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The Random Effect Transformation for Three Regularity Classes

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Abstract: We continue the analysis of the influence of the random effect transformation on the regularity of distribution functions. The paper considers three regularity classes: heavy-tailed distributions, distributions with consistently varying tails, and exponential-like-tailed distributions. We apply the random effect transformation to the primary distribution functions from these classes and investigate whether the resulting distribution function remains in the same class. We find that the random effect transformation has the greatest impact on exponential-like-tailed distributions. We establish that any heavy-tailed distribution subjected to a random effect transformation remains heavy-tailed, and any distribution with a consistently varying tail remains with a consistently varying tail after the random effect transformation. Meanwhile, different cases are possible when an exponential-like-tailed class of distributions is subjected to a random effect transformation. Sometimes, depending on the structure of a random effect, the resulting distribution remains exponential-like-tailed, and sometimes that distribution regularly varies. All of the derived theoretical results are illustrated with several examples.

Keywords: random effect; distribution function; distribution transformation; heavy tail; consistent variation; exponential-like-tailed distribution

MSC: 60E05; 62E15; 62N02

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1. Introduction

Random variables (r.v.s) are tools to describe and understand the random processes in various applied fields, in particular, in insurance, banking, and finance. In other words, they comprise a quantity that depends on randomness and acquires real values. It can be described by a measurable function $X : \mathbf{W} \to E$, where X is a random variable, $\mathbf{W}$ is the set of elementary events or the sample space, and $E \subset \mathbb{R}$ is the set of possible outcomes of this random variable. $\mathbf{W}$ must be part of the probability space $(\mathbf{W}, \mathcal{F}, \mathbf{P})$, where $\mathbf{W}$ is the set of all elementary events or the sample space, $\mathcal{F}$ is the σ-algebra of possible events, and $\mathbf{P}$ is the probability measure that is defined for all elements of the given σ-algebra $\mathcal{F}$. The distribution function (d.f.) is the main function for describing the behavior of a random variable. Suppose we have an r.v. X. The d.f. of this r.v. is defined by the formula:

$$F_X(x) := \mathbf{P}(w \in \mathbf{W} : X(w) \leqslant x),\ x \in \mathbb{R}.$$

It is well known that any d.f. F has the following properties:

- $0 \leqslant F(x) \leqslant 1, x \in \mathbb{R}$;
- $\lim\limits_{x\to-\infty} F(x) = 0,\ \lim\limits_{x\to+\infty} F(x) = 1$;
- Function F is non-decreasing, i.e., $F(x_1) \leqslant F(x_2)$ if $x_1 < x_2$;
- Function $F(x)$ is right-continuous on $\mathbb{R}$, i.e., $\lim\limits_{x\to a+} F(x) = F(a)$ for any $a \in \mathbb{R}$.

The expressions of the distribution functions (d.f.s) for discrete and continuous random variables are different. When the random variable X is discrete, its d.f. can be written as follows:

$$F_X(x) = \sum_{x_i \leqslant x} \mathbb{P}(X = x_i),\ x \in \mathbb{R},$$

where $\{x_i, i \in I\}$ is a finite or countable set of possible values of X.

The r.v. X is called absolutely continuous if its d.f. is absolutely continuous. In this case, d.f. F_X can be written in the following form:

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt,\ x \in \mathbb{R},$$

where f_X is an integrable nonnegative density function.

Along with the d.f. F, we consider the tail of the d.f. $\overline{F} = 1 - F$. Tail functions are mainly used in the analysis of the insurance business. Particularly in life insurance, the tail function is called the survival function. This function measures the probability that an individual will survive for at least x years:

$$\overline{F}_T(x) = \mathbb{P}(T > x),$$

where T is an absolutely continuous, nonnegative random variable representing the life expectancy of an individual. We notice that in life insurance, the survival function is usually supposed to be absolutely continuous because a person's life span T can acquire any positive value [1]. In this paper, we refer to the function $\overline{F}_X$ as the tail of the d.f. F_X or the tail function (t.f.) of a random variable X.

Various studies of the tail functions (t.f.s) are fascinating and especially important in the life insurance field because t.f.s help to describe human mortality and have a major impact in evaluating insurance premiums, technical provisions, and other key characteristics in the life insurance business. A detailed description of how survival functions are used to derive other actuarial characteristics can be found in books [2–7].

In non-life insurance, t.f.s typically describe the probability that the impending loss will exceed a certain value. In non-life insurance, it is essential to determine how the cumulative loss tail behaves because the probability of survival or ruin of the risk process that describes the business depends on it. A detailed description of such a connection can be found in the books [8–13].

For the reasons above, various transformations of d.f.s and t.f.s are considered. This paper considers new t.f.s derived using a special transformation through a random effect Z. This r.v. with d.f. F_Z can be defined on a probability space other than the primary r.v. X with d.f. $F = F_X$. Let us denote this new space by $(\Omega, \mathcal{A}, \mathbb{P})$. The random effect transformation was firstly proposed by Vaupel [14] and later studied by Manton et al. [15,16], Yashin et al. [17], Moger and Aalen [18], Hougaard [19], and Pitacco [20,21], among others.

The following equality defines the t.f. with the random effect transformation:

$$\overline{F^{(Z)}}(x) := \mathbb{E}\big((\overline{F}(x))^Z\big), \quad x \in \mathbb{R}.$$

The d.f. of this t.f. is

$$F^{(Z)}(x) = 1 - \overline{F^{(Z)}}(x) = 1 - \mathbb{E}\big((\overline{F}(x))^Z\big), \quad x \in \mathbb{R}.$$

As was shown in [1], the transformed d.f. is also a d.f. In addition, according to Theorem 1 of [22], transformed d.f. $F^{(Z)}$ is absolutely continuous if such is the primary d.f. F. These results show that the transformed d.f. can be examined in terms of regularity.

The random effect transformation is the direct generalization of the transformation defined by a randomly stopped minimum. Namely, if Z is a positive integer-valued r.v.

and independent of the sequence of i.i.d. r.v.s $\{X_1, X_2, \dots\}$ with d.f. F, then the t.f. of the randomly stopped minimum has the following form:

$$\begin{aligned}\mathbb{P}\big(\min\{X_1, X_2, \dots, X_Z\} > x\big) &= \sum_{n=1}^{\infty} \mathbb{P}\big(\min\{X_1, X_2, \dots, X_n\} > x\big)\mathbb{P}(Z = n)\\ &= \sum_{n=1}^{\infty} \big(\overline{F}(x)\big)^n \mathbb{P}(Z = n)\\ &= \mathbb{E}\big((\overline{F}(x))^Z\big)\end{aligned}$$

In papers [23–27], various results are derived for this particular case of the random effect transformation.

As we can see from the results of Section 4, the regularity of the function $F^{(Z)}$ often depends on the support of the random variable Z. The support of an object generally means the set of points where this object is nonzero. *More formally, if* $Z : \Omega \to \mathbb{R}$ *is an r.v. on the probability space* $(\Omega, \mathcal{A}, \mathbb{P})$*, then the support* $\text{supp}\{Z\}$ *of* Z *is the smallest closed set* R *such that* $\mathbb{P}(Z \in R) = 1$.

The rest of the article is organized as follows. In Section 2, the regularity classes are introduced. Afterward, some of the already known results are discussed in Section 3. In Section 4, the main results of this paper are presented together with illustrations of the results derived. The proofs of the main results are displayed in Sections 5–8. Finally, Section 9 deals with the conclusions of the whole analysis.

2. Regularity Classes

Distribution functions, according to their tail behavior, are categorized into various groups, commonly referred to as regularity classes. In this section, we describe a few such classes. A fairly extensive description of regularity classes and the properties of these classes can be found in the book [28]. In this article, we will deal with several classes of regularity: $\mathcal{H}, \mathcal{L}_\gamma$ with $\gamma > 0$, $\mathcal{OL}, \mathcal{S}, \mathcal{L}, \mathcal{D}, \mathcal{C}$ and $\mathcal{R}_\alpha$ with $\alpha \geqslant 0$. Below, we present definitions of these classes.

- *A d.f. F is heavy-tailed, denoted $F \in \mathcal{H}$, if for every fixed $\delta > 0$,*

$$\int_{-\infty}^{\infty} \mathrm{e}^{\delta x} \mathrm{d}F(x) = \infty.$$

  In the opposite case, we say that a d.f. F is light-tailed and denote $F \in \mathcal{H}^c$.
- *A d.f. F is exponential-like-tailed, with index $\gamma > 0$, denoted $F \in \mathcal{L}_\gamma$, if*

$$\lim_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} = \mathrm{e}^{\gamma y} \ \textit{for all} \ y > 0.$$

- *A d.f. F belongs to the class of generalized long-tailed distributions $\mathcal{OL}$, if for any $y > 0$,*

$$\limsup_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} < \infty.$$

- *A d.f. F is said to be subexponential, denoted $F \in \mathcal{S}$ if*

$$\lim_{x\to\infty} \frac{\overline{F^+ * F^+}(x)}{\overline{F^+}(x)} = 2,$$

  where $F^+(x) = F(x)\,\mathbf{I}_{[0,\infty)}(x)$.

- *A d.f. F is said to be long-tailed, denoted* $F \in \mathcal{L}$*, if for every* $y > 0$*,*

$$\overline{F}(x-y) \underset{x\to\infty}{\sim} \overline{F}(x).$$

- *A d.f. F is said to have dominatedly varying tail, denoted* $F \in \mathcal{D}$*, if for any* $y \in (0,1)$

$$\limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty.$$

- *A d.f. F has a consistently varying tail, denoted* $F \in \mathcal{C}$*, if*

$$\lim_{y\uparrow 1} \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1.$$

- *A d.f. F is said to be regularly varying with index* $\alpha \geqslant 0$*, denoted* $F \in \mathcal{R}_\alpha$*, if for any* $y > 0$*,*

$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha}.$$

It is known, see for example [9,28], that these classes admit the following relations:

$$\mathcal{R} := \bigcup_{\alpha\geqslant 0} \mathcal{R}_\alpha \subset \mathcal{C} \subset \mathcal{L}\cap\mathcal{D} \subset \mathcal{S} \subset \mathcal{L} \subset \mathcal{H}, \qquad \mathcal{D} \subset \mathcal{H}.$$

These inclusions are illustrated in Figure 1.

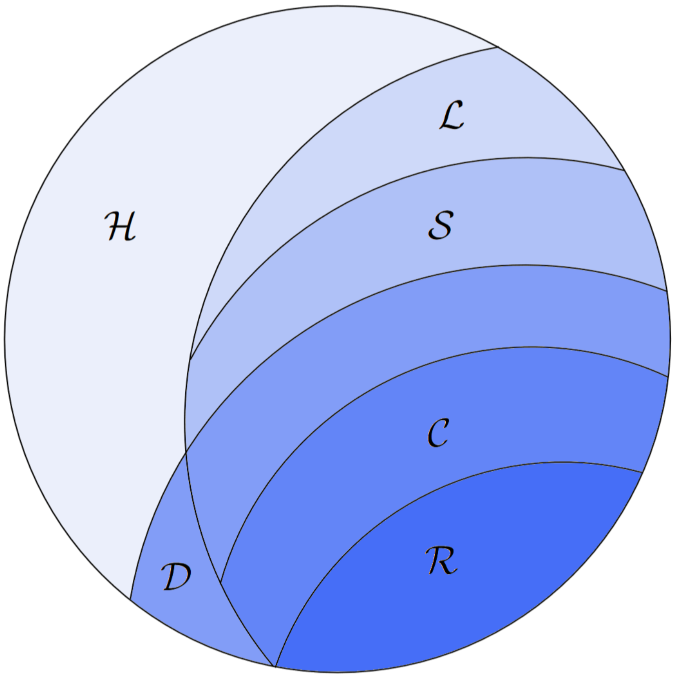

Figure 1. Relationships between the heavy-tailed distribution classes.

In addition, the above definitions imply that

$$\mathcal{L} \subset \mathcal{OL}, \qquad \bigcup_{\gamma>0} \mathcal{L}_\gamma \subset \mathcal{OL}.$$

3. Some Known Results

In paper [1], the influence of the random effect transformations was studied for four classes of d.f.s: regularly varying $\mathcal{R}$, dominatedly varying $\mathcal{D}$, long-tailed $\mathcal{L}$, and generalized long-tailed $\mathcal{OL}$. The random effect transformation was applied to the primary d.f.s from each class, and the resulting d.f.s were examined to determine if they retain their initial classification. The transformation was found to have the most significant impact on regularly varying d.f.s. The following assertion was proved in Theorem 1 of [1].

Let F belong to the class of regularly varying d.f.s $\mathcal{R}_\alpha$ with $\alpha \geqslant 0$, and let Z be a positive r.v. with the greatest lower bound $b > 0$, i.e.,

$$\mathbb{P}(Z \geqslant b) = 1, \quad \mathbb{P}(Z \geqslant b + \delta) < 1$$

for each $\delta > 0$. Then, the transformed d.f. belongs to class $\mathcal{R}_{b\alpha}$.

Further in [1], it is derived that for d.f.s from the classes $\mathcal{D}$, $\mathcal{L}$, and $\mathcal{OL}$, the transformation preserves the class of the d.f. as long as the r.v. on which the random effect acts is strictly positive.

In this paper, we consider the other three classes regularity of d.f.s, $\mathcal{H}, \mathcal{C}$ and $\mathcal{L}_\gamma$ with $\gamma > 0$. The random effect is chosen for all classes as some positive r.v. because only, in this case, the transformed d.f. remains d.f. The results of Section 4 show that the distributions from classes $\mathcal{H}$ and $\mathcal{C}$ were affected by a random effect remain in the same classes as they were. Meanwhile, when a random effect affects a distribution from the exponential-like-tailed class, that distribution can stay in a similar class. Still, it can also become so heavy that it ends up in the class of regularly varying distributions. In this case, the transformation's impact depends on the random effect's structure. The classes of primary d.f.s considered both in this paper and in paper [1] differ significantly in their structure. Therefore, when studying the impact of the random effect on each class, it is necessary to find a specific method. This can be seen from the proofs of this article, and article [1].

4. Main Results

As was mentioned earlier in this work, we focus on three classes of regularity: $\mathcal{H}$; $\mathcal{C}$; and $\mathcal{L}_\gamma$, with $\gamma > 0$. According to the results of paper [1], sometimes d.f. with the random effect preserves the properties of the initial d.f. However, sometimes, the random effect changes, more or less, the regularity of the initial d.f. The changes depend on the regularity class under consideration and the properties of the random effect. In this section, we present the main results of the paper together with several conclusions and illustrations.

4.1. Transformation of the Heavy-Tailed Distribution Function

In this subsection, we establish how the random effect affects the tail of the d.f. from the class of heavy-tailed distributions $\mathcal{H}$. The assertion below states that a heavy tail preserves its heaviness under the random effect transformation.

Theorem 1. *Let a d.f. F belong to the class of heavy-tailed distributions $\mathcal{H}$, and let Z be a positive random effect. Then, the transformed d.f. $F^{(Z)}$ belongs to class $\mathcal{H}$ as well.*

We remark that the analogous assertion fails for class $\mathcal{H}^c$ of light-tailed distributions, i.e., there are d.f.s belonging to class $\mathcal{H}^c$ and positive random effects Z for which $F^{(Z)} \notin \mathcal{H}^c$. This can be seen from Theorem 3 below and examples in Section 4.3.

4.2. Transformation of the Consistently Varying Distribution Function

The central assertion of this subsection states that the random effect transformation preserves the regularity property of class $\mathcal{C}$.

Theorem 2. *Let a d.f. F belong to the class of consistently varying functions $\mathcal{C}$, and let Z be a positive random effect. Then, transformed by this random effect d.f. $F^{(Z)}$ also belongs to class $\mathcal{C}$.*

We present the proof of this theorem in Section 6. Here, we present two examples illustrating how the random effect changes the initial d.f. from class $\mathcal{C}$.

Example 1. *Let us take the Cai-Tang r.v. (see [29]) with parameter $p \in (0,1)$:*

$$X = (1+Y)2^N,$$

where Y and N are independent r.v.s, Y is uniformly distributed on $[0,1]$, and N is the geometric r.v. for which

$$\mathbb{P}(N=k) = (1-p)p^k, \ \ k = 0,1,\ldots$$

Let F be the d.f. of r.v. X. For $x \geqslant 1$, the d.f. of r.v. X is

$$\begin{aligned} F(x) &= (1-p)\sum_{k=0}^{\infty} \mathbb{P}\Big(Y \leqslant \frac{x}{2^k} - 1\Big)p^k \\ &= (1-p)\sum_{k=0}^{\infty} p^k\bigg(\Big(\frac{x}{2^k}-1\Big)\mathbb{I}_{[0,1)}\Big(\frac{x}{2^k}-1\Big) + \mathbb{I}_{[1,\infty)}\Big(\frac{x}{2^k}-1\Big)\bigg) \\ &= (1-p)\bigg(\frac{x}{2^{\lfloor \log_2 x\rfloor}} - 1\bigg)p^{\lfloor \log_2 x\rfloor} + 1 - p^{\lfloor \log_2 x\rfloor}, \end{aligned}$$

see also Equation (6) in [30], and the t.f. for $x \geqslant 1$ is

$$\overline{F}(x) = p^{\lfloor \log_2 x\rfloor}\bigg(2 - p - (1-p)\frac{x}{2^{\lfloor \log_2 x\rfloor}}\bigg).$$

Due to the considerations on page 124 of [29], the d.f. F belongs to class $\mathcal{C}$. Namely, for $2^n < x \leqslant 2^{n+1}$ and $y \in (1/2,1)$, we have that $\lfloor \log_2 x\rfloor = \lfloor \log_2(xy)\rfloor = n$ and, therefore,

$$\begin{aligned} \frac{\overline{F}(xy)}{\overline{F}(x)} &= \frac{2-p-(1-p)\frac{xy}{2^n}}{2-p-(1-p)\frac{x}{2^n}} \\ &= 1 + \frac{(1-y)(1-p)\frac{x}{2^n}}{2-p-(1-p)\frac{x}{2^n}} \\ &\leqslant 1 + \frac{2(1-y)(1-p)}{p}. \end{aligned}$$

The last estimate shows that

$$\limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \leqslant 1 + \frac{2(1-y)(1-p)}{p}$$

for any $y \in (1/2,1)$, and, consequently d.f. F consistently varies according to the definition of class $\mathcal{C}$.
Now, let the random effect Z be a r.v. that takes values from the set $\{1,2,3\}$ with probabilities

$$\mathbb{P}(Z=1) = \mathbb{P}(Z=2) = \mathbb{P}(Z=3) = \frac{1}{3}.$$

Then, the transformed t.f. takes the form:

$$\begin{aligned} \overline{F^{(Z)}}(x) &= \mathbb{E}\Big(\big(\overline{F}(x)\big)^Z\Big) = \frac{1}{3}\Big(\big(\overline{F}(x)\big)^1 + \big(\overline{F}(x)\big)^2 + \big(\overline{F}(x)\big)^3\Big) \\ &= \frac{p^{\lfloor \log_2 x\rfloor}}{3}\bigg(2-p-(1-p)\frac{x}{2^{\lfloor \log_2 x\rfloor}}\bigg) \\ \times &\Bigg(1 + p^{\lfloor \log_2 x\rfloor}\bigg(2-p-(1-p)\frac{x}{2^{\lfloor \log_2 x\rfloor}}\bigg) + p^{2\lfloor \log_2 x\rfloor}\bigg(2-p-(1-p)\frac{x}{2^{\lfloor \log_2 x\rfloor}}\bigg)^2\Bigg). \end{aligned}$$

According to Theorem 2, d.f. $F^{(Z)} = 1 - \overline{F^{(Z)}}$ belongs to class $\mathcal{C}$ together with the initial d.f. F. The difference between $\overline{F}$ and $\overline{F^{(Z)}}$ in case $p = \frac{1}{3}$ is shown in Figure 2.

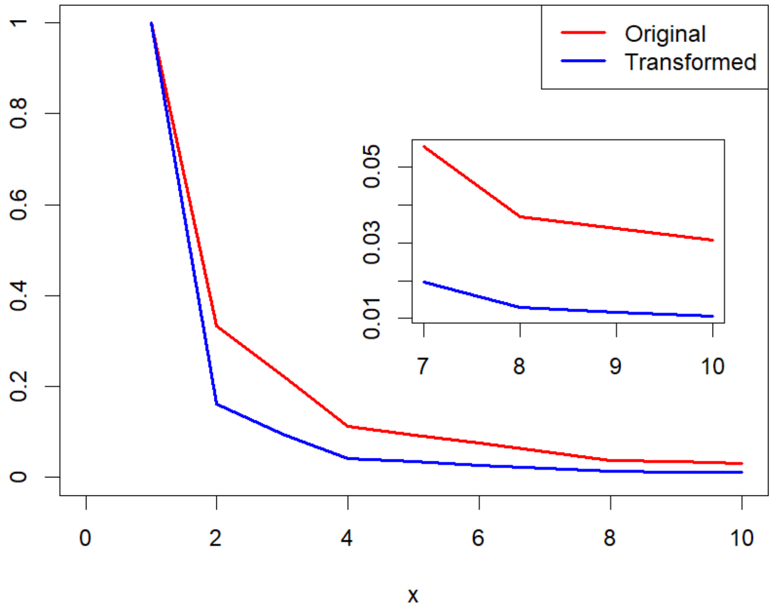

Figure 2. Comparison of tails $\overline{F}$ and $\overline{F^{(Z)}}$ from Example 1.

We can observe that, in this case, the tail of d.f. F is heavier than the tail $\overline{F^{(Z)}}$. However, both functions F and $F^{(Z)}$ belong to class $\mathcal{C}$.

Example 2. *Let us consider the Cauchy distribution with parameters $x_0 = 2, \gamma = 3$. The t.f. of such a distribution is the following:*

$$\overline{F}(x) = \frac{1}{2} - \frac{1}{\pi}\arctan\left(\frac{x-2}{3}\right),\ x \in \mathbb{R}.$$

The d.f.

$$F(x) = 1 - \overline{F}(x) = \frac{1}{\pi}\arctan\left(\frac{x-2}{3}\right) + \frac{1}{2}$$

belongs to class $\mathcal{C}$ because the t.f. satisfies condition of the class $\mathcal{C}$ definition:

$$\begin{aligned}
\lim_{y\uparrow 1}\limsup_{x\to\infty}\frac{\overline{F}(xy)}{\overline{F}(x)} &= \lim_{y\uparrow 1}\limsup_{x\to\infty}\frac{\left(\frac{1}{2}-\frac{1}{\pi}\arctan\left(\frac{xy-2}{3}\right)\right)'}{\left(\frac{1}{2}-\frac{1}{\pi}\arctan\left(\frac{x-2}{3}\right)\right)'} \\
&= \lim_{y\uparrow 1}\limsup_{x\to\infty}\frac{13y + x^2y - 4xy}{13 + x^2y^2 - 4xy} \\
&= \lim_{y\uparrow 1}\frac{1}{y} = 1.
\end{aligned}$$

Let us take a random effect Z uniformly distributed in the interval $[0, 1]$*. Then, the transformed t.f. is the following:*

$$\overline{F^{(Z)}}(x) = \int_0^1 \left(\frac{1}{2} - \frac{1}{\pi}\arctan\left(\frac{x-2}{3}\right)\right)^u \mathrm{d}u$$
$$= \frac{\left(\frac{1}{2} - \frac{1}{\pi}\arctan\left(\frac{x-2}{3}\right)\right)^u}{\log\left(\frac{1}{2} - \frac{1}{\pi}\arctan\left(\frac{x-2}{3}\right)\right)}\Bigg|_0^1$$
$$= \frac{-\left(\frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{x-2}{3}\right)\right)}{\log\left(\frac{1}{2} - \frac{1}{\pi}\arctan\left(\frac{x-2}{3}\right)\right)}.$$

According to Theorem 2, we have that d.f. $F^{(Z)} = 1 - \overline{F^{(Z)}}$ belongs to class $\mathcal{C}$ together with the initial d.f. F. The difference between $\overline{F}$ and $\overline{F^{(Z)}}$ is shown in Figure 3.

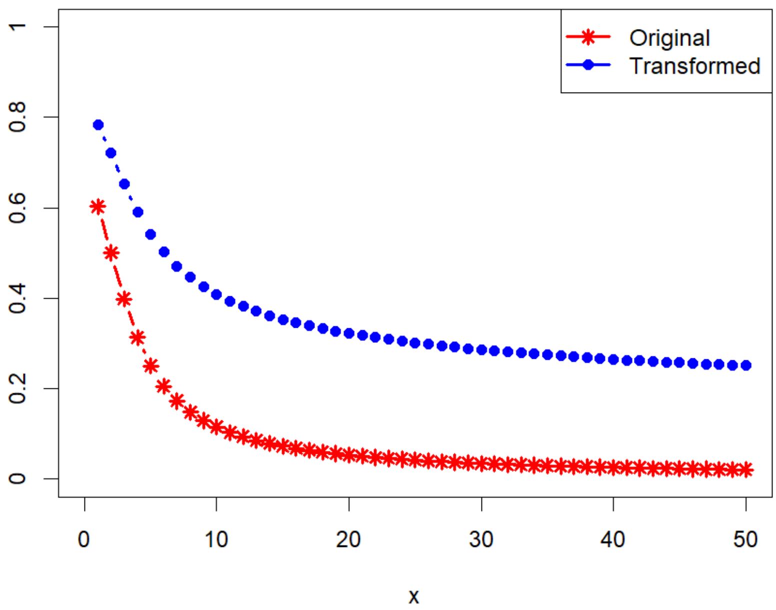

Figure 3. Comparison of tails $\overline{F}$ and $\overline{F^{(Z)}}$ from Example 2.

We note that in this case the tail of d.f. $F^{(Z)}$ is significantly heavier than the tail $\overline{F}$. However, both functions F and $F^{(Z)}$ belong to class $\mathcal{C}$, according to Theorem 2.

4.3. Transformation of the Exponential-like-Tailed Distribution Function

The main statements below show that the class of d.f.s $\mathcal{L}_\gamma, \gamma > 0$ is significantly more sensitive to the random effect transformation. In Theorem 3, we describe a reasonably general case where the random effect transformation of exponential-like-tailed d.f. remains in the same class by changing only the value of the parameter. Meanwhile, in Theorem 4, we show that a random effect of a specific form transfers the exponential-like-tailed d.f. to the class of regularly varying functions. In Sections 7 and 8, we present, respectively, the proofs of the main Theorems 3 and 4 of this subsection.

Theorem 3. *Let F be a d.f. from class* $\mathcal{L}_\gamma$ *with* $\gamma > 0$*, and let Z be a positive r.v. having the support of the special form*

$$\text{supp}\{Z\} = \{a\} \cup B$$

with $\inf\{B\} = b > a > 0$*. Then, the transformed d.f.* $F^{(Z)}$ *belongs to class* $\mathcal{L}_{\gamma a}$*.*

Corollary 1. *If $F \in \mathcal{L}_\gamma$, $\gamma > 0$, and Z is an arithmetic positive r.v. with $a = \min\{\text{supp}(Z)\}$, then transformed d.f. $F^{(Z)} \in \mathcal{L}_{\gamma a}$.*

Proof. According to conditions of the corollary,

$$\text{supp}\{Z\} = \{a, 2a, 3a, ...\}$$

with some positive a. Therefore,

$$\text{supp}\{Z\} = \{a\} \cup A,$$

where $A = \{2a, 3a, \ldots\}$ with $\inf\{A\} = 2a > a$. Now, the assertion of the corollary follows from Theorem 3 immediately. □

Corollary 2. *If $F \in \mathcal{L}_+ := \bigcup\limits_{\gamma > 0} \mathcal{L}_\gamma$ and Z is a positive r.v. such that*

$$\text{supp}\{Z\} = \{a\} \cup B, \ \ \inf\{B\} = b > a,$$

then the transformed d.f. $F^{(Z)}$ belongs to class $\mathcal{L}_+$ as well.

Proof. We derive directly from Theorem 3 that

$$F \in \mathcal{L}_+ \Rightarrow F \in \mathcal{L}_\gamma \text{ for some } \gamma > 0 \ \Rightarrow \ F^{(Z)} \in \mathcal{L}_{\gamma a} \ \Rightarrow \ F^{(Z)} \in \mathcal{L}_+.$$

These implications prove the assertion of the corollary. □

Theorem 4. *Let d.f. F belong to class $\mathcal{L}_\gamma$ with $\gamma > 0$, and let the random effect Z be distributed according to the gamma distribution with positive parameters α and β, i.e.,*

$$F_Z(u) = \mathbb{P}(Z \leqslant u) = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^u y^{\alpha-1} \mathrm{e}^{-\beta y} \mathrm{d}y, \ u \geqslant 0.$$

Then, the transformed d.f. $F^{(Z)}$ belongs to the regularity class $\mathcal{R}_\alpha$.

From the above theorem, we obtain the following statement when the random effect is exponentially distributed.

Corollary 3. *If d.f. F belongs to class $\mathcal{L}_\gamma$ with $\gamma > 0$, and the random effect Z is distributed according to the exponential law*

$$F_Z(u) = \mathbb{P}(Z \leqslant u) = \left(1 - \mathrm{e}^{-\beta u}\right) \mathbf{1}_{[0,\infty)}(u)$$

with parameter $\beta > 0$, then the transformed d.f. $F^{(Z)}$ belongs to the regularity class $\mathcal{R}_1$.

As already mentioned, we present the proofs of the main Theorems 3 and 4 in Sections 7 and 8. At the end of this subsection, we present three examples showing how the selected d.f. from class $\mathcal{L}_1$ can be transformed by different random effects.

Example 3. *Let us consider the exponential distribution with parameter 1, i.e.,*

$$F(x) = (1 - \mathrm{e}^{-x}) \mathbb{I}_{[0,\infty)}(x), \ \ \overline{F}(x) = \mathbb{I}_{(-\infty,0)}(x) + \mathrm{e}^{-x} \mathbb{I}_{[0,\infty)}(x).$$

The d.f. F belongs to class $\mathcal{L}_1$ because

$$\lim_{x \to \infty} \frac{\overline{F}(x - y)}{\overline{F}(x)} = \mathrm{e}^y$$

for any positive y. Let us take a positive random effect distributed according to the shifted Poisson law, i.e.,

$$\mathbb{P}(Z=k)=\frac{\mathrm{e}^{-1}}{(k-1)!},\ k\in\{1,2,\ldots\}.$$

Then, the tail of the transformed d.f. has the following expression:

$$\begin{aligned}\overline{F^{(Z)}}(x)&=\mathbb{E}(\overline{F}(x))^{Z}\\&=\sum_{k=1}^{\infty}\left(\overline{F}(x)\right)^{k}\mathbb{P}(Z=k)\\&=\mathrm{e}^{-1}\sum_{k=1}^{\infty}\frac{\left(\overline{F}(x)\right)^{k}}{(k-1)!}\\&=\frac{\overline{F}(x)}{\mathrm{e}}\sum_{k=0}^{\infty}\frac{\left(\overline{F}(x)\right)^{k}}{k!}\\&=\overline{F}(x)\exp\left\{\overline{F}(x)-1\right\}.\end{aligned}$$

We observe that d.f. $F^{(Z)}$ belongs to class $\mathcal{L}_1$, which is consistent with the statement of Theorem 3. The difference between $\overline{F}$ and $\overline{F^{(Z)}}$ is shown in Figure 4.

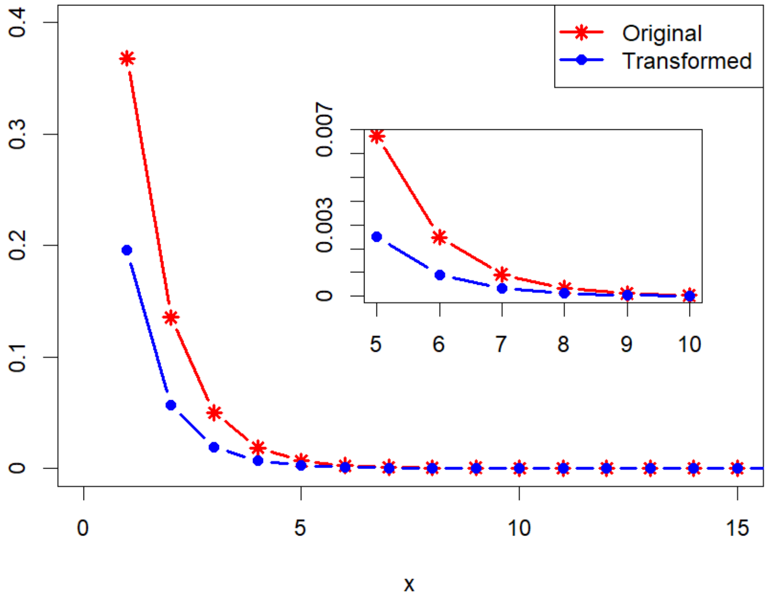

Figure 4. Comparison of tails $\overline{F}$ and $\overline{F^{(Z)}}$ from Example 3.

We observe that, in this case, the tails of d.f.s $F^{(Z)}$ and F are of similar heaviness.

Example 4. *Let us again consider exponential distribution with parameter* 1 *as in the example before. But in this case, let us take r.v. Z having exponential distribution with parameter* 2*. Then, the transformed t.f. has the following form:*

$$\overline{F^{(Z)}}(x) = 2\int_0^\infty \left(e^{-x}\right)^u e^{-2u} du$$
$$= 2\int_0^\infty e^{-(x+2)u} du$$
$$= \frac{2}{2+x},\ x \geqslant 0.$$

According to Theorem 4 and Corollary 3, we have that d.f. $F^{(Z)} = 1 - \overline{F^{(Z)}}$ belongs to class $\mathcal{R}_1$. The difference between $\overline{F}$ and $\overline{F^{(Z)}}$ is shown in Figure 5.

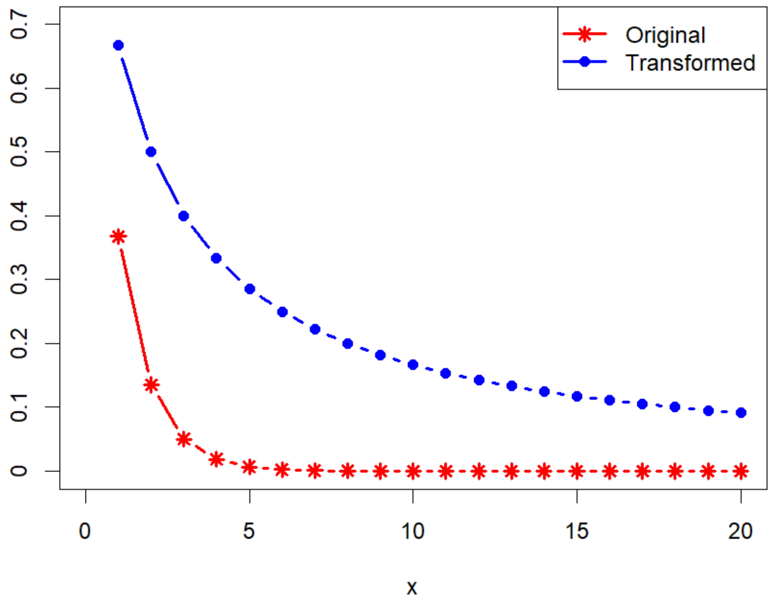

Figure 5. Comparison of tails $\overline{F}$ and $\overline{F^{(Z)}}$ from Example 4.

We observe that, in this case, the tail of d.f. $F^{(Z)}$ is noticeably heavier than the tail of F because in this case F is light-tailed and $F^{(Z)}$ is heavy-tailed.

Example 5. *Let us consider again the exponential distribution with parameter* 1 *as in Examples 3 and 4. As previously established, we have that d.f.*

$$F(x) = 1 - e^{-x},\ x \geqslant 0.$$

belongs to class $\mathcal{L}_1$. Now, let us take Z with the d.f.

$$F_Z(u) = \frac{1}{2}\mathbb{I}_{[2,3)}(u) + \left(\frac{1}{2}u - 1\right)\mathbf{1}_{[3,4)}(u) + \mathbb{I}_{[4,\infty)}(u).$$

Then, the transformed d.f., for $x \geqslant 0$, has the following form:

$$\overline{F^{(Z)}}(x) = \mathbb{E}(\overline{F}(x))^Z = e^{-2x}\,\mathbb{P}(Z=2) + \int_3^4 e^{-xu} d\left(\frac{1}{2}u - 1\right)$$
$$= \frac{1}{2}e^{-2x} + \frac{1}{2}\frac{e^{-4x}(e^x - 1)}{x}.$$

According to Theorem 3, we have that d.f. $F^{(Z)}$ belongs to class $\mathcal{L}_2$ because the support of the random effect is $\{2\} \cup [3, 4]$. This can also be observed from the obtained expressions of the tails of the d.f.s. The difference between $\overline{F}$ and $\overline{F^{(Z)}}$ is shown in Figure 6.

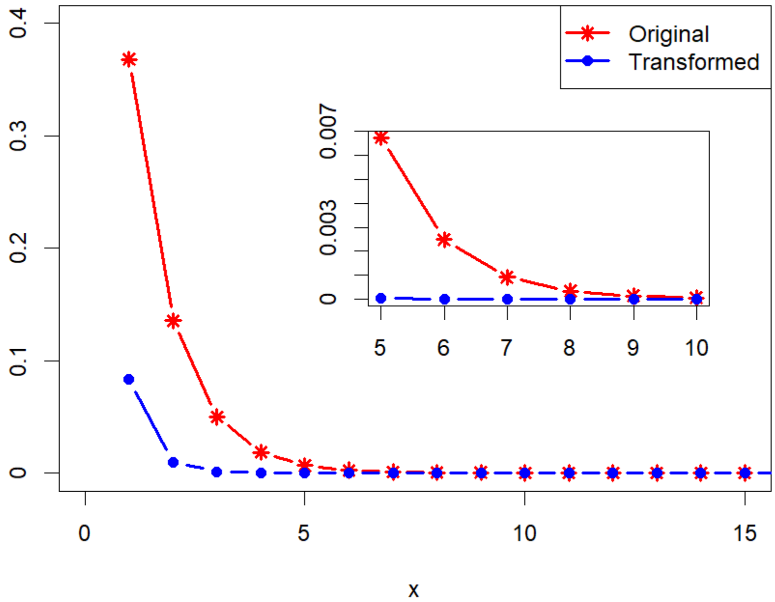

Figure 6. Comparison of tails $\overline{F}$ and $\overline{F^{(Z)}}$ from Example 5.

We observe that, in this case, the tails of d.f.s $F^{(Z)}$ and F are of similar heaviness. In fact, function F belongs to class $\mathcal{L}_1$ and $F^{(Z)}$ belongs to class $\mathcal{L}_2$, which is consistent with Theorem 3. We remind the reader here that both of the above-mentioned classes belong to class $\mathcal{L}_+$.

5. Proof of Theorem 1

For the proof of the theorem, we need one auxiliary lemma.

Lemma 1. *Let F be a d.f. of a real-valued r.v. Then, the following statements are equivalent:*

$$\text{(i) } F \text{ is heavy-tailed,}$$
$$\text{(ii) } \limsup_{x\to\infty} \mathrm{e}^{\lambda x}\,\overline{F}(x) = \infty \text{ for any } \lambda > 0.$$

Proof of the Lemma. (i) ⇒ (ii). Let on the contrary

$$\limsup_{x\to\infty} \mathrm{e}^{\lambda^* x}\,\overline{F}(x) < \infty$$

for some $\lambda^* > 0$. Then, c and $x_c > 0$ exist such that

$$\overline{F}(x) \leqslant c\,\mathrm{e}^{-\lambda^* x} \text{ for } x \geqslant x_c.$$

According to the alternative expectation formula (see [31], for instance), for any $\Delta \in (0, \lambda^*)$, we obtain the contradiction to $F \in \mathcal{H}$ because

$$\begin{aligned}\int_{[0,\infty)} \mathrm{e}^{\Delta x}\mathrm{d}F(x) &= 1+\Delta\int_0^{\infty} \mathrm{e}^{\Delta x}\,\overline{F}(x)\mathrm{d}x\\ &= 1+\Delta\int_0^{x_c} \mathrm{e}^{\Delta x}\,\overline{F}(x)\mathrm{d}x+\Delta\int_{x_c}^{\infty} \mathrm{e}^{\Delta x}\,\overline{F}(x)\mathrm{d}x\\ &\leqslant \Delta\, x_c\,\mathrm{e}^{\Delta x_c}+c\,\Delta\int_{x_c}^{\infty} \mathrm{e}^{-(\lambda^*-\Delta)x}\mathrm{d}x.\end{aligned}$$

(ii) $\Rightarrow$ (i). If $\lambda > 0$ and $x > 0$, then

$$\begin{aligned}\int_{-\infty}^{\infty} \mathrm{e}^{\lambda y}\mathrm{d}F(y) &\geqslant \int_{(x,\infty)} \mathrm{e}^{\lambda y}\mathrm{d}F(y)\\ &\geqslant \mathrm{e}^{\lambda x}\,\overline{F}(x),\end{aligned}$$

implying that

$$\int_{-\infty}^{\infty} \mathrm{e}^{\lambda y}\mathrm{d}F(y) \geqslant \limsup_{x\to\infty} \mathrm{e}^{\lambda x}\,\overline{F}(x) = \infty.$$

Therefore, $F \in \mathcal{H}$, and Lemma 1 is proved. □

Proof of Theorem 1. Due to Lemma 1, it is sufficient to prove that

$$\limsup_{x\to\infty} \mathrm{e}^{\lambda x}\,\overline{F^{(Z)}}(x) = \infty$$

for any positive λ.

Let $b > 0$ be such that $\mathbb{P}(Z \in [0,b]) > 0$. For this, b we have

$$\begin{aligned}\overline{F^{(Z)}}(x) &= \int_{[0,\infty)} \big(\overline{F}(x)\big)^u \mathrm{d}F_Z(u)\\ &\geqslant \int_{[0,b]} \big(\overline{F}(x)\big)^u \mathrm{d}F_Z(u)\\ &\geqslant \big(\overline{F}(x)\big)^b \mathbb{P}\big(Z \in [0,b]\big).\end{aligned}$$

Therefore, for any $\lambda > 0$

$$\limsup_{x\to\infty} \mathrm{e}^{\lambda x}\overline{F^{(Z)}}(x) \geqslant \limsup_{x\to\infty} \Big(\mathrm{e}^{\lambda x/b}\,\overline{F}(x)\Big)^b \mathbb{P}\big(Z \in [0,b]\big) = \infty$$

because condition $F \in \mathcal{H}$ implies

$$\limsup_{x\to\infty} \mathrm{e}^{\lambda x/b}\,\overline{F}(x) = \infty$$

according to Lemma 1. Theorem 1 is proved. □

6. Proof of Theorem 2

Proof. From the definition of class $\mathcal{C}$, we have that

$$\lim_{y\uparrow 1} \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1. \tag{1}$$

Let us choose $\varepsilon \in (0,1/2)$ and $\delta = \delta(\varepsilon) \in (0,1/2)$ so that $\mathbb{P}(Z \in [0,1/\varepsilon)) > 0$ and

$$\limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \leqslant 1+\epsilon^2,$$

for all $y \in (1-\delta, 1)$. For such ε and for positive x, y, we have

$$\begin{aligned}
\frac{\overline{F^{(Z)}}(xy)}{\overline{F^{(Z)}}(x)} &= \frac{\int\limits_{[0,\infty)} \big(\overline{F}(xy)\big)^u \mathrm{d}F_Z(u)}{\int\limits_{[0,\infty)} \big(\overline{F}(x)\big)^u \mathrm{d}F_Z(u)} \\
&= \frac{\int\limits_{[0,1/\varepsilon)} \big(\overline{F}(xy)\big)^u \mathrm{d}F_Z(u)}{\int\limits_{[0,\infty)} \big(\overline{F}(x)\big)^u \mathrm{d}F_Z(u)} + \frac{\int\limits_{[1/\varepsilon,\infty)} \big(\overline{F}(xy)\big)^u \mathrm{d}F_Z(u)}{\int\limits_{[0,\infty)} \big(\overline{F}(x)\big)^u \mathrm{d}F_Z(u)} \\
&=: J_{1\varepsilon} + J_{2\varepsilon},
\end{aligned} \tag{2}$$

where F_Z denotes the d.f. of the random effect Z.
If $y \in (1-\delta, 1)$, then

$$\begin{aligned}
\limsup_{x\to\infty} J_{1\varepsilon} &\leqslant \limsup_{x\to\infty} \frac{\int\limits_{[0,1/\varepsilon)} \big(\overline{F}(xy)\big)^u \frac{(\overline{F}(x))^u}{(\overline{F}(x))^u} \mathrm{d}F_Z(u)}{\int\limits_{[0,1/\varepsilon)} \big(\overline{F}(x)\big)^u \mathrm{d}F_Z(u)} \\
&\leqslant \limsup_{x\to\infty} \max_{0\leqslant u\leqslant 1/\varepsilon} \left(\frac{\overline{F}(xy)}{\overline{F}(x)}\right)^u \\
&\leqslant \limsup_{x\to\infty} \left(\frac{\overline{F}(xy)}{\overline{F}(x)}\right)^{1/\varepsilon} \\
&\leqslant \Big(1+\varepsilon^2\Big)^{1/\varepsilon}.
\end{aligned}$$

Also, for the selected ε and for $y \in (1-\delta, 1)$ we obtain

$$\begin{aligned}
\limsup_{x\to\infty} J_{2\epsilon} &\leqslant \frac{\int\limits_{[1/\varepsilon,\infty)} \big(\overline{F}(xy)\big)^u dF_Z(u)}{\int\limits_{[0,1/\varepsilon]} \big(\overline{F}(x)\big)^u dF_Z(u)} \\
&\leqslant \limsup_{x\to\infty} \frac{\big(\overline{F}(xy)\big)^{1/\varepsilon} \mathbb{P}(Z \geqslant \frac{1}{\varepsilon})}{\big(\overline{F}(x)\big)^{1/\varepsilon} \mathbb{P}(Z \in [0, \frac{1}{\varepsilon}])} \\
&\leqslant \Big(1+\varepsilon^2\Big)^{1/\varepsilon} \frac{\mathbb{P}(Z \geqslant \frac{1}{\varepsilon})}{\mathbb{P}(Z \in [0, \frac{1}{\varepsilon}])}.
\end{aligned}$$

The derived estimates and equality (2) imply that

$$\lim_{y\uparrow 1} \limsup_{x\to\infty} \frac{\overline{F^{(Z)}}(xy)}{\overline{F^{(Z)}}(x)} \leqslant \Big(1+\varepsilon^2\Big)^{1/\varepsilon} \left(\frac{\mathbb{P}(Z \geqslant \frac{1}{\varepsilon})}{\mathbb{P}(Z \in [0, \frac{1}{\varepsilon})} + 1\right)$$

for all ε satisfying the condition $\mathbb{P}(Z \in [0, \frac{1}{\varepsilon})) > 0$.
Therefore, by passing ε to zero we obtain

$$\lim_{y\uparrow 1} \limsup_{x\to\infty} \frac{\overline{F^{(Z)}}(xy)}{\overline{F^{(Z)}}(x)} \leqslant \lim_{\varepsilon\downarrow 0} \Big(1+\varepsilon^2\Big)^{1/\varepsilon} \left(1 + \frac{\mathbb{P}(Z \geqslant \frac{1}{\varepsilon})}{\mathbb{P}(Z \in [0, \frac{1}{\varepsilon}))}\right).$$

Since

$$\lim_{\varepsilon\downarrow 0} \Big(1+\varepsilon^2\Big)^{1/\varepsilon} = 1,$$

we have

$$\lim_{y\uparrow 1}\limsup_{x\to\infty}\frac{\overline{F^{(Z)}}(xy)}{\overline{F^{(Z)}}(x)}\leqslant 1. \tag{3}$$

In addition, if $y\in(0,1)$ and $x>0$ we have

$$\limsup_{x\to\infty}\frac{\overline{F^{(Z)}}(xy)}{\overline{F^{(Z)}}(x)}\geqslant\liminf_{x\to\infty}\frac{\overline{F^{(Z)}}(xy)}{\overline{F^{(Z)}}(x)}\geqslant 1,$$

for all $y\in(0,1)$. Consequently,

$$\lim_{y\uparrow 1}\limsup_{x\to\infty}\frac{\overline{F^{(Z)}}(xy)}{\overline{F^{(Z)}}(x)}\geqslant 1$$

which, together with the estimate (3), implies that

$$\lim_{y\uparrow 1}\limsup_{x\to\infty}\frac{\overline{F^{(Z)}}(xy)}{\overline{F^{(Z)}}(x)}= 1.$$

This means that the transformed d.f. $F^{(Z)}\in\mathcal{C}$. Theorem 2 is proved. □

7. Proof of Theorem 3

Proof. From the definition of class $\mathcal{L}_\gamma$ we have

$$F\in\mathcal{L}_\gamma \quad\Leftrightarrow\quad \lim_{x\to\infty}\frac{\overline{F}(x-y)}{\overline{F}(x)}=e^{\gamma y}\ \text{ for all }\ y>0. \tag{4}$$

If x is positive, then

$$\begin{aligned}\overline{F^{(Z)}}(x)&=\mathbb{E}\Big(\big(\overline{F}(x)\big)^Z\Big)\\&=\mathbb{E}\Big(\big(\overline{F}(x)\big)^Z\mathbf{1}_{\{Z=a\}}+\big(\overline{F}(x)\big)^Z\mathbf{1}_{\{Z\in B\}}\Big)\\&=\big(\overline{F}(x)\big)^a\mathbb{P}(Z=a)+\mathbb{E}\Big(\big(\overline{F}(x)\big)^Z\mathbf{1}_{\{Z\geqslant b\}}\Big).\end{aligned}$$

Hence, for any fixed $y>0$, we obtain

$$\begin{aligned}&\limsup_{x\to\infty}\frac{\overline{F^{(Z)}}(x-y)}{\overline{F^{(Z)}}(x)}\\&=\limsup_{x\to\infty}\frac{\big(\overline{F}(x-y)\big)^a\mathbb{P}(Z=a)+\mathbb{E}\Big(\big(\overline{F}(x-y)\big)^Z\mathbf{1}_{\{Z\geqslant b\}}\Big)}{\big(\overline{F}(x)\big)^a\mathbb{P}(Z=a)+\mathbb{E}\Big(\big(\overline{F}(x)\big)^Z\mathbf{1}_{\{Z\geqslant b\}}\Big)}\\&\leqslant\limsup_{x\to\infty}\frac{\big(\overline{F}(x-y)\big)^a\mathbb{P}(Z=a)+\big(\overline{F}(x-y)\big)^b\mathbb{P}(Z\geqslant b)}{\big(\overline{F}(x)\big)^a\mathbb{P}(Z=a)}\\&\leqslant\limsup_{x\to\infty}\left(\frac{\overline{F}(x-y)}{\overline{F}(x)}\right)^a+\limsup_{x\to\infty}\left(\frac{\overline{F}(x-y)}{\overline{F}(x)}\right)^b\big(\overline{F}(x)\big)^{b-a}\frac{\mathbb{P}(Z\geqslant b)}{\mathbb{P}(Z=a)}\\&=\mathrm{e}^{\gamma ay}.\end{aligned} \tag{5}$$

Similarly, we obtain

$$
\begin{aligned}
\liminf_{x\to\infty} \frac{\overline{F^{(Z)}}(x-y)}{\overline{F^{(Z)}}(x)} &\geqslant \frac{\big(\overline{F}(x-y)\big)^a \mathbb{P}(Z=a)}{\big(\overline{F}(x)\big)^a \mathbb{P}(Z=a) + \big(\overline{F}(x)\big)^b \mathbb{P}(Z\geqslant b)} \\
&= \liminf_{x\to\infty} \frac{\Big(\frac{\overline{F}(x-y)}{\overline{F}(x)}\Big)^a \mathbb{P}(Z=a)}{\mathbb{P}(Z=a) + \big(\overline{F}(x)\big)^{b-a}\mathbb{P}(Z\geqslant b)} \\
&= \liminf_{x\to\infty} \left(\frac{\overline{F}(x-y)}{\overline{F}(x)}\right)^a \\
&= e^{\gamma a y}. \qquad (6)
\end{aligned}
$$

The estimates (5) and (6) imply that $F^{(Z)} \in \mathcal{L}_{\gamma a}$. This finishes the proof of Theorem 3. □

8. Proof of Theorem 4

For the proof of Theorem 4, we need an auxiliary statement related to the representation theorem for the exponential-like-tailed d.f.s.

Lemma 2. *Let d.f. F belong to class $\mathcal{L}_\gamma$ with $\gamma > 0$. Then, for any $\varepsilon \in (0, \gamma/2)$ sufficiently large $x^* = x^*(\varepsilon)$ exists such that*

$$
\mathrm{e}^{-(\gamma+\varepsilon)x} \leqslant \overline{F}(x) \leqslant \mathrm{e}^{-(\gamma-\varepsilon)x}
$$

for all $x \geqslant x^*$.

Proof of the Lemma. D.f. F belongs to class $\mathcal{L}_\gamma$ with $\gamma > 0$. Therefore, for fixed $y > 0$, we have

$$
\frac{(yx)^\gamma \overline{F}(\log yx)}{x^\gamma \overline{F}(\log x)} = y^\gamma \frac{\overline{F}(\log y + \log x)}{\overline{F}(\log x)} \underset{x\to\infty}{\to} y^\gamma \mathrm{e}^{-\gamma \log y} = 1.
$$

This implies that the function $x^\gamma \overline{F}(\log x)$ slowly varies. According to the well-known Karamata's representation theorem, see [32], Theorem 1.2 in [33], or Theorem 1.3.1 together with Corollary 1.3.5 in [34],

$$
x^\gamma \overline{F}(\log x) = h(x) \exp\left\{-\int_1^x \frac{g(u)}{u}\mathrm{d}u\right\}, \qquad (7)
$$

where h and $g \geqslant 0$ are measurable functions such that

$$
h(x) \underset{x\to\infty}{\to} h > 0, \text{ and } g(x) \underset{x\to\infty}{\to} 0.
$$

Changing $\log x$ to x, we obtain from (7) that

$$
\begin{aligned}
\overline{F}(x) &= h(\mathrm{e}^x)\,\mathrm{e}^{-\gamma x} \exp\left\{-\int_1^{\mathrm{e}^x} \frac{g(u)}{u}\mathrm{d}u\right\} \\
&= h(\mathrm{e}^x)\,\mathrm{e}^{-\gamma x} \exp\left\{-\int_0^x g(\mathrm{e}^u)\mathrm{d}u\right\}.
\end{aligned}
$$

Let $\varepsilon \in (0, \gamma/2)$ be fixed. From the last expression, we obtain that

$$
\overline{F}(x) \leqslant \max\{1, 2h\}\,\mathrm{e}^{-\gamma x} \leqslant \mathrm{e}^{-(\gamma-\varepsilon)x} \qquad (8)
$$

if $x \geqslant \dfrac{1}{\varepsilon} \log \max\{1, 2h\}$.
Similarly, the condition $g(x) \underset{x\to\infty}{\to} 0$ implies that

$$\begin{aligned}\overline{F}(x) &= h\big(\mathrm{e}^x\big)\,\mathrm{e}^{-\gamma x} \exp\bigg\{-\int_0^{x_1} g\big(\mathrm{e}^u\big)\mathrm{d}u - \int_{x_1}^{x} g\big(\mathrm{e}^u\big)\mathrm{d}u\bigg\} \\ &\geqslant h\big(\mathrm{e}^x\big)\,\mathrm{e}^{-\gamma x} \exp\bigg\{-\int_0^{x_1} g\big(\mathrm{e}^u\big)\mathrm{d}u - \frac{\varepsilon}{2}(x - x_1)\bigg\} \\ &\geqslant h\big(\mathrm{e}^x\big)\,\mathrm{e}^{-(\gamma+\varepsilon/2)x} \exp\bigg\{-\int_0^{x_1} g\big(\mathrm{e}^u\big)\mathrm{d}u\bigg\},\ x \geqslant x_1,\end{aligned}$$

where x_1 is such that $g\big(\mathrm{e}^u\big) \leqslant \varepsilon/2$ for $u \geqslant x_1$.
If x is sufficiently large $(x \geqslant x_2 > x_1)$, then

$$h\big(\mathrm{e}^x\big) \geqslant \frac{h}{2} \geqslant \mathrm{e}^{-\varepsilon x/4}$$

by condition $h(x) \underset{x\to\infty}{\to} h > 0$ and

$$\exp\bigg\{-\int_0^{x_1} g\big(\mathrm{e}^u\big)\mathrm{d}u\bigg\} \geqslant \mathrm{e}^{-\varepsilon x/4},$$

due to the integrability of function g.
The derived above estimates imply that

$$\overline{F}(x) \geqslant \mathrm{e}^{-(\gamma+\varepsilon)x} \tag{9}$$

for all $x \geqslant x_2$.

It follows from the estimates (8) and (9) that the assertion of the lemma holds with $x^* = \max\bigg\{\dfrac{1}{\varepsilon}\log\max\{1, 2h\}, x_2\bigg\}$. Lemma 2 is proved. □

Proof of Theorem 4. Let $y > 0$ be fixed. For an arbitrary $x > 0$

$$\begin{aligned}\frac{\overline{F^{(Z)}}(xy)}{\overline{F^{(Z)}}(x)} &= \frac{\int_{[0,\infty)}\big(\overline{F}(xy)\big)^u \mathrm{d}F_Z(u)}{\int_{[0,\infty)}\big(\overline{F}(x)\big)^u \mathrm{d}F_Z(u)} \\ &= \frac{\int_0^\infty \big(\overline{F}(xy)\big)^u u^{\alpha-1}\mathrm{e}^{-\beta u}\mathrm{d}u}{\int_0^\infty \big(\overline{F}(x)\big)^u u^{\alpha-1}\mathrm{e}^{-\beta u}\mathrm{d}u}.\end{aligned}$$

Since $F \in \mathcal{L}_\gamma$ with $\gamma > 0$, we have from Lemma 2 that

$$\begin{aligned}\mathrm{e}^{-(\gamma+\varepsilon)xy} &\leqslant \overline{F}(xy) \leqslant \mathrm{e}^{-(\gamma-\varepsilon)xy}, \\ \mathrm{e}^{-(\gamma+\varepsilon)x} &\leqslant \overline{F}(x) \leqslant \mathrm{e}^{-(\gamma-\varepsilon)x}\end{aligned}$$

for $\varepsilon \in (0, \gamma/2)$ and sufficiently large x.
Therefore,

$$\begin{aligned}\limsup_{x\to\infty} \frac{\overline{F^{(Z)}}(xy)}{\overline{F^{(Z)}}(x)} &\leqslant \limsup_{x\to\infty} \frac{\int_0^\infty u^{\alpha-1}\,\mathrm{e}^{-\big((\gamma-\varepsilon)xy+\beta\big)u}\mathrm{d}u}{\int_0^\infty u^{\alpha-1}\,\mathrm{e}^{-\big((\gamma+\varepsilon)x+\beta\big)u}\mathrm{d}u} \\ &= \limsup_{x\to\infty}\bigg(\frac{(\gamma+\varepsilon)x+\beta}{(\gamma-\varepsilon)xy+\beta}\bigg)^\alpha \\ &= \bigg(\frac{\gamma+\varepsilon}{\gamma-\varepsilon}\bigg)^\alpha y^{-\alpha},\end{aligned} \tag{10}$$

and, similarly,

$$\begin{aligned}\liminf_{x\to\infty}\frac{\overline{F^{(Z)}}(xy)}{\overline{F^{(Z)}}(x)} &\geqslant \liminf_{x\to\infty}\frac{\int_0^\infty u^{\alpha-1}\,\mathrm{e}^{-\left((\gamma+\varepsilon)xy+\beta\right)u}\mathrm{d}u}{\int_0^\infty u^{\alpha-1}\,\mathrm{e}^{-\left((\gamma-\varepsilon)x+\beta\right)u}\mathrm{d}u}\\ &= \liminf_{x\to\infty}\left(\frac{(\gamma-\varepsilon)x+\beta}{(\gamma+\varepsilon)xy+\beta}\right)^{\alpha}\\ &= \left(\frac{\gamma-\varepsilon}{\gamma+\varepsilon}\right)^{\alpha} y^{-\alpha}. \end{aligned} \tag{11}$$

Because of the arbitrariness of $\varepsilon \in (0, \gamma/2)$, estimates (10) and (11) imply that

$$\lim_{x\to\infty}\frac{\overline{F^{(Z)}}(xy)}{\overline{F^{(Z)}}(x)} = y^{-\alpha}.$$

This finishes the proof of Theorem 4. □

9. Conclusions

In this paper, we examine how the random effect transformation changes the regularity properties of the distribution function. We consider three classes of distributions: heavy-tailed distributions, distributions with consistently varying tails, and exponential-like-tailed distributions. We found that the class of exponential-like-tailed distributions is most sensitive to the random effect transformation. In some cases, a random effect transforms distributions from this subclass of light-tailed distributions into heavy-tailed ones. Also, the random effect transformation is one way to construct distributions with specific properties. This method perfectly complements other methods of such construction discussed in many articles, see, for example, refs. [35–43] and references there in. In this paper and in paper [1], it is established how a random effect affects distributions from seven regularity classes: $\mathcal{R}, \mathcal{L}, \mathcal{L}_+, \mathcal{OL}, \mathcal{D}, \mathcal{C}$, and $\mathcal{H}$. According to the book [28], there are more regularity classes. In the future, it would be interesting to find out how a random effect changes distributions from other classes, especially those defined by the convolution.

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