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Smooth interpolation with biangle surface patches

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1. Introduction

Two-sided region is natural in design of surfaces of complex shape. In [5] and [6] they were investigated by M. Sabin from the different points of view. In [3] a new control point scheme was developed for a biangle surface patches of degree $2n$. Their properties (convex hull, subdivision, degree elevation, etc.) are more or less similar to the well-known properties of the classical Bézier tensor product and triangular patches. Any biangle region on any oval (but only oval) quadric, bounded by two conics, can be represented by a biangle patch of degree 2. (Oval quadric by a definition is a quadric projectively equivalent to a sphere). So the following question arises. How to modify a definition of a biangle surface patch of degree $2n$, that: main properties are still valid; any biangle surface patch on any quadric, bounded by two conics, can be represented by a new biangle patch of degree 2. So modified biangle patch would be no doubt more flexible.

In Section 2 a new definition of a biangle surface patch of degree $2n$ is given. Its main properties are formulated in Section 3. Two methods of smooth filling two-sided holes between Bézier patches are described in Section 4. Here also is presented a special case of a biangle patch, suitable for inclusion in a B-spline surface.

In this article we do not give the proofs. They can be found in [4]. There are also some interesting examples of smooth joining biangle patches of degree 2 (quadrics).

2. Biangle of degree $2n$

2.1. *Preliminary notations*

Let D be a domain triangle in a parameter plane with the vertices V_0, V_1, V_2. Any point V in a parameter plane is uniquely represented in a form $V = u_0V_0 + u_1V_1 + u_2V_2$ with $u_0 + u_1 + u_2 = 1$. So u_0, u_1, u_2 are barycentric coordinates. We also fix a real number a and define the functions h_0, h_1, h_2, h_3 by the formulas

$$h_0 = u_0^2,\ h_1 = u_0u_2,\ h_2 = u_0u_1,\ h_3 = u_1^2 + 2au_1u_2 + u_2^2.$$

The functions h_0, h_1, h_2, h_3 vanish simultaneously at the intersection points (maybe complex conjugate) of a degenerate conic $h_3 = 0$ and a line $u_0 = 0$.

2.2. *Basis polynomials and definition*

We define basis polynomials f_{ij}^n, $n \geqslant 1$, $i, j = 0, \dots, n$ by the formula

$$f_{ij}^n = \begin{cases} k_{ij}^n h_0^{n-i-j} h_1^j h_2^i, & i+j \leqslant n, \\ k_{ij}^n h_3^{i+j-n} h_1^{n-i} h_2^{n-j}, & i+j \geqslant n. \end{cases}$$

The coefficients k_{ij}^n, $i, j = 0, \dots, n$ are defined recurrently. It is convenient to assume $k_{ij}^n = 0$, if $i \notin \{0, \dots, n\}$ or $j \notin \{0, \dots, n\}$. For $n = 1$ let $k_{00}^1 = 1$, $k_{01}^1 = 2$, $k_{10}^1 = 2$, $k_{11}^1 = 1$,. For $n > 1$ we define

(1) $\quad k_{ij}^n = k_{ij}^{n-1} + k_{i-1,j}^{n-1} + k_{i,j-1}^{n-1} + k_{i,j-2}^{n-1} + k_{i-2,j}^{n-1} + 2ak_{i-1,j-1}^{n-1}, \quad i+j < n;$

(2) $\quad k_{ij}^n = k_{i-1,j}^{n-1} + k_{i,j-1}^{n-1} + k_{i,j-2}^{n-1} + k_{i-2,j}^{n-1} + 2ak_{i-1,j-1}^{n-1}$
$\quad + k_{n-j+1,n-i-1}^{n-1} + k_{n-j-1,n-i+1}^{n-1} + 2ak_{n-j,n-i}^{n-1}, \quad i+j = n;$

(3) $\quad k_{ij}^n = k_{i-1,j-1}^{n-1} + k_{i-1,j}^{n-1} + k_{i,j-1}^{n-1} + k_{i+1,j-1}^{n-1} + k_{i-1,j} + 2ak_{i,j+1}^{n-1}, \quad i+j > n.$

The coefficients k_{ij}^n satisfy symmetry relations $k_{ij}^n = k_{ji}^n$, $k_{ij}^n = k_{n-j,n-i}^n$. For example ($n = 2, 3$) we have
$k_{00}^2 = 1$, $k_{01}^2 = 3$, $k_{02}^2 = 4$, $k_{11}^2 = 4 + 4a$.
$k_{00}^3 = 1$, $k_{01}^3 = 4$, $k_{02}^3 = 8$, $k_{03}^3 = 10$, $k_{11}^3 = 10 + 6a$, $k_{12}^3 = 14 + 16a$.
The rest of the coefficients are determined by the symmetry relations.

It is easy to check that basis functions f_{ij}^n, $i, j = 0, \dots, n$, are linearly independent.

Definition. The rational biangle surface patch of degree $2n$ with control points $P_{ij} \in \mathbb{R}^3$, $i, j = 0, \dots, n$, and weights w_{ij}, $i, j = 0, \dots, n$, is an image $F(D)$ of a map $F : D \to \mathbb{R}^3$ given by the formula

$$F(p) = \frac{\sum_{i=0}^n \sum_{j=0}^n w_{ij} P_{ij} f_{ij}^n(p)}{\sum_{i=0}^n \sum_{j=0}^n w_{ij} f_{ij}^n(p)}. \tag{2.1}$$

The map F is defined over the domain triangle D. We get a biangle surface patch, since an edge $\overline{V_1 V_2}$ is contracted to a point. In general contracting an edge causes singularity at the image point. Nonsingularity of the biangle surface patch at the point $F(\overline{V_1 V_2})$ is insured by the symmetry of a biangle representation. Here is a precise formulation of the symmetry property.

By g we denote a birational transformation of domain triangle D given by the formula

$$g(u_0, u_1, u_2) = \Big(\big((u_1 + u_2)^2 + 2(a-1)u_1 u_2\big)/q,\ u_0 u_1/q,\ u_0 u_2/q\Big),$$

where $q = (u_1 + u_2)^2 + u_0(u_1 + u_2) + 2(a-1)u_1 u_2$.

PROPOSITION 1. (*Symmetry*). *Let $P'_{ij} = P_{n-j,n-i}$, $w_{ij} = w_{n-j,n-i}$ and $F' : D \to \mathbb{R}^3$ – a map defined by the formula 2.1 with control points P'_{ij} and weights w'_{ij}. Then $F' = F \circ g$.*

3. Properties of the biangle of degree $2n$

(i) *Convex hull property.* The patch is contained in a convex hull of the control points P_{ij}, $i, j = 0, \ldots, n$, if all weights are positive.

(ii) *Boundary.* The patch has a boundary composed from two rational Bézier curves of degree $2n$ with control points coinciding with boundary points of the control net $P_{00}, P_{01}, \ldots, P_{0n}, P_{1n}, P_{2n}, \ldots P_{nn}$ and $P_{00}, P_{10}, \ldots, P_{n0}, P_{n1}, P_{n2}, \ldots P_{nn}$ but with slightly different (if $n > 1$) weights

$$\tilde{w}_{ij} = \frac{k_{ij}}{\binom{2n}{i+j}} w_{ij}.$$

(iii) *Subdivision*

$|a| < 1$. The biangle of degree $2n$ can be subdivided into two Bézier triangles of degree $2n$ and a smaller biangle of the same degree.

$|a| = 1$. The biangle of degree $2n$ can be subdivided into two Bézier triangles of degree $2n$.

$|a| > 1$. The biangle of degree $2n$ is composed from two Bézier triangles of degree $2n$ intersecting each other by a smaller biangle of the same degree.

(iv) *Reparameterization.* The patch will be the same if the weights are changed according to $\hat{w}_{ij} = \lambda^{i+j} w_{ij}$ for arbitrary $\lambda > 0$.

(v) *Implicit degree.* Implicit degree of the patch of degree $2n$ does not exceed $2n^2$.

(vi) $n = 1$ (quadrics). Let x_0, x_1, x_2, x_3 be barycentric coordinates associated with vertices $P_{00}, P_{01}, P_{10}, P_{11}$. Then an implicit equation of a biangle patch of degree 2 is

$$\frac{x_1^2}{w_{01}^2} + \frac{2ax_1x_2}{w_{01}w_{10}} + \frac{x_2^2}{w_{10}^2} - \frac{4x_0x_3}{w_{00}w_{11}} = 0.$$

We get from this equation all information about quadric. For example, the biangle patch is:

- oval quadric if $|a| < 1$;
- conic quadric (projectively equivalent to a cone) if $|a| = 1$;
- hyperbolic quadric (projectively equivalent to a hyperbolic paraboloid) if $|a| > 1$.

More precise classification of biangles of degree 2 and conditions for their smooth joining can be found in [4].

(vii) *Degree elevation.* For a patch of order $2n$ we denote by P_{ij}^n, w_{ij}^n, $i, j = 1, \ldots, n$ its control points and weights. Setting $\underline{P}_{ij}^n = (w_{ij}P_{ij}^n, w_{ij}^n)$ we represent them in a homogeneous form. We also assume $\underline{P}_{ij}^n = (0, 0, 0, 0)$ if $i \notin \{0, \ldots, n\}$ or $j \in \{0, \ldots, n\}$. We set $H_{ij}^n = k_{ij}\underline{P}_{ij}^n$, $i, j = 0, \ldots, n$, and define homogeneous control points $\underline{P}_{ij}^{n+1}$, $i, j = 0, \ldots, n+1$, by the formulas

(1) $\underline{P}_{ij}^{n+1} = (H_{i,j}^n + H_{i-1,j}^n + H_{i,j-1}^n + H_{i,j-2}^n + H_{i-2,j}^n + 2aH_{i-1,j-1}^n)/k_{ij}^{n+1}, \quad i + j < n + 1;$

(2) $\underline{P}_{ij}^{n+1} = (H_{i-1,j}^n + H_{i,j-1}^n + H_{i,j-2}^n + H_{i-2,j}^n + 2aH_{i-1,j-1}^n + H_{n-j+2,n-i}^n + H_{n-j,n-i+2}^n + 2aH_{n-j+1,n-i+1}^n)/k_{ij}^{n+1}, \quad i + j = n + 1;$

(3) $\underline{P}_{ij}^{n+1} = (H_{i-1,j-1}^n + H_{i-1,j}^n + H_{i,j-1}^n + H_{i+1,j-1}^n + H_{i-1,j+1}^n + 2aH_{i,j}^n)/k_{ij}^{n+1}, \quad i + j > n + 1.$

PROPOSITION 2. *Control points P_{ij}^{n+1} and weights w_{ij}^{n+1}, $i, j = 0, \ldots, n$, corresponding to the homogeneous control points $\underline{P}_{ij}^{n+1} = (w_{ij}^{n+1} p_{ij}^{n+1}, \ w_{ij}^{n+1})$ represent an initial biangle patch of degree $2n$ as a biangle path of degree $2n + 2$.*

When n increases the control point nets tend to the patch.

4. Filling two-sided holes

4.1. *Gregory's biangle*

Let $\underline{P}_{ij} = (w_{ij} P_{ij}, \ w_{ij})$, $i, j = 0, \ldots, n$, be the homogeneous control points of a biangle patch of degree $2n$. A rational triangle Bézier patch of degree m we also represent in a homogeneous form with control points $\underline{T}_{ijk}$, $i + j + k = m$, $i, j, k \geqslant 0$. For a fixed triangle patch of degree m we consider the biangle patches of degree $2n$ with common boundary curve and common tangent planes at the ends of this curve. Computing conditions for tangent plane continuous join of the patches we get $3m + 2n - 6$ linear equations for homogeneous coordinates of inner control points $\underline{P}_{ij}$. In general, a system of these linear equations is solvable if $n \geqslant m/2 + 1$.

Suppose we want to fill smoothly two-sided hole between two Bézier triangles with common tangent planes at common vertices. In this case we solve two systems of linear equations for homogeneous coordinates of inner control points of biangle patch of degree $2n$. (There are also explicit formulas for unknown control points.) As a rule, twist incompatibility arises at the corner points of a patch. It means, that $\underline{P}_{11} \neq \underline{\tilde{P}}_{11}$ and $\underline{P}_{n-1,n-1} \neq \underline{\tilde{P}}_{n-1,n-1}$, where $\underline{P}_{ij}$ and $\underline{\tilde{P}}_{ij}$ are the solutions of a systems of linear equations insuring tangent plane continuous join along the boundary curves with $u_1 = 0$ and $u_2 = 0$ correspondingly. This problem is solved by a standard at this moment method of variable control points (see [1], [2]). We replace incompatible control points $\underline{P}_{11}$, $\underline{\tilde{P}}_{11}$ and $\underline{P}_{n-1,n-1}$, $\underline{\tilde{P}}_{n-1,n-1}$ by the variable homogeneous control points

$$\underline{G}_{11} = \frac{u_1 \underline{P}_{11} + u_2 \underline{\tilde{P}}_{11}}{u_1 + u_2}$$

and

$$\underline{G}_{n-1,n-1} = \frac{u_1 \underline{P}_{n-1,n-1} + u_2 \underline{\tilde{P}}_{n-1,n-1}}{u_1 + u_2}.$$

4.2. Filling holes between Steiner patches

Suppose we want to fill smoothly two-sided hole between two Bézier triangles of degree 2 (Steiner patches) with common tangent planes at the common vertices with a of degree $2n$. In this case we solve two systems of linear equations. A lowest suitable n is equal to 2, since each system is solvable if $n \geqslant m/2+1$ (see 4.1). So the degree of boundary conics must be elevated to 4 and a hole can be smoothly filled with a Gregory's biangle. Gregory's patches are geometrically smooth, but have parametric singularities at the corners. Since parametric singularities are, in general, unavoidable we take a special degree elevation, which produces a parametric singularity at one corner. It allows to fill smoothly two-sided hole between Steiner patches with two biangle patches of degree 4 that have an implicit degree 4 too (a general biangle patch of degree 4 has an implicit degree 8). An implicit degree of quadratically parameterizable surfaces is $\leqslant 4$ (see [7]). Combining Steiner patches with these special filling biangles we get an interpolating scheme with the following properties:

- composed surface interpolates arbitrary fixed points in $\mathbb{R}^3$ and arbitrary fixed tangent planes at these points;
- parametric and implicit degrees of the patches are $\leqslant 4$.

Now we give a formal definition of these special biangle patches of degree 4. For positive numbers e_1 and e_2 we set

$$\tilde{f}_0 = 2e_1h_0h_1 + 2e_2h_0h_2 + h_1^2 + h_2^2;\ \tilde{f}_1 = 4e_1h_1^2 + 2h_1h_3;\ \tilde{f}_2 = 4e_2h_2^2 + 2h_2h_3;$$

$$\tilde{f}_3 = 2e_1h_1h_3 + 2e_2h_2h_3 + h_3^2;\ \tilde{f}_4 = h_1h_2.$$

(The functions h_0, h_1, h_2, h_3 are defined in 2.1.)
For the control points $\tilde{P}_0$, $\tilde{P}_1$, $\tilde{P}_2$, $\tilde{P}_3$, $\tilde{P}_4$ and weights $\tilde{w}_0$, $\tilde{w}_1$, $\tilde{w}_2$, $\tilde{w}_3$, $\tilde{w}_4$ we define a map $\tilde{F} : D \to \mathbb{R}^3$ by the formula

$$\tilde{F}(p) = \frac{\sum_{i=0}^4 \tilde{w}_i \tilde{P}_i \tilde{f}_i(p)}{\sum_{i=0}^4 \tilde{w}_i \tilde{f}_i(p)}.$$

The patch is an image $\tilde{F}(D)$.

PROPOSITION 3. *The patch $\tilde{F}(D)$ is geometrically smooth if the control points $\tilde{P}_0$, $\tilde{P}_1$, $\tilde{P}_2$, $\tilde{P}_4$ are coplanar.*

The patch $\tilde{F}(D)$ with coplanar control points $\tilde{P}_0$, $\tilde{P}_1$, $\tilde{P}_2$, $\tilde{P}_4$ we call $BS4$ biangle (Biangle Special 4 - for both parametric and implicit degrees).

PROPOSITION 4. *Suppose two rational Bézier patches of degree 2 have common tangent planes at two common vertices. Then exists 5-parameter family of a pair of $BS4$ biangles smoothly filling two-sided hole between Bézier patches.*

Filling $BS4$ biangles touch along a common conic a quadric cone.

4.3. Biangle patch for B-spline surface

We use exactly the same B-spline control point scheme as in [6]. Here we define only a biangle patch, suitable for an inclusion into this scheme.

Using notations from 2.2 (definition of a biangle) we set $g_{ij} = f_{ij}^2/k_{ij}^2, i, j = 0, 1, 2$, $\overline{f}_0 = g_{00}+g_{20}+g_{02}, \overline{f}_1 = 2g_{10}+2g_{21}, \overline{f}_2 = 2g_{01}+2g_{12}, \overline{f}_3 = g_{22}+g_{20}+g_{02}, \overline{f}_4 = 4g_{11}$. For the control points $\overline{P}_0, \overline{P}_1, \overline{P}_2, \overline{P}_3, \overline{P}_4$ and weights $\overline{w}_0, \overline{w}_1, \overline{w}_2, \overline{w}_3, \overline{w}_4$ we define a map $\overline{F} : D \to \mathbb{R}^3$ by the formula

$$\overline{F}(p) = \frac{\sum_{i=0}^4 \overline{w}_i \overline{P}_i \overline{f}_i(p)}{\sum_{i=0}^4 \overline{w}_i \overline{f}_i(p)}.$$

The patch is an image $\overline{F}(D)$ with boundary conics definied by control points (and corresponding weights) $\overline{P}_0, \overline{P}_1, \overline{P}_3$ and $\overline{P}_0, \overline{P}_2, \overline{P}_3$. If $a = 1$ we get exactly Sabin's biangle from [6]. Changing the parameter a adds shape flexibility to Sabin's B-spline surface.

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Glodus interpoliavimas dvikampėmis paviršių skiautėmis

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Straipsnyje pateikiama racionalių dvikampių skiaučių schema, apibendrinanti darbe [3] sukonstruotus dvikampius paviršius. Taip pat parodoma, kaip naujo tipo skiautėmis glodžiai užpildomas dvikampis plyšys tarp Šteinerio paviršių.