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On intuitionistic branching tense logic with weak induction

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1. Introduction

As it is well known, the classical first-order linear and branching tense logics with the induction axiom $(A \wedge \Box(A \supset \bigcirc A)) \supset \Box A$ are incomplete (see, e.g., [3, 4]). The same holds for intuitionistic variants of the logics as well. In [1], the completeness and semantical admissibility of cut are proved with respect to the classical first-order branching tense logic with the weak induction, that is to say with the axiom $(A \wedge \bigcirc\Box A) \supset \Box A$ instead of the induction axiom $(A \wedge \Box(A \supset \bigcirc A)) \supset \Box A$. In the paper, we prove: the syntactical admissibility of cut, Harrop's theorem and the interpolation theorem for a calculus LBJ. In the construction of the calculus LBJ we use an intuitionistic variant of a sequent calculus – LJ – without structural rules:

1. Axioms: $\Gamma, A \to A$; $\Gamma, f \to \Delta$
2. Derivation rules:

$$\frac{\Gamma, A \to B}{\Gamma \to A \supset B}(\to\supset) \qquad \frac{A \supset B, \Gamma \to A;\ \ B, \Gamma \to \Delta}{A \supset B, \Gamma \to \Delta}(\supset\to)$$

$$\frac{\Gamma \to A;\ \ \Gamma \to B}{\Gamma \to A \wedge B}(\to \wedge) \qquad \frac{A, B, \Gamma \to \Delta}{A \wedge B, \Gamma \to \Delta}(\wedge \to)$$

$$\frac{\Gamma \to A \text{ or } \Gamma \to B}{\Gamma \to A \vee B}(\to \vee) \qquad \frac{A, \Gamma \to \Delta;\ \ B, \Gamma \to \Delta}{A \vee B, \Gamma \to \Delta}(\vee \to)$$

$$\frac{\Gamma \to A(b)}{\Gamma \to \forall x A(x)}(\to \forall) \qquad \frac{A(t), \forall x A(x), \Gamma \to \Delta}{\forall x A(x), \Gamma \to \Delta}(\forall \to)$$

$$\frac{\Gamma \to A(t)}{\Gamma \to \exists x A(x)}(\to \exists) \qquad \frac{A(b), \Gamma \to \Delta}{\exists x A(x), \Gamma \to \Delta}(\exists \to)$$

Here: f denotes 'false'; A, B denote arbitrary formulae; $\Delta \in \{\varnothing, D\}$ (D is an arbitrary formula); Γ denotes a finite, possibly empty multiset of formulae; x denotes a bound variable; t denotes a term which is free for x in $A(x)$; b denotes a free variable which does not occur in conclussions of the rules $(\to \forall)$, $(\exists \to)$; we use different letters to denote free and bound variables so that a variable can be only free or only bound; we do not have rules for negation: $\neg A =_{\text{def}} A \supset f$.

The calculus LBJ is obtained from the calculus LJ by adding the following four rules for handling '□' (always) and '○'(next) operators:

$$\frac{\Gamma \to \Delta}{\Pi, \bigcirc\Gamma \to \bigcirc\Delta}\ (\bigcirc) \qquad \frac{\Gamma \to A;\ \Gamma \to \bigcirc\Box A}{\Gamma \to \Box A}\ (\to \Box)$$

$$\frac{A, \bigcirc\Box A, \Gamma \to \Delta}{\Box A, \Gamma \to \Delta}\ (\Box \to) \qquad \frac{\Box\Gamma \to A}{\Pi, \Box\Gamma \to \Box A}\ (\Box)$$

Here: Γ, Δ, A are as in LJ; if $\Gamma = A_1, A_2, \ldots, A_n$, then $\bigcirc\Gamma = \bigcirc A_1, \bigcirc A_2, \ldots, \bigcirc A_n$ and $\Box\Gamma = \Box A_1, \Box A_2, \ldots, \Box A_n$; Π denotes an arbitrary finite, possibly empty multiset of formulae. The rule $(\to \Box)$ corresponds to the weak induction axiom $(A \wedge \bigcirc\Box A) \supset \Box A$.

2. Cut Elimination

Since our sequents are of multiset type and have no more than one formula in succedent: the structural rule of permutation has no sense here; the structural rule of contraction is impossible in succedent.

LEMMA 1. *Any LBJ derivable sequent has an atomic derivation, i.e., every axiom obtained in the derivation has the $\Gamma, E \to E$ shape, where E is an atomic formula.*

Proof. The lemma is proved by the complexity of the formulae.

LEMMA 2. *Let $(i) \in \{(\to\supset), (\supset\to), (\vee \to), (\to \wedge), (\exists \to), (\to \forall), (\Box \to), (\to \Box)\}$ and S be the sequent having the shape of the rule (i). Let S^* be the sequent having the shape of a premiss of the rule $(i_1) \in \{(\to\supset), (\vee \to), (\to \wedge), (\exists \to), (\to \forall), (\Box \to), (\to \Box)\}$ or the right premiss of $(\supset\to)$, then $LBJ \vdash^D S \Rightarrow LBJ \vdash^{D^*} S^*$ and $h(D^*) \leqslant h(D)$, where D stands for an atomic derivation and $h(D)$ stands for the height of D.*

Proof. The lemma is proved by induction on $h(D)$.

LEMMA 3. *If there exists an LBJ derivation with the structural rule of contraction in antecedent of a sequent, then there exists an LBJ derivation without the structural rule of contraction in antecedent with the same end sequent.*

Proof. We consider only atomic derivations here. The lemma is proved by induction on the ordered triplet $< n, \mathcal{G}, h >$, where n denotes the number of contraction applications in a derivation, $\mathcal{G}$ denotes the complexity of the main formula of contraction and is inductively defined as follows:

1) $\mathcal{G}(E) = 0$;
2) $\mathcal{G}(A\Theta D) = \mathcal{G}(A) + \mathcal{G}(D) + 1$ $(\Theta \in \{\supset, \wedge, \vee\})$;
3) if $B = \bigcirc\Box A$, then $\mathcal{G}(\bigcirc\Box A) = \mathcal{G}(\Box A)$, otherwise
   $\mathcal{G}(QD) = \mathcal{G}(D) + 1$ $(Q \in \{\forall x, \exists x, \Box, \bigcirc\})$;

here: E denotes an atomic formula; A, D denote arbitrary formulae; x denotes a bound variable,

h denotes the height of the derivation of the sequent to which contraction was applied for the first time with respect to the initial derivation. We make use, as well, of Lemmas 1, 2.

LEMMA 4. *If there exists an LBJ derivation with the structural rule of weakening of a sequent, then there exists an LBJ derivation without the structural rule of weakening of the same sequent.*

Proof. The lemma is proved by induction on the ordered pair $< n, h >$ where n stands for the number of weakening applications in a derivation, and h stands for the height of a derivation of the sequent to which weakening was applied for the first time with respect to the initial derivation.

LEMMA 5. *If the final step in a LBJ derivation of a sequent is the rule of cut and there are no more cuts in the derivation, then there exists a LBJ derivation without cut with the same end sequent.*

Proof. We have:

$$\frac{V_1\left\{\overline{\Pi \to C}\;(i)\;;\quad V_2\left\{\overline{C, \Gamma \to \Delta}\;(k)\right.\right.}{\Pi, \Gamma \to \Delta}$$

V_1, V_2 stand here for the derivations of $\Pi \to C$ and $C, \Gamma \to \Delta$, respectively. The heights of V_1, V_2 are denoted by $h(V_1)$, $h(V_2)$.

The complexity of a formula B, denoted by $\mathcal{G}(B)$, is defined as in the proof of Lemma 3.

We prove the lemma by induction on the ordered triplet $< \mathcal{G}, P, H >$. The parameter $\mathcal{G}$ stands for the complexity of the cut formula; P stands for the number of applications of the rules $(\Box \to)$, $(\to \Box)$ in a derivation; $H = h(V_1) + h(V_2)$.

1) (i), (k) are the applications of logical rules: these cases are considered in the traditional way (using the hypothesis on G or H);

2) (i) or (k) is an application of a logical rule: these cases are considered by the hypothesis on H;

3) $(i) = (\Box)$, $(k) = (\Box \to)$:

$$\frac{\dfrac{\Box\Pi \to A}{\Pi_1, \Box\Pi \to \Box A}(\Box); \quad \dfrac{A, \bigcirc\Box A, \Gamma \to \Delta}{\Box A, \Gamma \to \Delta}(\Box \to)}{\Pi_1, \Box\Pi, \Gamma \to \Delta}\;(cut).$$

Applying $(\Box)$ to $\Box\Pi \to A$ we get $\Box\Pi \to \Box A$; applying $(\bigcirc)$ to this sequent we obtain $S_1 = \Pi, \Pi_1, \bigcirc\Box\Pi \to \bigcirc\Box A$; applying (cut) and the hypothesis on P to S_1 and $A, \bigcirc\Box A, \Gamma \to \Delta$ we obtain $S_2 = \Pi, \Pi_1, \bigcirc\Box\Pi, \Gamma, A \to \Delta$; by applying (cut) and the hypothesis on $\mathcal{G}$ to $\Box\Pi \to A$ and S_2 we get $\Box\Pi, \Pi, \Pi_1, \bigcirc\Box\Pi, \Gamma \to \Delta$; applying $(\Box \to)$ to the last sequent we get $\Pi_1, \Box\Pi, \Gamma \to \Delta$;

4) $(i) = (\to \Box)$, $(k) = (\Box \to)$:

$$\frac{\dfrac{\Pi \to A; \;\; \Pi \to \bigcirc\Box A}{\Pi \to \Box A}(\to \Box); \quad \dfrac{A, \bigcirc\Box A, \Gamma \to \Delta}{\Box A, \Gamma \to \Delta}(\Box \to)}{\Pi, \Gamma \to \Delta}\;(cut).$$

By applying (*cut*) and the hypothesis on H to $\Pi \to \bigcirc\Box A$ and $A, \bigcirc\Box A, \Gamma \to \Delta$ we get $S_1 = \Pi, A, \Gamma \to \Delta$; applying (*cut*), the hypothesis on $\mathcal{G}$, $(C \to)$ ($(C \to)$ denotes an application of the antecedent contraction rule) and Lemma 4 to $\Pi \to A$ and S_1 we get $\Pi, \Gamma \to \Delta$;

5) $(i) = (\to \Box)$, $(k) = (\Box)$:

$$\dfrac{\dfrac{\Pi \to B;\ \Pi \to \bigcirc\Box B}{\Pi \to \Box B}(\to \Box); \quad \dfrac{\Box B, \Box\Gamma \to A}{\Pi_1, \Box B, \Box\Gamma \to \Box A}(\Box)}{\Pi, \Pi_1 \Box\Gamma \to \Box A}(cut).$$

Applying (*cut*) and the hypothesis on H to $\Pi \to \Box B$ and $\Box B, \Box\Gamma \to A$ we get $S_1 = \Pi, \Box\Gamma \to A$; applying $(\Box)$ to $\Box B, \Box\Gamma \to A$ we get $\Box B, \Box\Gamma \to \Box A$; applying $(\bigcirc)$ to this sequent we get $S_2 = \Gamma, \Pi_1, \bigcirc\Box B, \bigcirc\Box\Gamma \to \bigcirc\Box A$; applying (*cut*) and the hypothesis on P to $\Pi \to \bigcirc\Box B$ and S_2 we get $S_3 = \Gamma, \Pi, \Pi_1, \bigcirc\Box\Gamma \to \bigcirc\Box A$; applying $(W \to)$ ($(W \to)$ denotes an application of the antecedent weakening rule) and Lemma 4 to S_1 we get $S_4 = \Pi_1, \Pi, \Box\Gamma \to A$; applying $(\Box \to)$ to S_3 we get $S_5 = \Box\Gamma, \Pi, \Pi_1 \to \bigcirc\Box A$; applying $(\to \Box)$ to S_4 and S_5 we get $\Pi, \Pi_1, \Box\Gamma \to \Box A$;

6) the rest cases when $(i), (k) \in \{(\Box), (\Box \to), (\to \Box), (\bigcirc)\}$ are considered by the hypothesis on H.

THEOREM. *If a sequent is derivable in $LBJ + (cut)$, then the sequent is derivable in LBJ.*

Proof. The theorem is proved by induction on the number of applications of the cut rule and by using Lemma 5.

3. Harrop's Theorem

THEOREM (Harrop). *Let us assume that Γ is such a finite multiset of formulae that every occurrence of '$\vee$', '$\exists$' in it is within the scope of '$\bigcirc$', or within the left scope of '$\supset$' then*:

1) *the sequent $\Gamma \to A \vee B$ is derivable in LBJ iff $\Gamma \to A$ or $\Gamma \to B$ is derivable in LBJ*,

2) *the sequent $\Gamma \to \exists x F(x)$ is derivable in LBJ iff, for some term t, $\Gamma \to F(t)$ is derivable in LJ.*

Although the theorem differs a bit from the original version of Harrop's theorem, it still may well be called Harrop's theorem.

Proof. We prove the theorem by induction on the height of derivation. The first part of the theorem:

$(\Leftarrow)$ case is obvious.

$(\Rightarrow)$:

$$\dfrac{\Gamma_1 \to A \vee B}{\Gamma \to A \vee B}(i)$$

When $(i) \in \{(\supset\rightarrow), (\wedge \rightarrow), (\vee \rightarrow), (\forall \rightarrow), (\exists \rightarrow), (\rightarrow \vee)\}$, the proof is well known (see, e.g., [2]). It follows from the shape of the LBJ rules and the given sequent that the only possible remaining case is $(i) = (\Box \rightarrow)$:

$$\frac{C, \bigcirc\Box C, \Gamma \rightarrow A \vee B}{\Box C, \Gamma \rightarrow A \vee B} (\Box \rightarrow).$$

Obviously, we can apply the hypothesis of induction and get that $C, \bigcirc\Box C, \Gamma \rightarrow A$ or $C, \bigcirc\Box C, \Gamma \rightarrow B$ is derivable in LBJ. This implies that $\Box C, \Gamma \rightarrow A$ or $\Box C, \Gamma \rightarrow B$ is derivable in LBJ.

The second part of the theorem:

($\Leftarrow$) case is obvious.

($\Rightarrow$):

$$\frac{\Gamma_1 \rightarrow \exists x F(x)}{\Gamma \rightarrow \exists x F(x)} (i).$$

Again, when $(i) \in \{(\supset\rightarrow), (\wedge \rightarrow), (\vee \rightarrow), (\forall \rightarrow), (\exists \rightarrow), (\rightarrow \exists)\}$, the proof is well known. It follows from the shape of LBJ rules and the given sequent that the only possible remaining case is $(i) = (\Box \rightarrow)$:

$$\frac{A, \bigcirc\Box A, \Gamma \rightarrow \exists x F(x)}{\Box A, \Gamma \rightarrow \exists x F(x)} (\Box \rightarrow).$$

By the induction hypothesis $A, \bigcirc\Box A, \Gamma \rightarrow F(t)$ is derivable in LBJ for some t. That implies the derivability of $\Box A, \Gamma \rightarrow F(t)$ in LBJ.

Example. Using Harrop's theorem we show that $\bigcirc(A \vee B) \rightarrow \bigcirc A \vee \bigcirc B$ is non-derivable in LBJ. Instead of $\bigcirc(A \vee B) \rightarrow \bigcirc A \vee \bigcirc B$, by Harrop's theorem we can consider the following two sequents: $\bigcirc(A \vee B) \rightarrow \bigcirc A$ and $\bigcirc(A \vee B) \rightarrow \bigcirc B$ (we consider them in bottom-up fashion):

$$\cfrac{\cfrac{A \rightarrow A;\ B \rightarrow A}{A \vee B \rightarrow A} (\vee \rightarrow)}{\bigcirc(A \vee B) \rightarrow \bigcirc A} (\bigcirc) \qquad \cfrac{\cfrac{A \rightarrow B;\ B \rightarrow B}{A \vee B \rightarrow B} (\vee \rightarrow)}{\bigcirc(A \vee B) \rightarrow \bigcirc B} (\bigcirc)$$

In general, the sequents $B \rightarrow A$, $A \rightarrow B$ are non-derivable in LBJ.

4. Interpolation Theorem

LEMMA. *Suppose that $\Gamma \rightarrow \Delta$ is derivable in LBJ. Let us consider an arbitrary partition $\{\Gamma_1; \Gamma_2\}$ of Γ: it is possible that Γ_1 or (and) Γ_2 is (are) empty; $\Gamma_1 \cup \Gamma_2 = \Gamma$. Then there exists a formula 'C' called the interpolant of the partition $\{\Gamma_1; \Gamma_2\}$ such that:*

1) $\Gamma_1 \rightarrow C$ and $C, \Gamma_2 \rightarrow \Delta$ are derivable in LBJ

2) all free variables, constants, and predicate symbols, except perhaps 'f', of C occur in Γ_1 and $\Gamma_2 \cup \Delta$.

Proof. The lemma is proved by the height of the derivation of $\Gamma \to \Delta$:

$$\frac{}{\Gamma \to \Delta}\,(i)$$

1) when (i) is a logical rule, see [2].

2) $(i) = (\to \Box)$:

$$\frac{\Gamma \to A;\ \Gamma \to \bigcirc\Box A}{\Gamma \to \Box A}\,(\to \Box).$$

Suppose that the partition of Γ is $\{\Gamma_1; \Gamma_2\}$. By the induction hypothesis we have that $\Gamma_1 \to C_1$ and $C_1, \Gamma_2 \to A$; $\Gamma_1 \to C_2$ and $C_2, \Gamma_2 \to \bigcirc\Box A$ are derivable in LBJ. This implies that $\Gamma_1 \to C_1 \wedge C_2$ and $C_1 \wedge C_2, \Gamma_2 \to \Box A$ are derivable in LBJ. Thus the required interpolant of the partition $\{\Gamma_1; \Gamma_2\}$ is $C_1 \wedge C_2$.

3) $(i) = (\bigcirc)$:

$$\frac{\Gamma \to \Delta}{\Pi, \bigcirc\Gamma \to \bigcirc\Delta}\,(\bigcirc)$$

Suppose that the partition of $\Pi,\ \bigcirc\Gamma$ is $\{\Pi_1, \bigcirc\Gamma_1; \Pi_2, \bigcirc\Gamma_2\}$. By the induction hypothesis we have that $\Gamma_1 \to C$ and $C, \Gamma_2 \to \Delta$ are derivable in LBJ. This immediately implies that $\Pi_1, \bigcirc\Gamma_1 \to \bigcirc C$ and $\Pi_2, \bigcirc\Gamma_2, \bigcirc C \to \bigcirc\Delta$ are derivable in LBJ. Thus the required interpolant of the partition $\{\Gamma_1; \Gamma_2\}$ is $\bigcirc C$.

4) $(i) = (\Box)$: the case is considered in the same way as 3).

5) $(i) = (\Box \to)$:

$$\frac{A, \bigcirc\Box A, \Gamma \to \Delta}{\Box A, \Gamma \to \Delta}\,(\Box \to)$$

The two possible kinds of the partition of $\Box A, \Gamma$ are: $\{\Box A, \Gamma_1; \Gamma_2\}$ and $\{\Gamma_1; \Box A, \Gamma_2\}$. By the hypothesis of induction we have that $A, \bigcirc\Box A, \Gamma_1 \to C_1$; $C_1, \Gamma_2 \to \Delta$; $\Gamma_1 \to C_2$ and $C_2, A, \bigcirc\Box A, \Gamma_2 \to \Delta$ are derivable in LBJ. That immediately implies the LBJ derivability of $\Box A, \Gamma_1 \to C_1$ and $C_2, \Box A, \Gamma_2 \to \Delta$. Thus the interpolant of the partition $\{\Box A, \Gamma_1; \Gamma_2\}$ is C_1 and that of the partition $\{\Gamma_1; \Box A, \Gamma_2\}$ is C_2.

THEOREM (interpolation theorem). *Suppose that the formula $A \supset B$ is derivable in LBJ. Then there exists a formula 'C' called the interpolant of the formula $A \supset B$ such that formulas $A \supset C$ and $C \supset B$ are derivable in LBJ and constants, variables, and predicate symbols (except for 'f') which do not occur in both A and B, are absent in C as well.*

Proof. The proof of the theorem follows from the above lemma. See also [2].

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In the paper, the first-order branching tense logic calculus is given: LBJ with the weak induction, that is to say with the axiom $(A \wedge \bigcirc\Box A) \supset \Box A$ instead of the induction axiom $(A \wedge \Box(A \supset \bigcirc A)) \supset \Box A$. The syntactical cut elimination theorem, Harrop's theorem and the interpolation theorem is proved here with respect to the LBJ calculus.