

On large deviations for the negative binomial law

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1. Preliminaries

Let E_a denote the distribution concentrated at a point a , $E \equiv E_0$. Products and powers of measures are defined in the convolution sense: $FG = F * G$, $F^n = F^{*n}$, $F^0 = E$. For any signed measure of bounded variation W we denote by $\exp\{W\} = \sum_{k=0}^{\infty} W^k/k!$ its exponential measure, by $|W| = \sup_x |W\{(-\infty, x)\}|$ the analogue of the uniform distance, $\widehat{W}(t) = \int_{-\infty}^{\infty} \exp\{itx\} dW$ its Fourier–Stieltjes transform.

Let F be a distribution concentrated on $0, 1, 2, \dots$. We denote its factorial cumulants by Γ_k . Note that, for F having s finite absolute moments, we have

$$\ln \widehat{F}(t) = \sum_{k=1}^{s-1} \frac{\Gamma_k}{k!} (e^{it} - 1)^k + o(|t|^s), \quad \text{as } t \rightarrow 0.$$

We use notation C for absolute positive constants. The symbol θ is used for all quantities satisfying $|\theta| \leq 1$.

Let ξ be a lattice random variable concentrated on non-negative integers, having distribution F and $\mathbb{E}\xi = \lambda > 0$. We say that ξ satisfies (\tilde{S}) condition if, for some $\Delta \geq 1$,

$$|\Gamma_k| \leq \frac{k!\lambda}{\Delta^{k-1}}, \quad k = 2, 3, \dots \quad (\tilde{S})$$

Obviously, (\tilde{S}) condition is a lattice analogue of the Statulevičius (S) condition for cumulants – see, for example, [12, 13]. It was introduced in [1].

We note that, in 1976, Bikelis and Žemaitis [4] formulated the following analogue of (S):

$$|\Gamma_k| \leq \Pi(\Delta) \frac{k!\lambda^k}{\Delta^{k-1}}, \quad k = 2, 3, \dots \quad (P)$$

Here $\Pi(\Delta)$, for $0 < \Delta < \infty$, is some non-negative and bounded function.

Estimates under (P) condition were considered in [5, 8–11] and under (\tilde{S}) in [1, 2, 6]. Statulevičius (S) condition for other infinitely divisible approximations was considered in [3].

The aim of this note is a demonstration of the fact that, under (\tilde{S}) condition, the negative binomial approximation can be used as well. Note that, in [11], the negative binomial distribution was also considered, but only as an approximated law.

2. Results

Let $S_n = \xi_1 + \xi_2 + \dots + \xi_n$ be a sum of independent identically distributed random variables. Let ξ_1 be concentrated on $0, 1, \dots$ and have a distribution satisfying condition (\tilde{S}). Let

$$\mathbf{E}\xi_1 = \mu, \quad \lambda = n\mu, \quad y = (x - \lambda)/(n\mathbf{D}\xi_1). \quad (1)$$

Further on we assume that, for all x such that $n\mu \leq x \leq n\mu\Delta/(100\pi\sqrt{6})$, the following relations hold

$$n\mu \rightarrow \infty, \quad x = o(n), \quad \mu = o(1), \quad 1/\Delta = o(1), \quad \mu\Delta \rightarrow \infty, \quad (2)$$

as $n \rightarrow \infty$.

Let G be a negative binomial distribution with the Fourier–Stieltjes transform:

$$\widehat{G}^n(t) = (1 - \mu(e^{it} - 1))^{-n}. \quad (3)$$

In theorems 1, 2 we assume that x is an integer number.

THEOREM 1. *The following relation holds*

$$\frac{1 - F^n(x)}{1 - G^n(x)} = e^{L(x)} \left(1 + O \left(\mu\sqrt{x} + \frac{\sqrt{x}}{\Delta} + y\sqrt{x} + \frac{x\sqrt{x}}{n} \right) \right). \quad (4)$$

Here $L(x)$ denotes Cramer series.

Due to the lattice structure of F and G we can obtain a local estimate.

THEOREM 2. *The following relation holds*

$$\frac{F^n\{x\}}{G^n\{x\}} = e^{L(x)} (1 + O(x/n)). \quad (5)$$

Remark. Conditions (1) and (2) can be weakened. In principle, it suffices to take small μ and Δ , not necessarily vanishing. However, then the proof becomes longer.

Example. Let $F = (1 - p)E + pE_1$, i.e. let F^n be a binomial distribution. Then $\Delta = 1/p$, $\mu = p$. Let $np^2 \rightarrow \infty$, $np^3 \rightarrow 0$, $x = np + o(\sqrt{np})$ and $x \geq np$. Then it can be established that $1 - F^n(x) = (1 - G^n(x))(1 + o(1))$. In this situation the same relation holds for the standard Poisson approximation, see [7].

3. Proofs

As usual we employ conjugate distributions. Set

$$F_h\{k\} = e^{hk} F\{k\} / \sum_{j=0}^{\infty} e^{hj} F\{j\}, \quad Q_z\{k\} = e^{zk} G\{k\} / \sum_{j=-\infty}^{\infty} e^{zj} G\{j\}.$$

We have

$$1 - F^n(x) = e^{nK(h)-hx} \sum_{k \geq x} e^{-h(k-x)} F_h^n\{k\}, \tag{6}$$

$$1 - G^n(x) = e^{nM(z)-zx} \sum_{k \geq x} e^{-z(k-x)} Q_z^n\{k\}. \tag{7}$$

Here $K(h) = \ln E \exp\{h\xi\}$, $M(z) = -\ln(1 - \mu(e^z - 1))$. Quantities h and z are chosen from the saddle point equations: $x = n(K(h))'$ and $x = n(M(z))'$. Note that

$$x = \frac{n\mu e^z}{1 - \mu(e^z - 1)}, \quad \widehat{Q}_z^n(t) = (1 - (x/n)(e^{it} - 1))^{-n}. \tag{8}$$

From (6) and (7) we have

$$\begin{aligned} \frac{1 - F^n(x)}{1 - G^n(x)} &= e^{nK(h)-hx-nM(z)+zx} \\ &\times \left(1 + \frac{\sum_{k \geq x} e^{-h(k-x)} (F_h^n\{k\} - Q_z^n\{k\}) + \sum_{k \geq x} (e^{-h(k-x)} - e^{-z(k-x)}) Q_z^n\{k\}}{\sum_{k \geq x} e^{-z(k-x)} Q_z^n\{k\}} \right) \\ &= e^{L(x)} \left(1 + \frac{A_1(x) + A_2(x)}{A_3(x)} \right). \end{aligned} \tag{9}$$

By Abel's summation formula we obtain

$$|A_1(x)| \leq 2|F_h^n - Q_z^n|. \tag{10}$$

Further on we assume that $y \leq \Delta/(50e)$, $\lambda \geq 1$, $\Delta \geq 12e$. Then we can use some auxiliary estimates, which can be derived from the estimates in [1].

LEMMA 1. *The following inequalities hold*

$$e^h < 3x/(2\lambda), \quad |e^h - 1| \leq 7y, \tag{11}$$

$$e^h - 1 = y + \sum_{k=2}^{\infty} d_k y^k, \quad d_k = 3\theta(6e/\Delta)^{k-1}, \tag{12}$$

$$n \ln \widehat{F}_h(t) = x(e^{it} - 1) + \theta \frac{11x^2}{\lambda\Delta} \sin^2(t/2), \tag{13}$$

$$|\widehat{F}_h(t)| \leq \exp\{-(3x/2n) \sin^2(t/2)\}. \tag{14}$$

From the definition of Q , for all sufficiently large n , we get

$$\begin{aligned} |\widehat{Q}_z(t)| &\leq \left| \exp\left\{ \frac{x}{n}(e^{it} - 1) + \sum_{j=2}^n \left(\frac{x}{n}\right)^j \frac{1}{j} (e^{it} - 1)^j \right\} \right| \\ &\leq \exp\left\{ -2\frac{x}{n} \sin^2(t/2) + 2 \sin^2(t/2) \left(\frac{x}{n}\right)^2 \sum_{j=2}^n \left(\frac{2x}{n}\right)^{j-2} \right\} \\ &\leq \exp\{-(3x/2n) \sin^2(t/2)\}. \end{aligned} \tag{15}$$

Consequently, for all large n ,

$$\begin{aligned} |\widehat{F}_h^n(t) - \widehat{Q}_z^n(t)| &\leq \exp\{-(3x/2n) \sin^2(t/2)\} n |\ln \widehat{F}_h(t) - \ln \widehat{Q}_z(t)| \\ &\leq C \exp\{-(3x/2n) \sin^2(t/2)\} \left(\frac{x^2}{\lambda\Delta} + \frac{x^2}{n} \right) \sin^2(t/2). \end{aligned} \tag{16}$$

By the formula of inversion and Tsaregradskii inequality we have

$$|F_h^n - Q_z^n| = O\left(\frac{x}{n} \left(\frac{1}{\mu\Delta} + 1\right)\right), \tag{17}$$

$$\sup_k |F_h^n\{k\} - Q_z^n\{k\}| = O\left(\frac{\sqrt{x}}{n} \left(\frac{1}{\mu\Delta} + 1\right)\right). \tag{18}$$

By conditioning and the formula of inversion

$$I = \sup_k |Q_z^n\{k\} - \exp\{x(E_1 - E)\}| = o(\sqrt{x}). \tag{19}$$

Consequently,

$$A_3(x) \geq Q_z^n\{x\} \geq \exp\{x(E_1 - E)\}\{x\} - I = C\sqrt{x}(1 + o(1)). \tag{20}$$

From (8) it follows that

$$e^z - 1 = y \left(1 + O\left(\frac{1}{\Delta} + \mu\right) \right). \tag{21}$$

Hence

$$A_2(x) \leq C|e^h - e^z|(\min(h, z))^{-1} \sum_{k \geq x} Q_z^n\{k\} = O(1/\Delta + \mu + y). \tag{22}$$

Combining (9), (10), (20) and (22) we get the assertion of theorem 1. Theorem 2 can be proved similarly. Note that $L(x) = nK(h) - hx + zy - nM(z)$. Moreover, $nK(h) - hx$ can be expanded in the powers of y (see [1]).

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Apie didelius nuokrypius neigiamam binominiam dėsniai

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Parodyta, kad esant reguliariam faktorialinių kumuliantų nykimui (Statulevičiaus (S) sąlygos analogui) didelių nuokrypių rezultatus galima gauti ne tik Puasono dėsniai, bet ir neigiamam binominiam skirstiniui.