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Bergström expansion for mixtures of lattice distributions

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Let $\mathcal{F}$ be the set of all probability measures, $\mathcal{M}$ be the set of all measures of bounded variation on $\mathbb{R}$. If $W \in \mathcal{M}$ then, due to the Jordan-Hahn decomposition, $W = W^+ - W^-$.

We denote by $\| W \|$ the total variation norm of W, i.e., $\| W \| = W^+\{\mathbb{R}\} + W^-\{\mathbb{R}\}$. Let E_a be the distribution concentrated at a point a (i.e. $E_a\{a\} = 1$), $E \equiv E_0$. The notation $C(\cdot)$ will be used for different positive constants depending on the indicated argument only. Products and powers of measures will be understood in the convolution sense: $FG = F * G$, $W^n = W^{*n}$, $W^0 = E$. For $W \in \mathcal{M}$ we shall denote its Fourier-Stieltjes transform by $\widehat{W}(t) = \int_{\mathbb{R}} \exp\{itx\}\, W\{dx\}$, $t \in \mathbb{R}$ and the analogue of the uniform distance by

$$|W| = \sup_{x\in\mathbb{R}} |W\{(-\infty, x)\}| = \sup_{x\in\mathbb{R}} |W(x)|\,.$$

Let $\mathbb{N}$ be the set of all natural numbers, Z be the set of all integer numbers.

H. Bergström [1] used asymptotic expansions based on the following identity

$$F^n = \sum_{j=0}^{s} \binom{n}{j} G^{n-j}(F-G)^j + r_n^{(s+1)},$$

with

$$r_n^{(s+1)} = \sum_{\mu=s+1}^{n} \binom{\mu-1}{s} F^{n-\mu}(F-G)^{s+1}G^{\mu-s-1}, \tag{1}$$

Bergström expansion was applied in [1]–[11]. In [3], [4], two generalizations of (1) for the convolutions of non-identical distributions were given. We shall use the generalization of (1) from [8] (see also [6]).

Let $F_1, F_2, \ldots, F_n \in \mathcal{M}$, $G_1, \ldots, G_n \in \mathcal{M}$, $0 \leqslant s \leqslant n-1$. Analogously to (1) we have

$$\prod_{j=1}^{n} F_j = \sum_{\nu=0}^{s} \Delta_\nu + \sum_{\nu=s+1}^{n} \Delta_\nu = \sum_{\nu=0}^{s} \Delta_\nu + R_n^{(s+1)}, \tag{2}$$

$$\Delta_\nu = {\sum}_{n,\mu}^{\nu} \prod_{m=1}^{n} G_m^{1-\mu_m}(F_m - G_m)^{\mu_m}. \tag{3}$$

$$R_n^{(s+1)} = \sum_{j=s+1}^{n} (F_j - G_j) \prod_{i=j+1}^{n} {\sum}_{j-1,\mu}^{s} \prod_{m=1}^{j-1} G_m^{1-\mu_m}(F_m - G_m)^{\mu_m}. \tag{4}$$

Here $\sum_{n,\mu}^{\nu}$ means summation over all possible $\mu_1, \mu_2, \ldots, \mu_n \in \{0, 1\}$ such that $\mu_1 + \ldots + \mu_n = \nu$, i.e.

$$\sum\nolimits_{n,\mu}^{\nu} = \sum\{\mu_1 + \ldots + \mu_n = \nu,\ \mu_m \in \{0,1\},\ m = 1, \ldots, n\}.$$

Let $F \in \mathcal{F}$, $i = 1, \ldots, n$,

$$\varphi_i(F) = \sum_{j=0}^{\infty} p_{ij} F^j, \qquad \psi_i(F) = \sum_{j=0}^{\infty} q_{ij} F^j, \tag{5}$$

$$\sum_{j=0}^{\infty} p_{ij} = \sum_{j=0}^{\infty} q_{ij} = 1, \quad \sum_{j=0}^{\infty} |p_{ij}| < \infty, \quad \sum_{j=0}^{\infty} |q_{ij}| < \infty. \tag{6}$$

Note that if $p_{ij}, q_{ij} \geqslant 0$,then $\varphi_i(F)$, $\psi_i(F)$ are distributions of the sums of a random number of i.i.d.r.v. In general, we deal with signed measures. We shall say that, $\varphi_i(F)$ and $\psi_i(F)$ satisfy condition (λ_i), if there exists $\lambda_i < C$ such that

$$\max\{|\varphi_i(\widehat{F}(t))|, |\psi_i(\widehat{F}(t))|\} \leqslant \exp\{\lambda_i(Re\widehat{F}(t) - 1)\}. \tag{7}$$

Here $Re\widehat{F}(t)$ denotes the real part of $\widehat{F}(t)$ and

$$\varphi_i(\widehat{F}(t)) = \sum_{j=0}^{\infty} p_{ij}(\widehat{F}(t))^j = \widehat{\varphi_i(F)}(t).$$

The following Lemma asserts that the class of measures satisfying condition (λ) is large enough.

LEMMA 1. *Let* $F \in \mathcal{F}$,

$$\varphi(F) = \sum_{j=0}^{\infty} p_j F^{j\prime}, \quad \sum_{j=0}^{\infty} p_j = 1, \quad \sum_{j=0}^{\infty} |p_j| < \infty, \quad \beta_2(\varphi(E_1)) < \infty.$$

Then, for all $t \in \mathbb{R}$,

$$|\varphi(\widehat{F}(t))| \leqslant \exp\{\big(\alpha_1(\varphi(E_1)) - \alpha_1^2(\varphi(E_1)) - \beta_2(\varphi(E_1))\big)(Re\widehat{F}(t) - 1)\}.$$

Proof. By the Bergström identity and definition of $\varphi(E_1)$ we get

$$\begin{aligned} &|\varphi(\widehat{F}(t)) - 1 - \alpha_1(\varphi(E_1))(\widehat{F}(t) - 1)| \\ &= \Big|\sum_{j=0}^{\infty} p_j(\widehat{F}^j(t) - 1 - j(\widehat{F}(t) - 1))\Big| = \Big|\sum_{j=2}^{\infty} p_j \sum_{\mu=2}^{j} (\mu - 1)\widehat{F}^{j-\mu}(t)(\widehat{F}(t) - 1)^2\Big| \\ &\leqslant \sum_{j=2}^{\infty} |p_j| \binom{j}{2} |\widehat{F}(t) - 1|^2 \leqslant \beta_2(\varphi(E_1))|\widehat{F}(t) - 1|^2/2. \end{aligned} \tag{8}$$

Therefore

$$|\varphi(\widehat{F}(t))| \leqslant |1+\alpha_1(\varphi(E_1))(\widehat{F}(t)-1)| + \beta_2(\varphi(E_1))|\widehat{F}(t)-1|^2/2. \tag{9}$$

Taking into account that

$$\widehat{F}(t) = Re\widehat{F}(t) + i\,Im\widehat{F}(t), \qquad (Im\widehat{F}(t))^2 \leqslant 1 - \big(Re\widehat{F}(t)\big)^2,$$

we get from (9)

$$\begin{aligned} |\varphi(\widehat{F}(t))| &\leqslant |1+2(Re\widehat{F}(t)-1)\big(\alpha_1(\varphi(E_1)) - \alpha_1^2(\varphi(E_1))\big)|^{1/2} + \beta_2(\varphi(E_1))(1-Re\widehat{F}(t)) \\ &\leqslant \exp\{\big(\alpha_1(\varphi(E_1)) - \alpha_1^2(\varphi(E_1)) - \beta_2(\varphi(E_1))\big)(Re\widehat{F}(t)-1)\}. \qquad \square \end{aligned}$$

Now we shall formulate the main result of this note. Let us denote a summand of the Bergström expansion by

$$\Delta_\nu(F) = \sum\nolimits_{n,\mu}^{\nu} \prod_{j=1}^{n} \psi_j^{1-\mu_j}(F)\big(\varphi_j(F) - \psi_j(F)\big)^{\mu_j}.$$

THEOREM 1. *Let* $F \in \mathcal{F}$, $F\{\mathrm{Z}\} = 1$ *and let, for* $m \geqslant 2$, $i = 1, 2\ldots, n$, $k = 1, 2, \ldots, m-1$, *the following conditions be satisfied*

$$\alpha_k(\varphi_i(E_1) - \psi_i(E_1)) = 0, \qquad \beta_m(\varphi_i(E_1) - \psi_i(E_1)) < \infty,$$

$$\max\{\beta_2(\varphi(E_1)),\ \beta(\psi(E_1))\} < \infty, \qquad \lambda_i \geqslant 0.$$

Then, for all $s \leqslant n-1$, $\nu = 1, 2, \ldots, s$, *the following inequalities hold*

$$\begin{aligned} &\sup_{x\in\mathrm{Z}} |\Delta_\nu(F)\{x\}| \\ &\quad\leqslant C(m,\nu)h^{-\nu/2}\Big(\sum\nolimits_{n,\mu}^{\nu}\prod_{i=1}^{n}\beta_m(\varphi_i(E_1)-\psi_i(E_1))^{\mu_i}\Big)\min\big\{1,\,(h(1-F\{0\}))^{1/2}\big\} \qquad (10) \\ &\quad\leqslant C(m,\nu)h^{-\nu/2}\min\big\{1,\,(h(1-F\{0\}))^{1/2}\big\}\Big(\sum_{i=1}^{n}\beta_m(\varphi_i(E_1)-\psi_i(E_1)\Big)^{\nu}, \end{aligned}$$

$$\begin{aligned} \sup_{x\in\mathrm{Z}}\left|\prod_{i=1}^{n}\varphi_i(F)\{x\} - \sum_{\nu=0}^{s}\Delta_\nu(F)\{x\}\right| &\leqslant C(m,s)\left(\sum_{i=1}^{n}\beta_m(\varphi_i(E_1)-\psi_i(E_1)\right)^{s+1} \qquad (11) \\ &\quad\times h^{-(s+1)/2}\min\big\{1,\,(h(1-F\{0\}))^{1/2}\big\}. \end{aligned}$$

Here $h = \max\big\{1, \sum_{i=1}^{n}\lambda_i\big\}$.

Proof. Analogously to the proof of (8) we get

$$\begin{aligned}|\varphi_i(\widehat{F}(t)) - \psi_i(\widehat{F}(t))| &\leqslant C(m)\beta_m(\varphi_i(E_1) - \psi_i(E_1))|\widehat{F}(t) - 1|^m \\ &\leqslant C(m)\beta_m(\varphi_i(E_1) - \psi_i(E_1))(1 - Re\widehat{F}(t))^{m/2}.\end{aligned} \tag{12}$$

Noting that, if $\lambda_i > 0$ then $\lambda_i \leqslant 1$, we get

$$\begin{aligned}&\left|\sum\nolimits_{n,\mu}^{\nu} \prod_{i=1}^{n} \psi_i^{1-\mu_i}(\widehat{F}(t))\big(\varphi_i(\widehat{F}(t)) - \psi_i(\widehat{F}(t))\big)^{\mu_i}\right| \\ &\leqslant C(m,\nu)\big(1 - Re\widehat{F}(t)\big)^{\nu/2} \sum\nolimits_{n,\mu}^{\nu} \exp\left\{\sum_{i=1}^{n}(1-\mu_i)\lambda_i(Re\widehat{F}(t) - 1)\right\} \\ &\times \prod_{i=1}^{n} \beta_m^{\mu_i}(\varphi_i(E_1) - \psi_i(E_1)) \\ &\leqslant C(m,\nu)\exp\left\{\sum_{l=1}^{n}\lambda_l(Re\widehat{F}(t) - 1)/2\right\} h^{-\nu/2} \sum\nolimits_{n,\mu}^{\nu} \prod_{i=1}^{n} \beta_m^{\mu_i}(\varphi_i(E_1) - \psi_i(E_1)).\end{aligned}$$

By the formula of inversion

$$\begin{aligned}|\Delta_\nu(F)| &\leqslant C(m,\nu)h^{-\nu/2} \sum\nolimits_{n,\mu}^{\nu} \prod_{i=1}^{n} \beta_m^{\mu_i}(\varphi_i(E_1) - \psi_i(E_1)) \\ &\times \int_{-\pi}^{\pi} \exp\left\{\sum_{l=1}^{n}\lambda_l(Re\widehat{F}(t) - 1)/2\right\} dt.\end{aligned} \tag{13}$$

Note that

$$Re\widehat{F}(t) = \widehat{F}(t)/2 + \widehat{F}(-t)/2,$$

i.e., $Re\widehat{F}(t)$ is a characteristic function. To end the proof of (10) one should apply the following inequality:

$$\int_{-\pi}^{\pi} \exp\{a(\widehat{F}(t) - 1)\}\, dt \leqslant C(1 - F\{0\})^{-1/2} a^{-1/2}.$$

The proof of (11) is similar. *Q.E.D.*

Example. Let $0 \leqslant p \leqslant 1/2$, $F \in \mathcal{F}$, $F\{\mathbb{Z}\} = 1$, $n \in \mathbb{N}$. Then

$$\begin{aligned}&\sup_{x\in\mathbb{Z}} |\big((1-p)E + pF\big)^n\{x\} - \exp\{np(F - E) - np^2(F - E)^2/2\}\{x\}| \\ &\leqslant Cn^{-1}(1 - F\{0\})^{-1/2}.\end{aligned}$$

To prove this inequality one must note that

$$\beta_3\big(((1-p)E + pF)^n - \exp\{np(F - E) - np^2(F - E)^2/2\}\big) \leqslant Cp^3,$$

and

$$\max\{|1 + p(e^{it} - 1)|,\ |\exp\{p(e^{it} - 1) - p^2(e^{it} - 1)^2/2\}|\} \leqslant \exp\{-Cp\sin^2(t/2)\}$$

and apply Theorem 1.

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Bergstremo skleidiniai gardelinių skirstinių mišiniams

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Lyginame dviejų atsitiktinių dydžių sumų pasiskirstymų artumą lokalioje metrikoje. Kiekvienos sumos dėmenys savo ruožtu yra atsitiktinių dydžių su atsitiktiniais rėžiais sumos. Sąlygos keliamos atsitiktiniams rėžiams.