LIETUVOS MATEMATIKOS RINKINYS
Proc. of the Lithuanian Mathematical Society, Ser. A
Vol. 65, 2024, pages 9–17
https://doi.org/10.15388/LMR.2024.37367

Multi-succedent sequent calculus for intuitionistic epistemic logic

Romas Alonderis

Institute of Data Science and Digital Technologies, Vilnius University
Akademijos str. 4, LT-08412 Vilnius, Lithuania
E-mail: romas.alonderis@mif.vu.lt

Received May 13, 2024; published online December 10, 2024

Abstract. A multi-succedent sequent calculus for intuitionistic epistemic logic (**IEL**) is introduced in the paper. It is proved that the structural rules of weakening and contraction and the rule of cut are admissible in the calculus. It is also proved that any sequent with at most one formula in succedent is derivable in the calculus, iff it is derivable in the standard non-multi-succedent sequent calculus of **IEL**.

Keywords: intuitionistic epistemic logic; sequent calculus

AMS Subject Classification: 03B45

1 Introduction

Various intuitionistic modal [3, 7, 9] and temporal [1, 5] logics are consider in the literature. Such logics are applied in various disciplines, ranging from economics to computer science and mathematics. In [4], the intuitionistic epistemic logic (**IEL**) is introduced. It is an intuitionistic modal logic where belief and knowledge are considered from an intuitionistic point of view. An arbitrary proposition A is intuitionistically true if there is a direct proof of A. Hence $\neg\neg A$ does not imply A because a proof of $\neg\neg A$ is not a proof of A, i.e., the formula $\neg\neg A \to A$ is not intuitionistically valid. The formula KA is understood in **IEL** as follows: given a proof P of A, an agent knows, can verify, whether P is indeed a proof of A. The formula $A \to KA$ (co-reflection) is valid. It states that if there is a proof of A, then an agent can always verify that proof. On the other hand, the formula $KA \to A$ (reflection) is valid in classical epistemic logics, but not in **IEL**. In the classical case it states that if an agent knows A, then A is true. In the intuitionistic case it states that the fact that an agent can verify any proof of A implies that A is provable. The latter a statement can not of course be held true.

A sequent calculus $\mathbf{IEL}_{\mathrm{G}}^{-}$ for **IEL** is introduced in [6]. Sequent calculi are handy tools for validity check of formulas and sequents. In the present paper, we introduce a multi-succedent sequent calculus $\mathbf{IEL}_{\mathrm{G}}^{*}$ for **IEL**. Multi-succedent intuitionistic calculi provide more flexible backward proof-search and are more convenient for implementation. $\mathbf{IEL}_{\mathrm{G}}^{*}$ has more invertible rules rules than $\mathbf{IEL}_{\mathrm{G}}^{-}$; consequently backward proof-search using $\mathbf{IEL}_{\mathrm{G}}^{*}$ requires less backtracking in comparison with $\mathbf{IEL}_{\mathrm{G}}^{-}$.

The rest of the paper is organized as follows. Syntax and sequent calculi of the intuitionistic epistemic logic are in Section 2. Admissibility of the structural rules of weakening and contraction and invertibility of $\mathbf{IEL}_{\mathrm{G}}^{*}$ rules are proved in Section 3. In section 4, equivalence of $\mathbf{IEL}_{\mathrm{G}}^{*}$ and $\mathbf{IEL}_{\mathrm{G}}^{-}$ for intuitionistic sequents as well as admissibility of the rule of cut are proved. Some concluding remarks are in Section 5.

2 Syntax and sequent calculi

The language of **IEL** contains a set of propositional symbols, the constant $\bot$, propositional connectives $\vee, \wedge, \rightarrow$, and the unary modal operator K. The constant $\bot$ or a propositional symbol is called an *atomic formula*. We use the letter A to denote an arbitrary atomic formula. Formulas are constructed traditionally from atomic formulas using the propositional connectives and modal operator. The letters F, G, and H denote arbitrary formulas. We do not include the negation symbol '$\neg$' into syntax. A formula $\neg F$ is expressed by $F \rightarrow \bot$.

Sequents are objects of the shape $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite multisets of formulas. A sequent where $\Delta \in \{F, \emptyset\}$ is called intuitionistic.

We recall the calculus $\mathbf{IEL}_{\mathrm{G}}^{-}$ introduced in [6]:

1. Axioms: $\Gamma, A \Rightarrow A$ and $\Gamma, \bot \Rightarrow \Delta$.

2. Rules:

$$\frac{F, G, \Gamma \Rightarrow \Delta}{F \wedge G, \Gamma \Rightarrow \Delta}\ (\wedge \Rightarrow), \qquad \frac{\Gamma \Rightarrow F \quad \Gamma \Rightarrow G}{\Gamma \Rightarrow F \wedge G}\ (\Rightarrow \wedge),$$

$$\frac{F, \Gamma \Rightarrow \Delta \quad G, \Gamma \Rightarrow \Delta}{F \vee G, \Gamma \Rightarrow \Delta}\ (\vee \Rightarrow), \qquad \frac{\Gamma \Rightarrow F_i}{\Gamma \Rightarrow F_1 \vee F_2}\ (\Rightarrow \vee), \quad i \in \{1, 2\},$$

$$\frac{F \rightarrow G, \Gamma \Rightarrow F \quad G, \Gamma \Rightarrow \Delta}{F \rightarrow G, \Gamma \Rightarrow \Delta}\ (\rightarrow\Rightarrow), \qquad \frac{\Gamma, F \Rightarrow G}{\Gamma \Rightarrow F \rightarrow G}\ (\Rightarrow\rightarrow),$$

$$\frac{\Gamma, \Pi, K\Pi \Rightarrow \theta'}{\Gamma, K\Pi \Rightarrow \theta}\ (KI).$$

Here: all sequents are intuitionistic. $\theta = KF$ ($\theta \neq KF$) and $\theta' = F$ ($\theta' = \emptyset$, respectively). $K\Pi$ denotes a multiset of formulas of the form KH. It is required that formulas of the shape KG do not occur in Γ in the rule (KI). We have slightly modified the calculus $\mathbf{IEL}_{\mathrm{G}}^{-}$ in [6] by replacing the rules (KI_1) and (U) by the rule (KI). If $\theta = KF$ ($\theta \neq KF$), then (KI) is used instead of (KI_1) (instead of (U), respectively) in backward proof-search of sequents.

The calculus $\mathbf{IEL}_{\mathrm{G}}^{*}$ is defined as follows:

1. Axioms: $\Gamma, A \Rightarrow A, \Delta$ and $\Gamma, \bot \Rightarrow \Delta$.
2. Rules:

$$\frac{F, G, \Gamma \Rightarrow \Delta}{F \wedge G, \Gamma \Rightarrow \Delta}\ (\wedge \Rightarrow), \qquad \frac{\Gamma \Rightarrow F, \Delta \quad \Gamma \Rightarrow G, \Delta}{\Gamma \Rightarrow F \wedge G, \Delta}\ (\Rightarrow \wedge),$$

$$\frac{F, \Gamma \Rightarrow \Delta \quad G, \Gamma \Rightarrow \Delta}{F \vee G, \Gamma \Rightarrow \Delta}\ (\vee \Rightarrow), \qquad \frac{\Gamma \Rightarrow F, G, \Delta}{\Gamma \Rightarrow F \vee G, \Delta}\ (\Rightarrow \vee),$$

$$\frac{F \to G, \Gamma \Rightarrow F, \Delta \quad G, \Gamma \Rightarrow \Delta}{F \to G, \Gamma \Rightarrow \Delta}\ (\to\Rightarrow), \qquad \frac{\Gamma, F \Rightarrow G}{\Gamma \Rightarrow F \to G, \Delta}\ (\Rightarrow\to),$$

$$\frac{\Gamma, \Pi, K\Pi \Rightarrow \theta'}{\Gamma, K\Pi \Rightarrow \theta}\ (KI).$$

Here: Δ denotes an arbitrary multiset of formulas. $\theta = (KF, \Delta)$ ($\theta \neq (KF, \Delta)$) and $\theta' = F$ ($\theta' = \emptyset$, respectively). It is required that $\Gamma \neq (KG, \Gamma')$ in the rule (KI).

A proof-search of a sequent S in a sequent calculus (SC) is performed by subsequently applying backwards derivation rules of SC to S and the generated sequents, obtaining a proof-search tree with S in the root, $V(S)$ in notation. $V(S)$ all branches of which end up in axioms is called a derivation tree, and S is called derivable in SC ($SC \vdash^V S$ in notation). The height of $V(S)$ ($hV(S)$ in notation) is the length of its longest branch, where the length of a branch is measured by the number of rule applications on it.

3 Some proof-theoretical properties of $\mathbf{IEL}^*_{\mathbf{G}}$

Let

$$\frac{S_1 \ldots S_n}{S}\ (r)$$

be a $n > 0$ premise derivation rule. The rule is called *height-preserving admissible* in a sequent calculus SC, if $SC \vdash^{V^i} S_i$ implies $SC \vdash^V S$, where $hV^i(S_i) \leqslant hV(S)$, for all $1 \leqslant i \leqslant n$. If it is not required that $hV(S) \leqslant hV^i(S_i)$, then (r) is called *admissible* in SC. Let (r) be in SC. The rule (r) is called *height-preserving invertible*, if $SC \vdash^V S$ implies $SC \vdash^{V^i} (S_i)$, where $hV^i(S_i) \leqslant hV(S)$, for all $1 \leqslant i \leqslant n$.

It is proved in [6] that the rule

$$\frac{\Gamma \Rightarrow F \quad F, \Pi \Rightarrow \theta}{\Gamma, \Pi \Rightarrow \theta}\ (cut)$$

is admissible in $\mathbf{IEL}^-_{\mathrm{G}}$, where F is an arbitrary formula and $\theta \in \{\emptyset, G\}$.

Lemma 1. *The rule of weakening*

$$\frac{\Gamma \Rightarrow \Delta}{\Pi, \Gamma \Rightarrow \Delta, \Lambda}\ (W)$$

is height-preserving admissible in $\boldsymbol{IEL}^*_G$; *here* Π *and* Λ *are arbitrary multisets of formulas.*

Proof. Let $IEL^*_{\mathrm{G}} \vdash^V (\Gamma \Rightarrow \Delta)$. The lemma is proved by induction on hV. If $hV = 0$, then the premise of (W) is an axiom and the proof is obtained. Let $hV > 0$ and V be as follows:

$$\dfrac{\begin{matrix} V' \\ F, G, \Gamma' \Rightarrow \Delta \end{matrix}}{F \wedge G, \Gamma' \Rightarrow \Delta}\ (\wedge \Rightarrow).$$

The derivation height of $F, G, \Gamma' \Rightarrow \Delta'$ is by one less than hV. We apply the inductive hypothesis to that sequent and obtain $IEL^*_{\mathrm{G}} \vdash \Pi, F, G, \Gamma' \Rightarrow \Delta, \Lambda$. Hence

$$\dfrac{\Pi, F, G, \Gamma' \Rightarrow \Delta, \Lambda}{\Pi, F \wedge G, \Gamma' \Rightarrow \Delta, \Lambda}\ (\wedge \Rightarrow).$$

The remaining cases of V are considered similarly, see e.g., [1, 8]. □

It is proved in [6] that the rule of weakening

$$\dfrac{\Gamma \Rightarrow \theta}{\Pi, \Gamma \Rightarrow \theta'}\ (W)'$$

is admissible in $\mathbf{IEL}^-_{\mathrm{G}}$. Here Π is an arbitrary multiset of formulas, and $\theta = \theta' = F$ or $\theta = \emptyset$ and $\theta' \in \{\emptyset, F\}$.

Lemma 2. *All* $\boldsymbol{IEL}^*_G$*, rules except* $(\Rightarrow\rightarrow)$ *and* (KI)*, are height-preserving invertible.*

Proof. This lemma is proved by induction on conclusion derivation height. Let us consider, e.g., the rule $(\Rightarrow\rightarrow)$. If the conclusion is an axiom, then the premise is an axiom as well. Let the conclusion be derived as follows:

$$\dfrac{\begin{matrix} V \\ \Gamma \Rightarrow F \rightarrow G, F_1, F_2, \Delta \end{matrix}}{\Gamma \Rightarrow F \rightarrow G, F_1 \vee F_2, \Delta}\ (\Rightarrow \vee).$$

We apply the inductive hypothesis to the premise of $(\Rightarrow \vee)$ and obtain $IEL^*_{\mathrm{G}} \vdash \Gamma, F \Rightarrow G$. The required sequent is obtained as follows:

$$\dfrac{\Gamma, F \Rightarrow G}{\Gamma, F \Rightarrow G, F_1 \vee F_2, \Delta}\ (W).$$

Hence the proof follows from Lemma 1.

Let the conclusion be derived as follows:

$$\dfrac{\begin{matrix} V \\ F_1, F_2, \Gamma \Rightarrow F \rightarrow G, \Delta \end{matrix}}{F_1 \wedge F_2, \Gamma \Rightarrow F \rightarrow G, \Delta}\ (\wedge \Rightarrow)$$

$IEL^*_{\mathrm{G}} \vdash F_1, F_2, \Gamma, F \Rightarrow G$. Using this fact, the required sequent is obtained as follows:

$$\dfrac{\dfrac{F_1, F_2, \Gamma, F \Rightarrow G}{F_1, F_2, \Gamma, F \Rightarrow G, \Delta}\ (W)}{F_1 \wedge F_2, \Gamma, F \Rightarrow G, \Delta}\ (\wedge \Rightarrow).$$

The proof follows from the fact that the rule of weakening is height-preserving admissible in $\mathbf{IEL}^*_{\mathrm{G}}$, according to Lemma 1.

The remaining cases and rules of $\mathbf{IEL}^*_{\mathrm{G}}$ are considered similarly. We skip the details and refer to the analogous lemmas in [1, 2, 8]. □

Theorem 1. *The rule of contraction*

$$\frac{\Gamma' \Rightarrow \Delta'}{\Gamma \Rightarrow \Delta}\ (C)$$

is admissible in $\boldsymbol{IEL}^*_{G}$. *Here:*

1) $\Gamma' = \Gamma$ *or* $\Gamma' = F, F, \Gamma_1$ *and* $\Gamma = F, \Gamma_1$ *and*

2) $\Delta' = \Delta$ *or* $\Delta' = G, G, \Delta_1$ *and* $\Delta = G, \Delta_1$.

Proof. The theorem is proved by induction on the height h of derivation of the premise. If $h = 0$, then the conclusion is an axiom and the proof is obtained. Let $\Gamma' = (F \wedge G, F \wedge G, \Gamma_1)$, $\Delta' = (H, H, \Delta_1)$, $IEL^*_{\mathrm{G}} \vdash^V \Gamma' \Rightarrow \Delta'$ and V be as follows:

$$\frac{\begin{array}{c}V' \\ F, G, F \wedge G, \Gamma_1 \Rightarrow H, H, \Delta_1\end{array}}{F \wedge G, F \wedge G, \Gamma_1 \Rightarrow H, H, \Delta_1}\ (\wedge \Rightarrow).$$

According to Lemma 2, the rule $(\wedge \Rightarrow)$ is height-preserving invertible. Hence $IEL^*_{\mathrm{G}} \vdash^{V_1} F, G, F, G, \Gamma_1 \Rightarrow H, H, \Delta_1$ and $hV_1 \leqslant hV$. We apply the inductive hypothesis to this sequent twice, obtaining $IEL^*_{\mathrm{G}} \vdash F, G, \Gamma_1 \Rightarrow H, \Delta_1$. The required sequent is inferred as follows:

$$\frac{F, G, \Gamma_1 \Rightarrow H, \Delta_1}{F \wedge G, \Gamma_1 \Rightarrow H, \Delta_1}\ (\wedge \Rightarrow).$$

Let V be as follows:

$$\frac{\begin{array}{c}V' \\ F, F, \Gamma_1, G \Rightarrow H\end{array}}{F, F, \Gamma_1 \Rightarrow G \to H, G \to H, \Delta_1}\ (\Rightarrow\to).$$

According to the inductive hypothesis, $IEL^*_{\mathrm{G}} \vdash F, \Gamma_1, G \Rightarrow H$. The required sequent is inferred as follows:

$$\frac{F, \Gamma_1, G \Rightarrow H}{F, \Gamma_1 \Rightarrow G \to H, \Delta_1}\ (\Rightarrow\to).$$

The remaining cases of V are considered similarly. □

4 Equivalence of $\mathbf{IEL}^*_{\mathbf{G}}$ and $\mathbf{IEL}^-_{\mathbf{G}}$ for intuitionistic sequents

Lemma 3. *If* $IEL^-_G \vdash^V S$, *then* $IEL^*_G \vdash S$.

Proof. The lemma is proved by induction on hV. If $hV = 0$, then S is an axiom. Let $hV > 0$. We consider cases of the first from bottom rule applications (r) in $V(S)$. Let $(r) = (\Rightarrow \vee)$:

$$\frac{\begin{matrix}\dots\\ \Gamma \Rightarrow F_i\end{matrix}}{S:\ \Gamma \Rightarrow F_1 \vee F_2}\ (\Rightarrow \vee)$$

where $i \in \{1,2\}$. According to the inductive hypothesis, $IEL^*_{\mathrm{G}} \vdash (\Gamma \Rightarrow F_i)$. We have $IEL^*_{\mathrm{G}} \vdash (\Gamma \Rightarrow F_1, F_2)$, based on Lemma 1. Hence we infer S in $\mathbf{IEL}^*_{\mathrm{G}}$:

$$\frac{\Gamma \Rightarrow F_1, F_2}{S:\ \Gamma \Rightarrow F_1 \vee F_2}\ (\Rightarrow \vee).$$

Let $(r) = (\Rightarrow\rightarrow)$:

$$\frac{\begin{matrix}\dots\\ \Gamma, F \Rightarrow G\end{matrix}}{S:\ \Gamma \Rightarrow F \rightarrow G}\ (\Rightarrow\rightarrow).$$

We have $IEL^*_{\mathrm{G}} \vdash (\Gamma, F \Rightarrow G)$ by the inductive hypothesis. Hence we infer S in $\mathbf{IEL}^*_{\mathrm{G}}$:

$$\frac{\Gamma, F \Rightarrow G}{S:\ \Gamma \Rightarrow F \rightarrow G}\ (\Rightarrow\rightarrow).$$

The remaining cases of (r) are considered in the same way, using the inductive hypothesis on hV. $\square$

Let $\Delta = (F_1, \dots, F_n)$, where $n \geqslant 0$ (we assume that Δ is empty if $n = 0$). If $n \in \{0,1\}$, then $\vee\Delta = \Delta$; otherwise, $\vee\Delta = F_1 \vee \ldots \vee F_n$.

Lemma 4. *If $IEL^*_G \vdash^V (\Gamma \Rightarrow \Delta)$, then $IEL^-_G \vdash (\Gamma \Rightarrow \vee\Delta)$.*

Proof. The lemma is proved by induction on hV. If $hV = 0$, then
1) Γ has a member $\perp$ or
2) both Γ and Δ have some member A,
i.e., $\Gamma = (A, \Gamma')$ and $\Delta = (A, \Delta')$, where Γ' and Δ' are some multisets of formulas. The sequent $\Gamma \Rightarrow \vee\Delta$ is an axiom in case 1. In case 2, $\Gamma \Rightarrow \vee\Delta$ is an axiom if $\Delta' = \emptyset$; otherwise $\Gamma \Rightarrow \vee\Delta$ is derived in $\mathbf{IEL}^-_{\mathrm{G}}$ by using rule $(\Rightarrow \vee)$:

$$\frac{A, \Gamma' \Rightarrow A}{A, \Gamma' \Rightarrow \vee(A, \Delta')}\ (\Rightarrow \vee).$$

Let $hV > 0$. We consider cases of the first from bottom rule applications (r) in V. Let $(r) = (\Rightarrow \vee)$:

$$\frac{\begin{matrix}\dots\\ \Gamma \Rightarrow F, G, \Delta\end{matrix}}{S:\ \Gamma \Rightarrow F \vee G, \Delta}\ (\Rightarrow \vee).$$

We have $IEL^-_{\mathrm{G}} \vdash \Gamma \Rightarrow \vee(F, G, \Delta)$, according to the inductive hypothesis. The proof follows from the fact that $\vee(F, G, \Delta) = \vee(F \vee G, \Delta)$.

Let $(r) = (\Rightarrow \wedge)$:

$$\frac{\begin{matrix}\dots\\ \Gamma \Rightarrow F, \Delta\end{matrix} \qquad \begin{matrix}\dots\\ \Gamma \Rightarrow G, \Delta\end{matrix}}{\Gamma \Rightarrow F \wedge G, \Delta}\ (\Rightarrow \wedge).$$

According to the inductive hypothesis, $IEL^{-}_{\mathrm{G}} \vdash \Gamma \Rightarrow \vee(F, \Delta)$ and $IEL^{-}_{\mathrm{G}} \vdash \Gamma \Rightarrow \vee(G, \Delta)$. We have

$$\frac{\Gamma \Rightarrow \vee(F, \Delta) \qquad \Gamma \Rightarrow \vee(G, \Delta)}{\Gamma \Rightarrow \vee(F, \Delta) \wedge \vee(G, \Delta)} \, (\Rightarrow \wedge).$$

It is easy to check that $IEL^{-}_{\mathrm{G}} \vdash \big(\vee(F, \Delta) \wedge \vee(G, \Delta) \Rightarrow \vee(F \wedge G, \Delta)\big)$. Hence

$$\frac{\Gamma \Rightarrow \vee(F, \Delta) \wedge \vee(G, \Delta) \qquad \vee(F, \Delta) \wedge \vee(G, \Delta) \Rightarrow \vee(F \wedge G, \Delta)}{\Gamma \Rightarrow \vee(F \wedge G, \Delta)} \, (\mathrm{cut}).$$

The proof follows from the fact that the rule (cut) is admissible in $\mathbf{IEL}^{-}_{\mathrm{G}}$.

Let $(r) = (KI)$:

$$\frac{\begin{array}{c}\ldots\\ \Gamma, \Pi, K\Pi \Rightarrow F\end{array}}{\Gamma, \Pi, K\Pi \Rightarrow KF, \Delta} \, (KI).$$

We have

$$\frac{\dfrac{\Gamma, \Pi, K\Pi \Rightarrow F}{\Gamma, \Pi, K\Pi \Rightarrow KF} \, (KI) \qquad KF \Rightarrow \vee(KF, \Delta)}{\Gamma, \Pi, K\Pi \Rightarrow \vee(KF, \Delta)} \, (cut).$$

The proof follows from the facts that the rule (cut) is admissible and the sequent $KF \Rightarrow \vee(KF, \Delta)$ is derivable in $\mathbf{IEL}^{-}_{\mathrm{G}}$.

The remaining cases of (r) are proved similarly. □

Theorem 2. *$IEL^{-}_{G} \vdash \Gamma \Rightarrow \theta$, iff $IEL^{*}_{G} \vdash \Gamma \Rightarrow \theta$, where $\theta \in \{\emptyset, F\}$.*

Proof. The proof follows from Lemmas 3 and 4. □

Corollary 1. ***IEL^{*}_{G}*** *is sound and complete for intuitionistic sequents.*

Proof. The proof follows from the fact that $\mathbf{IEL}^{-}_{\mathrm{G}}$ is sound and complete for intuitionistic sequents. □

According to Theorem 2, the calculi $\mathbf{IEL}^{*}_{\mathrm{G}}$ and $\mathbf{IEL}^{-}_{\mathrm{G}}$ are equivalent for intuitionistic sequents. Making use of this theorem and the fact that the rule of cut is admissible in $\mathbf{IEL}^{-}_{\mathrm{G}}$, we also prove in this section that the rule of cut is admissible in $\mathbf{IEL}^{*}_{\mathrm{G}}$.

Proposition 1. *If $IEL^{-}_{G} \vdash^{V} \Gamma \Rightarrow \vee\Delta$, then $IEL^{-}_{G} \vdash \Gamma \Rightarrow \vee(\vee\Delta, \Lambda)$, where Λ is any multiset of formulas.*

Proof. We apply $(\Rightarrow \vee)$ backwards to $\Gamma \Rightarrow \vee(\vee\Delta, \Lambda)$ and get

$$\frac{\begin{array}{c}V\\ \Gamma \Rightarrow \vee\Delta\end{array}}{\Gamma \Rightarrow \vee(\vee\Delta, \Lambda)} \, (\Rightarrow \vee). \quad \square$$

Lemma 5. *The rule*

$$\frac{\Gamma \Rightarrow \vee(F, \Delta) \quad F, \Pi \Rightarrow \theta}{\Gamma, \Pi \Rightarrow \vee(\theta, \Delta)} \; (cut)'$$

is admissible in $\boldsymbol{IEL}_{\mathrm{G}}^{-}$. *Here* F *is an arbitrary formula,* Δ *is an arbitrary multiset of formulas and* $\theta \in \{\emptyset, G\}$.

Proof. If $\Delta = \emptyset$, then the proof follows from the fact that (cut) is admissible in $\mathbf{IEL}_{\mathrm{G}}^{-}$. Assume that some formulas occur in Δ. Let $IEL_{\mathrm{G}}^{-} \vdash^{V} \Gamma \Rightarrow \vee(F, \Delta)$. The lemma is proved by induction on hV. If $hV = 0$, then the conclusion of $(cut)'$ is an axiom. Let $hV > 0$. We consider cases of the first from bottom rule applications (r) in V. Let $(r) = (\Rightarrow \vee)$:

$$\frac{\Gamma \Rightarrow F}{\Gamma \Rightarrow \vee(F, \Delta)} \; (\Rightarrow \vee) \quad \text{or} \quad \frac{\Gamma \Rightarrow \vee\Delta}{\Gamma \Rightarrow \vee(F, \Delta)} \; (\Rightarrow \vee).$$

The first case: $IEL_{\mathrm{G}}^{-} \vdash \Gamma, \Pi \Rightarrow \theta$, according to the inductive hypothesis. Hence $IEL_{\mathrm{G}}^{-} \vdash \Gamma, \Pi \Rightarrow \vee(\theta, \Delta)$ by Proposition 1. The second case: $IEL_{\mathrm{G}}^{-} \vdash \Gamma, \Pi \Rightarrow \vee(\theta, \Delta)$ based on the fact that the rule of weakening is admissible in $\mathbf{IEL}_{\mathrm{G}}^{-}$ and Proposition 1.

The remaining cases of (r) are considered by the inductive hypothesis. □

Theorem 3. *The rule of cut*

$$\frac{\Gamma \Rightarrow F, \Delta \quad F, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Delta, \Lambda} \; (cut)^{*}$$

is admissible in $\boldsymbol{IEL}_{\mathrm{G}}^{*}$, *where* F *is an arbitrary formula.*

Proof. Assume that the premises of $(cut)^{*}$ are derivable in $\mathbf{IEL}_{\mathrm{G}}^{*}$. We get $IEL_{\mathrm{G}}^{-} \vdash \Gamma \Rightarrow \vee(F, \Delta)$ and $IEL_{\mathrm{G}}^{-} \vdash \Gamma \Rightarrow \vee(\Lambda)$, according to Lemma 4. Hence $IEL_{\mathrm{G}}^{-} \vdash \Gamma, \Pi \Rightarrow \vee(\vee\Lambda, \Delta)$ by Lemma 5. This yields $IEL_{\mathrm{G}}^{*} \vdash \Gamma, \Pi \Rightarrow \vee(\vee\Lambda, \Delta)$, according to Theorem 2. We obtain $IEL_{\mathrm{G}}^{*} \vdash \Gamma, \Pi \Rightarrow \Lambda, \Delta$, based on the fact that the rule $(\Rightarrow \vee)$ is invertible, according to Lemma 2. □

5 Conclusion

The multi-succedent sequent calculus $\mathbf{IEL}_{\mathrm{G}}^{*}$ for intuitionistic epistemic logic has been presented in the paper. The following proof-theoretical properties of $\mathbf{IEL}_{\mathrm{G}}^{*}$ have been proved: the structural rules of weakening and contraction and the rule of cut are admissible in $\mathbf{IEL}_{\mathrm{G}}^{*}$; all rules of $\mathbf{IEL}_{\mathrm{G}}^{*}$, except $(\Rightarrow\rightarrow)$ and (KI), are invertible. It has also been proved that $\mathbf{IEL}_{\mathrm{G}}^{*}$ and $\mathbf{IEL}_{\mathrm{G}}^{-}$ are equivalent for the class of intuitionistic sequents, i.e., sequents with at most one formula in the succedent. This fact shows that the requirement that premises and conclusions of all rules in a sequent calculus have at most one formula in the succedent is not essential for the intuitionistic epistemic logic.

References

[1] R. Alonderis. Proof-theoretical investigation of temporal logic with time gaps. *Lith. Math. J.*, **40**(3):255–276, 2000.

[2] R. Alonderis, J. Sakalauskaitė. A labelled sequent calculus for half-order modal logic. *IfCoLoG J. Log. their Appl.*, **5**(1):121–163, 2018.

[3] G. Amati, F. Pirri. A uniform tableau method for intuitionistic modal logics I. *Stud. Log.*, **53**(1):29–60, 1994.

[4] S. Artemov, T. Protopopescu. Intuitionistic epistemic logic. *Rev. Symb. Log.*, **9**:266–298, 2016.

[5] P. Balbiani, J. Boudou. Intuitionistic linear temporal logics. *ACM Trans. Comput. Log.*, **21**(2):1–32, 2019.

[6] V.N. Krupski, A. Yatmanov. Sequent calculus for intuitionistic epistemic logic IEL. In *International Symposium on Logical Foundations of Computer Science*, pp. 187–201. Springer, 2016.

[7] Z. Lin, M. Ma. Gentzen sequent calculi for some intuitionistic modal logics. *Log. J. IGPL*, **27**(4):596–623, 2019.

[8] A.S. Troelstra, H. Schwichtenberg. *Basic Proof Theory,* second edition. Cambridge University Press, 2000.

[9] D. Wijesekera. Constructive modal logics I. *Ann. Pure Appl. Log.*, **50**:271–301, 1990.

REZIUMĖ

Multisukcedentinis sekvencinis skaičiavimas intuicionistinei epsiteminei logikai

R. Alonderis

Straipsnyje yra pateiktas daugiasukcedentinis sekvencinis skaičiavimas intuicionistinei episteminei logikai. Įrodytas struktūrinių ir pjūvio taisyklių leistinumas šiame skaičivime. Taip pat įrodytas šio skaičiavimo bei tradicinio intuicionistinio skaičiavimo ekvivalentumas intuicionistinių sekvencijų atžvilgiu.

Raktiniai žodžiai: intuicionistinė episteminė logika; sekvencinis skaičiavimas