 2nd ed., Springer, New York, 2013, https://doi.org/10.1007/978-1-4614-7101-1.
14. J. Galambos, I. Simonelli, *Products of Random Variables: Applications to Problems of Physics and to Arithmetical Functions*, Taylor & Francis, Boca Raton, FL, 2004, https://doi.org/10.1201/9781482276633.
15. R. Gaunt, Stein's method and the distribution of the product of zero mean correlated normal random variables, *Commun. Stat. Theory Methods*, **50**:280–285, 2019, https://doi.org/10.1080/03610926.2019.1634210.
16. E. Hashorva, Z. Weng, Tail asymptotic of Weibull-type risks, *Statistics*, **48**(5):1155–1165, 2014, https://doi.org/10.1080/02331888.2013.800520.
17. J.L. Jensen, *Saddlepoint Approximations*, Oxford Univ. Press, Oxford, 1995, https://doi.org/10.1093/oso/9780198522959.001.0001.

18. D. Konstantinides, R. Leipus, J. Šiaulys, A note on product-convolution for generalized subexponential distributions, *Nonlinear Anal. Model. Control*, **27**(6):1054–1067, 2022, https://doi.org/10.15388/namc.2022.27.29405.
19. D.G. Konstantinides, *Risk Theory: A Heavy Tail Approach*, World Scientific, Hackensack, NJ, 2018.
20. R. Leipus, J. Šiaulys, M. Dirma, R. Zovė, On the distribution-tail behaviour of the product of normal random variables, *J. Inequal. Appl.*, **2023**:32, 2023, https://doi.org/10.1186/s13660-023-02941-1.
21. R. Leipus, J. Šiaulys, D. Konstantinides, *Closure Properties for Heavy-Tailed and Related Distributions*, Springer, Cham, 2023, https://doi.org/10.1007/978-3-031-34553-1.
22. Y. Liu, Q. Tang, The subexponential product convolution of two Weibull-type distributions, *J. Aust. Math. Soc.*, **89**(2):277–288, 2010, https://doi.org/10.1017/S1446788710000182.
23. S. Ly, K.H. Pho, S. Ly, W.K. Wong, Determining distribution for the product of random variables by using copulas, *Risks*, **7**:23, 2019, https://doi.org/10.3390/risks7010023.
24. H.J. Malik, Exact distribution of the product of independent generalized gamma variables with the same shape parameter, *Ann. Math. Stat.*, **39**(5):1751–1752, 1968, https://doi.org/10.1214/aoms/1177698159.
25. T. Mikosch, *Non-Life Insurance Mathematics: An Introduction with the Poisson Process*, Springer, Berlin, Heidelberg, 2009, https://doi.org/10.1007/978-3-540-88233-6.
26. G. Mikutavičius, J. Šiaulys, Product convolution of generalized subexponential distributions, *Mathematics*, **11**(1):248, 2023, https://doi.org/10.3390/math11010248.
27. S. Nadarajah, S. Kotz, On the product and ratio of gamma and beta random variables, *Allg. Stat. Arch.*, **89**:435–449, 2005, https://doi.org/10.1007/s10182-005-0214-9.
28. S. Nadarajah, S. Kotz, On the product and ratio of gamma and Weibull random variables, *Econom. Theory*, **22**(2):338–344, 2006, https://doi.org/10.1017/S0266466606060154.
29. J. Nair, A. Wierman, B. Zwart, *The Fundamentals of Heavy Tails: Properties, Emergence and Estimation*, Cambridge Univ. Press, Cambridge, 2022.
30. E.J.G. Pitman, Subexponential distribution functions, *J. Aust. Math. Soc., Ser. A*, **29**:337–347, 1980, https://doi.org/10.1017/S1446788700021340.
31. A.P. Prudnikov, Y.U. Brychkov, O.I. Marichev, Evaluation of integrals and the Mellin transform, *J. Math. Sci.*, **54**:1239–1341, 1991, https://doi.org/10.1007/BF01373648.
32. A. Seijas-Macías, A. Oliveira, An approach to distribution of the product of two normal variables, *Discuss. Math. Probab. Stat.*, **32**(1–2):87–99, 2012, https://doi.org/10.7151/dmps.1146.
33. A. Seijas-Macías, A. Oliveira, T.A. Oliveira, V. Leiva, Approximating the distribution of the product of two normally distributed random variables, *Symmetry*, **12**(8):1201, 2010, https://doi.org/10.3390/sym12081201.

34. M.D. Springer, *The Algebra of Random Variables*, Wiley, New York, 1979.
35. M.D. Springer, W.E. Thompson, The distribution of products of beta, gamma and Gaussian random variables, *SIAM J. Appl. Math.*, **18**:721–737, 1970, https://doi.org/10.1137/0118065.
36. Q. Tang, From light tails to heavy tails through multiplier, *Extremes*, **11**(4):379–391, 2008, https://doi.org/10.1007/s10687-008-0063-5.
37. R. Wong, *Asymptotic Approximations of Integrals*, Academic Press, Boston, 1989.