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Viscosity solutions to the Dirichlet problem of Bellman equation

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1. Introduction

Bellman equations that are, in general, fully nonlinear degenerate integrodifferential equations of the second order, play an important role in the theory of controlled continuous time Markov processes. In many cases, the payoff functions are classical or generalized solutions of Bellman equations.

The class of viscosity solutions introduced by M.G. Crandall and P.L. Lions [1] is an important class of generalized solutions of Bellman equations. Analysis of viscosity solutions usually requires minimum regularity both of initial data and of solutions.

There are some difficulties in defining the boundary conditions for boundary value problems of degenerate Bellman equations. For example, the Dirichlet problem may have no viscosity solution continuous up to the boundary and coinciding with a given function on the entire boundary. In such a case, the boundary condition is understood in a generalized viscosity sense (see [2], Section 7).

In this paper, we consider the existence of viscosity solutions to the Dirichlet problem of degenerate Bellman equations with a generalized boundary condition. The approach is based on a probabilistic representation of a viscosity solution.

2. Notation, definitions, and the main result

Let D be an open, bounded, and connected set in $\mathbb{R}^d$ with the boundary ∂D and the closure $\overline{D}$. Let $A = \{1, 2, \ldots\}$, and let $\Pi(dz) = dz/|z|^{d+1}$.

Suppose that we are given measurable functions $\sigma = \{\sigma_{ij}^{\alpha}(x)\}$, $b = \{b_i^{\alpha}(x)\}$, $c = \{c_i^{\alpha}(x,z)\}$, $r = r^{\alpha}(x)$, $f = f^{\alpha}(x)$, and $g = g(x)$; $i, j = 1, \ldots, d$; $x, z \in \mathbb{R}^d$; $\alpha \in A$, satisfying the following assumption.

Assumption 1. (i). There exists a constant K such that, for any $x, y \in \mathbb{R}^d$, $\alpha \in A$,

$$\|\sigma^{\alpha}(x)\| + |b^{\alpha}(x)| + \int |c^{\alpha}(x,z)|^2 \Pi(dz) \leqslant K,$$

$$\|\sigma^{\alpha}(x) - \sigma^{\alpha}(y)\|^2 + |b^{\alpha}(x) - b^{\alpha}(y)|^2 + \int |c^{\alpha}(x,z) - c^{\alpha}(y,z)|^2 \Pi(dz) \leqslant K|x-y|^2,$$

(ii) r, f, and g are bounded, continuous in x uniformly with respect to α, and $r \geqslant \lambda$ for some constant $\lambda > 0$.

Denote by $C(\Gamma)$ the class of bounded and continuous functions on Γ, and denote by $C^2(\Gamma)$ the class of bounded and continuous functions on Γ, together with their first and second order partial derivatives.

Denote $a = \frac{1}{2}\sigma\sigma^*$ and introduce the operators L^α, F^α, $\alpha \in A$, and F by

$$\begin{aligned} L^\alpha u(x) &= \sum_{i,j=1}^{d} a_{ij}^\alpha(x) u_{x_i x_j}(x) + \sum_{i=1}^{d} b_i^\alpha(x) u_{x_i}(x) \\ &\quad + \int \Big[u(x + c^\alpha(x,z)) - u(x) - \sum_{i=1}^{d} u_{x_i}(x) c_i^\alpha(x,z) \Big] \Pi(dz), \\ F^\alpha[u](x) &= L^\alpha u(x) - r^\alpha(x) u(x) + f^\alpha(x), \\ F[u](x) &= \sup_{\alpha \in A} F^\alpha[u](x). \end{aligned}$$

We consider the Dirichlet problem for the Bellman equation

$$F[u](x) = 0, \quad x \in D, \tag{1}$$

$$u(x) = g(x), \quad x \in \mathbb{R}^d \backslash D, \tag{2}$$

in the sense of viscosity solutions (see Definition 1 and Remark 1 below).

Definition 1. A function $u \in C(\overline{D})$, $u = g$ on $\mathbb{R}^d \backslash \overline{D}$, is called a viscosity solution to the problem (1)–(2) with generalized boundary condition if it possesses the following properties:

(i) if $x \in D$ and there exists a function $\phi \in C^2(\mathbb{R}^d)$ such that $\phi(x) = u(x)$ and $\phi \geqslant u$ (resp. $\phi \leqslant u$) on $\mathbb{R}^d$, then $F[\phi](x) \geqslant 0$ (resp. $F[\phi](x) \leqslant 0$).

(ii) if $x \in \partial D$, $u(x) > g(x)$ (resp. $u(x) < g(x)$), and there exists a function $\phi \in C^2(\mathbb{R}^d)$ such that $\phi(x) = u(x)$ and $\phi \geqslant u$ (resp. $\phi \leqslant u$) on $\mathbb{R}^d$, then $F[\phi](x) \geqslant 0$ (resp. $F[\phi](x) \leqslant 0$).

Remark 1. We say that boundary condition (2) is satisfied in the strong viscosity sense if $u = g$ on $\mathbb{R}^d \backslash D$. In this case, in order to prove that a function u is a viscosity solution to problem (1)–(2), it remains to verify property (i) only.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space with a complete right-continuous filtration $\mathbb{F} = \{\mathcal{F}_t,\ t \geqslant 0\}$, $\mathcal{F} = \bigvee_{t \geqslant 0} \mathcal{F}_t$. Denote by $\mathfrak{A}$ the set of all $\mathcal{F}_t$-progressively measurable functions $\alpha = \alpha_t(\omega)$ on $[0, \infty) \times \Omega$ with values in A.

Consider the strong solutions $X_t = X_t^{\alpha,x}$, $\alpha \in \mathfrak{A}$, $x \in \mathbb{R}^d$, of Ito equations

$$X_t = x + \int_0^t \sigma^{\alpha_u}(X_u) dW_u + \int_0^t b^{\alpha_u}(X_u) du + \int_0^t \int c^{\alpha_u}(X_u, z) q(du dz), \tag{3}$$

where $(w_t, \mathcal{F}_t)$ is a standard Wiener process in $\mathbb{R}^d$, $q(dt, dz) = p(dtdz) - \Pi(dz)dt$, and $(p(dt, dz), \mathcal{F}_t)$ is a Poisson measure with the compensator $\Pi(dz)dt$. As is well known, equation (3) has a unique solution for any $\alpha \in \mathfrak{A}$ and $x \in \mathbb{R}^d$.

Introduce the payoff function

$$v(x) = \sup_{\alpha\in\mathfrak{A}} \mathbf{E}\left\{ \int_0^{\tau_D} e^{-\varphi_t} f^{\alpha_t}(X_t)dt + e^{-\varphi_{\tau_D}} g(X_{\tau_D}) \mathrm{I\!I}_{\{\tau_D<\infty\}} \right\},$$

where $\tau_D = \tau_D^{\alpha,x} = \inf\{t \geqslant 0 : X_t^{\alpha,x} \notin \overline{D}\}$, $\tau_D = \infty$ if the set in brackets is empty, $X_t = X_t^{\alpha,x}$, and $\varphi_t = \varphi_t^{\alpha,x} = \int_0^t r^{\alpha_s}(X_s^{\alpha,x})ds$.

It is not difficult to prove that v is measurable and bounded. Obviously, $v = g$ on $\mathbb{R}^d\setminus\overline{D}$, but not necessarily on ∂D.

Introduce the following assumption.

Assumption 2. For all $\alpha \in \mathfrak{A}$ and $x \in \overline{D}$, the process

$$\xi_t^{\alpha,x} := e^{-\varphi_{t\wedge\tau_D}} v(X_{t\wedge\tau_D}) + \int_0^{t\wedge\tau_D} e^{-\varphi_s} f(X_s)ds, \quad t \geqslant 0,$$

where $\tau_D = \tau_D^{\alpha,x}$, $\varphi_t = \varphi_t^{\alpha,x}$, and $X_t = X_t^{\alpha,x}$, is a right-continuous $\mathcal{F}_t$-supermartingale.

Remark 2. Assumption 2 implies the Bellman principle

$$v(x) = \sup_{\alpha\in\mathfrak{A}} \mathbf{E}\left\{ \int_0^{\tau_D\wedge t} e^{-\varphi_s} f^{\alpha_s}(X_s)ds + e^{-\varphi_{\tau_D\wedge t}} v(X_{\tau_D\wedge t}) \right\}, \tag{4}$$

for all $t \geqslant 0$.

Indeed,

$$v(x) = \mathbb{E}\xi_0^{\alpha,x} \geqslant \mathbb{E}\xi_t^{\alpha,x} \geqslant \lim_{t\to\infty} \mathbb{E}\xi_t^{\alpha,x} = v^\alpha(x).$$

Taking here the upper bounds over $\alpha \in \mathfrak{A}$, we get (4).

The main result of the paper is the following theorem.

THEOREM 1. *Let Assumptions* 1 *and* 2 *be satisfied, and let* $v \in C(\overline{D})$. *Then* v *is a viscosity solution to problem* (1)–(2).

Remark 3. It is not easy to give nontrivial sufficient conditions on the initial data, which assure that $v \in C(\overline{D})$. For this purpose, appropriate barrier functions are investigated (cf. [3]). Moreover, it is usually assumed that the boundary ∂D consists of two parts with nonintersecting closures, one of which is accessible and the other one is totally inaccessible. Assumption 2 can be verified in many cases if $v \in C(\overline{D})$.

3. Proof of Theorem 1

We give here a sketch of the proof only.

Let $x \in D$, and let there exist a fuction $\phi \in C^2(\mathbb{R}^d)$ such that $\phi(x) = v(x)$ and $\phi \geqslant v$ on $\mathbb{R}^d$. By the Ito formula we have, for all $\alpha \in \mathfrak{A}$ and $t \geqslant 0$,

$$\phi(x) = \mathbf{E}\left\{ \int_0^{t\wedge\tau_D} e^{-\varphi_s}(-L^{\alpha_s}\phi + r^{\alpha_s}\phi)(X_s)ds + e^{-\varphi_{\tau_D\wedge t}} \phi(X_{\tau_D\wedge t}) \right\}, \tag{5}$$

where $\tau_D = \tau_D^{\alpha,x}$, $\varphi_t = \varphi_t^{\alpha,x}$, $X_t = X_t^{\alpha,x}$. Together with the Bellman principle, this yields

$$0 = v(x) - \phi(x) = \sup_{\alpha \in \mathfrak{A}} \mathbf{E}\Big\{ \int_0^{\tau_D \wedge t} e^{-\varphi_s} F^{\alpha_s}[\phi](X_s)ds + e^{-\varphi_{\tau_D \wedge t}} (v - \phi)(X_{\tau_D \wedge t}) \Big\} \leqslant \sup_{\alpha \in \mathfrak{A}} \mathbf{E} \int_0^{\tau_D \wedge t} e^{-\varphi_s} F[\phi](X_s)ds.$$

Using this inequality and Assumption 1, it is not difficult to prove that $F[\phi] \in C(\mathbb{R}^d)$ and

$$0 \leqslant \lim_{t \downarrow 0} \frac{1}{t} \sup_{\alpha \in \mathfrak{A}} \mathbf{E} \int_0^{\tau_D \wedge t} e^{-\varphi_s} F[\phi](X_s)ds = F[\phi](x).$$

Let $x \in \partial D$ and $v(x) > g(x)$. Fix arbitrary sequences $\varepsilon_m \downarrow 0$ and $t_m \downarrow 0$, $m \uparrow \infty$. Obviously, there exists a sequence $\alpha_m \in \mathfrak{A}$, $m = 1, 2, \ldots$, such that

$$v(x) \leqslant v^{\alpha_m}(x) + \varepsilon_m t_m.$$

The following two cases are possible:

$$\text{(i)} \quad \varliminf_{m \to \infty} \mathbf{P}\Big\{\tau_D^{\alpha_m,x} > 0\Big\} = 0,$$

$$\text{(ii)} \quad \varliminf_{m \to \infty} \mathbf{P}\Big\{\tau_D^{\alpha_m,x} > 0\Big\} > 0.$$

In the case (i), we have $v(x) \leqslant g(x)$, by the definition of v. Therefore, it remains to consider the case (ii). By Assumption 2 and Ito formula, we have

$$\begin{aligned} 0 =& v(x) - \phi(x) \leqslant v^{\alpha_m}(x) - \phi(x) + \varepsilon_m t_m \\ \leqslant& \mathbf{E}\Big\{ \int_0^{\tau_D \wedge t_m} e^{-\varphi_s} f^{\alpha_{ms}}(X_s)ds + e^{-\varphi_{\tau_D \wedge t_m}} v(X_{\tau_D \wedge t_m}) \Big\} \\ &- \phi(x) + \varepsilon_m t_m \leqslant \mathbf{E} \int_0^{\tau_D \wedge t_m} e^{-\varphi_s} F^{\alpha_{ms}}[\phi](X_s)ds + \varepsilon_m t_m \\ \leqslant& \mathbf{E} \int_0^{\tau_D \wedge t_m} e^{-\varphi_s} F[\phi](X_s)ds + \varepsilon_m t_m, \end{aligned}$$

where $\tau_D = \tau_D^{\alpha_m,x}$, $\varphi_t = \varphi_t^{\alpha_m,x}$, and $X_t = X_t^{\alpha_m,x}$.

The last inequality, together with the continuity of $F[\phi]$, yields

$$0 \leqslant F[\phi](x) \varliminf_{m \to \infty} \mathbf{E} \frac{\tau_D^{\alpha_m,x}}{t_m} \wedge 1.$$

Hence $F[\phi](x) \geqslant 0$.

Let us prove the converse inequalities (see Definition 1).

Let $x \in D$, and let there exist a function $\phi \in C^2(\mathbb{R}^d)$ such that $\phi(x) = v(x)$ and $\phi \leqslant v$ on $\mathbb{R}^d$. Fix an arbitrary $k \in A$. Using (5), written for the constant strategy $\alpha_t = k \in A$, and the Bellman principle, we get

$$\begin{aligned}
0 &= v(x) - \phi(x) \\
&\geqslant E\left\{ \int_0^{\tau_D \wedge t} e^{-\varphi_s} F^k[\phi](X_s)\, ds + e^{-\varphi_{\tau_D \wedge t}} (v - \phi)(X_{\tau_D \wedge t}) \right\} \\
&\geqslant E \int_0^{\tau_D \wedge t} e^{-\varphi_s} F^k[\phi](X_s)\, ds,
\end{aligned}$$

where $\tau_D = \tau_D^{k,x}$, $\varphi_t = \varphi_t^{k,x}$, and $X_t = X_t^{k,x}$. Using this inequality and Assumption 1, it is not difficult to prove that $F^k[\phi] \in C(\mathbb{R}^d)$, $k \in A$, and

$$0 \geqslant \lim_{t \downarrow 0} \frac{1}{t} E \int_0^{\tau_D \wedge t} e^{-\varphi_s} F^k[\phi](X_s)\, ds = F^k[\phi](x).$$

Since k was arbitrary, this yields

$$F[\phi](x) \leqslant 0.$$

Let $x \in \partial D$ and $v(x) < g(x)$. Fix an arbitrary $k \in A$. The following two cases are possible:

$$\text{(i)} \quad P\left\{ \tau_D^{k,x} > 0 \right\} = 0,$$

$$\text{(ii)} \quad P\left\{ \tau_D^{k,x} > 0 \right\} > 0.$$

In the case (i), we have $v(x) \geqslant g(x)$, by the definition of v. Therefore, it remains to consider the case (ii). By the Bellman principle and the Ito formula, we have

$$\begin{aligned}
0 &= v(x) - \phi(x) \\
&\geqslant E\left\{ \int_0^{\tau_D \wedge t} e^{-\varphi_s} f^k(X_s)\, ds + e^{-\varphi_{\tau_D \wedge t}} v(X_{\tau_D \wedge t}) - \phi(x) \right\} \\
&= E\left\{ \int_0^{\tau_D \wedge t} e^{-\varphi_s} F^k[\phi](X_s)\, ds + e^{-\varphi_{\tau_D \wedge t}} (v - \phi)(X_{\tau_D \wedge t}) \right\} \\
&\geqslant E \int_0^{\tau_D \wedge t} e^{-\varphi_s} F^k[\phi](X_s)\, ds,
\end{aligned}$$

where $\tau_D = \tau_D^{k,x}$, $\varphi_t = \varphi_t^{k,x}$, and $X_t = X_t^{k,x}$. This inequality, together with the continuity of $F^k[\phi]$, yields

$$0 \geqslant \lim_{t \downarrow 0} E\left(\frac{\tau_D^{k,x}}{t} \wedge 1\right) F^k[\phi](x).$$

Hence,

$$F^k[\phi](x) \leqslant 0.$$

Since $k \in A$ was arbitrary, we get

$$F[\phi](x) \leqslant 0.$$

The theorem is proved.

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Belmano lygties dirichlė uždavinio viskoziniai sprendiniai

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Aprašytos pakankamos integrodiferencialinės Belmano lygties Košy–Dirichlė uždavinio viskozinio sprendinio egzistavimo sąlygos. Tyrimas pagrįstas Markovo procesų optimaliojo valdymo teorijos metodais.