ON THE SPEISER EQUIVALENT FOR THE RIEMANN HYPOTHESIS

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Abstract. A. Speiser showed that the Riemann hypothesis is equivalent to the absence of non-trivial zeros of the derivative of the Riemann zeta-function left of the critical line. The quantitative version of this result was obtained by N. Levinson and H. Montgomery. This result (or the quantitative version of this result proved by N. Levinson and H. Montgomery) were generalized for many zeta-functions for which the Riemann hypothesis is expected. Here we generalize the Speiser equivalent for zeta-functions. We also investigate the relationship between the non-trivial zeros of the extended Selberg class functions and of their derivatives in this region. This class contains zeta functions for which Riemann hypothesis is not true. As an example, we study the relationship between the trajectories of zeros of linear combinations of Dirichlet L-functions and of their derivatives computationally.

Extended Selberg class, derivatives of zeta-functions, zeros of zeta-functions

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1. Introduction

In the first part of the 20th century, A. Speiser [7] studies the relationship between the location of zeros of the derivative of the Riemann zeta-function and the Riemann hypothesis. His result, achieved by geometric means, is that the Riemann hypothesis is equivalent to the absence of non-real zeros of the derivative of the Riemann zeta-function left of the critical line.

Later on, N. Levinson and H. Montgomery [5] investigate the relationship between the zeros of the Riemann zeta-function and its derivative analytically. They prove the quantitative version of Speiser's result, namely, that the Riemann zeta-function and its derivative have approximately the same number of zeros left of the critical line. In this paper, we prove a similar result to a certain subset of functions from the extended Selberg class (Šležiavičienė [12], Brase [3]).

We use the notation $\overline{f}(s) := \overline{f(\overline{s})}$. A not identically vanishing Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

which converges absolutely for $\sigma > 1$ belongs to the *extended Selberg class* $S^{\#}$ if:

(i) (Meromorphic continuation) There exists $k \in \mathbb{N}$ such that $(s-1)^k F(s)$ is an entire function of finite order.

(ii) (Functional equation) $F(s)$ satisfies the functional equation:

$$\Phi(s) = \omega \overline{\Phi(1-\overline{s})}, \tag{1}$$

where $\Phi(s) := F(s) Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j)$, with $Q > 0$, $\lambda_j > 0$, $\Re(\mu_j) \geq 0$ and $|\omega| = 1$.

A not identically vanishing function $F(s)$ belongs to the extended Selberg class $S^{\#}$ if and only if it satisfies the following conditions:

- The function F is of the form $\sum_{n=1}^{\infty} a_n/n^s$, where $a_n \ll n^{\epsilon}$ for any $\epsilon > 0$. Here the implicit constant may depend on ϵ.
- There exists a non-negative integer k such that $(s-1)^k F(s)$ is an entire function. The smallest such k is denoted k_F and called the polar order of F.
- F satisfies the functional equation:

$$\Phi(s) = \omega \overline{\Phi(1-\overline{s})}. \tag{2}$$

Here $\Phi(s) := F(s) Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j)$, $Q, \lambda_j \in \mathbb{R}$, $\omega, \mu_j \in \mathbb{C}$, $|\omega| = 1$, $Q > 0$, $\lambda_j > 0$, $\Re(\mu_j) \geq 0$.

While the functional equation need not be unique for a given function, the value $d_F = 2\sum_{j=1}^{r} \lambda_j$ is an invariant. It is called the *degree* of F.

For the positive degree the zeros of $F(s)$ located at the poles of the Gamma functions of the functional equation (2), i.e. at $s = -\frac{\mu_j+k}{\lambda_j}$ with $k = 0, 1, 2, \dots$ and $j = 1, \dots, r$, are called the *trivial zeros*. If degree is equal to zero then the functional equation of $F(s)$ has no gamma factors and thus $F(s)$ has no trivial zeros. For any degree there is σ_0 such that $F(s) \neq 0$ in the right-half-plane $\sigma \geq \sigma_0$. Then by functional equation (2) we see that in the left-half-plane $\sigma \leq 1 - \sigma_0$ the function $F(s)$ has only trivial zeros.

Next we consider a zero free regions of $F'(s)$ in the left-half-plane. If degree is equal to zero then by the functional equation

$$F'(s) = -\omega\overline{F'(1-\overline{s})}Q^{1-2s} - 2\omega\overline{F(1-\overline{s})}Q^{1-2s}\log Q.$$

We see that $\overline{F'(1-\overline{s})} \to 0$ and $\overline{F(1-\overline{s})} \to 1$ as $\sigma \to -\infty$. Therefore for $d_F = 0$ there is σ_1 such that $F'(s) \neq 0$ if $\sigma \leq \sigma_1$. For $d_F > 0$ a zero free region of $F'(s)$ is described by the next proposition.

Proposition 1. *Let $F(s) \in S^{\#}$ and $d_F > 0$. Let σ_0 be such that $F(s) \neq 0$ for $\sigma \geq \sigma_0$. There is $\tau \geq 0$ such that $F'(s) \neq 0$ in $\sigma \leq 1 - \sigma_0$, $|t| \geq \tau$.*

From the proof of Proposition 1 we see that for a given function $F(s)$ the explicit upper bound for τ can be calculated.

In this paper T always tends to plus infinity. The main results of this article are the following:

Theorem 2. *Let $F(s) \in S^{\#}$ and $d_F > 0$. Let τ be the same as in Proposition 1. Let $N(T)$ and $N_1(T)$ respectively denote the number of zeros of $F(s)$ and $F'(s)$ in the region $\tau < t < T$, $\sigma < 1/2$. Then*

$$N(T) = N_1(T) + O(\log T).$$

Moreover, if $N(T) < T/(2\sigma_0 - 1) + O(1)$ then there is a monotone sequence $\{T_j\}$, $T_j \to \infty,\ j \to \infty$ such that

$$N(T_j) - N(T_1) = N_1(T_j) - N_1(T_1).$$

Theorem 3. *Let $F(s) \in S^{\#}$ and $d_F = 0$. Let $N(T)$ and $N_1(T)$ respectively denote the number of zeros of $F(s)$ and $F'(s)$ in the region $0 < t < T$, $\sigma < 1/2$. Then*

$$N(T) = N_1(T) + O(1).$$

It is well known that $\zeta'(1/2 + it) \neq 0$ if $\zeta(1/2 + it) \neq 0$, see Spira [8, Corollary 3]. Analogous statement is true for the functions from the extended Selberg class.

Proposition 4. *Let $F(s) \in S^{\#}$. Then there is $\tau \geq 0$ such that, for $t \geq \tau$,*

$$F'(1/2 + it) \neq 0 \quad \text{if} \quad F(1/2 + it) \neq 0.$$

Moreover, if $d_F = 0$ *then* $\tau = 0$.

In [2], E. Balanzario and J. Sánchez-Ortiz calculate the locations of the zeros of the Davenport-Heilbronn zeta-function. This function is famous for the fact that it satisfies a symmetric functional equation but the Riemann hypothesis for it fails. However, one of the results of our paper is that the difference between the number of zeros of this function and its derivative still satisfies similar conditions to those of the Riemann zeta-function.

In order to carry out their calculations, E. Balanzario and J. Sánchez-Ortiz study convex combinations of two zeta-functions, one of them being the Davenport-Heilbronn zeta-function. They observe that a symmetric functional equation prevents the zeros from leaving the critical line unles as they move along it depending on a parameter, they meet another zero. We note that, in addition, it must be the case that a zero of the derivative, moving from the right, meets the point where the trajectories of the two zeros of the original function touch each other.

Earlier in [1], E. Balanzario provides a way to construct Dirichlet series from already known ones. Of special interest are the Dirichlet series constructed from the Riemann zeta-function. In the same paper, the author offers a way to construct functional equations for such series. Based on these results, it is easy to find linearly indepedent Dirichlet series satisfying functional equations similar to that of the Riemann zeta-function. For such linearly independent Dirichlet series, other Dirichlet series can be made which could have a zero at any preassigned place in the complex plane.

In what follows, we first present the graph showing how the trajectory of the zero of the derivative crosses the critical line as it moves from right to left depending on the parameter at the point where, again depending on the parameter, the trajectories of the zeros of the function itself meet each other. This suggests that the number of zeros of the function left of the critical line should be equal to the number of the zeros of the derivative. Later on, we shift our attention to the proofs of the theorems.

2. Computations

In this section we compute the trajectories of zeros depending on a parameter. We investigate the function of the following form:

$$f(s,\tau) := f_0(s)\cdot(1-\tau) + f_1(s)\cdot\tau. \tag{3}$$

Here $\tau \in [0,1]$, $f_0(s) := (1+\sqrt{5}/5^s)\zeta(s)$, $\zeta(s)$ is the Riemann zeta-function, and $f_1(s) := L(s,\chi_2^{(5)})$, which is a Dirichlet L-function given by the following sum:

$$L(s,\chi_2^{(5)}) = 1 - \frac{1}{2^s} - \frac{1}{3^s} + \frac{1}{4^s} + \frac{0}{5^s} + \cdots. \tag{4}$$

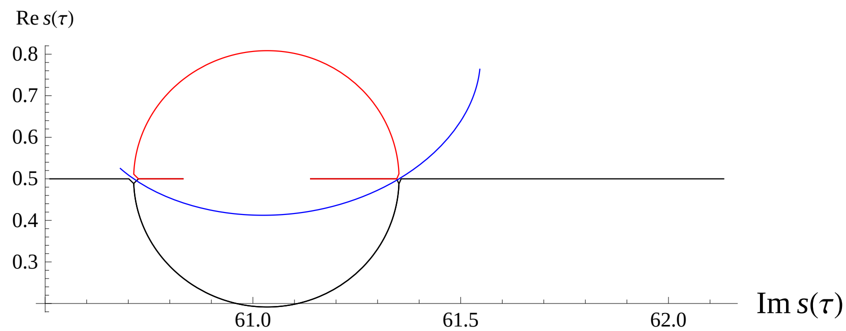

FIGURE 1. Red zero trajectory $s_r(\tau)$: $f(s_r(\tau),\tau) = 0$, $s_r(0) = 0.5 + i60.84$, $s_r(1) = 0.5 + i61.14$. Black zero trajectory $s_b(\tau)$: $f(s_b(\tau),\tau) = 0$, $s_b(0) = 0.5 + i60.51$, $s_b(1) = 0.5 + i62.13$. Blue derivative zero trajectory $s_1(\tau)$: $f'_s(s_1(\tau),\tau) = 0$, $s_1(0) = 0.52 + i60.68$, $s_1(1) = 0.76 + i61.55$.

It is quite likely that all the non-trivial zeros of f_0 and f_1 are located on the line $\sigma = 1/2$. However, this is not true in general of f_τ.

E. Balanzario and J. Sánchez-Ortiz [2] prove that given a zero ρ of f_0, for small $\tau_0 > 0$ and $\delta > 0$, there exists a zero of f_{τ_0} in the δ-neighborhood of ρ. By continuing the procedure of deforming τ, we get a trajectory of a zero as a function of τ.

Both f_0 and f_1 satisfy the functional equation

$$f(s) = T^{-s+1/2}\chi_1(s)f(1-s), \tag{5}$$

here $\chi_1 := 2(2\pi)^{s-1}\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)$ and $T = 5$.

It follows that provided the Riemann hypothesis holds for f_0 and if we have its simple zero, the symmetric functional equation forbids this zero from leaving the critical line as τ increases unless it meets another zero. Only in the case of zeros of degree two they can leave the critical line. Moreover, their trajectories are symmetric with respect to it. In addition, the derivative of f_τ must vanish at this meeting point. Thus the trajectory of a zero of the derivative of f_τ with respect to τ meets the trajectories of the simple zeros at their meeting point.

Figures 1 and 2 are parametric plots of the trajectories of the zeros of f_τ and its derivative, solid and dotted lines respectively. As we have already mentioned, the trajectory of the derivative crosses from right to left at the meeting point of the zeros of f_τ. Let

$$f'_s(s,\tau) = \frac{\partial f(s,\tau)}{\partial s}.$$

We consider solutions (zero trajectories) $s(\tau)$ and $s_1(\tau)$, $0 \le \tau \le 1$, of

$$f(s(\tau),\tau) = 0 \quad \text{and} \quad f'_s(s_1(\tau),\tau) = 0.$$

To find $s(\tau)$ and $s_1(\tau)$ we solve differential equations

$$\frac{\partial s(\tau)}{\partial \tau} = -\frac{\frac{\partial f(s,\tau)}{\partial \tau}}{\frac{\partial f(s,\tau)}{\partial s}} \quad \text{and} \quad \frac{\partial s_1(\tau)}{\partial \tau} = -\frac{\frac{\partial^2 f(s,\tau)}{\partial s \partial \tau}}{\frac{\partial^2 f(s,\tau)}{\partial s^2}}$$

For initial conditions some zeros of $f(s,0) = (1+\frac{\sqrt{5}}{5^s})\zeta(s)$ and $f'_s(s,0)$ are used.

Further we consider the Davenport-Heilbronn zeta-function defined by

$$\ell(s) := \frac{1}{2\cos\alpha}\left(e^{-i\alpha}L(s,\chi_2) + e^{i\alpha}L(s,\overline{\chi_2})\right),$$

where $\chi_2 \bmod 5$, $\chi_2(2) = i$, and $\tan\alpha = \frac{\sqrt{10-2\sqrt{5}-2}}{\sqrt{5}-1}$. Functional equation

$$\ell(s) = 5^{1/2-s}2(2\pi)^{s-1}\Gamma(1-s)\cos\left(\frac{\pi s}{2}\right)\ell(1-s).$$

$$\ell(s) \in S^{\#}.$$

Titchmarsh: $\ell(s)$ has zeros with $\sigma > 1$ and has infinitely zeros on the critical line. S.M. Voronin: $\ell(s)$ has zeros in $1/2 < \sigma < 1$. It belongs to the extended Selberg class and hence falls within the class of functions for which our theorems hold. It should be noted that the zeros of the Davenport-Heilbronn zeta-function have been subject to much analysis. R. Spira in [9] calculates the Davenport-Heilbronn zeta-function zeros off the critical line $\sigma = 1/2$ in the region $0 \le t \le 200$. He does not find any zeros of its derivative left of $\sigma = 1/2$ in this region although he does find several locations of the zeros of the function itself. This would go against our result there are approximately the same number of zeros of the functions of the extended Selberg class and of their

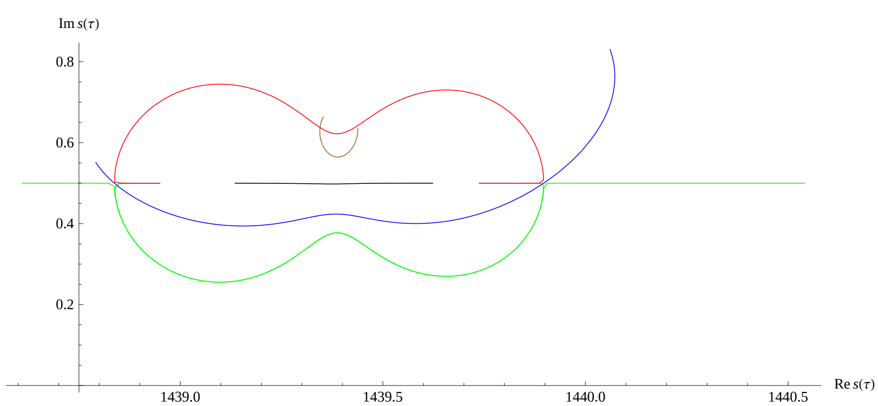

FIGURE 2. Red, green, and black are zero trajectories ($s(\tau)$ is a zero trajectory if $f(s(\tau),\tau) = 0$, $0 \le \tau \le 1$). Blue and brown are derivative zero trajectories($s_1(\tau)$ is a derivative zero trajectory if $f'_s(s(\tau),\tau) = 0$, $0 \le \tau \le 1$).

derivatives. However, we recalculated the zeros of the derivative of the Davenport-Heilbronn zeta-function and we did find zeros left of the critical line with imaginary parts less that 200.

For $0 \le t \le 200$ R. Spira calculated 8 zeros of $\ell(s)$ off the critical line:

0.80+i 85.69,

0.65+i 114.16,

0.57+i 166.47,

0.72+i 176.70

and claimed that 'no zeros of $\ell'(s)$ were found in $\sigma < 1/2$, $0 \le t \le 200$'. We found 4 zeros of $\ell'(s)$ in $\sigma < 1/2$, $0 \le t \le 200$:

0.40+i 85.70,

0.47+i 114.15,

0.49+i 166.47,

0.43+i 176.70.

The Davenport-Heilbronn zeta-function belongs to the extended Selberg class.

3. Proofs

Proof of Proposition 1 . Let $\tau' > \tau$ and $\sigma' < 1 - \sigma_0$. Let R and $\overline{R}$ be two rectangles with vertices $1 - \sigma_0 + i\tau$, $1 - \sigma_0 + i\tau'$, $\sigma' + i\tau'$, $\sigma' + i\tau$ and $1 - \sigma_0 - i\tau$, $1 - \sigma_0 - i\tau'$, $\sigma' - i\tau'$, $\sigma' - i\tau$ respectively. Using Lemma 7 we will show that there is τ such that for any τ' and any σ' the inequality

$$\Re \frac{F'}{F}(s) < 0 \tag{6}$$

holds for $s \in R$ and $s \in \overline{R}$. By argument principle from this it follows that $F'(s)$ and $F(s)$ have the same number of zeros inside of the rectangle R (also in $\overline{R}$). This will prove the proposition since for sufficiently large τ the function $F(s)$ has no zeros inside of the rectangles R and $\overline{R}$ (see the note before Lemma 7).

By the definition of the extended Selberg class there is an integer n_F such that the function $G(s) = s^{n_F}(s-1)^{n_F}\Phi(s)$ is an entire function and $G(1) \neq 0$. By functional equation (2) we have that $G(0) \neq 0$. Moreover $G(s)$ is an entire function of order 1 (see Lemma 3.3 and a comment below the proof of Lemma 3.3 in Smajlović [6]). Applying Hadamard's factorization theorem to the function $G(s)$ analogously as in Šležiavičienė [12, Proof of Theorem 3, formula (6)] (see also Smajlović [6, formulas (8), (10)]) we have that

$$\Re\frac{F'}{F}(s)=\sum_{\substack{\rho \text{ nontrivial}\\ \rho\neq 0,1}}\frac{\sigma-\beta}{|s-\rho|^2}-\frac{n_F\sigma}{|s|^2}-\frac{n_F(\sigma-1)}{|s-1|^2}-\log Q \tag{7}$$
$$-\Re\sum_{j=1}^{r}\lambda_j\frac{\Gamma'}{\Gamma}(\lambda_j s+\mu_j),$$

where the summation is over nontrivial zeros of $F(s)$ except possible nontrivial zeros at $s=0$ and $s=1$.

Next we will prove inequality (6). In view of $\Gamma(z)\Gamma(1-z)=\pi/\sin(\pi z)$ we can write

$$\sum_{j=1}^{r}\lambda_j\frac{\Gamma'}{\Gamma}(\lambda_j s+\mu_j)=\sum_{j=1}^{r}\left(\lambda_j\frac{\Gamma'}{\Gamma}(1-\lambda_j s-\mu_j)-\lambda_j\cot\left(\pi(\lambda_j s+\mu_j)\right)\right).$$

Recall that $\lambda_j>0$, $j=1,\dots,r$ and $\sum_{j=1}^{r}\lambda_j>0$. Then by formulas

$$\frac{\Gamma'}{\Gamma}(s)=\log s+O\left(|s|^{-1}\right)\quad(\Re(s)\geq 0,\ |s|\to\infty), \tag{8}$$
$$\cot z=1+O\left(e^{-|\Im z|}\right)\quad(\Im z\to\pm\infty),$$

$\Gamma'/\Gamma(s+1)=\Gamma'/\Gamma(s)+1/s$ and equality (7) we have that there is τ such that, for any τ' and σ', inequality (6) is true if $s\in R$ or $s\in\overline{R}$. This proves Proposition 1.

□

Proof of Theorem 2. Let

$$R=\left\{s\in\mathbb{C}:\tau<t<T,1-\sigma_0<\sigma<\frac{1}{2}\right\},$$

where τ and σ_0 are the same as in Proposition 1. To prove the theorem it is enough to consider the difference of the number of zeros of $F(s)$ and $F'(s)$ in the region R.

Without loss of generality we assume that $F(\sigma+iT)\neq 0$ and $F'(\sigma+iT)\neq 0$ for $1-\sigma_0\leq\sigma\leq 1/2$. We consider the change of $\arg F'/F(s)$ along the appropriately intended boundary R' of the region R. More precisely upper, left, and lower sides of R' coincide with upper, left, and lower boundaries of R. To obtain the right-hand side of the contour R' we take the the right-hand side boundary of R and deform it to bypass the zeros of Z(1/2+it) by left semicircles with an arbitrarily small radius.

To prove the first part of Theorem 2 we will show that the change of $\arg F'(s)/F(s)$ along the contour R' is $O(\log T)$.

By formula (7), similarly as in the proof of Proposition 1, we have that

$$\Re\frac{F'}{F}(1-\sigma_0+it)<0,$$

where $\tau\leq t\leq T$.

We switch to the right hand side of R'. For this we evaluate terms of equality (7). In view of the symmetry of zeros in the respect of the critical line we consider

$$\frac{\sigma-\beta}{|s-\rho|^2}+\frac{\sigma-1+\beta}{|s-1+\overline{\rho}|^2}=-2\left(\frac{1}{2}-\sigma\right)\frac{(t-\gamma)^2+(\sigma-\frac{1}{2})^2-(\frac{1}{2}-\beta)^2}{|s-\rho|^2|s-1+\overline{\rho}|^2}.$$

Let

$$I_1:=2\sum_{\beta<1/2}\frac{(t-\gamma)^2+(\sigma-\frac{1}{2})^2-(\frac{1}{2}-\beta)^2}{|s-\rho|^2|s-1+\overline{\rho}|^2}+\sum_{\beta=1/2}\frac{1}{|s-\rho|^2}. \tag{9}$$

Then

$$I:=\sum_{\rho\ \text{nontrivial}}\frac{\sigma-\beta}{|s-\rho|^2}=-\left(\frac{1}{2}-\sigma\right)I_1. \tag{10}$$

Suppose that $s=1/2+it$ is not a zero of $F(s)$. When $I=0$ (see equation (10)). Again, similarly as in the proof of Proposition 1, we see that $\Re F'/F(s)<0$. Let $\rho_0=1/2+i\gamma_0$ is a zero of $F(s)$. Then I_1 (see formula (9)) can be made arbitrarily large as we move along a left semicircle with an arbitrarily small radius and center at ρ_0. This is because the term $1/|s-\rho_0|^2\to\infty$ as $|s-\rho_0|\to 0$. Hence on the right hand side of R' we again have $\Re F'(s)/F(s)<0$.

By the Phragmén-Lindelöf principle and the functional equation we have that for any σ' there is $A>0$ such that

$$F(\sigma+iT)=O(T^A) \tag{11}$$

uniformly in $\sigma\geq\sigma'$ (cf. Steuding [10, Theorem 6.8]). By the Cauchy differentiation formula and by bound (11) we have that the bound analogous to (11) is true also for $F'(s)$. Then using Jensen's theorem it is possible to show that the change of $\arg F(s)$ and $\arg F'(s)$ along the vertical sides of R' is $O(\log T)$ (cf. Šležiavičienė [12, Proof of Theorem 1] or Titchmarsh [11, Section 9.4]). This proves the first part of Theorem 2.

We will prove the second part of Theorem 2. Suppose there is a monotone sequence $\{T_j\}$, $T_j\to\infty$, $j\to\infty$ with the property $\Re(F'/F(\sigma+iT_j))<0$, here $\sigma_0<\sigma<1/2$. Then by the first part of the proof we have that $N(T_j)-N(T_1)=N_1(T_j)-N_1(T_1)$.

Suppose there is no such sequence $\{T_j\}$. Then for sufficiently high t there is $1-\sigma_0\leq\sigma\leq 1/2$ such that $\Re F'/F(s)\geq 0$. Thus $I>0$ and $I_1<0$. Then at least one term in I_1 must be negative. Hence there is a zero $\rho=\beta+i\gamma$ with $1-\sigma_0<\beta<1/2$ such that

$$\left(\frac{1}{2}-\beta\right)^2>(t-\gamma)^2+\left(\sigma-\frac{1}{2}\right)^2.$$

It follows that $|t-\gamma|<\sigma_0-1/2$. Thus if for sufficiently high t we divide the imaginary line into intervals of length $2\sigma_0-1$, it would follow that for every interval there will be at least one zero whose imaginary part falls into that interval. Since we started with sufficiently high t, it follows that in this case $F(s)$ has more than $T/(2\sigma_0 - 1) + O(1)$ zeros in the region R. This concludes the proof.

$\square$

Proof of Theorem 3. Let us denote the set of degree zero functions of the extended Selberg class $S_0^{\#}$. Let $F(s) \in S_0^{\#}$ and let $q = \sqrt{Q}$, where Q is from functional equation (2) of $F(s)$. By Kaczorowski and Perelli [4] we have that q is a positive integer and

$$F(s) = \sum_{n|q} \frac{a_n}{n^s},$$

where

$$a(n) = \frac{\omega n}{\sqrt{q}} \overline{a\left(\frac{q}{n}\right)},$$

moreover, if $\sqrt{q} \in \mathbb{N}$ then $a(\sqrt{q}) = \varepsilon b$ with $b \in \mathbb{R}$, where ε denotes a fixed square root of ω.

The fact that $d_F = 0$ means that there are no Gamma factors in the functional equation. Hence

$$\frac{F'}{F}(s) = -2\log Q - \overline{\frac{F'}{F}(1-\overline{s})}. \tag{12}$$

Let σ_1 be a real number such that $F(s) \neq 0$ and $F'(s) \neq 0$ for $\sigma \leq \sigma_1$ (see the comment before Proposition 1). Let R be a rectangle with vertices $1/2-\delta$, $1/2-\delta+iT$, σ_1+iT, σ_1, where $\delta > 0$ is as small as we like and it will be decided later. Without the loss of generality we assume that $F(s) \neq 0$ and $F'(s) \neq 0$ on the rectangle R. To prove the theorem it is enough to show that the change of $\arg F'/F(s)$ along the rectangle R is $O(1)$ as $T \to \infty$.

Easy to see that

$$\lim_{\sigma\to\infty} \frac{F'}{F}(\sigma + it) = 0.$$

Suppose that s' is on the left-hand side of R and suppose that $\Re s'$ is small. Then $\Re F'/F(s') < 0$.

Similarly as in the proof of Theorem 2, on the horizontal sides the change in argument is $O(1)$ since $F(s)$ is bounded on vertical strips.

We consider the right-hand side $1/2 - \delta + it$, $0 \leq t \leq T$ of R. By equality (12) we see that

$$\Re\frac{F'}{F}\left(\frac{1}{2} + it\right) = -\log Q$$

if $1/2+it$ is not a zero of $F(s)$. We claim that there is a sufficiently small $\delta=\delta(T)$ such that, for $0\le t\le T$,

$$\Re\frac{F'}{F}\left(\frac{1}{2}-\delta+it\right)\le-\frac{\log Q}{2}. \tag{13}$$

To prove this inequality it is enough to consider the case then t is in the neighborhood of a zero $\rho=1/2+i\gamma$. We have

$$\frac{F'(s)}{F(s)}=\frac{m}{s-\rho}+m'+O(|s-\rho|), \tag{14}$$

where m is the multiplicity of ρ. Hence taking $s=1/2-\delta+it$ we see that

$$\Re\frac{F'}{F}(s)=-\frac{m\delta}{|s-\rho|^2}+\Re(m')+O(|s-\rho|). \tag{15}$$

Thus $\Re m'=-\log Q$. This proves the inequality (13). Therefore, the argument change along the right side of the contour is $O(1)$. This gives the proof of Theorem 3.

□

Proof of Proposition 4. Let a degree $d_F>0$. Assume the contrary, that there is a large number t such that $F'(1/2+it)=0$ and $F(1/2+it)\neq0$. Then by Hadamard's type formula (7), Gamma function property (8), and using that $\sigma=1/2$ in (10) we obtain the contradiction

$$0=\Re\frac{F'}{F}(1/2+it)<0.$$

This proves the proposition for $d_F>0$. If $d_F=0$ then the proposition follows by formula (12).

□

4. Ending notes

The basic intention of this paper consists of the study of certain zeta-functions and their derivatives. Our analysis shows that the zeros of the two are connected in a deep way. We hope that the results of this kind would prove helpful when looking at a bigger picture: the Riemann hypothesis.

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