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Micropolar Fluid-Thin Elastic Structure Interaction: Variational Analysis

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Abstract. We consider the non-stationary flow of a micropolar fluid in a thin channel with an impervious wall and an elastic stiff wall, motivated by applications to blood flows through arteries. We assume that the elastic wall is composed of several layers with different elastic characteristics and that the domains occupied by the two media are infinite in one direction and the problem is periodic in the same direction. We provide a complete variational analysis of the two dimensional interaction between the micropolar fluid and the stratified elastic layer. For a suitable data regularity, we prove the existence, the uniqueness and the regularity of the solution to the variational problem associated to the physical system. Increasing the data regularity, we prove that the fluid pressure is unique, we obtain additional regularity for all the unknown functions and we show that the solution to the variational problem is solution for the physical system, as well.

Keywords: fluid-structure interaction, micropolar fluid, stratified elastic layer, periodic flow, existence, uniqueness, regularity.

AMS Subject Classification: 74F10; 76A05; 35Q35; 35Q74.

1 Introduction

The interaction between a fluid and an elastic structure (FSI) is a widely spread phenomenon in real life. Due to this practical interest and to the difficulty of the mathematical model as well, FSI problems have been studied in many recent articles. A survey on mathematical modeling, analysis and simulation techniques for FSI problems is given in [23] and we may highlight different results related to FSI problems with the following selection of the latest works: [5, 16, 21, 26, 27]. An example of FSI problem is represented by a viscous flow in a thin elastic tube structure. It is motivated by applications in biology (blood circulation in a network of vessels) as well as in the engineering of industrial installations such as pipelines. Mostly the fluid rheology is Newtonian, described by the Navier-Stokes or Stokes system. However, several fluids exhibit a non-Newtonian behavior. In particular, the modeling of the motion of colloids needs to take into consideration the rotation of small particles. In hemodynamics the red blood cells, platelets and other (smaller) cells flow in plasma and in order to take into account the rotations of these cells one may use the micropolar fluid model. It was introduced firstly in [2, 15] and then studied extensively by mathematicians (see [19] and the bibliography therein, [9, 10, 11, 12, 13, 17, 18, 25, 29, 30]). In the case of thin structures the asymptotic analysis of micropolar flows was performed for various geometries and various boundary conditions in: [3, 4, 6, 7, 8, 14, 22]. The importance of the micropolar fluids from the physical viewpoint generated also numerous articles presenting applications of the micropolar fluid model in biomedicine and blood flow modeling: see e.g., [1, 31].

Periodic flows have many applications in fluid mechanics. This model is used, e.g., to study the behaviour of fluids in pipes and channels or to describe the blood flows through arteries.

In the present article we consider the periodic flow of a micropolar fluid in an infinite thin channel with an impervious wall and an elastic stiff wall in the context of FSI problems for thin structures. For the Newtonian rheology (without micro-rotations) this topic was studied in [21]. To our knowledge, the present work is the first article on the micropolar flow in a thin channel with elastic stiff wall. The nondimensionalization and various scalings of the Newtonian fluid-structure interaction model are discussed in [20]. The model considered in the present paper solves a more challenging problem, taking into consideration the non-Newtonian rheology of the blood corresponding to micro-rotations of the blood cells in the plasma, which introduces an additional unknown and coupling in the description of the problem. From the mathematical point of view, we have to overcome new difficulties generated by the additional coupling velocity-microrotation.

In our study the elastic wall is composed of several layers with different elastic characteristics. As an example from the real life of such a material we mention the vessels walls. The walls of both arteries and veins have the same three distinct elastic layers: the outer layer of the wall, composed of collagen fibers and elastic tissue, the middle layer made up of smooth muscle cells, elastic tissue and collagen fibres and the inner layer mainly made up of endothelial cells (see e.g., [32]). This last layer sometimes contains one-way valves, especially in the veins of the arms and legs.

The purpose of the present article is to provide a complete variational analysis of the two dimensional interaction between a micropolar fluid and a stratified elastic layer when the domains occupied by the two media are infinite in one direction and the problem is periodic in the same direction. The problem depends on a small parameter, ε, defined as the ratio between the thickness of the elastic structure and that of the fluid layer. This small parameter will play an important role in a forthcoming article, dealing with the asymptotic analysis of the physical problem described above.

The outline of this article is as follows: in Section 2 we present the geometry of the problem, the characteristics of the two media and the assumptions of our study. Then we give the non stationary, coupled system that describes the interaction between the micropolar fluid and the stratified elastic structure. This system contains linear equations for the micropolar fluid and for the elastic medium, junction conditions between the elastic layers and between the fluid and the elastic structure, as well, periodicity, boundary and initial conditions. The unknowns of the physical system are: the fluid velocity ($\mathbf{v}_\varepsilon$), the microrotation (ω_ε), the fluid pressure (p_ε) and the displacement of the elastic structure ($\mathbf{u}_\varepsilon$). In addition to the hypothesis of periodicity of the motion, we also suppose that the deformation of the elastic structure is small, assumption that allows us to write the fluid equations and conditions on the undeformed fluid domain. Section 3 deals with the variational analysis of the problem. For a suitable data regularity, we prove the existence, the uniqueness and the regularity of the solution to the variational problem associated to the physical system and we define the weak solution of the problem, that involves the unknowns: $\mathbf{u}_\varepsilon$, $\mathbf{v}_\varepsilon$ and ω_ε. In the last part of this section we introduce the fourth unknown, the micropolar fluid pressure, and we establish some properties of this function. In Section 4, we increase the data regularity in such a way that we succeed in proving that the fluid pressure is unique, as the other three unknowns, in obtaining additional regularity for the unknowns and in showing that the solution to the variational problem is solution for the physical system, as well.

2 Description of the physical problem

Consider an infinite layer $L_\varepsilon = \{(x_1, x_2)/x_1 \in \mathbb{R}, x_2 \in (-1, \varepsilon)\}$ divided into two parts: $L^- = \{(x_1, x_2)/x_1 \in \mathbb{R}, x_2 \in (-1, 0)\}$ representing the layer occupied by a micropolar fluid and $L_\varepsilon^+ = \{(x_1, x_2)/x_1 \in \mathbb{R}, x_2 \in (0, \varepsilon)\}$ representing an elastic layer. The boundaries associated with this geometry are denoted by:

$$\begin{cases} F^- = \{(x_1, -1)/x_1 \in \mathbb{R}\}, \quad F_\varepsilon^+ = \{(x_1, \varepsilon)/x_1 \in \mathbb{R}\}, \\ F^0 = \{(x_1, 0)/x_1 \in \mathbb{R}\}, \end{cases}$$

with F^- an impervious boundary, F_ε^+ an elastic boundary and F^0 the interface between the micropolar fluid and the elastic layer. We suppose that the solid structure is composed by p elastic layers each of them being characterized by constant density and matrix-valued elasticity coefficients. The constant values of these physical characteristics, denoted $\rho^{+,s}$ and A^s_{ij}, respectively, $s = 1, ..., p$, are different from one layer to another, generating jumps on each boundary that separates the elastic layers. So, the elastic medium is described by a piecewise constant density, denoted ρ^+, and by piecewise constant elasticity coefficients, denoted A_{ij}, with the properties:

$$(\exists)\, \rho^+_{\min},\, \rho^+_{\max} \text{independent of}\, \varepsilon, 0 < \rho^+_{\min} \leq \rho^+(x_2/\varepsilon) \leq \rho^+_{\max}\, (\forall)\, x_2 \in [0, \varepsilon] \tag{2.1}$$

and

$$\begin{aligned}
&A_{ij} = (a^{kl}_{ij})_{1\leq k,l\leq 2} = \left(a^{kl}_{ij}\left(\frac{x_2}{\varepsilon}\right)\right)_{1\leq k,l\leq 2},\ i,j \in \{1,2\},\\
&a^{kl}_{ij} = \frac{E}{2(1+\nu)}\left(\frac{2\nu}{1-2\nu}\delta_{ik}\delta_{jl} + \delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}\right),\\
&(\exists)\, \kappa > 0 \text{ independent of } \varepsilon \text{ s.t. } \sum_{i,j,k,l=1}^{2} a^{kl}_{ij}\left(\frac{x_2}{\varepsilon}\right)\eta^l_j\eta^k_i \geq \kappa \sum_{j,l=1}^{2}(\eta^l_j)^2,\\
&\qquad (\forall)\, x_2 \in [0,\varepsilon], (\forall)\, \eta = (\eta^l_j)_{1\leq j,l\leq 2}, \eta^l_j = \eta^j_l,
\end{aligned} \tag{2.2}$$

with $E = E\left(\frac{x_2}{\varepsilon}\right)$ the Young's modulus and $\nu = \nu\left(\frac{x_2}{\varepsilon}\right)$ the Poisson's ratio.

Denote in what follows $\xi_2 = x_2/\varepsilon$.

We give below a more precise description of the elastic medium. We suppose that there exist $\zeta_1, \zeta_2, ..., \zeta_p$ with $0 < \zeta_1 < ... < \zeta_{p-1} < \zeta_p = 1$ such that $\rho^+(\xi_2) = \rho^{+,s}$, $A_{ij}(\xi_2) = A^s_{ij}$ in $L^{+,s}_\varepsilon$ $(\forall)\, s \in \{1, ..., p\}$ with $\rho^{+,s}$ positive constants and A^s_{ij} matrices with constant elements and

$$L^{+,s}_\varepsilon = \{(x_1, x_2)/x_1 \in \mathbb{R}, \zeta_{s-1}\varepsilon < x_2 < \zeta_s\varepsilon\},\ \ s \in \{1, ..., p\}$$

representing the elastic layers. By convention, $\zeta_0 = 0$.

The interface between two consecutive elastic layers, $L^{+,s}_\varepsilon$ and $L^{+,s+1}_\varepsilon$, $s \in \{1, ..., p-1\}$ is defined by $F^{+,s}_\varepsilon = \{(x_1, \zeta_s\varepsilon)/x_1 \in \mathbb{R}\}$.

Remark 1. All the characteristics of the elastic medium, namely ρ^+, E, ν, depend on ε, but for the sake of notational simplicity, we omit it.

The characteristics of the micropolar fluid, are several positive constants independent of ε: ρ^- its density, χ, μ viscosity coefficients, j, γ constants related to the microrotation. The nonstationary interaction between the micropolar fluid and the elastic layer in a given time interval $(0, T)$ is described by the following coupled system

$$
\begin{cases}
\rho^{+,s}\dfrac{\partial^2 \mathbf{u}_\varepsilon}{\partial t^2} - c_\varepsilon \displaystyle\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(A_{ij}^s \frac{\partial \mathbf{u}_\varepsilon}{\partial x_j}\right) = \mathbf{h}_\varepsilon \quad \text{in } L_\varepsilon^{+,s}\times(0,T),\ s\in\{1,...,p\},\\
\begin{cases}
\rho^{-}\dfrac{\partial \mathbf{v}_\varepsilon}{\partial t} - 2(\mu+\chi)\mathrm{div}\,(D(\mathbf{v}_\varepsilon)) + \nabla p_\varepsilon - \chi \mathrm{curl}\,\omega_\varepsilon = \mathbf{f}\\
\mathrm{div}\ \mathbf{v}_\varepsilon = 0 \\
j\dfrac{\partial \omega_\varepsilon}{\partial t} - \gamma \Delta\omega_\varepsilon + 2\chi\omega_\varepsilon - \chi \mathrm{curl}\ \mathbf{v}_\varepsilon = g
\end{cases} \quad \text{in } L^{-}\times(0,T),\\
\displaystyle\sum_{j=1}^{2} A_{2j}^p \frac{\partial \mathbf{u}_\varepsilon}{\partial x_j} = \mathbf{0} \quad \text{on } F_\varepsilon^{+}\times(0,T),\\
\begin{cases}
[\mathbf{u}_\varepsilon]_s = \mathbf{0},\\
\displaystyle\sum_{j=1}^{2} A_{2j}^s \frac{\partial \mathbf{u}_\varepsilon}{\partial x_j} = \sum_{j=1}^{2} A_{2j}^{s+1} \frac{\partial \mathbf{u}_\varepsilon}{\partial x_j}
\end{cases} \quad \text{on } F_\varepsilon^{+,s}\times(0,T),\ s\in\{1,...,p-1\},\\
\mathbf{v}_\varepsilon = \mathbf{0},\ \omega_\varepsilon = 0 \quad \text{on } F^{-}\times(0,T),\\
\begin{cases}
\mathbf{v}_\varepsilon = \dfrac{\partial \mathbf{u}_\varepsilon}{\partial t}\\
\omega_\varepsilon = 0\\
-p_\varepsilon \mathbf{e}_2 + 2(\mu+\chi)D(\mathbf{v}_\varepsilon)\mathbf{e}_2 = c_\varepsilon \displaystyle\sum_{j=1}^{2} A_{2j}^1 \frac{\partial \mathbf{u}_\varepsilon}{\partial x_j}
\end{cases} \quad \text{on } F^{0}\times(0,T),\\
\rho^{+,s}\frac{\partial^2 \mathbf{u}_\varepsilon}{\partial t^2} - c_\varepsilon \displaystyle\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(A_{ij}^s \frac{\partial \mathbf{u}_\varepsilon}{\partial x_j}\right) = \mathbf{h}_\varepsilon \quad \text{in } L_\varepsilon^{+,s}\times(0,T),\ s\in\{1,...,p\},
\end{cases}
\tag{2.3}
$$

where $[\mathbf{u}_\varepsilon]_s$ represents the jump of the function $\mathbf{u}_\varepsilon$ on $F_\varepsilon^{+,s}\times(0,T)$, $s\in\{1,...,p-1\}$ and c_ε is a positive constant depending on ε that expresses the dependence of the elasticity coefficients on the small parameter ε. Since the elasticity coefficients are great, in general c_ε is a negative power of ε. More precisely, in a forthcoming article dealing with the asymptotic analysis of the considered problem we take $c_\varepsilon = \varepsilon^{-3}$. In addition to the characteristics of the two media, the other data of the problem are the forces $\mathbf{h}_\varepsilon, \mathbf{f}$, and g. The unknowns of problem (2.3) are: the displacement of the elastic medium, $\mathbf{u}_\varepsilon$ and the velocity, the microrotation, the pressure of the fluid, $\mathbf{v}_\varepsilon, \omega_\varepsilon, p_\varepsilon$, respectively. We denoted in the previous system

$$
D(\mathbf{v}) = \frac{1}{2}(\nabla\mathbf{v} + (\nabla\mathbf{v})^T),\ \ \mathrm{curl}\,\omega = \frac{\partial\omega}{\partial x_2}\mathbf{e}_1 - \frac{\partial\omega}{\partial x_1}\mathbf{e}_2,\ \mathrm{curl}\,\mathbf{v} = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2},
$$

with $\mathbf{e}_1, \mathbf{e}_2$ the coordinate axes vectors.

Remark 2. Relations $(2.3)_{6,7}$ represent the continuity of the displacement and of the normal stresses, respectively, at the interface between two consecutive elastic layers. In the case when the characteristics of the elastic medium are continuous, i.e. it has only one layer, these conditions disappear.

Remark 3. The coupling conditions between the two media, $(2.3)_{9,11}$, represent the continuity of velocities and of normal stresses on the interface. In a more accurate micropolar fluid-elastic structure interaction description, the junction conditions should be imposed on the deformed interface, e.g., $\mathbf{v}_\varepsilon(x+\mathbf{u}_\varepsilon(x),t) = \frac{\partial \mathbf{u}_\varepsilon}{\partial t}(x,t)$, instead of $(2.3)_9$, since for the fluid we use the Eulerian description, while for the structure, the Lagrangeian one. However, when the deformation of the elastic structure is small, that represents an assumption of our model, the equations for the two media can be written with a good approximation in the initial corresponding domains and the junction conditions on the fixed interface separating these domains (see e.g., [23]).

Condition $(2.3)_{12}$ is related to another assumption of our approach, that the interaction problem is 1−periodic with respect to x_1 variable. This means that both the data and the unknown functions are 1−periodic in x_1.

For any $a \in \mathbb{R}$ we may define as periodicity domains

$$\begin{cases} D_a^- = (a,a+1)\times(-1,0), \;\; D_{a,\varepsilon}^+ = (a,a+1)\times(0,\varepsilon), \\ D_{a,\varepsilon}^{+,s} = (a,a+1)\times(\zeta_{s-1}\varepsilon,\zeta_s\varepsilon), \;\; s\in\{1,...,p\}, \\ D_{a,\varepsilon} = (a,a+1)\times(-1,\varepsilon) \end{cases} \tag{2.4}$$

and as periodicity boundaries

$$\begin{cases} \Gamma_a^- = (a,a+1)\times\{-1\}, \;\; \Gamma_{a,\varepsilon}^+ = (a,a+1)\times\{\varepsilon\}, \\ \Gamma_{a,\varepsilon}^{+,s} = (a,a+1)\times\{\zeta_s\varepsilon\}, \;\; s\in\{1,...,p-1\}, \\ \Gamma_a^0 = (a,a+1)\times\{0\}. \end{cases} \tag{2.5}$$

3 Variational analysis of the problem

This section is devoted to the analysis of the weak formulation for (2.3). In the first part we present the functional framework of this analysis. For a suitable data regularity, we prove the existence, the uniqueness and the regularity of the solution to the variational problem associated with the physical system (2.3). For obtaining and studing this variational problem it is more convenient to reduce first the number of unknown functions. To this aim, we replace two unknowns of the problem, namely the velocity of the fluid, $\mathbf{v}_\varepsilon$, and the displacement of the elastic layer, $\mathbf{u}_\varepsilon$, with one function, $\mathbf{w}_\varepsilon$: $\bar{L}_\varepsilon \times [0,T] \mapsto \mathbb{R}^2$, defined as follows:

$$\mathbf{w}_\varepsilon(x,t) = \begin{cases} \mathbf{v}_\varepsilon(x,t) & \text{if } (x,t)\in \bar{L}^- \times [0,T], \\ \dfrac{\partial \mathbf{u}_\varepsilon}{\partial t}(x,t) & \text{if } (x,t)\in \bar{L}_\varepsilon^+ \times [0,T]. \end{cases}$$

Coupling condition $(2.3)_9$ ensures that the function $\mathbf{w}_\varepsilon$ is well defined on F^0.

3.1 Data regularity and functional spaces

In order to perform the variational analysis of the previous problem we present the regularity of the data and the functional spaces corresponding to the unknown functions.

As we previously said, we consider that the density and the elastic coefficients characterizing the elastic medium, ρ^+, ν and E are piecewise-constant functions of ξ_2. It is possible to extend our results for the case when these functions are piecewise-smooth in ξ_2.

For fixing the ideas, we will work with the periodicity domains $D_0^- =: D^-$, $D_{0,\varepsilon}^+ =: D_\varepsilon^+$, $D_{0,\varepsilon}^{+,s} =: D_\varepsilon^{+,s}$, $D_{0,\varepsilon} =: D_\varepsilon$ and the periodicity boundaries $\Gamma_0^- =: \Gamma^-$, $\Gamma_{0,\varepsilon}^+ =: \Gamma_\varepsilon^+$, $\Gamma_{0,\varepsilon}^{+,s} =: \Gamma_\varepsilon^{+,s}$, $\Gamma_0^0 =: \Gamma^0$.

For the given forces we begin with the minimum regularity necessary for establishing our first results, which is:

$$\mathbf{h}_\varepsilon \in H^1(0,T;(L^2_\#(D_\varepsilon^+))^2), \mathbf{f} \in H^1(0,T;(L^2_\#(D^-))^2), g \in H^1(0,T;L^2_\#(D^-)). \quad (3.1)$$

Introduce next the following functional spaces:

$$\begin{cases} W_\varepsilon = \left\{ \boldsymbol{\varphi} \in (H^1_\#(D_\varepsilon))^2 \Big/ \operatorname{div} \boldsymbol{\varphi} = 0 \text{ in } D^-, \boldsymbol{\varphi} = \mathbf{0} \text{ on } \Gamma^- \right\}, \\ W^- = \left\{ \boldsymbol{\varphi} \in (H^1_\#(D^-))^2 \Big/ \operatorname{div} \boldsymbol{\varphi} = 0 \text{ in } D^-, \boldsymbol{\varphi} = \mathbf{0} \text{ on } \Gamma^- \right\}, \\ V = \left\{ z \in H^1_\#(D^-) \Big/ z = 0 \text{ on } \Gamma^- \cup \Gamma^0 \right\}, \\ H_{W_\varepsilon} = \left\{ \mathbf{w} \in L^2(0,T;W_\varepsilon) \Big/ \frac{\partial \mathbf{w}}{\partial t} \in L^2(0,T;(L^2_\#(D_\varepsilon))^2) \right\}, \\ H_{W^-} = \left\{ \mathbf{w} \in L^2(0,T;W^-) \Big/ \frac{\partial \mathbf{w}}{\partial t} \in L^2(0,T;(L^2_\#(D^-))^2) \right\}, \\ H_V = \left\{ \omega \in L^2(0,T;V) \Big/ \frac{\partial \omega}{\partial t} \in L^2(0,T;L^2_\#(D^-)) \right\}. \end{cases}$$

Taking into account the properties of the unknowns $\mathbf{u}_\varepsilon, \mathbf{v}_\varepsilon, \omega_\varepsilon$ and the definition of $\mathbf{w}_\varepsilon$, we notice that the spaces W_ε, H_{W_ε} correspond to the unknown $\mathbf{w}_\varepsilon$, the spaces W^-, H_{W^-} correspond to the unknown $\mathbf{v}_\varepsilon$ and the spaces V, H_V correspond to the unknown ω_ε. The additional regularity in time for the unknown functions, included in the definition of the spaces H_{W_ε}, H_V, is a consequence of the regularity of the data with respect to the same variable, given by (3.1).

Remark 4. Here and below the symbol # appearing as index of a space H^s, with $s \in \mathbb{N}^*$ means that this space represents the closure of C^∞-space of functions 1−periodic in x_1 with respect to the norm of the corresponding space. $L^2_\#(D^-) = \{q : L^- \mapsto \mathbb{R}/q \in L^2(D^-), q(x_1+k,x_2) = q(x_1,x_2)$ a.e. for $(x_1,x_2) \in D^-, (\forall)\, k \in \mathbb{Z}\}$ and a similar definition holds also for the other periodicity domains.

Any function F, belonging to a space $H^s_\#$, $s \in \mathbb{N}^*$, can be extended from its periodicity domain, for example D_ε, to the corresponding infinite layer, L_ε, by putting $F(x_1+k,x_2) = F(x_1,x_2)$ a.e. for $(x_1,x_2) \in D_\varepsilon$, $(\forall)\, k \in \mathbb{Z}$ and the extension will receive the same notation as the corresponding function.

3.2 Existence, uniqueness and regularity of the weak solution

The unknowns $\mathbf{v}_\varepsilon, \mathbf{u}_\varepsilon$ being replaced with "the global" unknown $\mathbf{w}_\varepsilon$, we are leaded to the study of the following variational problem

$$\begin{cases} \text{Find } (\mathbf{w}_\varepsilon, \omega_\varepsilon) \in H_{W_\varepsilon} \times H_V \text{ such that} \\ \displaystyle\int_{D_\varepsilon} \rho_\pm \frac{\partial \mathbf{w}_\varepsilon}{\partial t}(t) \cdot \boldsymbol{\varphi} + c_\varepsilon \sum_{i,j=1}^{2} \int_{D_\varepsilon^+} A_{ij} \frac{\partial}{\partial x_j} \left(\int_0^t \mathbf{w}_\varepsilon(s)\, \mathrm{d}s \right) \cdot \frac{\partial \boldsymbol{\varphi}}{\partial x_i} \\ \quad +2(\mu+\chi) \displaystyle\int_{D^-} D(\mathbf{w}_\varepsilon(t)) : D(\boldsymbol{\varphi}) - \chi \int_{D^-} \operatorname{curl} \omega_\varepsilon(t) \cdot \boldsymbol{\varphi} + j \int_{D^-} \frac{\partial \omega_\varepsilon}{\partial t}(t) z \\ \quad +\gamma \displaystyle\int_{D^-} \nabla \omega_\varepsilon(t) \cdot \nabla z + 2\chi \int_{D^-} \omega_\varepsilon(t)\, z - \chi \int_{D^-} z \operatorname{curl} \mathbf{w}_\varepsilon(t) \\ = \displaystyle\int_{D_\varepsilon} \mathbf{f}_\varepsilon(t) \cdot \boldsymbol{\varphi} + \int_{D^-} g(t) z \quad (\forall)\, (\boldsymbol{\varphi}, z) \in W_\varepsilon \times V, \text{ in } L^2(0,T), \\ \mathbf{w}_\varepsilon(0) = \mathbf{0} \text{ in } (L^2(D_\varepsilon))^2, \\ \omega_\varepsilon(0) = 0 \text{ in } L^2(D^-), \end{cases} \tag{3.2}$$

where we denoted

$$\rho_\pm = \rho^+ \chi(D_\varepsilon^+) + \rho^- \chi(D^-), \mathbf{f}_\varepsilon = \chi(D_\varepsilon^+)\, \mathbf{h}_\varepsilon + \chi(D^-)\, \mathbf{f},$$

$\chi(S)$ representing the characteristic function of the set S. As in the case of the elastic structure, $\rho_\pm$ depends on ε.

Since the test function $\boldsymbol{\varphi}$ is defined, as $\mathbf{w}_\varepsilon$, on the whole domain D_ε, the functions that appear in the second and third terms of $(3.2)_2$ represent the restrictions of the corresponding functions to D_ε^+ and D^-, respectively.

Remark 5. Due to the 1−periodicity in x_1 of all functions, the results presented below hold if we replace the particular domains $D^-, D_\varepsilon, D_\varepsilon^+$ and their boundaries with the general ones given by (2.4), (2.5).

The first main result of this section is given by

Theorem 1. *Let us assume that the data have the regularity (3.1). Then the variational problem (3.2) has an unique solution* $(\mathbf{w}_\varepsilon, \omega_\varepsilon)$.

Proof. We begin by obtaining the existence result by means of the Galerkin's method. To this aim, consider $\{\boldsymbol{\varphi}_n\}_{n\in\mathbb{N}^*}$ and $\{z_m\}_{m\in\mathbb{N}^*}$ bases for the separable spaces W_ε and V, respectively and define the approximate functions

$$\begin{cases} \mathbf{w}_n(x,t) = \displaystyle\sum_{k=1}^{n} a_k^n(t) \boldsymbol{\varphi}_k(x), \quad (x,t) \in D_\varepsilon \times (0,T), \\ \omega_m(x,t) = \displaystyle\sum_{i=1}^{m} b_i^m(t) z_i(x), \quad (x,t) \in D^- \times (0,T), \end{cases} \tag{3.3}$$

with a_k^n, $b_i^m : [0,T] \to \mathbb{R}$. These coefficients will be determined by considering for $(\mathbf{w}_n, \omega_m)$ a problem of the same type as (3.2), namely

$$\begin{cases} \displaystyle\int_{D_\varepsilon} \rho_\pm \frac{\partial \mathbf{w}_n}{\partial t}(t)\cdot\boldsymbol{\varphi} + c_\varepsilon \sum_{i,j=1}^{2} \int_{D_\varepsilon^+} A_{ij} \frac{\partial}{\partial x_j} \left(\int_0^t \mathbf{w}_n(s)\,\mathrm{d}s\right)\cdot \frac{\partial \boldsymbol{\varphi}}{\partial x_i} \\ \quad +2(\mu+\chi)\displaystyle\int_{D^-} D(\mathbf{w}_n(t)) : D(\boldsymbol{\varphi}) - \chi\int_{D^-} \operatorname{curl}\omega_m(t)\cdot\boldsymbol{\varphi} + j\int_{D^-} \frac{\partial \omega_m}{\partial t}(t) z \\ \quad +\gamma \displaystyle\int_{D^-} \nabla\omega_m(t)\cdot\nabla z + 2\chi\int_{D^-} \omega_m(t)\, z - \chi\int_{D^-} z \operatorname{curl}\mathbf{w}_n(t) \\ = \displaystyle\int_{D_\varepsilon} \mathbf{f}_\varepsilon(t)\cdot\boldsymbol{\varphi} + \int_{D^-} g(t) z \quad (\forall)\,(\boldsymbol{\varphi}, z)\in W_n\times V_m,\ (\forall)\, t\in[0,T], \\ \mathbf{w}_n(0) = \mathbf{0} \text{ in } W_n,\ \ \omega_m(0) = 0 \text{ in } V_m, \end{cases} \tag{3.4}$$

with $W_n = \operatorname{span}\{\boldsymbol{\varphi}_1, ..., \boldsymbol{\varphi}_n\}$ and $V_m = \operatorname{span}\{z_1, ..., z_m\}$.

Replacing in (3.4) the functions $\mathbf{w}_n, \omega_m$ with their expressions given by (3.3) and taking as test functions $(\boldsymbol{\varphi}_1, 0), ..., (\boldsymbol{\varphi}_n, 0)$; $(\mathbf{0}, z_1), ..., (\mathbf{0}, z_m)$ we obtain a linear system of $n+m$ integro-differential equations for the unknown functions $a_k^n, b_i^m, k = 1, ..., n, i = 1, ..., m$. Denoting

$$c_l^n(t) = \int_0^t a_l^n(s)\mathrm{d}s, \qquad l = 1, ..., n$$

we transform the previous system into a linear system of $2n + m$ differential equations of order 1. We obtain the existence and the uniqueness of the functions $a_k^n, b_i^m, k = 1, ..., n, i = 1, ..., m$ as a consequence of the fact that the matrix of the derivatives is non degenerate, which follows since $\{\boldsymbol{\varphi}_k\}_{k\in\mathbb{N}^*}$ and $\{z_i\}_{i\in\mathbb{N}^*}$ are linear independent systems. In addition, due to the regularity (3.1) of the data, we obtain $a_k^n, b_i^m \in H^2(0,T)$, $(\forall)\, k = 1, ..., n, i = 1, ..., m$.

We obtain next the estimates that will allow us to pass to the limit in the approximate problem. To this aim, we take as test function in $(3.4)_1$ written for a fixed value of $t\in(0,T)$ $(\boldsymbol{\varphi}, z) = (\mathbf{w}_n(t), \omega_m(t))$ and we use the obvious equality

$$\int_{D^-} \operatorname{curl}\,\omega\cdot\mathbf{w} = \int_{D^-} \omega \operatorname{curl}\,\mathbf{w} \qquad (\forall)\ \mathbf{w}\in W_\varepsilon,\ (\forall)\ \omega\in V.$$

Using the classical Young's inequality and integrating from 0 to θ, with $\theta\in(0,T]$, we obtain

$$\begin{aligned} &\int_{D_\varepsilon} \rho_\pm \mathbf{w}_n^2(\theta) + c_\varepsilon \sum_{i,j=1}^{2} \int_{D_\varepsilon^+} A_{ij} \frac{\partial \mathbf{u}_n(\theta)}{\partial x_j}\cdot\frac{\partial \mathbf{u}_n(\theta)}{\partial x_i} + 4(\mu+\chi)\int_0^\theta\int_{D^-} (D(\mathbf{w}_n(\theta)))^2 \\ &+j\int_{D^-} \omega_m^2(\theta) + \gamma\int_0^\theta\int_{D^-} (\nabla\omega_m)^2 + 4\chi\int_0^\theta\int_{D^-} (\omega_m)^2 \le \frac{4\chi^2}{\gamma}\int_0^\theta\int_{D^-} (\mathbf{w}_n)^2 \\ &+\int_0^\theta\int_{D_\varepsilon} (\mathbf{w}_n)^2 + \int_0^\theta\int_{D^-} (\omega_m)^2 + \|\mathbf{f}_\varepsilon\|^2_{L^2(0,T;(L^2_\#(D_\varepsilon))^2)} + \|g\|^2_{L^2(0,T;L^2_\#(D^-))} \\ &(\forall)\, n, m\in\mathbb{N}^*,\ (\forall)\, \theta\in[0,T], \end{aligned} \tag{3.5}$$

where

$$\mathbf{u}_n(x,t) = \int_0^t \mathbf{w}_n(x,s)\mathrm{d}s \quad \text{in } D_\varepsilon^+\times(0,T). \tag{3.6}$$

Using the properties (2.1) and (2.2) of the elastic layer characteristics and applying Gronwall's lemma we obtain from (3.5) a first set of estimates in the form

$$\begin{cases} \|\mathbf{w}_n\|_{L^\infty(0,T;(L^2_\#(D_\varepsilon))^2)} \leq CE(\mathbf{f}_\varepsilon, g), \\ \|D(\mathbf{w}_n)\|_{L^2(0,T;(L^2_\#(D^-))^{2\times 2})} \leq CE(\mathbf{f}_\varepsilon, g), \\ \|\mathcal{E}(\mathbf{u}_n)\|_{L^\infty(0,T;(L^2_\#(D^+_\varepsilon))^{2\times 2})} \leq Cc_\varepsilon^{-1/2}E(\mathbf{f}_\varepsilon, g), \\ \|\omega_m\|_{L^\infty(0,T;L^2_\#(D^-))} \leq CE(\mathbf{f}_\varepsilon, g), \\ \|\nabla\omega_m\|_{L^2(0,T;(L^2_\#(D^-))^2)} \leq CE(\mathbf{f}_\varepsilon, g), \end{cases} \tag{3.7}$$

with $\mathcal{E}$ representing the strain tensor, C a positive constant independent of n, m, ε and $E(\mathbf{f}_\varepsilon, g) = \|\mathbf{f}_\varepsilon\|_{L^2(0,T;(L^2_\#(D_\varepsilon))^2)} + \|g\|_{L^2(0,T;L^2_\#(D^-))}$.

Remark 6. Even if the function $\mathbf{w}_n$ is defined on $D_\varepsilon \times (0,T)$, the previous estimates give its H^1-bounds in x only on D^-. For obtaining H^1-bounds of $\mathbf{w}_n$ on the whole D_ε we need additional estimates that rely on the data regularity (3.1).

For obtaining these additional estimates, we return to problem (3.4) and, using the data regularity (3.1), we derivate $(3.4)_1$ with respect to t. Taking into account the definition (3.3), derivating in t the left hand side of $(3.4)_1$ means to derivate the functions of t, a_k^n, b_i^m, $k = 1, ..., n, i = 1, ..., m$, which is allowed, due to their regularity. In this way we obtain a (3.4)-type problem, with every function depending on t replaced with its time derivative, but with non homogeneous initial conditions for $\dfrac{\partial \mathbf{w}_n}{\partial t}, \dfrac{\partial \omega_n}{\partial t}$. Taking $(\boldsymbol{\varphi}, z) = \left(\dfrac{\partial \mathbf{w}_n(t)}{\partial t}, \dfrac{\partial \omega_n(t)}{\partial t}\right)$ as test function in $\dfrac{\mathrm{d}}{\mathrm{d}t}(3.4)$ and integrating from 0 to θ, $\theta \in (0,T]$ we obtain, instead of (3.5):

$$\begin{cases} \displaystyle\int_{D_\varepsilon} \rho_\pm \left(\frac{\partial \mathbf{w}_n(\theta)}{\partial t}\right)^2 + c_\varepsilon \sum_{i,j=1}^2 \int_{D^+_\varepsilon} A_{ij} \frac{\partial \mathbf{w}_n(\theta)}{\partial x_j} \cdot \frac{\partial \mathbf{w}_n(\theta)}{\partial x_i} \\ \displaystyle\quad +4(\mu+\chi)\int_0^\theta \int_{D^-} \left(D\left(\frac{\partial \mathbf{w}_n}{\partial t}\right)\right)^2 + j\int_{D^-}\left(\frac{\partial \omega_m(\theta)}{\partial t}\right)^2 \\ \displaystyle\quad +\gamma \int_0^\theta\int_{D^-}\left(\nabla\left(\frac{\partial \omega_m}{\partial t}\right)\right)^2 + 4\chi\int_0^\theta\int_{D^-}\left(\frac{\partial \omega_m}{\partial t}\right)^2 \leq \int_{D_\varepsilon}\rho_\pm\left(\frac{\partial \mathbf{w}_n(0)}{\partial t}\right)^2 \\ \displaystyle\quad + j\int_{D^-}\left(\frac{\partial \omega_m(0)}{\partial t}\right)^2 + \frac{4\chi^2}{\gamma}\int_0^\theta\int_{D^-}\left(\frac{\partial \mathbf{w}_n}{\partial t}\right)^2 + \int_0^\theta\int_{D_\varepsilon}\left(\frac{\partial \mathbf{w}_n}{\partial t}\right)^2 \\ \displaystyle\quad + \int_0^\theta\int_{D^-}\left(\frac{\partial \omega_m}{\partial t}\right)^2 + \left\|\frac{\partial \mathbf{f}_\varepsilon}{\partial t}\right\|^2_{L^2(0,T;(L^2_\#(D_\varepsilon))^2)} + \left\|\frac{\partial g}{\partial t}\right\|^2_{L^2(0,T;L^2_\#(D^-))} \\ \qquad\qquad (\forall)\, n, m \in \mathbb{N}^*, (\forall)\, \theta \in [0,T]. \end{cases} \tag{3.8}$$

For estimating the first two terms of the right hand side of (3.8) we make $t = 0$ in $(3.4)_1$, we take as test function $(\boldsymbol{\varphi}, z) = \left(\dfrac{\partial \mathbf{w}_n(0)}{\partial t}, \dfrac{\partial \omega_m(0)}{\partial t}\right)$ and we use $(3.4)_{2,3}$; this yields:

$$\left\|\frac{\partial \mathbf{w}_n(0)}{\partial t}\right\|_{(L^2_\#(D_\varepsilon))^2} + \left\|\frac{\partial \omega_m(0)}{\partial t}\right\|_{L^2_\#(D^-)} \leq CE_1(\mathbf{f}_\varepsilon, g),$$

with $E_1(\mathbf{f}_\varepsilon, g) = \|\mathbf{f}_\varepsilon\|_{H^1(0,T;(L^2_\#(D_\varepsilon))^2)} + \|g\|_{H^1(0,T;L^2_\#(D^-))}$ and C a positive constant independent of n, m, ε. So, we obtain the second set of estimates presented below:

$$\begin{cases} \left\|\dfrac{\partial \mathbf{w}_n}{\partial t}\right\|_{L^\infty(0,T;(L^2_\#(D_\varepsilon))^2)} \le C(\mathbf{f}_\varepsilon, g), \\ \left\|D\left(\dfrac{\partial \mathbf{w}_n}{\partial t}\right)\right\|_{L^2(0,T;(L^2_\#(D^-))^{2\times 2})} \le C(\mathbf{f}_\varepsilon, g), \\ \|\mathcal{E}(\mathbf{w}_n)\|_{L^\infty(0,T;(L^2_\#(D^+_\varepsilon))^{2\times 2})} \le c_\varepsilon^{-1/2} C(\mathbf{f}_\varepsilon, g), \\ \left\|\dfrac{\partial \omega_m}{\partial t}\right\|_{L^\infty(0,T;L^2_\#(D^-))} \le C(\mathbf{f}_\varepsilon, g), \\ \left\|\nabla\left(\dfrac{\partial \omega_m}{\partial t}\right)\right\|_{L^2(0,T;(L^2_\#(D^-))^2)} \le C(\mathbf{f}_\varepsilon, g), \end{cases} \tag{3.9}$$

where

$$C(\mathbf{f}_\varepsilon, g) = C\left(\|\mathbf{f}_\varepsilon\|_{H^1(0,T;(L^2_\#(D_\varepsilon))^2)} + \|g\|_{H^1(0,T;L^2_\#(D^-))}\right),$$

with C independent of n, m and ε.

From $(3.7)_{1,2}$, $(3.9)_{1,2}$ and Korn's inequality in the fluid domain we get

$$\|\mathbf{w}_n\|_{H^1(0,T;(H^1_\#(D^-))^2)} \le C(\mathbf{f}_\varepsilon, g). \tag{3.10}$$

Using next $(3.9)_3$ and applying Korn's inequality in the elastic domain we obtain the missing H^1-regularity which finally gives

$$\|\mathbf{w}_n\|_{L^\infty(0,T;(H^1_\#(D_\varepsilon))^2)} \le C(\mathbf{f}_\varepsilon, g). \tag{3.11}$$

The estimates (3.7) and (3.9) provide us weakly convergent subsequences with respect to the corresponding norms. For any bounded sequence $\{\bullet_n\}_n$ or $\{\bullet_m\}_m$ appearing in (3.7) and (3.9) we will denote by $\{\bullet_{n_q}\}_q$ or $\{\bullet_{m_p}\}_p$ its weakly convergent subsequence in the corresponding space. Consider $\tau \in L^2(0,T)$ an arbitrary function. Let us calculate $\displaystyle\int_0^T (3.4)_1\, \tau \mathrm{d}t$, with $(3.4)_1$ corresponding to the weakly convergent subsequences $\{\bullet_{n_q}\}_q$ and $\{\bullet_{m_p}\}_p$ and to test functions $(\boldsymbol{\varphi}, z) \in W_r \times V_r$, with r fixed:

$$\begin{aligned} &\int_0^T \tau \int_{D_\varepsilon} \rho_\pm \frac{\partial \mathbf{w}_{n_q}}{\partial t} \cdot \boldsymbol{\varphi} + c_\varepsilon \sum_{i,j=1}^2 \int_0^T \tau \int_{D^+_\varepsilon} A_{ij} \frac{\partial \mathbf{u}_{n_q}}{\partial x_j} \cdot \frac{\partial \boldsymbol{\varphi}}{\partial x_i} \\ &+2(\mu+\chi) \int_0^T \tau \int_{D^-} D(\mathbf{w}_{n_q}) : D(\boldsymbol{\varphi}) - \chi \int_0^T \tau \int_{D^-} \mathrm{curl}\ \omega_{m_p} \cdot \boldsymbol{\varphi} \\ &+j \int_0^T \tau \int_{D^-} \frac{\partial \omega_{m_p}}{\partial t} z + \gamma \int_0^T \tau \int_{D^-} \nabla \omega_{m_p} \cdot \nabla z + 2\chi \int_0^T \tau \int_{D^-} \omega_{m_p} z \\ &-\chi \int_0^T \tau \int_{D^-} z\, \mathrm{curl}\ \mathbf{w}_{n_q} = \int_0^T \tau \int_{D_\varepsilon} \mathbf{f}_\varepsilon \cdot \boldsymbol{\varphi} + \int_0^T \tau \int_{D^-} g\, z \\ &\qquad\qquad (\forall)\, (\boldsymbol{\varphi}, z) \in W_r \times V_r,\ (\forall)\, \tau \in L^2(0,T). \end{aligned} \tag{3.12}$$

Remark 7. In the previous relation it was possible to take the test function in $W_r \times V_r$ with r fixed since it is obvious that $(\boldsymbol{\varphi}, z) \in W_r \times V_r$ yields $(\boldsymbol{\varphi}, z) \in W_k \times V_l$ for any $k, l \geq r$. In particular, as $n_q, m_p \to \infty$ when $q, p \to \infty$, it follows that $(\exists)\, q_0, p_0$ such that $(\boldsymbol{\varphi}, z) \in W_{n_q} \times V_{m_p}$ $(\forall)\, q \geq q_0,\, p_q \geq p_0$.

The estimates (3.7), (3.9) and (3.11) provide us all the weak convergences necessary for passing to the limit in (3.12) with $q, p \to \infty$, after discussing the convergence of the second term of (3.12), more precisely the convergence of $\left\{\int_0^t \mathbf{w}_{n_q}(s)\mathrm{d}s\right\}_q = \{\mathbf{u}_{n_q}\}_q$. Denote $\mathbf{w}_\varepsilon$, $\mathbf{u}_\varepsilon$, ω_ε the weak limits of $\{\mathbf{w}_{n_q}\}_q$, $\{\mathbf{u}_{n_q}\}_q$, ω_{m_p}, respectively and define the space

$$H^\infty = \left\{\mathbf{w} \in L^\infty(0,T;(H^1_\#(D_\varepsilon))^2) \Big/ \frac{\partial \mathbf{w}}{\partial t} \in L^\infty(0,T;(L^2_\#(D_\varepsilon))^2)\right\}.$$

From $(3.9)_1$ and (3.11) it follows that $\{\mathbf{w}_n\}_n \subset H^\infty$ and

$$\mathbf{w}_{n_q} \rightharpoonup \mathbf{w}_\varepsilon \text{ weakly star in } H^\infty \text{ as } q \to \infty. \tag{3.13}$$

Using a result of [24] we obtain that the embedding $H^\infty \subset C^0([0,T];(L^2_\#(D_\varepsilon))^2)$ is compact. This property and the convergence (3.13) give (on a subsequence of $\mathbf{w}_{n_q}$, that we denote in the same way, for simplicity)

$$\mathbf{w}_{n_q} \to \mathbf{w}_\varepsilon \text{ strongly in } C^0([0,T];(L^2_\#(D_\varepsilon))^2) \text{ when } q \to \infty \tag{3.14}$$

and so

$$\int_0^t \mathbf{w}_{n_q}(s)\mathrm{d}s \to \int_0^t \mathbf{w}_\varepsilon(s)\mathrm{d}s \text{ strongly in } C^1([0,T];(L^2_\#(D_\varepsilon))^2) \text{ as } q \to \infty. \tag{3.15}$$

Combining (3.15), (3.6) and the weak convergenge in $L^2(0,T;(H^1_\#(D^+_\varepsilon))^2)$ of $\{\mathbf{u}_{n_q}\}_q$ to $\mathbf{u}_\varepsilon$ we get

$$\mathbf{u}_\varepsilon = \int_0^t \mathbf{w}_\varepsilon(s)\mathrm{d}s \text{ in } L^2(0,T;(H^1_\#(D^+_\varepsilon))^2) \cap C^1([0,T];(L^2_\#(D^+_\varepsilon))^2).$$

Moreover, for $q \to \infty$ we also have

$$\frac{\partial \mathbf{u}_{n_q}}{\partial x_j} \rightharpoonup \frac{\partial}{\partial x_j}\left(\int_0^t \mathbf{w}_\varepsilon(s)\mathrm{d}s\right) \text{ weakly in } L^2(0,T;(L^2_\#(D^+_\varepsilon))^2),\ j = 1,2.$$

We are now in a position to pass to the limit in (3.12) with $q, p \to \infty$ and we obtain

$$\begin{aligned}
&\int_0^T \tau \int_{D_\varepsilon} \rho_\pm \frac{\partial \mathbf{w}_\varepsilon}{\partial t} \cdot \boldsymbol{\varphi} + c_\varepsilon \sum_{i,j=1}^2 \int_0^T \tau \int_{D^+_\varepsilon} A_{ij} \frac{\partial}{\partial x_j}\left(\int_0^t \mathbf{w}_\varepsilon(s)\mathrm{d}s\right) \cdot \frac{\partial \boldsymbol{\varphi}}{\partial x_i} \\
&+2(\mu+\chi)\int_0^T \tau \int_{D^-} D(\mathbf{w}_\varepsilon) : D(\boldsymbol{\varphi}) - \chi \int_0^T \tau \int_{D^-} \operatorname{curl} \omega_\varepsilon \cdot \boldsymbol{\varphi} \\
&+j\int_0^T \tau \int_{D^-} \frac{\partial \omega_\varepsilon}{\partial t} z + \gamma \int_0^T \tau \int_{D^-} \nabla \omega_\varepsilon \cdot \nabla z + 2\chi \int_0^T \tau \int_{D^-} \omega_\varepsilon\, z \\
&-\chi \int_0^T \tau \int_{D^-} z \operatorname{curl} \mathbf{w}_\varepsilon = \int_0^T \tau \int_{D_\varepsilon} \mathbf{f}_\varepsilon \cdot \boldsymbol{\varphi} + \int_0^T \tau \int_{D^-} g\, z \\
&\qquad\qquad\qquad\qquad (\forall)\, (\boldsymbol{\varphi}, z) \in W_r \times V_r,\ (\forall)\, \tau \in L^2(0,T).
\end{aligned}$$

Taking into account that τ is an arbitrary function of $L^2(0,T)$ and that $\bigcup_{r\geq 1}(W_r \times V_r)$ is dense in $W_\varepsilon \times V$, it follows that $(\mathbf{w}_\varepsilon, \omega_\varepsilon)$ verifies $(3.2)_{1,2}$. In addition, all the estimates (3.7), (3.9), (3.10) and (3.11) hold for $(\mathbf{w}_\varepsilon, \omega_\varepsilon)$.

For achieving the proof of the existence result it remains to show the initial conditions $(3.2)_{3,4}$. The initial condition $(3.2)_3$ is obtained as a consequence of (3.14) and $\mathbf{w}_{n_q}(0) = \mathbf{0}$ $(\forall)\, n_q$. The initial condition for the microrotation follows in a similar way. Hence $(\mathbf{w}_\varepsilon, \omega_\varepsilon)$ is solution for (3.2).

For obtaining the uniqueness result, consider $(\mathbf{w}^i_\varepsilon, \omega^i_\varepsilon)$, $i = 1, 2$ two solutions of problem (3.2) and denote $(\mathbf{w}_\varepsilon, \omega_\varepsilon) = (\mathbf{w}^1_\varepsilon, \omega^1_\varepsilon) - (\mathbf{w}^2_\varepsilon, \omega^2_\varepsilon)$. Subtracting the relations $(3.2)_2$ corresponding to the two solutions and taking in the resulting equation as test function $(\boldsymbol{\varphi}, z) = (\mathbf{w}_\varepsilon(t), \omega_\varepsilon(t))$, we obtain for $(\mathbf{w}_\varepsilon, \omega_\varepsilon)$ estimates of the same type as (3.7), but now with a right hand side equal to zero. This leads to the desired uniqueness result and the previous convergences, established before only on subsequences, hold now for the whole sequences. □

Remark 8. As an obvious consequence of $(3.2)_1$ we have

$$\mathbf{w}_\varepsilon \in C^0([0,T]; (L^2_\#(D_\varepsilon))^2),\ \omega_\varepsilon \in C^0([0,T]; L^2_\#(D^-)).$$

Taking into account the definition of $\mathbf{w}_\varepsilon$ we have

$$\begin{cases} \mathbf{u}_\varepsilon = \displaystyle\int_0^t \mathbf{w}_\varepsilon(s)ds \quad \text{in } L^+_\varepsilon \times (0,T), \\ \mathbf{v}_\varepsilon = \mathbf{w}_\varepsilon \quad \text{in } L^- \times (0,T), \end{cases} \tag{3.16}$$

relation $(3.16)_1$ corresponding to a homogeneous initial condition for the displacement. The regularity with respect to the space and time variables of the functions with physical meaning, the displacement of the elastic medium and the fluid velocity, is an obvious consequence of (3.16) and of the regularity of "the global" function $\mathbf{w}_\varepsilon$ previously established.

We end of this subsection with a definition that expresses the relation between the physical system and the variational problem presented before.

Definition 1. We say that $(\mathbf{u}_\varepsilon, \mathbf{v}_\varepsilon, \omega_\varepsilon)$ is a weak solution for the coupled system (2.3) if $(\mathbf{w}_\varepsilon, \omega_\varepsilon)$ is the unique solution to the variational problem (3.2) and $\mathbf{u}_\varepsilon, \mathbf{v}_\varepsilon$ are given by (3.16).

3.3 The micropolar fluid pressure

As we can see, the previous definition does not involve the fourth unknown of the problem, namely the fluid pressure p_ε. In this subsection we introduce the fluid pressure and we establish some properties for this unknown.

Lemma 1. *Let $(\mathbf{w}_\varepsilon, \omega_\varepsilon)$ be the unique solution to the variational problem (3.2). Then there exists an unique function $q_{0,\varepsilon} \in L^2(0,T; L^2(D^-))$ such that*

$$\rho^- \frac{\partial \mathbf{v}_\varepsilon}{\partial t} - 2(\mu+\chi) div\ (D(\mathbf{v}_\varepsilon)) - \chi curl \omega_\varepsilon - \mathbf{f} = -\nabla q_{0,\varepsilon} \quad in\ L^2(0,T; (H^{-1}(D^-))^2) \tag{3.17}$$

and

$$\int_{D^-} q_{0,\varepsilon}(t)\,dx = 0 \quad a.e.\ in\ (0,T). \tag{3.18}$$

Proof. Define the space $\mathcal{V} = \{\boldsymbol{\psi} \in (C_0^\infty(D^-))^2/\mathrm{div}\ \boldsymbol{\psi} = 0\}$ and consider $\boldsymbol{\psi} \in \mathcal{V}$. Every function $\boldsymbol{\varphi} = \boldsymbol{\psi}$ in D^-, extended by $\mathbf{0}$ in D_ε^+ and then extended by periodicity to the whole layer L_ε has the property that $(\boldsymbol{\varphi}, 0)$ is a test function for $(3.2)_2$, that gives:

$$\rho^- \int_{D^-} \frac{\partial \mathbf{v}_\varepsilon}{\partial t}(t) \cdot \boldsymbol{\psi} + 2(\mu+\chi)\int_{D^-} D(\mathbf{v}_\varepsilon(t)) : D(\boldsymbol{\psi}) - \chi \int_{D^-} \mathrm{curl}\, \omega_\varepsilon(t) \cdot \boldsymbol{\psi} \\ = \int_{D^-} \mathbf{f}(t) \cdot \boldsymbol{\psi} \quad (\forall)\, \boldsymbol{\psi} \in \mathcal{V}, \ \text{a.e. in } (0,T).$$

De Rham's theorem provides the existence of a distribution $q_\varepsilon(t)$ such that we have:

$$\rho^- \frac{\partial \mathbf{v}_\varepsilon}{\partial t}(t) - 2(\mu+\chi)\mathrm{div}\ (D(\mathbf{v}_\varepsilon))(t) - \chi \mathrm{curl}\, \omega_\varepsilon(t) - \mathbf{f}(t) = -\nabla q_\varepsilon(t) \\ \text{in } (H^{-1}(D^-))^2, \ \text{a.e. in } (0,T). \tag{3.19}$$

The space that appears in (3.19) is given by the regularity of the left hand side of this equality, the term $\mathrm{div}(D(\mathbf{v}_\varepsilon))(t)$ having the lowest regularity, namely $(H^{-1}(D^-))^2$. Hence $\nabla q_\varepsilon(t) \in (H^{-1}(D^-))^2$ a.e. in $(0,T)$ and, using e.g., Proposition 1.2., Chap. 1 of [28], we obtain $q_\varepsilon(t) \in L^2(D^-)$ a.e. in $(0,T)$. Notice that, if $q_\varepsilon(t)$ verifies (3.19) then, for any function λ depending only on t, $q_\varepsilon(t) + \lambda(t)$ verifies (3.19). This means that there exists at least a function $q_{0,\varepsilon}(t) \in L^2(D^-)$ with the property (3.18) that verifies the relation (3.19).

Suppose next that there exist two functions $q^1_{0,\varepsilon}$ and $q^2_{0,\varepsilon}$ with these properties. Since D^- is a connected set, subtracting (3.19) corresponding to these two functions we obtain

$$q^1_{0,\varepsilon} - q^2_{0,\varepsilon} = \alpha(t) \tag{3.20}$$

with α independent of x. The uniqueness of $q_{0,\varepsilon}(t)$ follows integrating (3.20) on D^- and using (3.18).

As a consequence of Proposition 1.2., Chap. 1 of [28] it follows that the gradient operator is an isomorphism from $L^2(D^-)/\mathbb{R}$ into $(H^{-1}(D^-))^2$ which means that the mapping

$$\rho^- \frac{\partial \mathbf{v}_\varepsilon}{\partial t}(t) - 2(\mu+\chi)\mathrm{div}\ (D(\mathbf{v}_\varepsilon))(t) - \chi \mathrm{curl}\, \omega_\varepsilon(t) - \mathbf{f}(t) \mapsto q_{0,\varepsilon}(t)$$

is linear and continuous from $(H^{-1}(D^-))^2$ into $L^2(D^-)/\mathbb{R}$ and, since

$$\rho^- \frac{\partial \mathbf{v}_\varepsilon}{\partial t} - 2(\mu+\chi)\mathrm{div}\ (D(\mathbf{v}_\varepsilon)) - \chi \mathrm{curl}\, \omega_\varepsilon - \mathbf{f} \in L^2(0,T;(H^{-1}(D^-))^2),$$

it follows that $q_{0,\varepsilon} \in L^2(0,T;L^2(D^-))$. $\square$

In what follows we extend the unique function $q_{0,\varepsilon}$, defined on D^-, to the whole layer L^-. We recall that all the other unknowns and given functions are defined on the infinite layers, according to Remark 4. To this aim, we take as test function in $(3.2)_2$ $\boldsymbol{\varphi} \in W_\varepsilon$, $\boldsymbol{\varphi} = \mathbf{0}$ in $\mathbb{R}^2 \backslash L^-$ and, using the 1-periodicity in x_1 of all functions, we get a.e. in $(0,T)$, $(\forall)\,\boldsymbol{\varphi}$ with the previous properties

$$\rho^- \int_{D_k^-} \frac{\partial \mathbf{v}_\varepsilon}{\partial t}(t)\cdot\boldsymbol{\varphi} + 2(\mu+\chi)\int_{D_k^-} D(\mathbf{v}_\varepsilon(t)) : D(\boldsymbol{\varphi}) - \chi\int_{D_k^-} \mathrm{curl}\,\omega_\varepsilon(t)\cdot\boldsymbol{\varphi} = \int_{D_k^-} \mathbf{f}(t)\cdot\boldsymbol{\varphi},$$

where D_k^- is defined by $(2.4)_1$, $k \in \mathbb{Z}$. Proceeding as before, we introduce the unique function $q_{0,\varepsilon}^k$ with the properties

$$\rho^- \frac{\partial \mathbf{v}_\varepsilon}{\partial t} - 2(\mu+\chi)\mathrm{div}\,(D(\mathbf{v}_\varepsilon)) - \chi \mathrm{curl}\,\omega_\varepsilon - \mathbf{f} = -\nabla q_{0,\varepsilon}^k \quad \text{in } L^2(0,T;(H^{-1}(D_k^-))^2), \tag{3.21}$$

$$q_{0,\varepsilon}^k \in L^2(0,T;L^2(D_k^-)), \tag{3.22}$$

$$\int_{D_k^-} q_{0,\varepsilon}^k(t)\mathrm{d}x = 0 \quad \text{a.e. in } (0,T). \tag{3.23}$$

Lemma 2. *The function $q_{0,\varepsilon}^k$ has the property*

$$q_{0,\varepsilon}^k(x_1,x_2,t) = q_{0,\varepsilon}(x_1-k,x_2,t) \quad \textit{a.e. for } (x_1,x_2,t) \in D_k^- \times (0,T).$$

Proof. From (3.17) we infer that:

$$\rho^-\int_{D^-} \frac{\partial \mathbf{v}_\varepsilon}{\partial t}(t)\cdot\boldsymbol{\varphi} + 2(\mu+\chi)\int_{D^-} D(\mathbf{v}_\varepsilon(t)) : D(\boldsymbol{\varphi}) - \int_{D^-} q_{0,\varepsilon}(t)\,\mathrm{div}\,\boldsymbol{\varphi} - \chi\int_{D^-} \mathrm{curl}\,\omega_\varepsilon(t)\cdot\boldsymbol{\varphi} = \int_{D^-} \mathbf{f}(t)\cdot\boldsymbol{\varphi}\ (\forall)\,\boldsymbol{\varphi} \in (H_0^1(D^-))^2, \text{ a.e. in } (0,T). \tag{3.24}$$

Let us suppose that the variable of integration in (3.24) is denoted by (y_1,y_2). With the change of variable $(y_1,y_2) = (x_1-k,x_2)$ and using the periodicity in x_1 of the functions $\mathbf{v}_\varepsilon, \omega_\varepsilon, \mathbf{f}, \boldsymbol{\varphi}$ the previous relation becomes

$$\rho^-\int_{D_k^-} \frac{\partial \mathbf{v}_\varepsilon}{\partial t}(t)\cdot\boldsymbol{\varphi} + 2(\mu+\chi)\int_{D_k^-} D(\mathbf{v}_\varepsilon(t)) : D(\boldsymbol{\varphi}) - \int_{D_k^-} q_{0,\varepsilon}(x_1-k,x_2,t)\,\mathrm{div}\boldsymbol{\varphi} - \chi\int_{D_k^-} \mathrm{curl}\,\omega_\varepsilon(t)\cdot\boldsymbol{\varphi} = \int_{D_k^-} \mathbf{f}(t)\cdot\boldsymbol{\varphi}\ \ (\forall)\,\boldsymbol{\varphi} \in (H_0^1(D_k^-))^2, \text{ a.e. in } (0,T).$$

Denoting $q_{0,\varepsilon,-k}(x_1,x_2,t) = q_{0,\varepsilon}(x_1-k,x_2,t)$ a.e. for $(x_1,x_2,t) \in D_k^- \times (0,T)$, the previous relation gives

$$\rho^- \frac{\partial \mathbf{v}_\varepsilon}{\partial t} - 2(\mu+\chi)\mathrm{div}\,(D(\mathbf{v}_\varepsilon)) - \chi \mathrm{curl}\,\omega_\varepsilon - \mathbf{f} = -\nabla q_{0,\varepsilon,-k} \quad \text{in } L^2(0,T;(H^{-1}(D_k^-))^2). \tag{3.25}$$

Moreover

$$q_{0,\varepsilon,-k} \in L^2(0,T;L^2(D_k^-)) \tag{3.26}$$

from the definition of this function and

$$\int_{D_k^-} q_{0,\varepsilon,-k}(t)\mathrm{d}x = \int_{D^-} q_{0,\varepsilon,}(t)\mathrm{d}x = 0. \tag{3.27}$$

We notice that (3.25)–(3.27) represent (3.21)–(3.23) that gives, together with the uniqueness of $q^k_{0,\varepsilon}$, $q^k_{0,\varepsilon} = q_{0,\varepsilon,-k}$ and the proof is completed. □

In this way we obtained an unique function $q_{0,\varepsilon} : L^- \times (0,T) \mapsto \mathbb{R}$ with the properties

$$\rho^- \frac{\partial \mathbf{v}_\varepsilon}{\partial t} - 2(\mu+\chi)\mathrm{div}\ (D(\mathbf{v}_\varepsilon)) - \chi \mathrm{curl}\, \omega_\varepsilon - \mathbf{f} = -\nabla q_{0,\varepsilon} \quad \text{in } L^2(0,T;(H^{-1}(D_k^-))^2),$$

$$q_{0,\varepsilon} \in L^2(0,T;L^2_\#(D^-)), \quad \int_{D_k^-} q_{0,\varepsilon}(t)\mathrm{d}x = 0 \quad \text{a.e. in } (0,T)$$

for all $k \in \mathbb{Z}$.

4 Improvement of the regularity with respect to the space variable

In order to obtain more properties of the unknown functions and return from the variational problem (3.2) to the physical coupled system (2.3), it is necessary to increase the regularity of the data.

4.1 Regularity of microrotation, velocity, pressure

Remark 9. If the regularity of the data is given by (3.1), then the maximum regularity for $\mathbf{w}_\varepsilon$ and q_ε in space is given by

$$\mathbf{w}_\varepsilon \in H_{W_\varepsilon},\ q_{0,\varepsilon} \in L^2(0,T;L^2_\#(D^-)). \tag{4.1}$$

Unlike for the velocity and the pressure, the microrotation ω_ε is more regular than $\omega_\varepsilon \in H_V$. Indeed, for $\boldsymbol{\varphi} = \mathbf{0}$ in $(3.2)_2$ the problem for ω_ε may be written as:

$$\begin{cases} -\gamma \Delta \omega_\varepsilon = g - j\dfrac{\partial \omega_\varepsilon}{\partial t} - 2\chi\omega_\varepsilon + \chi \mathrm{curl}\ \mathbf{v}_\varepsilon & \text{in } D^- \times (0,T), \\ \omega_\varepsilon = 0 & \text{on } (\Gamma^0 \cup \Gamma^-) \times (0,T), \\ \omega_\varepsilon & 1-\text{periodic in } x_1, \end{cases} \tag{4.2}$$

with the regularity of the right hand side of $(4.2)_1$, $L^2(0,T;L^2_\#(D^-))$, given by Theorem 1.

Applying for (4.2) classical regularity results for Poisson's equation with periodicity conditions in x_1 and homogeneous Dirichlet boundary conditions on $\Gamma^0 \cup \Gamma^-$ we get $\omega_\varepsilon \in L^2(0,T;H^2_\#(D^-))$.

For obtaining more regularity for $\mathbf{w}_\varepsilon$ and $q_{0,\varepsilon}$ it is necessary to consider smoother data. In what follows we suppose that the data satisfy, in addition to (3.1)

$$\frac{\partial \mathbf{h}_\varepsilon}{\partial x_1} \in H^1(0,T;(L^2_\#(D^+_\varepsilon))^2), \quad \frac{\partial \mathbf{f}}{\partial x_1} \in H^1(0,T;(L^2_\#(D^-))^2),$$
$$\frac{\partial g}{\partial x_1} \in H^1(0,T;L^2_\#(D^-)). \tag{4.3}$$

The first result relying on this improvement of the data regularity is given by

Theorem 2. *If the data have the regularity (3.1) and (4.3),* $(\mathbf{w}_\varepsilon, \omega_\varepsilon)$ *is the unique solution to (3.2) and* $q_{0,\varepsilon}$ *is the unique function introduced in Lemma 1, then*

$$\frac{\partial \mathbf{w}_\varepsilon}{\partial x_1} \in H_{W_\varepsilon}, \quad \frac{\partial q_{0,\varepsilon}}{\partial x_1} \in L^2(0,T;L^2(D^-)). \tag{4.4}$$

Proof. Let $h > 0$ be a small parameter. Let us consider the domains $D^+_{h,\varepsilon}$, D^-_h, $D_{h,\varepsilon}$ given by (2.4), obtained from D^+_ε, D^-, D_ε, respectively, with a shift on x_1 axis. For any arbitrary element $(\boldsymbol{\varphi}, z) \in W_\varepsilon \times V$ we use in what follows the same notation for the functions extended by periodicity to the infinite layers (see Remark 4). If we define $\boldsymbol{\psi}(x_1,x_2) = \boldsymbol{\varphi}(x_1+h,x_2)$ a.e. for $(x_1,x_2) \in L_\varepsilon$, $\zeta(x_1,x_2) = z(x_1+h,x_2)$ a.e. for $(x_1,x_2) \in L^-$, then $(\boldsymbol{\psi},\zeta) \in W_\varepsilon \times V$. We take in $(3.2)_2$ as test function $(\boldsymbol{\psi},\zeta)$ and, denoting $(x_1,x_2)=x$, $(x_1+h,x_2)=x_{+h}$, $(y_1-h,x_2)=y_{-h}$, we obtain:

$$\begin{aligned}
&\int_{D_\varepsilon} \rho_\pm \frac{\partial \mathbf{w}_\varepsilon}{\partial t}(x,t)\cdot\boldsymbol{\varphi}(x_{+h}) + c_\varepsilon \sum_{i,j=1}^{2} \int_{D^+_\varepsilon} A_{ij} \frac{\partial}{\partial x_j}\left(\int_0^t \mathbf{w}_\varepsilon(x,s)\,\mathrm{d}s\right)\cdot\frac{\partial \boldsymbol{\varphi}}{\partial x_i}(x_{+h}) \\
&\quad +2(\mu+\chi)\int_{D^-} D(\mathbf{w}_\varepsilon(x,t)) : D(\boldsymbol{\varphi}(x_{+h})) - \chi\int_{D^-} \operatorname{curl}\omega_\varepsilon(x,t)\cdot\boldsymbol{\varphi}(x_{+h}) \\
&\quad +j\int_{D^-}\frac{\partial \omega_\varepsilon}{\partial t}(x,t)z(x_{+h}) + \gamma\int_{D^-}\nabla\omega_\varepsilon(x,t)\cdot\nabla z(x_{+h}) + 2\chi\int_{D^-}\omega_\varepsilon(x,t)\,z(x_{+h}) \\
&\quad -\chi\int_{D^-} z(x_{+h})\operatorname{curl}\mathbf{w}_\varepsilon(x,t) = \int_{D_\varepsilon}\mathbf{f}_\varepsilon(x,t)\cdot\boldsymbol{\varphi}(x_{+h}) + \int_{D^-} g(x,t)z(x_{+h}).
\end{aligned} \tag{4.5}$$

We make the change of variable $x_1+h=y_1$ in (4.5), we denote $f(y_1,x_2)=f_{y,x}$ for any function f and, taking into account that the densities and the matrix-valued elasticity coefficients are independent of x_1 it follows that:

$$\begin{aligned}
&\int_{D_{h,\varepsilon}} \rho_\pm \frac{\partial \mathbf{w}_\varepsilon}{\partial t}(y_{-h},t)\cdot\boldsymbol{\varphi}_{y,x} + c_\varepsilon\Big(\int_{D^+_{h,\varepsilon}} A_{11}\frac{\partial}{\partial y_1}\Big(\int_0^t \mathbf{w}_\varepsilon(y_{-h},s)\,\mathrm{d}s\Big)\cdot\frac{\partial \boldsymbol{\varphi}_{y,x}}{\partial y_1} \\
&\quad +A_{12}\frac{\partial}{\partial x_2}\Big(\int_0^t \mathbf{w}_\varepsilon(y_{-h},s)\,\mathrm{d}s\Big)\cdot\frac{\partial \boldsymbol{\varphi}_{y,x}}{\partial y_1} + A_{21}\frac{\partial}{\partial y_1}\Big(\int_0^t \mathbf{w}_\varepsilon(y_{-h},s)\,\mathrm{d}s\Big)\cdot\frac{\partial \boldsymbol{\varphi}_{y,x}}{\partial x_2} \\
&\quad + A_{22}\frac{\partial}{\partial x_2}\Big(\int_0^t \mathbf{w}_\varepsilon(y_{-h},s)\,\mathrm{d}s\Big)\cdot\frac{\partial \boldsymbol{\varphi}_{y,x}}{\partial x_2}\Big) \\
&\quad +2(\mu+\chi)\int_{D^-_h} D_{y_1,x_2}\mathbf{w}_\varepsilon(y_{-h},t) : D_{y_1,x_2}\boldsymbol{\varphi}_{y,x} - \chi\int_{D^-_h}\operatorname{curl}_{y_1,x_2}\omega_\varepsilon(y_{-h},t)\cdot\boldsymbol{\varphi}_{y,x}
\end{aligned}$$

$$
\begin{aligned}
&+j\int_{D_h^-}\frac{\partial\omega_\varepsilon}{\partial t}(y_{-h},t)z_{y,x}+\gamma\int_{D_h^-}\nabla_{y_1,x_2}\omega_\varepsilon(y_{-h},t)\cdot\nabla_{y_1,x_2}z_{y,x}\\
&+2\chi\int_{D_h^-}\omega_\varepsilon(y_{-h},t)\,z_{y,x}-\chi\int_{D_h^-}z_{y,x}\,\mathrm{curl}_{y_1,x_2}\mathbf{w}_\varepsilon(y_{-h},t)\\
&=\int_{D_{h,\varepsilon}}\mathbf{f}_\varepsilon(y_{-h},t)\cdot\boldsymbol{\varphi}_{y,x}+\int_{D_h^-}g(y_{-h},t)z_{y,x}.
\end{aligned}
\tag{4.6}
$$

Using next the $1-$periodicity in x_1 we show that

$$
\int_{\diamond_h}\star\,\mathrm{d}y_1\mathrm{d}x_2=\int_{\diamond}\star\,\mathrm{d}x_1\mathrm{d}x_2,
\tag{4.7}
$$

where $\diamond$ is D_ε^+, D^- or D_ε and $\star$ is any integrand from (4.6). Let us show (4.7), e.g., for

$$
\begin{aligned}
&\int_{D_h^-}D_{y_1,x_2}(\mathbf{w}_\varepsilon(y_{-h},t)):D_{y_1,x_2}(\boldsymbol{\varphi}(y_1,x_2))\mathrm{d}y_1\mathrm{d}x_2\\
&\quad=\int_h^1\int_{-1}^0D_{y_1,x_2}(\mathbf{w}_\varepsilon(y_{-h},t)):D_{y_1,x_2}(\boldsymbol{\varphi}(y_1,x_2))\mathrm{d}y_1\mathrm{d}x_2\\
&\quad\quad+\int_1^{1+h}\int_{-1}^0D_{y_1,x_2}(\mathbf{w}_\varepsilon(y_{-h},t)):D_{y_1,x_2}(\boldsymbol{\varphi}(y_1,x_2))\mathrm{d}y_1\mathrm{d}x_2\\
&\quad=\int_h^1\int_{-1}^0D_{x_1,x_2}(\mathbf{w}_\varepsilon(x_1-h,x_2,t)):D_{x_1,x_2}(\boldsymbol{\varphi}(x_1,x_2))\mathrm{d}x\\
&\quad\quad+\int_0^h\int_{-1}^0D_{x_1,x_2}(\mathbf{w}_\varepsilon(x_1+1-h,x_2,t)):D_{x_1,x_2}(\boldsymbol{\varphi}(x_1+1,x_2))\mathrm{d}x\\
&\quad=\int_{D^-}D_{x_1,x_2}(\mathbf{w}_\varepsilon(x_1-h,x_2,t)):D_{x_1,x_2}(\boldsymbol{\varphi}(x_1,x_2))\mathrm{d}x_1\mathrm{d}x_2,
\end{aligned}
$$

due to the $1-$periodicity in x_1. With these calculations (4.6) becomes

$$
\begin{aligned}
&\int_{D_\varepsilon}\rho_\pm\frac{\partial\mathbf{w}_\varepsilon}{\partial t}(x_{-h},t)\cdot\boldsymbol{\varphi}(x)+c_\varepsilon\sum_{i,j=1}^2\int_{D_\varepsilon^+}A_{ij}\frac{\partial}{\partial x_j}\Big(\int_0^t\mathbf{w}_\varepsilon(x_{-h},s)\,\mathrm{d}s\Big)\cdot\frac{\partial\boldsymbol{\varphi}}{\partial x_i}(x)\\
&+2(\mu+\chi)\int_{D^-}D(\mathbf{w}_\varepsilon(x_{-h},t)):D(\boldsymbol{\varphi}(x))-\chi\int_{D^-}\mathrm{curl}\,\omega_\varepsilon(x_{-h},t)\cdot\boldsymbol{\varphi}(x)\\
&+j\int_{D^-}\frac{\partial\omega_\varepsilon}{\partial t}(x_{-h},t)z(x)+\gamma\int_{D^-}\nabla\omega_\varepsilon(x_{-h},t)\cdot\nabla z(x)+2\chi\int_{D^-}\omega_\varepsilon(x_{-h},t)\,z(x)\\
&-\chi\int_{D^-}z(x)\,\mathrm{curl}\,\mathbf{w}_\varepsilon(x_{-h},t)=\int_{D_\varepsilon}\mathbf{f}_\varepsilon(x_{-h},t)\cdot\boldsymbol{\varphi}(x)+\int_{D^-}g(x_{-h},t)z(x).
\end{aligned}
\tag{4.8}
$$

Denote

$$
\star_h=\frac{\star(x_1,x_2,t)-\star(x_1-h,x_2,t)}{h},
$$

where $\star$ is a known or unknowm function. Calculating next $\frac{(3.2)_2-(4.8)}{h}$ and using the previous notation we get:

$$\begin{aligned}&\int_{D_\varepsilon} \rho_\pm \frac{\partial \mathbf{w}_{\varepsilon,h}}{\partial t}(t)\cdot\boldsymbol{\varphi} + c_\varepsilon \sum_{i,j=1}^{2}\int_{D_\varepsilon^+} A_{ij}\frac{\partial}{\partial x_j}\left(\int_0^t \mathbf{w}_{\varepsilon,h}(s)\,\mathrm{d}s\right)\cdot\frac{\partial\boldsymbol{\varphi}}{\partial x_i}\\
&+2(\mu+\chi)\int_{D^-} D(\mathbf{w}_{\varepsilon,h}(t)) : D(\boldsymbol{\varphi}) - \chi\int_{D^-}\operatorname{curl}\omega_{\varepsilon,h}(t)\cdot\boldsymbol{\varphi} + j\int_{D^-}\frac{\partial\omega_{\varepsilon,h}}{\partial t}(t)z\\
&+\gamma\int_{D^-}\nabla\omega_{\varepsilon,h}(t)\cdot\nabla z + 2\chi\int_{D^-}\omega_{\varepsilon,h}(t)\,z - \chi\int_{D^-} z\operatorname{curl}\mathbf{w}_{\varepsilon,h}(t)\\
&=\int_{D_\varepsilon}\mathbf{f}_{\varepsilon,h}(t)\cdot\boldsymbol{\varphi} + \int_{D^-} g_h(t)z \quad (\forall)\,(\boldsymbol{\varphi},z)\in W_\varepsilon\times V,\ \text{in } L^2(0,T).\end{aligned} \tag{4.9}$$

On the other hand, let us consider the variational problem (3.2) corresponding to the data $\left(\frac{\partial \mathbf{h}_\varepsilon}{\partial x_1}, \frac{\partial \mathbf{f}}{\partial x_1}, \frac{\partial g}{\partial x_1}\right)$. Due to (4.3), these new data have at least the regularity (3.1) and the unique solution of (3.2) corresponding to these new data, denoted $(\mathbf{w}_\varepsilon^*, \omega_\varepsilon^*)$, has the regularity given by $(3.2)_1$. Calculating $(3.2)_2$ corresponding to $\left(\frac{\partial \mathbf{h}_\varepsilon}{\partial x_1}, \frac{\partial \mathbf{f}}{\partial x_1}, \frac{\partial g}{\partial x_1}\right)$–(4.9), using the estimates obtained in the proof of Theorem 1 and taking into account that

$$\begin{aligned}&\lim_{h\to 0}\left\|\mathbf{h}_{\varepsilon,h} - \frac{\partial \mathbf{h}_\varepsilon}{\partial x_1}\right\|_{L^2(0,T;(L^2_\#(D_\varepsilon^+))^2)} = 0,\\
&\lim_{h\to 0}\left\|\mathbf{f}_h - \frac{\partial \mathbf{f}}{\partial x_1}\right\|_{L^2(0,T;(L^2_\#(D^-)^2)} = 0,\quad \lim_{h\to 0}\left\|g_h - \frac{\partial g}{\partial x_1}\right\|_{L^2(0,T;L^2_\#(D^-))} = 0,\end{aligned}$$

we obtain $\mathbf{w}_{\varepsilon,h}\to\mathbf{w}_\varepsilon^*$, $\omega_{\varepsilon,h}\to\omega_\varepsilon^*$ as $h\to 0$ strongly with respect to all norms appearing in the estimates established in the proof of Theorem 1. This means that

$$\left(\frac{\partial \mathbf{w}_\varepsilon}{\partial x_1}, \frac{\partial \omega_\varepsilon}{\partial x_1}\right) = (\mathbf{w}_\varepsilon^*, \omega_\varepsilon^*) \in H_{W_\varepsilon}\times H_V. \tag{4.10}$$

Consider next $\boldsymbol{\psi}\in (C_0^\infty(D^-))^2$; then we can take $\boldsymbol{\varphi} = \frac{\partial \boldsymbol{\psi}}{\partial x_1}$ in (3.24) and, integrating by parts all the terms and using (4.10), we obtain

$$\begin{aligned}\rho^-\frac{\partial \mathbf{v}_\varepsilon^*}{\partial t}(t) - 2(\mu+\chi)\operatorname{div}(D(\mathbf{v}_\varepsilon^*(t)) - \chi\operatorname{curl}\omega_\varepsilon^*(t) - \frac{\partial \mathbf{f}}{\partial x_1}(t) = -\nabla\frac{\partial q_{0,\varepsilon}}{\partial x_1}(t)\\
\text{in } (H^{-1}(D^-))^2,\ \text{a.e. in } (0,T),\end{aligned} \tag{4.11}$$

where $\mathbf{v}_\varepsilon^* = \mathbf{w}^*_{\varepsilon/_{D^-\times(0,T)}}$. This gives $\frac{\partial q_{0,\varepsilon}}{\partial x_1}(t)\in L^2(D^-)$, a.e. in $(0,T)$. On the other hand, considering the problem (3.2) that corresponds to the data $\left(\frac{\partial \mathbf{h}_\varepsilon}{\partial x_1}, \frac{\partial \mathbf{f}}{\partial x_1}, \frac{\partial g}{\partial x_1}\right)$ and proceeding as in Lemma 1 we obtain the existence of an unique $q_{0,\varepsilon}^*\in L^2(0,T;L^2(D^-))$ such that:

$$\begin{aligned}&\rho^-\frac{\partial \mathbf{v}_\varepsilon^*}{\partial t}(t) - 2(\mu+\chi)\operatorname{div}(D(\mathbf{v}_\varepsilon^*(t)) - \chi\operatorname{curl}\omega_\varepsilon^*(t) - \frac{\partial \mathbf{f}}{\partial x_1}(t) = -\nabla q_{0,\varepsilon}^*(t)\\
&\qquad\qquad \text{in } L^2(0,T;(H^{-1}(D^-))^2),\\
&\int_{D^-} q_{0,\varepsilon}^*(t)\mathrm{d}x = 0 \text{ a.e. in } (0,T).\end{aligned} \tag{4.12}$$

Subtracting (4.11) and (4.12) and taking into account that D^- is a connected set it follows that for some $\alpha = \alpha(t)$ we have

$$\frac{\partial q_{0,\varepsilon}}{\partial x_1} = q^*_{0,\varepsilon} + \alpha \text{ in } L^2(D^-), \text{ a.e. in } (0,T). \tag{4.13}$$

We establish in what follows the regularity of the function α. To this aim, let us take $\boldsymbol{\varphi} = \varphi_1 \mathbf{e}_1$, $\varphi_1 \in H^1_0(D^-)$ as test function in (3.24). For the term that contains the pressure we proceed as follows:

$$-\int_{D^-} q_{0,\varepsilon}(t)\mathrm{div}\boldsymbol{\varphi} = -\int_{D^-} q_{0,\varepsilon}(t)\frac{\partial \varphi_1}{\partial x_1} = \left\langle \frac{\partial q_{0,\varepsilon}(t)}{\partial x_1}, \varphi_1 \right\rangle_{H^{-1}(D^-),H^1_0(D^-)}$$
$$= \int_{D^-} \frac{\partial q_{0,\varepsilon}(t)}{\partial x_1}\varphi_1,$$

the last equality being a consequence of the regularity given by (4.13). Using again (4.13) we get:

$$\rho^- \int_{D^-} \frac{\partial (v_\varepsilon)_1}{\partial t}(t)\varphi_1 + 2(\mu+\chi)\int_{D^-} D(\mathbf{v}_\varepsilon(t)) : D(\varphi_1 \mathbf{e}_1) - \chi \int_{D^-} \frac{\partial \omega_\varepsilon}{\partial x_2}(t)\varphi_1$$
$$-\int_{D^-} f_1(t)\varphi_1 = -\int_{D^-} \frac{\partial q_{0,\varepsilon}}{\partial x_1}(t)\varphi_1 = -\int_{D^-} \left(q^*_{0,\varepsilon}(t) + \alpha(t)\right)\varphi_1 \text{ a.e. in } (0,T)$$

or

$$-\alpha(t)\int_{D^-} \varphi_1 = \rho^- \int_{D^-} \frac{\partial (v_\varepsilon)_1}{\partial t}(t)\varphi_1 + 2(\mu+\chi)\int_{D^-} D(\mathbf{v}_\varepsilon(t)) : D(\varphi_1 \mathbf{e}_1)$$
$$-\chi \int_{D^-} \frac{\partial \omega_\varepsilon}{\partial x_2}(t)\varphi_1 - \int_{D^-} f_1(t)\varphi_1 + \int_{D^-} q^*_{0,\varepsilon}(t)\varphi_1 \ (\forall)\, \varphi_1 \in H^1_0(D^-), \tag{4.14}$$
$$\text{a.e. in } (0,T).$$

Choosing $\varphi_1 \in H^1_0(D^-)$ with $\int_{D^-} \varphi_1 \neq 0$ and taking into account that the right hand side of (4.14) is an element of $L^2(0,T)$ from $(3.2)_1$, $(3.1)_2$ and the regularity of $q^*_{0,\varepsilon}$, we conclude that $\alpha \in L^2(0,T)$ which gives, together with (4.13), the regularity $(4.4)_2$ that completes the proof. □

The main results concerning the regularity of the unknown functions are given by the next two theorems.

Theorem 3. *Let* $(\mathbf{w}_\varepsilon, \omega_\varepsilon)$ *be the unique solution to (3.2),* $\mathbf{v}_\varepsilon$ *defined by* $(3.16)_2$ *and* $q_{0,\varepsilon}$ *the unique function introduced in Lemma 1. Then,*

$$\mathbf{v}_\varepsilon \in L^2(0,T;(H^2(D^-))^2), \quad q_{0,\varepsilon} \in L^2(0,T;H^1(D^-)). \tag{4.15}$$

Proof. Notice that, due to $(4.4)_1$, it follows that $\frac{\partial \mathbf{v}_\varepsilon}{\partial x_1} \in L^2(0,T;(H^1_\#(D^-))^2)$; so, for obtaining $(4.15)_1$, it remains to show that $\frac{\partial^2 \mathbf{v}_\varepsilon}{\partial x_2^2} \in L^2(0,T;(L^2(D^-))^2)$. Similarly, due to $(4.4)_2$, $(4.15)_2$ is satisfied if $\frac{\partial q_{0,\varepsilon}}{\partial x_2} \in L^2(0,T;L^2(D^-))$. For obtaining the desired regularity for the velocity and for the pressure, we replace the vectorial equation (3.17) with two scalar equations in $L^2(0,T;H^{-1}(D^-))$, written in a convenient way, as below:

$$\begin{aligned}
-(\mu+\chi)\frac{\partial^2(v_\varepsilon)_1}{\partial x_2^2} &= -\rho^-\frac{\partial(v_\varepsilon)_1}{\partial t} + (\mu+\chi)\frac{\partial^2(v_\varepsilon)_1}{\partial x_1^2} + \chi\frac{\partial\omega_\varepsilon}{\partial x_2} + f_1 - \frac{\partial q_{0,\varepsilon}}{\partial x_1} \\
-(\mu+\chi)\frac{\partial^2(v_\varepsilon)_2}{\partial x_2^2} + \frac{\partial q_{0,\varepsilon}}{\partial x_2} &= -\rho^-\frac{\partial(v_\varepsilon)_2}{\partial t} + (\mu+\chi)\frac{\partial^2(v_\varepsilon)_2}{\partial x_1^2} - \chi\frac{\partial\omega_\varepsilon}{\partial x_1} + f_2
\end{aligned} \tag{4.16}$$

For passing from (3.17) to (4.16) we took into account that $2\mathrm{div}(D(\mathbf{v})) = \Delta\mathbf{v}$ if $\mathrm{div}\,\mathbf{v} = 0$.

Using the regularity provided by Theorem 1 and by (4.4) it follows that the right hand side of $(4.16)_1$ is an element of $L^2(0,T;L^2(D^-))$, that gives

$$\frac{\partial^2(v_\varepsilon)_1}{\partial x_2^2} \in L^2(0,T;L^2(D^-))$$

and so

$$(v_\varepsilon)_1 \in L^2(0,T;H^2(D^-)). \tag{4.17}$$

For obtaining the same regularity for the second component of the velocity it remains to prove that $\frac{\partial^2(v_\varepsilon)_2}{\partial x_2^2} \in L^2(0,T;L^2(D^-))$ since we already know that $\frac{\partial(v_\varepsilon)_2}{\partial x_1} \in L^2(0,T;H^1(D^-))$ from $(4.4)_1$. Consider $\Psi \in L^2(0,T;H_0^1(D^-))$ and denote $\langle\cdot,\cdot\rangle$ the duality pairing between $L^2(0,T;H^{-1}(D^-))$ and $L^2(0,T;H_0^1(D^-))$. Then,

$$\left\langle \frac{\partial^2(v_\varepsilon)_2}{\partial x_2^2}, \Psi \right\rangle = -\left\langle \frac{\partial(v_\varepsilon)_2}{\partial x_2}, \frac{\partial\Psi}{\partial x_2} \right\rangle = \left\langle \frac{\partial(v_\varepsilon)_1}{\partial x_1}, \frac{\partial\Psi}{\partial x_2} \right\rangle = -\left\langle \frac{\partial^2(v_\varepsilon)_1}{\partial x_1\partial x_2}, \Psi \right\rangle$$

$$(\forall)\,\Psi \in L^2(0,T;H_0^1(D^-)),$$

since $\mathrm{div}\,\mathbf{v}_\varepsilon = 0$ in D^-. This means that

$$\frac{\partial^2(v_\varepsilon)_2}{\partial x_2^2} = -\frac{\partial^2(v_\varepsilon)_1}{\partial x_1\partial x_2} \text{ in } L^2(0,T;H^{-1}(D^-))$$

and, with (4.17), we get

$$\frac{\partial^2(v_\varepsilon)_2}{\partial x_2^2} \in L^2(0,T;L^2(D^-))$$

and hence

$$(v_\varepsilon)_2 \in L^2(0,T;H^2(D^-)). \tag{4.18}$$

Returning to $(4.16)_2$ and using (4.18) we obtain

$$\frac{\partial q_{0,\varepsilon}}{\partial x_2} \in L^2(0,T;L^2(D^-)). \tag{4.19}$$

The regularity (4.15) follows as an obvious consequence of (4.17), (4.18), $(4.4)_2$ and (4.19), that achieves the proof. □

4.2 Regularity of the displacement

Recall that the elastic medium is composed by p horizontal elastic layers with constant densities and constant matrix valued elasticity coefficients. For any $s \in \{1, ..., p\}$ consider an arbitrary function $\boldsymbol{\psi}_s : D_\varepsilon \mapsto \mathbb{R}^2$ with the properties $\boldsymbol{\psi}_s \in (C^\infty(D_\varepsilon))^2$, supp $\boldsymbol{\psi}_s \subset\subset D_\varepsilon^{+,s}$ and take as test function in $(3.2)_2$ $(\boldsymbol{\varphi}, z) = (\boldsymbol{\psi}_s, 0)$. This gives:

$$\rho^{+,s} \frac{\partial^2 \mathbf{u}_\varepsilon}{\partial t^2} - c_\varepsilon \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} \left(A_{ij}^s \frac{\partial \mathbf{u}_\varepsilon}{\partial x_j} \right) = \mathbf{h}_\varepsilon \text{ in } L^2(0, T; (H^{-1}(D_\varepsilon^{+,s}))^2). \quad (4.20)$$

A first additional regularity for the displacement is obtained as a consequence of $(4.4)_1$ and of the definition of $\mathbf{u}_\varepsilon$, $(3.16)_1$, namely

$$\frac{\partial \mathbf{u}_\varepsilon}{\partial x_1} \in H^1(0, T; (H^1_\#(D_\varepsilon^+))^2). \quad (4.21)$$

Theorem 4. *If the regularity assumptions (3.1), (4.3) hold, then,*

$$\mathbf{u}_\varepsilon \in L^2(0, T; (H^2_\#(D_\varepsilon^{+,s}))^2) \quad (\forall)\, s \in \{1, ..., p\}. \quad (4.22)$$

Proof. We use the same ideas as in the previous theorem. We leave in the left hand side of (4.20) the term with unknown regularity, $A_{22}^s \frac{\partial^2 \mathbf{u}_\varepsilon}{\partial x_2^2}$, and we use for all the other terms the regularity provided by $(4.3)_1$, by Theorem 1 and by (4.21). □

Remark 10. The regularity (4.22) established in the previous theorem for the displacement allows us to write the equation (4.20) in $L^2(0, T; (L^2_\#(D_\varepsilon^{+,s}))^2)$ for every $s \in \{1, ..., p\}$.

4.3 Uniqueness of the fluid pressure, return to the physical system

In the end of this article we obtain the uniqueness of the pressure and, starting from the variational problem (3.2), we return to the physical system (2.3).

Theorem 5. *Suppose that the assumptions (3.1) and (4.3) are fulfilled. Then, there exists an unique element* $p_\varepsilon \in L^2(0, T; H^1(D^-)) \cap L^2(0, T; L^2_\#(D^-))$ *such that* $(\mathbf{u}_\varepsilon, \mathbf{v}_\varepsilon, \omega_\varepsilon, p_\varepsilon)$ *is solution to the physical problem (2.3), with* $(\mathbf{w}_\varepsilon, \omega_\varepsilon)$ *the unique solution to the variational problem (3.2) and* $\mathbf{u}_\varepsilon, \mathbf{v}_\varepsilon$ *defined by (3.16).*

Proof. The equation for microrotation coupled with velocity, represented by $(4.2)_1$, is

$$j \frac{\partial \omega_\varepsilon}{\partial t} - \gamma \Delta \omega_\varepsilon + 2\chi \omega_\varepsilon - \chi \text{curl } \mathbf{v}_\varepsilon = g \text{ in } L^2(0, T; L^2_\#(D^-)), \quad (4.23)$$

with all the terms having at least this regularity. As a consequence, we obtain the relation (4.23) a.e. in $D^- \times (0, T)$. By means of the 1−periodicity in x_1, we extend it to $L^- \times (0, T)$, obtaining in this way $(2.3)_4$.

The equations for the displacement, $(2.3)_1$, $s \in \{1, ..., p\}$ are obtained as a consequence of (4.20) written in $L^2(0,T;(L^2_\#(D_\varepsilon^{+,s}))^2)$ (see Remark 10). Indeed, from (4.20) we obtain the same equation a.e. in $D_\varepsilon^{+,s} \times (0,T)$ and, extending it by periodicity to the infinite layer $L_\varepsilon^{+,s}$ as before, we get $(2.3)_1$.

Since the boundaries that separate the elastic layers are of measure zero, we obtain, as a consequence of $(2.3)_1$

$$\rho^+ \frac{\partial^2 \mathbf{u}_\varepsilon}{\partial t^2} - c_\varepsilon \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(A_{ij}\frac{\partial \mathbf{u}_\varepsilon}{\partial x_j}\right) = \mathbf{h}_\varepsilon \text{ a.e. in } L_\varepsilon^+ \times (0,T).$$

We obtain next the junction conditions $(2.3)_7$. To this aim, consider $\boldsymbol{\psi}_s : D_\varepsilon \mapsto \mathbb{R}^2$ with the properties $\boldsymbol{\psi}_s \in (C^\infty(D_\varepsilon))^2$, supp $\boldsymbol{\psi}_s \subset\subset (D_\varepsilon^{+,s} \cup D_\varepsilon^{+,s+1} \cup \Gamma_\varepsilon^{+,s})$, $s \in \{1, ..., p-1\}$ and take as test function in $(3.2)_2$ $(\boldsymbol{\varphi}, z) = (\boldsymbol{\psi}_s, 0)$. This yields:

$$\begin{aligned}
&\int_{D_\varepsilon^{+,s}} \rho^{+,s} \frac{\partial^2 \mathbf{u}_\varepsilon}{\partial t^2}(t)\cdot \boldsymbol{\psi}_s + \int_{D_\varepsilon^{+,s+1}} \rho^{+,s+1} \frac{\partial^2 \mathbf{u}_\varepsilon}{\partial t^2}(t)\cdot \boldsymbol{\psi}_s \\
&\quad + c_\varepsilon \sum_{i,j=1}^{2} \int_{D_\varepsilon^{+,s}} A_{ij}^{s} \frac{\partial \mathbf{u}_\varepsilon}{\partial x_j}(t)\cdot \frac{\partial \boldsymbol{\psi}_s}{\partial x_i} + c_\varepsilon \sum_{i,j=1}^{2} \int_{D_\varepsilon^{+,s+1}} A_{ij}^{s+1} \frac{\partial \mathbf{u}_\varepsilon}{\partial x_j}(t)\cdot \frac{\partial \boldsymbol{\psi}_s}{\partial x_i} \\
&= \int_{D_\varepsilon^{+,s}\cup D_\varepsilon^{+,s+1}} \mathbf{h}_\varepsilon(t)\cdot \boldsymbol{\psi}_s \text{ a.e. in } (0,T).
\end{aligned}$$

Integrating by parts the third and the fourth terms of the previous relation and using the equations (4.20) in $L^2(0,T;(L^2(D_\varepsilon^{+,s}))^2)$ (see Remark 10) and the regularity (4.22) corresponding to s and to $s+1$, we obtain, for any $s \in \{1, ..., p-1\}$, the junction conditions $(2.3)_7$ verified a.e. on $\Gamma_\varepsilon^{+,s} \times (0,T)$ and next extended by periodicity to $F_\varepsilon^{+,s} \times (0,T)$.

We obtain in what follows the Stokes equation $(2.3)_2$ and the coupling condition between the fluid and the elastic medium, $(2.3)_{11}$. Taking into account the regularity (4.15), Equation (3.17) becomes:

$$\rho^- \frac{\partial \mathbf{v}_\varepsilon}{\partial t} - 2(\mu+\chi)\mathrm{div}(D(\mathbf{v}_\varepsilon)) - \chi \mathrm{curl}\, \omega_\varepsilon + \nabla q_{0,\varepsilon} = \mathbf{f} \text{ in } L^2(0,T;(L^2(D^-))^2), \quad (4.24)$$

all the terms of (4.24) having at least the regularity stated in this equation.

Consider $\boldsymbol{\varphi} \in (H_0^1(D_\varepsilon))^2$, $\boldsymbol{\varphi} = \mathbf{0}$ in $D_\varepsilon^+ \setminus D_\varepsilon^{+,1}$ and calculate $\int_{D_\varepsilon^{+,1}} (4.20) \cdot \boldsymbol{\varphi}_{/_{D_\varepsilon^{+,1}}}$, $\int_{D^-} (4.24) \cdot \boldsymbol{\varphi}_{/_{D^-}}$, where (4.20) is considered for $s = 1$ and written in $L^2(0,T;(L^2(D_\varepsilon^{+,1}))^2)$. Adding the corresponding relations, integrating by parts and using the regularity properties (4.15), (4.22), we get:

$$\begin{aligned}
&\int_{D_\varepsilon^{+,1}} \rho^{+,1} \frac{\partial^2 \mathbf{u}_\varepsilon}{\partial t^2}(t)\cdot \boldsymbol{\varphi} + c_\varepsilon \sum_{i,j=1}^{2} \int_{D_\varepsilon^{+,1}} A_{ij}^{1} \frac{\partial \mathbf{u}_\varepsilon}{\partial x_j}(t)\cdot \frac{\partial \boldsymbol{\varphi}}{\partial x_i} \\
&\quad + \int_{D^-} \rho^- \frac{\partial \mathbf{v}_\varepsilon}{\partial t}(t)\cdot \boldsymbol{\varphi} + 2(\mu+\chi)\int_{D^-} D(\mathbf{v}_\varepsilon(t)) : D(\boldsymbol{\varphi}) \\
&\quad - \chi \int_{D^-} \mathrm{curl}\, \omega_\varepsilon(t)\cdot \boldsymbol{\varphi} - \int_{D^-} q_{0,\varepsilon}(t) \mathrm{div}\, \boldsymbol{\varphi}
\end{aligned}$$

$$
\begin{aligned}
&-\int_{\Gamma^0}(-q_{0,\varepsilon}(t)\mathbf{e}_2+2(\mu+\chi)D(\mathbf{v}_\varepsilon(t))\mathbf{e}_2)\cdot\boldsymbol{\varphi}\,\mathrm{d}x_1\\
&+c_\varepsilon\int_{\Gamma^0}\sum_{j=1}^{2}A^1_{2j}\frac{\partial\mathbf{u}_\varepsilon}{\partial x_j}\cdot\boldsymbol{\varphi}\,\mathrm{d}x_1=\int_{D_\varepsilon}\mathbf{f}_\varepsilon(t)\cdot\boldsymbol{\varphi}\\
&\qquad(\forall)\,\boldsymbol{\varphi}\in(H^1_0(D_\varepsilon))^2,\boldsymbol{\varphi}=\mathbf{0}\text{ in }D^+_\varepsilon\backslash D^{+,1}_\varepsilon,\ \text{in }L^2(0,T).
\end{aligned}
\tag{4.25}
$$

Remark 11. It was possible to write the last two terms of the left hand side of (4.25) in this form due to (4.15) and (4.22) for $s=1$.

Let us denote

$$
\mathcal{F}_\varepsilon(t)=-q_{0,\varepsilon}(t)\mathbf{e}_2+2(\mu+\chi)D(\mathbf{v}_\varepsilon(t))\mathbf{e}_2-c_\varepsilon\sum_{j=1}^{2}A^1_{2j}\frac{\partial\mathbf{u}_\varepsilon}{\partial x_j}(t).
$$

Taking in (4.25) a test function $\boldsymbol{\varphi}$ with the additional property div $\boldsymbol{\varphi}=0$ in D^- and, using $(3.2)_2$ written for $(\boldsymbol{\varphi},0)$, we get:

$$
\begin{aligned}
&\int_{\Gamma^0}\mathcal{F}_\varepsilon(t)\cdot\boldsymbol{\varphi}\,\mathrm{d}x_1=0\ \ (\forall)\,\boldsymbol{\varphi}\in(H^1_0(D_\varepsilon))^2,\ \boldsymbol{\varphi}=\mathbf{0}\text{ in }D^+_\varepsilon\backslash D^{+,1}_\varepsilon,\\
&\qquad\text{div }\boldsymbol{\varphi}=0\text{ in }D^-,\ \text{in }L^2(0,T).
\end{aligned}
\tag{4.26}
$$

Define the space:

$$
(H^{1/2}(\Gamma^0))^2=\left\{\boldsymbol{\psi}\in(H^{1/2}(\partial D^-))^2\,/\boldsymbol{\psi}=\mathbf{0}\text{ on }\partial D^-\backslash\Gamma^0\right\},
$$

and, for any $\boldsymbol{\psi}\in(H^{1/2}(\Gamma^0))^2$ with $\int_{\Gamma^0}\psi_2(x_1,0)\mathrm{d}x_1=0$, consider the problem

$$
\begin{cases}
\tilde{\boldsymbol{\varphi}}\in(H^1(D^-))^2,\\
\text{div }\tilde{\boldsymbol{\varphi}}=0\text{ in }D^-,\\
\tilde{\boldsymbol{\varphi}}=\boldsymbol{\psi}\text{ on }\partial D^-.
\end{cases}
\tag{4.27}
$$

Using the known result that problem (4.27) has at least one solution $\tilde{\boldsymbol{\varphi}}$, we will construct a test function for (4.26) starting from $\tilde{\boldsymbol{\varphi}}$, as follows. Define $\hat{\boldsymbol{\varphi}}:D_\varepsilon\mapsto\mathbb{R}^2$ by

$$
\hat{\boldsymbol{\varphi}}(x_1,x_2)=\begin{cases}\tilde{\boldsymbol{\varphi}}(x_1,x_2) & \text{if }(x_1,x_2)\in D^-,\\ \tilde{\boldsymbol{\varphi}}(x_1,-x_2) & \text{if }(x_1,x_2)\in D_\varepsilon\backslash D^-.\end{cases}
$$

Consider next another function $\varphi^+_{\varepsilon,1}:[-1,\varepsilon]\mapsto\mathbb{R}$ with $\varphi^+_{\varepsilon,1}(x_2)=1$ if $x_2\in[-1,0]$, $\varphi^+_{\varepsilon,1}(x_2)=0$ if $x_2\in[\zeta_1\varepsilon,\varepsilon]$, $0\le\varphi^+_{\varepsilon,1}(x_2)\le1$ $(\forall)\,x_2\in[-1,\varepsilon]$ and $\varphi^+_{\varepsilon,1}\in C^\infty([-1,\varepsilon])$. It is easy to show that the function $\boldsymbol{\varphi}:D_\varepsilon\mapsto\mathbb{R}^2$, $\boldsymbol{\varphi}(x_1,x_2)=\varphi^+_{\varepsilon,1}(x_2)\hat{\boldsymbol{\varphi}}(x_1,x_2)$ $(\forall)(x_1,x_2)\in D_\varepsilon$ has all the properties for being test function for (4.26). So, relation (4.26) may be written as:

$$
\int_{\Gamma^0}\mathcal{F}_\varepsilon(t)\cdot\boldsymbol{\psi}\,\mathrm{d}x_1=0\,(\forall)\,\boldsymbol{\psi}\in(H^{1/2}(\Gamma^0))^2,\ \text{s. t.}\ \int_{\Gamma^0}\psi_2(x_1,0)\mathrm{d}x_1=0,\ \text{in }L^2(0,T).
$$

This means that $\varphi\mapsto\int_{\Gamma_0}\mathcal{F}_\varepsilon(t)\cdot\varphi dx_1$ is a linear and continuous functional on the space $(H^{1/2}(\Gamma^0))^2$ that vanishes on the subspace characterized by the linear contraint $\int_{\Gamma^0} \psi_2(x_1,0)\ \mathrm{d}x_1 = 0$. Hence, there exists an unique Lagrange's multiplier corresponding to this contraint, denoted $\lambda_\varepsilon(t)$, such that we have a.e. in (0,T):

$$\int_{\Gamma^0} \mathcal{F}_\varepsilon(t)\cdot\boldsymbol{\psi} - \lambda_\varepsilon(t)\int_{\Gamma^0}\psi_2(x_1,0)\ \mathrm{d}x_1 = 0\ (\forall)\,\boldsymbol{\psi}\in(H^{1/2}(\Gamma^0))^2 \text{ a.e. in } (0,T). \tag{4.28}$$

Due to the regularity of the first term of (4.28) we infer that

$$\lambda_\varepsilon \in L^2(0,T), \tag{4.29}$$

that allows us to write (4.28) in $L^2(0,T)$. Define

$$p_\varepsilon = q_{0,\varepsilon} + \lambda_\varepsilon. \tag{4.30}$$

The uniqueness of p_ε follows from the definition of $\mathcal{F}_\varepsilon$ and from the uniqueness of $\mathbf{u}_\varepsilon, \mathbf{v}_\varepsilon, q_{0,\varepsilon}, \lambda_\varepsilon$. The regularity $p_\varepsilon \in L^2(0,T;H^1(D^-)) \cap L^2(0,T;L^2_\#(D^-))$ is a consequence of $(4.1)_2$, $(4.15)_2$ and (4.29). Since the difference between p_ε and $q_{0,\varepsilon}$ is a function depending only on time, the Stokes equation for the micropolar fluid corresponding to $q_{0,\varepsilon}$, (4.24), is satisfied if we replace $q_{0,\varepsilon}$ with p_ε, which means that relation $(2.3)_2$ holds a.e. in $D^- \times (0,T)$. By means of Remark 5, we obtain the same relation a.e. in $D_k^- \times (0,T)$, $(\forall)\, k\in\mathbb{Z}$ and so, the Stokes equations for the micropolar fluid hold a.e. in $L^- \times (0,T)$.

We show next that relation (4.28) leads to the junction condition $(2.3)_{11}$ with p_ε defined by (4.30). To this aim, consider $\boldsymbol{\psi}_0 : [0,1] \mapsto \mathbb{R}^2$ with $\boldsymbol{\psi}_0 \in (C_0^\infty(0,1))^2$ and define $\boldsymbol{\varphi} : D^- \mapsto \mathbb{R}^2$, $\boldsymbol{\varphi}(x_1,x_2) = \boldsymbol{\psi}_0(x_1)(1+x_2)\ (\forall)\ (x_1,x_2)\in D^-$. If we define $\boldsymbol{\psi} = \boldsymbol{\varphi}_{/_{\partial D^-}}$, it is obvious that $\boldsymbol{\psi}\in(H^{1/2}(\Gamma^0))^2$. Considering this test function in (4.28) and denoting

$$\tilde{\mathcal{F}}_\varepsilon(t) = -p_\varepsilon(t)\mathbf{e}_2 + 2(\mu+\chi)D(\mathbf{v}_\varepsilon(t))\mathbf{e}_2 - c_\varepsilon\sum_{j=1}^{2} A^1_{2j}\frac{\partial \mathbf{u}_\varepsilon}{\partial x_j}(t),$$

we obtain

$$\int_{\Gamma^0}\tilde{\mathcal{F}}_\varepsilon(t)\cdot\boldsymbol{\psi}_0 \mathrm{d}x_1 = 0\ (\forall)\,\boldsymbol{\psi}_0\in(C_0^\infty(0,1))^2,\ \text{in } L^2(0,T), \tag{4.31}$$

and, by density of $(C_0^\infty(0,1))^2$ in $(L^2(\Gamma^0))^2$, it follows that

$$\tilde{\mathcal{F}}_\varepsilon = \mathbf{0} \text{ a.e. on } \Gamma^0\times(0,T),$$

i.e., the junction condition between the fluid and the elastic medium, on $\Gamma^0 \times (0,T)$, extended then by periodicity on $F^0 \times (0,T)$.

Remark 12. From the density $(C_0^\infty(0,1))^2$ in $(L^2(\Gamma^0))^2$ it is possible to take in (4.31) $\boldsymbol{\psi}_0 = \mathbf{e}_2$, that gives

$$\lambda_\varepsilon(t) = \int_{\Gamma^0}\mathcal{F}_\varepsilon(t)\cdot\mathbf{e}_2\mathrm{d}x \ \text{ in } L^2(0,T).$$

To achieve the proof of the theorem it remains to obtain the boundary condition $(2.3)_5$. For this purpose, we consider a function $\boldsymbol{\varphi} \in (H^1(D^{+,p}))^2$ with $\boldsymbol{\varphi} = \mathbf{0}$ on $\partial D^{+,p} \backslash \Gamma_\varepsilon^+$. Denoting also by $\boldsymbol{\varphi}$ the previous function extended by $\mathbf{0}$ in $D_\varepsilon \backslash D^{+,p}$, we obtain a pair $(\boldsymbol{\varphi}, 0)$ which is test function for $(3.2)_2$ and so:

$$\rho^{+,p} \int_{D_\varepsilon^{+,p}} \frac{\partial^2 \mathbf{u}_\varepsilon}{\partial t^2}(t) \cdot \boldsymbol{\varphi} + c_\varepsilon \sum_{i,j=1}^{2} \int_{D_\varepsilon^{+,p}} A_{ij}^p \frac{\partial \mathbf{u}_\varepsilon}{\partial x_j}(t) \cdot \frac{\partial \boldsymbol{\varphi}}{\partial x_i} = \int_{D_\varepsilon^{+,p}} \mathbf{h}_\varepsilon(t) \cdot \boldsymbol{\varphi}$$
$$(\forall)\, \boldsymbol{\varphi} \in (H^1(D^{+,p}))^2, \boldsymbol{\varphi} = \mathbf{0} \text{ on } \partial D^{+,p} \backslash \Gamma_\varepsilon^+, \quad \text{in } L^2(0,T). \tag{4.32}$$

Integrating by parts the second term of (4.32) and using (4.20) and (4.22) for $s = p$ we get the condition $(2.3)_5$ verified first a.e. on $\Gamma_\varepsilon^+ \times (0,T)$ and then extended by periodicity to $F_\varepsilon^+ \times (0,T)$. □

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