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# Joint Approximation by the Riemann and Hurwitz Zeta-Functions in Short Intervals

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**Abstract:** In this study, the approximation of a pair of analytic functions defined on the strip $\{s = \sigma + it \in \mathbb{C} : 1/2 < \sigma < 1\}$ by shifts $(\zeta(s + i\tau), \zeta(s + i\tau, \alpha))$, $\tau \in \mathbb{R}$, of the Riemann and Hurwitz zeta-functions with transcendental $\alpha$ in the interval $[T, T + H]$ with $T^{27/82} \leqslant H \leqslant T^{1/2}$ was considered. It was proven that the set of such shifts has a positive density. The main result was an extension of the Mishou theorem proved for the interval $[0, T]$, and the first theorem on the joint mixed universality in short intervals. For proof, the probability approach was applied.

**Keywords:** Hurwitz zeta-function; joint universality; Riemann zeta-function; weak convergence of probability measures

**MSC:** 11M06; 11M35

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## 1. Introduction

Denote by $s = \sigma + it$ is a complex variable and $0 < \alpha \leqslant 1$ is a fixed parameter. The Riemann and Hurwitz zeta-functions $\zeta(s)$ and $\zeta(s, \alpha)$, for $\sigma > 1$, are defined by the Dirichlet series as follows:

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} \quad \text{and} \quad \zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m + \alpha)^s},$$

The functions have analytic continuations to the whole complex plane, except for point $s = 1$, which is a simple pole with residues 1. Moreover, the function $\zeta(s)$, for $\sigma > 1$, can be defined by the Euler product as follows:

$$\zeta(s) = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1},$$

where $\mathbb{P}$ is the set of all prime numbers.

The functions $\zeta(s)$ and $\zeta(s, \alpha)$ are important tools for research in the analytic number theory. The function $\zeta(s)$ is the main tool for investigating the distribution of prime numbers in the set $\mathbb{N}$, while the function $\zeta(s, \alpha)$ with rational parameter $\alpha$ is applied for studying prime numbers in arithmetical progressions. However, the range of applications of the functions $\zeta(s)$ and $\zeta(s, \alpha)$ is significantly wider than the distribution of primes. They are used also in function theory, algebraic number theory, functional analysis, probability theory, and even in quantum mechanics, cosmology, and music [1–5].

One of the most interesting applications of the functions $\zeta(s)$ and $\zeta(s, \alpha)$ is connected to a very important problem of the function theory—the approximation of analytic functions. At present, it is known that analytic functions defined in the strip $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$ can be approximated by shifts $\zeta(s + i\tau)$ (the case of non-vanishing analytic functions) or by shifts $\zeta(s + i\tau, \alpha)$, $\tau \in \mathbb{R}$, for some classes of the parameter $\alpha$. The latter property of zeta-functions is called universality and, for the function $\zeta(s)$, was proved by S. M. Voronin in [6,7]. The initial form of the Voronin universality theorem was improved by various authors (see [8–14]), but its remains the same in essence: the set $\zeta(s+i\tau)$, $\tau \in \mathbb{R}$, is dense in the space of analytic functions. For the statement of a modern version of Voronin's theorem, the following notation is convenient. The class of compact sets of the strip $D$ with connected complements is denoted by $\mathcal{K}$, and the class of continuous functions that are analytic in the interior of $K$ by $H_0(K)$ with $K \in \mathcal{K}$. Moreover, let meas$A$ stand for the Lebesgue measure of a measurable set $A \subset \mathbb{R}$, and

$$L_T(\dots) = \frac{1}{T}\text{meas}\{\tau \in [0,T] : \dots\},$$

where in place of dots, a condition satisfied by $\tau$ is to be written. Then, we have the following statement [8–14]:

**Theorem 1.** *Let $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} L_T\left(\sup_{s\in K}|f(s)-\zeta(s+i\tau)| < \varepsilon\right) > 0.$$

*Moreover, the limit*

$$\lim_{T\to\infty} L_T\left(\sup_{s\in K}|f(s)-\zeta(s+i\tau)| < \varepsilon\right)$$

*exists and is positive for all but at most countably many $\varepsilon > 0$ .*

The problem of the approximation of analytic functions by shifts $\zeta(s+i\tau,\alpha)$ is more complicated and depends on the arithmetic of the parameter $\alpha$. The simplest case is of transcendental $\alpha$, i.e., when $\alpha$ is not a root of any polynomial $p(s) \not\equiv 0$ with rational coefficients. In this case, the set $\{\log(m+\alpha) : m \in \mathbb{N}_0\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, is linearly independent over $\mathbb{Q}$, and we have a certain analogy with the function $\zeta(s)$, where the linear independence of the set $\{\log p : p \in \mathbb{P}\}$ is applied. The case of rational parameter $\alpha = a/q$, $(a,q) = 1$, in virtue of the following representation:

$$\zeta\left(s, \frac{a}{q}\right) = \frac{q^s}{\varphi(q)} \sum_{\chi\ (\mathrm{mod}\, q)} \overline{\chi}(a) L(s,\chi),$$

where a summing runs over all Dirichlet characters modulus $q$, $L(s,\chi)$ denotes the Dirichlet $L$-functions, and $\varphi(q)$ is the Euler totient function, is reduced to the simultaneous approximation of a tuple of $\varphi(q)$ analytic functions by shifts $(L(s+i\tau,\chi_1),\dots,L(s+i\tau,\chi_{\varphi(q)}))$. More precisely, the following result by different methods was obtained in [8,9,14] (see also [12,15]). The class of continuous on $K$ functions that are analytic in the interior of $K$ is denoted by $H(K)$ with $K \in \mathcal{K}$.

**Theorem 2.** *Suppose that the parameter $\alpha$ is transcendental, or rational $\neq 1, 1/2$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} L_T\left(\sup_{s\in K}|f(s)-\zeta(s+i\tau,\alpha)| < \varepsilon\right) > 0.$$

*Moreover, the limit*

$$\lim_{T\to\infty} L_T\left(\sup_{s\in K}|f(s)-\zeta(s+i\tau,\alpha)| < \varepsilon\right)$$

*exists and is positive for all but at most countably many $\varepsilon > 0$ .*

The cases $\alpha = 1$ and $\alpha = 1/2$ are excluded in Theorem 2 because $\zeta(s, 1) = \zeta(s)$ and

$$\zeta\left(s, \frac{1}{2}\right) = (2^s - 1)\zeta(s),$$

and, for them, the statement of Theorem 1 with class $H_0(K)$ is valid.

The most complicated case is of algebraic irrational parameter $\alpha$. This case was studied in [16]. The degree of $\alpha$ is denoted by $d$. Let $\theta = 4(27(4.45)^2)^{-1}$ and $\beta = \theta d^{-2}$. Then, in [16], the following statement was proven to be true.

**Theorem 3.** *Suppose that the parameter $\alpha$ is algebraic irrational. Let $\gamma \in (0, \beta)$, $1 - \beta + \gamma \leqslant \sigma_0 \leqslant 1$, $s_0 = \sigma_0 + it_0$, and $f(s)$ be continuous functions on $|s - s_0| \leqslant r$, $r > 0$ and analytic in the interior of that disc. Moreover, let $0 < a < 1$ and $\varepsilon \in (0, |f(s_0)|)$. Then, for all but finitely many $\alpha \in [a, 1]$, of degree at most $d_0 - 2\theta_1/d_0^2 + \gamma$ with*

$$d_0 \leqslant \left(\frac{\theta}{1 - \sigma_0 + \gamma}\right)^{1/2},$$

*there exist $\tau \in [T, 2T]$ and $\delta = \delta(\varepsilon, f, T) > 0$ such that*

$$\max_{|s-s_0| \leqslant \delta r} |f(s) - \zeta(s + i\tau, \alpha)| < 3\varepsilon,$$

*where $T = T(\varepsilon, f, \alpha)$ is explicitly given, the set of exceptional $\alpha$ is effectively described, and $\delta$ is also effectively computable.*

Theorems 1–3 are devoted to the approximation of one function from a wide class of analytic functions. Also, there are the so-called joint universality theorems in which a tuple of analytic functions is approximated simultaneously by shifts of zeta-functions. The first joint universality result can also be found in Voronin [17] and deals with Dirichlet $L$-functions with pairwise non-equivalent characters (see also [9,18,19]). A joint universality theorem for a pair of Hurwitz zeta-functions was given in [20]. The joint approximation of analytic functions by shifts of Hurwitz zeta- functions involving imaginary parts of non-trivial zeros of the Riemann zeta-function was discussed in [21]. However, later, many joint universality theorems were obtained for functions of the same name (for more results, see [12]). For illustration purposes, we present one example. For $j = 1, \dots, r$, let $0 < \alpha_j \leqslant 1$, and

$$L(\alpha_1, \dots, \alpha_r) = \{\log(m + \alpha_j) : m \in \mathbb{N}_0,\ j = 1, \dots, r\}.$$

**Theorem 4** ([15])**.** *Suppose that the set $L(\alpha_1, \dots, \alpha_r)$ is linearly independent over $\mathbb{Q}$. For $j = 1, \dots, r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T \to \infty} L_T\left(\sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} |f_j(s) - \zeta(s + i\tau, \alpha_j)| < \varepsilon\right) > 0.$$

Also, some problems of joint universality for Hurwitz zeta-functions can be found in [22].

In [23], H. Mishou initiated a new type of joint mixed universality theorems; he proved a joint universality theorem for two functions of different types, for the Riemann zeta-function and Hurwitz zeta-function. Here, it is important to stress that $\zeta(s)$ has the Euler product, while $\zeta(s, \alpha)$ has no such a product for $\alpha \neq 1$ and $\alpha \neq 1/2$. Moreover, the function $\zeta(s)$ satisfies the symmetric functional equation

$$\xi(s) = \xi(1 - s),\ s \in \mathbb{C}, \qquad \xi(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

where $\Gamma(s)$ is the Euler gamma-function, while, for $\zeta(s, \alpha)$, the following non-symmetric equations connecting $s$ and $1-s$ are true:

$$\zeta(1-s, \alpha) = \frac{\Gamma(s)}{(2\pi)^s} \left( e^{-\pi i s/2} \sum_{m=1}^{\infty} \frac{e^{2\pi i m \alpha}}{m^s} + e^{\pi i s/2} \sum_{m=1}^{\infty} \frac{e^{-2\pi i m \alpha}}{m^s} \right), \quad \sigma > 1,$$

or

$$\zeta(s, \alpha) = \frac{2\Gamma(1-s)}{(2\pi)^{1-s}} \left( \sin \frac{\pi s}{2} \sum_{m=1}^{\infty} \frac{\cos(2\pi m \alpha)}{m^{1-s}} + \cos \frac{\pi s}{2} \sum_{m=1}^{\infty} \frac{\sin(2\pi m \alpha)}{m^{1-s}} \right), \quad \sigma < 0.$$

This is one of the causes of differences in the value distribution of $\zeta(s)$ and $\zeta(s, \alpha)$ and also reflects the approximate functional equation for $\zeta(s, \alpha)$, which is the main ingredient for the proof of the mean square estimate in short intervals [24].

**Theorem 5** ([23])**.** *Suppose that the parameter $\alpha$ is transcendental. Let $K_1, K_2 \in \mathcal{K}$ and $f_1(s) \in H_0(K_1)$, $f_2(s) \in H(K_2)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} L_T \left( \sup_{s\in K_1} |f_1(s) - \zeta(s+i\tau)| < \varepsilon, \sup_{s\in K_2} |f_2(s) - \zeta(s+i\tau, \alpha)| < \varepsilon \right) > 0.$$

The thesis [25] is devoted to joint discrete universality for the Riemann and Hurwitz zeta-functions. Mixed joint universality is studied also for more general zeta-functions. We mention the works [26–31]. The weighted versions of the Mishou theorem are proven in [32]. Theorems 1, 2, 4, and 5 have one common shortcoming: they imply that the set of approximating shifts is infinite; however, they do not provide any algorithm to find at least one approximating shift. In this sense, these theorems are ineffective. Of course, it is difficult to discuss concrete approximation shifts; however, some additional information on the efficacy of universality theorems is always useful. In Theorem 3, the efficacy of approximation is described by indication of explicitly given interval $[T, 2T]$ containing $\tau$ such that $\zeta(s+i\tau, \alpha)$ is an approximating shift. This is a very good step in the effectivization direction.

In contrast to Theorem 3, the proofs of Theorems 1, 2, 4, and 5 are based on measure theory; thus, it is impossible to find an explicitly given interval containing $\tau$ with the approximation property. Therefore, there is another method to consider approximating shifts with $\tau$ in the interval of lengths shorter than $T$ or, more precisely, $o(T)$ as $T \to \infty$. This method leads to universality theorems in short intervals. For the function $\zeta(s)$, the first universality theorem of such a type was obtained in [33]. Let

$$L_{T,H}(\dots) = \frac{1}{H} \text{meas}\{\tau \in [T, T+H] : \dots\},$$

where in place of dots, a condition satisfied by $\tau$ is to be written.

**Theorem 6** ([33])**.** *Suppose that $T^{1/3}(\log T)^{26/15} \leqslant H \leqslant T$. Let $K \in \mathcal{K}$, $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} L_{T,H} \left( \sup_{s\in K} |f(s) - \zeta(s+i\tau)| < \varepsilon \right) > 0.$$

*Moreover, "$\liminf$" can be replaced by "$\lim$" for all but at most countably many $\varepsilon > 0$.*

Recently, improvements in Theorem 6 were given in [34].

An analog of Theorem 6 for the Hurwitz zeta-function is given in [35].

**Theorem 7** ([35])**.** *Suppose that $T^{27/82} \leqslant H \leqslant T^{1/2}$, and the parameter $\alpha$ is transcendental. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} L_{T,H}\left(\sup_{s\in K} |f(s) - \zeta(s+i\tau,\alpha)| < \varepsilon\right) > 0.$$

*Moreover, the lower limit can be replaced by a limit for all but at most countably many $\varepsilon > 0$.*

The aim of this study is to obtain a version of Theorem 5 in short intervals.

**Theorem 8.** *Suppose that $T^{27/82} \leqslant H \leqslant T^{1/2}$, and the parameter $\alpha$ is transcendental. Let $K_1, K_2 \in \mathcal{K}$ and $f_1(s) \in H_0(K_1)$, $f_2(s) \in H(K_2)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} L_{T,H}\left(\sup_{s\in K_1} |f_1(s) - \zeta(s+i\tau)| < \varepsilon, \sup_{s\in K_2} |f_2(s) - \zeta(s+i\tau,\alpha)| < \varepsilon\right) > 0.$$

*Moreover, the lower limit can be replaced by a limit for all but at most countably many $\varepsilon > 0$.*

Using short intervals extends and improves the Mishou theorem on joint mixed universality for the functions $\zeta(s)$ and $\zeta(s,\alpha)$ and is the novel approach presented in this article.

For effectivization aims of approximation, the quantity of $H$ must be as small as possible. On the other hand, $H$ is closely connected to a very important but complicated problem of analytic number theory on the mean square estimates of the functions $\zeta(s)$ and $\zeta(s,\alpha)$ in short intervals. Unfortunately, at present, we only have a result of $H \geqslant T^{27/82}$ in the latter problem (see Lemmas 2 and 3 below).

Mean square estimates together with a joint probabilistic limit theorem for the pair $(\zeta(s), \zeta(s,\alpha))$ in the space of analytic functions occupy a central place in the proof of Theorem 8 in short intervals for the functions $\zeta(s)$ and $\zeta(s,\alpha)$.

## 2. Mean Square Estimates

The first results for the Riemann zeta-function in short intervals were obtained by D. R. Heath-Brown, J.-M. Deshouillers, A. Ivič, H. Iwaniec, M. Jutila, A. A. Karatsuba, G. Kolesnik (for references, see [36]). We recall one mean square estimate from [36].

**Lemma 1.** *Let $(\kappa, \lambda)$ be an exponential pair and $1/2 < \sigma < 1$ fixed. Then, for $T^{(\kappa+\lambda+1-2\sigma)/2(\kappa+1)} \times (\log T)^{(2+\kappa)/(\kappa+1)} \leqslant H \leqslant T$, $1 + \lambda - \kappa \geqslant 2\sigma$, we have uniformly in $H$*

$$\int_{T-H}^{T+H} |\zeta(\sigma+it)|^2 \, \mathrm{d}t \ll H.$$

**Lemma 2.** *Suppose that $\alpha \neq 1/2, 1$, and $1/2 < \sigma \leqslant 7/12$ is fixed. Then, for $T^{27/82} \leqslant H \leqslant T$, uniformly in $H$*

$$\int_{T-H}^{T+H} |\zeta(\sigma+it)|^2 \, \mathrm{d}t \ll_\sigma H.$$

**Proof.** The lemma follows from Lemma 1 by taking the exponential pair $(11/30, 16/30)$. □

**Lemma 3.** *Suppose that $\alpha \neq 1/2, 1$, and $1/2 < \sigma \leqslant 7/12$ is fixed. Then, for $T^{27/82} \leqslant H \leqslant T^\sigma$, uniformly in $H$*

$$\int_{T-H}^{T+H} |\zeta(\sigma+it,\alpha)|^2 \, \mathrm{d}t \ll_{\sigma,\alpha} H.$$

**Proof.** The lemma is Theorem 2 from [24], where its proof is presented. □

Let $\theta > 1/2$ be fixed, and, for $n \in \mathbb{N}$,

$$v_n(m) = \exp\left\{-\left(\frac{m}{n}\right)^\theta\right\}.$$

Define the series

$$\zeta_n(s) = \sum_{m=1}^{\infty} \frac{v_n(m)}{m^s}$$

which absolutely converges in any half-plane $\sigma > \sigma_0$ with finite $\sigma_0$.

**Lemma 4.** *Suppose that $K \subset D$ is a compact set and $T^{27/82} \leqslant H \leqslant T$. Then*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{H} \int_T^{T+H} \sup_{s\in K} |\zeta(s+i\tau) - \zeta_n(s+i\tau)| \, \mathrm{d}\tau = 0.$$

**Proof.** Let

$$l_n(s) = \frac{1}{\theta}\Gamma\left(\frac{s}{\theta}\right) n^s.$$

We use the following integral representation [10]:

$$\zeta_n(s) = \frac{1}{2\pi i} \int_{\theta-i\infty}^{\theta+i\infty} \zeta(s+z) l_n(z) \, \mathrm{d}z \tag{1}$$

which is a result of the classical Mellin formula that yields

$$v_n(m) = \frac{1}{2\pi i} \int_{\theta-i\infty}^{\theta+i\infty} \frac{1}{\theta}\Gamma\left(\frac{s}{\theta}\right)\left(\frac{m}{n}\right)^{-s} \mathrm{d}s.$$

Let $K \subset D$ be a fixed compact set. Then, $K$ is closed and bounded; hence, there exists a positive number $\delta < 1/12$ such that $1/2 + 2\delta \leqslant \sigma \leqslant 1 - \delta$ for all $s = \sigma + it \in K$. We take $\theta = 1/2 + \delta$ and $\widehat{\theta} = 1/2 + \delta - \sigma$. Then, $\widehat{\theta} < 0$ and $\widehat{\theta} \geqslant 1/2 + \delta - 1 + \delta = 2\delta - 1/2 > -1/2 - \delta = -\theta$. The integrand in (1) has only two simple poles in the strip $\widehat{\theta} < \mathrm{Re}z < \theta$, i.e., a pole at the point $z = 0$ of the function $\Gamma(s/\theta)$ and a pole at the point $z = 1 - s$ of the function $\zeta(s+z)$. Therefore, using the well-known estimate

$$\Gamma(\sigma + it) \ll \exp\{-c|t|\}, \quad c > 0, \tag{2}$$

which is uniform in $\sigma \in (\sigma_1, \sigma_2)$ with every $\sigma_1 < \sigma_2$, and replacing $\theta$ by $\widehat{\theta}$ in the line of integration in (1), via the residue theorem, we obtain, for $s \in K$,

$$\zeta_n(s) - \zeta(s) = \frac{1}{2\pi i} \int_{\widehat{\theta}-i\infty}^{\widehat{\theta}+i\infty} \zeta(s+z) l_n(z) \, \mathrm{d}z + l_n(1-s).$$

This gives, for $s \in K$,

$$\zeta_n(s) - \zeta(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \zeta\left(\sigma + it + \frac{1}{2} + \delta - \sigma + iu\right) l_n\left(\frac{1}{2} + \delta - \sigma + iu\right) \mathrm{d}u + l_n(1-s).$$

Hence, for $s \in K$,

$$\zeta_n(s+i\tau) - \zeta(s+i\tau) \ll \int_{-\infty}^{\infty} \left|\zeta\left(\frac{1}{2} + \delta + it + i\tau + iu\right)\right| \left|l_n\left(\frac{1}{2} + \delta - \sigma + iu\right)\right| \mathrm{d}u$$
$$+ \sup_{s\in K} |l_n(1-s-i\tau)|,$$

and, after change $t + u$ by $u$, we obtain

$$\sup_{s\in K}|\zeta_n(s+i\tau)-\zeta(s+i\tau)| \ll \int_{-\infty}^{\infty}\left|\zeta\left(\frac{1}{2}+\delta+i\tau+iu\right)\right|\sup_{s\in K}\left|l_n\left(\frac{1}{2}+\delta-s+iu\right)\right| \mathrm{d}u + \sup_{s\in K}|l_n(1-s-i\tau)|. \tag{3}$$

In view of (2), we have, for $s\in K$,

$$l_n\left(\frac{1}{2}+\delta-s+iu\right) \ll_\delta n^{1/2+\delta-\sigma}\exp\left\{-\frac{c}{\theta}|u-t|\right\} \ll_{\delta,K} n^{-\delta}\exp\{-c_1|u|\} \tag{4}$$

with $c_1>0$. Moreover, it is well known that, for $\sigma\geqslant 1/2$,

$$\zeta(\sigma+it)\ll t^{1-\sigma}, \quad t\geqslant 2.$$

This and (4) imply that

$$\begin{aligned}\left(\int_{-\infty}^{-\log^2 T}+\int_{\log^2 T}^{\infty}\right)&\left|\zeta\left(\frac{1}{2}+\delta+i\tau+iu\right)\right|\sup_{s\in K}\left|l_n\left(\frac{1}{2}+\delta-s+iu\right)\right|\mathrm{d}u\\ &\ll_{\delta,K} n^{-\delta}\left(\int_{-\infty}^{-\log^2 T}+\int_{\log^2 T}^{\infty}\right)(|\tau|+|u|)^{1/2}\exp\{-c_1|u|\}\,\mathrm{d}u\\ &\ll_{\delta,K} n^{-\delta}(|\tau|+1)^{1/2}\exp\{-c_2\log^2 T\}, \quad c_2>0.\end{aligned}$$

Therefore, via (3), we find

$$\begin{aligned}&\frac{1}{H}\int_T^{T+H}\sup_{s\in K}|\zeta(s+i\tau)-\zeta_n(s+i\tau)|\,\mathrm{d}\tau\\ &\ll_{\delta,K}\int_{-\log^2 T}^{\log^2 T}\left(\frac{1}{H}\int_T^{T+H}\left|\zeta\left(\frac{1}{2}+\delta+i\tau+iu\right)\right|\mathrm{d}\tau\right)\sup_{s\in K}\left|l_n\left(\frac{1}{2}+\delta-s+iu\right)\right|\mathrm{d}u\\ &\quad+\frac{1}{H}\int_T^{T+H}\sup_{s\in K}|l_n(1-s-i\tau)|\,\mathrm{d}\tau+\frac{n^{-\delta}\exp\{-c_2\log^2 T\}}{H}\int_T^{T+H}(|\tau|+1)^{1/2}\,\mathrm{d}\tau\\ &\stackrel{\mathrm{def}}{=} J_1+J_2+J_3.\end{aligned} \tag{5}$$

Using the Cauchy–Schwarz inequality gives

$$\begin{aligned}\int_T^{T+H}\left|\zeta\left(\frac{1}{2}+\delta+i\tau+iu\right)\right|\mathrm{d}\tau &\ll \left(\int_T^{T+H}\left|\zeta\left(\frac{1}{2}+\delta+i\tau+iu\right)\right|^2\mathrm{d}\tau\right)^{1/2}\\ &\ll\left(\frac{1}{H}\int_{T-H-|u|}^{T+H+|u|}\left|\zeta\left(\frac{1}{2}+\delta+i\tau\right)\right|^2\mathrm{d}\tau\right)^{1/2}.\end{aligned} \tag{6}$$

For $|u|\leqslant\log^2 T$ and large $T$,

$$H+|u|\leqslant T^{1/2}+\log^2 T\leqslant T.$$

Therefore, (6) and Lemma 2 show that, for $|u|\leqslant\log^2 T$,

$$\frac{1}{H}\int_T^{T+H}\left|\zeta\left(\frac{1}{2}+\delta+i\tau+iu\right)\right|\mathrm{d}\tau\ll_\delta\left(\frac{1}{H}(H+|u|)\right)^{1/2}\ll_\delta(1+|u|)^{1/2}.$$

This and (4) give the estimate

$$J_1\ll_{\delta,K} n^{-\delta}\int_{-\log^2 T}^{\log^2 T}\exp\{-c_1|u|\}(1+|u|)^{1/2}\,\mathrm{d}u\ll_{\delta,K} u^{-\delta}. \tag{7}$$

Similarly to (4), we obtain that, for $s \in K$,

$$l_n(1-s-i\tau) \ll n^{1-\sigma} \exp\{-c_1|t+\tau|\} \ll_K n^{1/2-2\delta} \exp\{-c_3|\tau|\}, \quad c_3 > 0.$$

Thus,

$$J_2 \ll_K n^{1/2-2\delta} \frac{1}{H} \int_T^{T+H} \exp\{-c_3|\tau|\} \, d\tau \ll_K \frac{n^{1/2-2\delta}}{H}. \tag{8}$$

It is easily seen that

$$J_3 \ll n^{-\delta} \exp\{-c_2 \log^2 T\} T^{1/2}.$$

This, (5), (7), and (8) lead to the following estimate:

$$\frac{1}{H} \int_T^{T+H} \sup_{s \in K} |\zeta(s+i\tau) - \zeta_n(s+i\tau)| \, d\tau \ll_{\delta,K} n^{-\delta} + n^{1/2-2\delta} H^{-1} + n^{-\delta} \exp\{-c_2 \log^2 T\} T^{1/2}.$$

Taking $T \to \infty$, and then $n \to \infty$, gives the equality of the lemma. □

Recall a metric in $H(D)$, inducing its topology [37]. There exists a sequence $\{K_l\}$ of embedded compact sets lying in $D$ such that

$$\bigcup_{l=1}^{\infty} K_l = D,$$

and every compact set $K \subset D$ lies in some set $K_l$. Then, for $g_1, g_2 \in H(D)$, denoting

$$\rho(g_1, g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K} |g_1(s) - g_2(s)|},$$

we have the metric $\rho$ that induces the topology of $H(D)$.

The latter formula with Lemma 4 yields the following statement.

**Lemma 5.** *Suppose that $T^{27/82} \leqslant H \leqslant T$. Then the equality*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{H} \int_T^{T+H} \rho(\zeta(s+i\tau), \zeta_n(s+i\tau)) \, d\tau = 0$$

*holds.*

A similar lemma for the Hurwitz zeta-function was obtained in [35]. For the same $\theta$ as above, define

$$v_n(m,\alpha) = \exp\left\{-\left(\frac{m+\alpha}{n}\right)^{\theta}\right\}.$$

and

$$\zeta_n(s,\alpha) = \sum_{m=0}^{\infty} \frac{v_n(m,\alpha)}{(m+\alpha)^s}$$

Then, the latter series, as $\zeta_n(s)$, is absolutely convergent for $\sigma > \sigma_0$, with arbitrary finite $\sigma_0$.

**Lemma 6.** *Suppose that $T^{27/82} \leqslant H \leqslant T^{1/2}$, and $\alpha \neq 1/2$ or $1$. Then*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{H} \int_T^{T+H} \rho(\zeta(s+i\tau,\alpha), \zeta_n(s+i\tau,\alpha)) \, d\tau = 0.$$

**Proof.** The lemma is Lemma 10 from [35], where its proof is given. □

For

$$\underline{g}_k = (g_{k1}, g_{k2}) \in H^2(D), \quad k = 1, 2,$$

set

$$\rho_2(\underline{g}_1, \underline{g}_2) = \max(\rho(g_{11}, g_{12}), \rho(g_{21}, g_{22})).$$

Then $\rho_2$ is a metric that induces the topology of $H^2(D)$. This definition of $\rho_2$ and Lemmas 5 and 6 imply the following lemma. For brevity, let

$$\underline{\zeta}(s, \alpha) = (\zeta(s), \zeta(s, \alpha))$$

and

$$\underline{\zeta}_n(s, \alpha) = (\zeta_n(s), \zeta_n(s, \alpha)).$$

**Lemma 7.** *Suppose that $T^{27/82} \leqslant H \leqslant T^{1/2}$, and $\alpha \neq 1/2$ or 1. Then*

$$\lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{H} \int_T^{T+H} \rho_2\Big(\underline{\zeta}(s+i\tau, \alpha), \underline{\zeta}_n(s+i\tau, \alpha)\Big)\, \mathrm{d}\tau = 0.$$

## 3. Limit Theorem

In this section, we will consider the weak convergence for

$$P_{T,H,\alpha}(A) = L_{T,H}\Big(\underline{\zeta}(s+i\tau, \alpha) \in A\Big), \quad A \in \mathcal{B}(H^2(D)),$$

as $T \to \infty$, with $H$ restricted in Lemma 7, and $\mathcal{B}(\mathbb{X})$ denotes the Borel $\sigma$-field of the space $\mathbb{X}$.

We start with the weak convergence of probability measures on a certain topological group. Let

$$\Omega_1 = \prod_{p\in\mathbb{P}} \{s \in \mathbb{C} : |s| = 1\}, \quad \Omega_2 = \prod_{m\in\mathbb{N}_0} \{s \in \mathbb{C} : |s| = 1\}, \quad \text{and} \quad \Omega = \Omega_1 \times \Omega_2.$$

Since $\Omega_1$ and $\Omega_2$ with the product topology and pointwise multiplication are compact topological groups, the Tikhonov theorem implies that $\Omega$ is again a compact topological group. Thus, on $(\Omega_1, \mathcal{B}(\Omega_1))$, $(\Omega_2, \mathcal{B}(\Omega_2))$, and $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measures $m_{1H}$, $m_{2H}$, and $m_H$, respectively, can be defined. We notice that $m_H = m_{1H} \times m_{2H}$, i. e., if $A = A_1 \times A_2$, $A_1 \in \mathcal{B}(\Omega_1)$, $A_2 \in \mathcal{B}(\Omega_2)$, then

$$m_H(A) = m_{1H}(A_1) m_{2H}(A_2).$$

For $\omega \in \Omega$, we have $\omega = (\omega_1, \omega_2)$ with $\omega_1 = (\omega_1(p) : p \in \mathbb{P})$ and $\omega_2 = (\omega_2(m) : m \in \mathbb{N}_0)$.

For $A \in \mathcal{B}(\Omega)$, set

$$P^{\Omega}_{T,H,\alpha}(A) = L_{T,H}\bigg(\Big(\Big(p^{-i\tau} : p \in \mathbb{P}\Big), \Big((m+\alpha)^{-i\tau} : m \in \mathbb{N}_0\Big)\Big) \in A\bigg).$$

**Lemma 8.** *Suppose that $T^{27/82} \leqslant H \leqslant T^{1/2}$, and $\alpha$ is transcendental. Then, $P^{\Omega}_{T,H,\alpha}$ converges weakly to the Haar measure $m_H$ as $T \to \infty$.*

**Proof.** We use similar arguments as in the case $\tau \in [0, T]$. Let $F_{T,H,\alpha}(\underline{k}_1, \underline{k}_2)$, $\underline{k}_1 = (k_{1p} : k_{1p} \in \mathbb{Z}, p \in \mathbb{P})$, $\underline{k}_2 = (k_{2m} : k_{2m} \in \mathbb{Z}, m \in \mathbb{N}_0)$, be the Fourier transform of $P^{\Omega}_{T,H,\alpha}$, i. e.,

$$F_{T,H,\alpha}(\underline{k}_1, \underline{k}_2) = \int_\Omega {\prod_{p\in\mathbb{P}}}^* \omega^{k_{1p}}(p) {\prod_{m\in\mathbb{N}_0}}^* \omega^{k_{2m}}(m)\, \mathrm{d}P^{\Omega}_{T,H,\alpha},$$

where the star $*$ shows that only a finite number of integers $k_{1p}$ and $k_{2m}$ are not zeroes. Thus, taking into account the definition of the measure $P^{\Omega}_{T,H,\alpha}$, we have

$$\begin{aligned} F_{T,H,\alpha}(\underline{k}_1,\underline{k}_2) &= \frac{1}{H}\int_T^{T+H} \prod_{p\in\mathbb{P}}^{*} p^{-i\tau k_{1p}} \prod_{m\in\mathbb{N}_0}^{*} (m+\alpha)^{-i\tau k_{2m}}\,\mathrm{d}\tau \\ &= \frac{1}{H}\int_T^{T+H} \exp\left\{-i\tau\left(\sum_{p\in\mathbb{P}}^{*} k_{1p}\log p + \sum_{m\in\mathbb{N}_0}^{*} k_{2m}\log(m+\alpha)\right)\right\}\mathrm{d}\tau. \end{aligned} \tag{9}$$

We have to show that

$$\lim_{T\to\infty} F_{T,H,\alpha}(\underline{k}_1,\underline{k}_2) = \begin{cases} 1 & \text{if } (\underline{k}_1,\underline{k}_2) = (\underline{0},\underline{0}), \\ 0 & \text{if } (\underline{k}_1,\underline{k}_2) \neq (\underline{0},\underline{0}), \end{cases} \tag{10}$$

where $\underline{0} = (0,0,\dots)$. Obviously, by (9),

$$F_{T,H,\alpha}(\underline{0},\underline{0}) = 1. \tag{11}$$

Therefore, only the case $(\underline{k}_1,\underline{k}_2) \neq (\underline{0},\underline{0})$ remains for consideration. Since $\alpha$ is transcendental, the set $L(\alpha) = \{\log(m+\alpha) : m \in \mathbb{N}_0\}$ is linearly independent over $\mathbb{Q}$. The set $\{\log p : p \in \mathbb{P}\}$ is also linearly independent over $\mathbb{Q}$. The linear independence over $\mathbb{Q}$ for the set $L(\mathbb{P},\alpha) \overset{\text{def}}{=} \{(\log p : p \in \mathbb{P}), (\log(m+\alpha) : m \in \mathbb{N}_0)\}$ is easily seen. Actually, if, for some non-zeroes $k_{1p_1},\dots,k_{1p_r},k_{2m_1},\dots,k_{2m_v}$,

$$k_{1p_1}\log p_1 + \cdots k_{1p_r}\log p_r + k_{2m_1}\log(m_1+\alpha) + \cdots k_{2m_v}\log(m_v+\alpha) = 0,$$

then

$$(m_1+\alpha)^{k_{21}}\cdots(m_v+\alpha)^{k_{2v}} = p_1^{-k_{11}}\cdots p_r^{-k_{1r}}.$$

From this, it follows that there exists a polynomial $p(s)$ with rational coefficients such that $p(\alpha) = 0$, and this contradicts the transcendence of $\alpha$.

The linear independence of the set $L(\mathbb{P},\alpha)$ shows that, in the case $(\underline{k}_1,\underline{k}_2) \neq (\underline{0},\underline{0})$,

$$A(\underline{k}_1,\underline{k}_2) \overset{\text{def}}{=} \sum_{p\in\mathbb{P}}^{*} k_{1p}\log p + \sum_{m\in\mathbb{N}_0}^{*} k_{2m}\log(m+\alpha) \neq 0.$$

Hence, in view of (9),

$$F_{T,H,\alpha}(\underline{k}_1,\underline{k}_2) = \frac{\exp\{-iTA(\underline{k}_1,\underline{k}_2)\} - \exp\{-i(T+H)A(\underline{k}_1,\underline{k}_2)\}}{iHA(\underline{k}_1,\underline{k}_2)}.$$

Thus, for $(\underline{k}_1,\underline{k}_2) \neq (\underline{0},\underline{0})$,

$$\lim_{T\to\infty} F_{T,H,\alpha}(\underline{k}_1,\underline{k}_2) = 0,$$

and with (11), we obtain (10). The lemma is proven to be true. □

Now, we are in position to consider the weak convergence for

$$P_{T,H,n,\alpha}(A) \overset{\text{def}}{=} L_{T,H}\Big(\underline{\zeta}_n(s+i\tau,\alpha) \in A\Big), \quad A \in \mathcal{B}(H^2(D)).$$

For this, define $u_{n,\alpha} : \Omega \in H^2(D)$ by

$$u_{n,\alpha}(\omega) = \underline{\zeta}_n(s,\omega,\alpha), \quad \omega \in \Omega,$$

where

$$\underline{\zeta}_n(s,\omega,\alpha) = (\zeta_n(s,\omega_1), \zeta_n(s,\omega_2,\alpha)),$$

$$\zeta_n(s,\omega_1) = \sum_{m=1}^{\infty} \frac{\omega_1(m), v_n(m)}{m^s}, \quad \omega_1(m) = \prod_{\substack{p^l | m \\ p^{l+1} \nmid m}} \omega_1^l(p),$$

and

$$\zeta_n(s,\omega_2,\alpha) = \sum_{m=0}^{\infty} \frac{\omega_2(m) v_n(m,\alpha)}{(m+\alpha)^s}.$$

Since the series for $\zeta_n(s,\omega_1)$ and $\zeta_n(s,\omega_2,\alpha)$ are absolutely convergent in every half-plane $\sigma > \sigma_0$, the mapping $u_{n,\alpha}$ is continuous; hence, $(\mathcal{B}(\Omega), \mathcal{B}(H^2(D)))$-measurable. Therefore, the measure $m_H$ defines, on $(H^2(D), \mathcal{B}(H^2(D)))$, the probability measure $m_H u_{n,\alpha}^{-1}$ by

$$m_H u_{n,\alpha}^{-1}(A) = m_H(u_{n,\alpha}^{-1} A), \quad A \in \mathcal{B}(H^2(D)).$$

For brevity, let $U_{n,\alpha} = m_H u_{n,\alpha}^{-1}$.

**Lemma 9.** *Suppose that $T^{27/82} \leqslant H \leqslant T^{1/2}$, and $\alpha$ is transcendental. Then, $P_{T,H,n,\alpha}$ converges weakly to $U_{n,\alpha}$ as $T \to \infty$.*

**Proof.** The definition of $u_{n,\alpha}$ yields

$$u_{n,\alpha}\Big(\Big(p^{-i\tau} : p \in \mathbb{P}\Big), \Big((m+\alpha)^{-i\tau} : m \in \mathbb{N}_0\Big)\Big) = \underline{\zeta}_n(s+i\tau,\alpha).$$

Therefore, by the definitions of $P^{\Omega}_{T,H,\alpha}$ and $P_{T,H,n,\alpha}$, we have

$$P_{T,H,n,\alpha}(A) = L_{T,H}\Big(\Big(\Big(p^{-i\tau} : p \in \mathbb{P}\Big), \Big((m+\alpha)^{-i\tau} : m \in \mathbb{N}_0\Big)\Big) \in u_{n,\alpha}^{-1} A\Big)$$

for all $A \in \mathcal{B}(H^2(D))$. Hence,

$$P_{T,H,n,\alpha} = P^{\Omega}_{T,H,\alpha} u_{n,\alpha}^{-1}.$$

This, the continuity of $u_{n,\alpha}$, Lemma 8, and Theorem 5.1 of [38] prove that $P_{T,H,n,\alpha}$ converges weakly to $U_{n,\alpha}$. □

On $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H^2(D)$-valued random element $\underline{\zeta}(s,\omega,\alpha)$ by

$$\underline{\zeta}(s,\omega,\alpha) = (\zeta(s,\omega_1), \zeta(s,\omega_2,\alpha)),$$

where

$$\zeta(s,\omega_1) = \sum_{m=1}^{\infty} \frac{\omega_1(m)}{m^s} \quad \text{and} \quad \zeta(s,\omega_2,\alpha) = \sum_{m=0}^{\infty} \frac{\omega_2(m)}{(m+\alpha)^s}.$$

We observe that the latter series are uniformly convergent on compact subsets of strip $D$ for almost all $\omega_1$ and $\omega_2$, respectively (see, for example, [10,15]). Let $P_{\underline{\zeta},\alpha}$ be the distribution of the random element $\underline{\zeta}(s,\omega,\alpha)$, i. e.,

$$P_{\underline{\zeta},\alpha}(A) = m_H\Big\{\omega \in \Omega : \underline{\zeta}(s,\omega,\alpha) \in A\Big\}, \quad A \in \mathcal{B}(H^2(D)).$$

In [23], for the proof of Theorem 5, a limit theorem for the functions $\zeta(s)$ and $\zeta(s,\alpha)$ with transcendental $\alpha$ was obtained. For $A \in \mathcal{B}(H^2(D))$, let

$$P_{T,\alpha}(A) = L_T\Big(\underline{\zeta}(s+i\tau,\alpha) \in A\Big).$$

Then, in [23], it was proved that $P_{T,\alpha}$, as $T \to \infty$, and $U_{n,\alpha}$, as $n \to \infty$ converges weakly to the same probability measure on $(H^2(D), \mathcal{B}(H^2(D)))$, and this measure is $P_{\underline{\zeta},\alpha}$. Thus, we have the following statement.

**Lemma 10.** *Suppose that $\alpha$ is transcendental. Then, $U_{n,\alpha}$ converges weakly to $P_{\underline{\zeta},\alpha}$ as $n \to \infty$.*

Now, we are ready to prove a limit theorem for $P_{T,H,\alpha}$.

**Theorem 9.** *Suppose that $T^{27/82} \leqslant H \leqslant T^{1/2}$, and $\alpha$ is transcendental. Then $P_{T,H,\alpha}$ converges weakly to $P_{\underline{\zeta},\alpha}$ as $T \to \infty$.*

**Proof.** Introduce a random variable $\xi_{T,H}$ defined on a certain probability space $(\widehat{\Omega}, \mathcal{A}, \mu)$ and uniformly distributed on $[T, T+H]$. Define the $H^2(D)$-valued random elements as follows:

$$\underline{\zeta}_{T,H,n,\alpha} = \underline{\zeta}_{T,H,n,\alpha}(s) = (\zeta_n(s + i\xi_{T,H}), \zeta_n(s + i\xi_{T,H}, \alpha))$$

and

$$\underline{\zeta}_{T,H,\alpha} = \underline{\zeta}_{T,H,\alpha}(s) = (\zeta(s + i\xi_{T,H}), \zeta(s + i\xi_{T,H}, \alpha)).$$

Moreover, let $\underline{\zeta}_{n,\alpha}$ denote the $H^2(D)$-valued random element with distribution $U_{n,\alpha}$. Further on, we will use the language of convergence in distribution ($\xrightarrow{\mathcal{D}}$), i. e., we say that a random element $\eta_n$, as $n \to \infty$, converges in distribution to $\eta$ if the distribution of $\eta_n$, as $n \to \infty$, converges weakly to that of $\eta$.

In virtue of Lemma 10, we have

$$\underline{\zeta}_{T,H,n,\alpha} \xrightarrow[T\to\infty]{\mathcal{D}} \underline{\zeta}_{n,\alpha}. \tag{12}$$

By Lemma 10,

$$\underline{\zeta}_{n,\alpha} \xrightarrow[n\to\infty]{\mathcal{D}} \underline{\zeta}(s,\alpha). \tag{13}$$

The definitions of $\underline{\zeta}_{T,H,n,\alpha}$, $\underline{\zeta}_{T,H,\alpha}$, and $\xi_{T,H}$ show that, for every $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} \limsup_{T\to\infty} \mu\left\{\rho_2\left(\underline{\zeta}_{T,H,\alpha}, \underline{\zeta}_{T,H,n,\alpha}\right) \geqslant \varepsilon\right\} \\
&\quad = \lim_{n\to\infty} \limsup_{T\to\infty} L_{T,H}\left(\rho_2\left(\underline{\zeta}(s+i\tau,\alpha), \underline{\zeta}_n(s+i\tau,\alpha)\right) \geqslant \varepsilon\right) \\
&\quad \leqslant \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{\varepsilon H} \int_T^{T+H} \rho_2\left(\underline{\zeta}(s+i\tau,\alpha), \underline{\zeta}_n(s+i\tau,\alpha)\right) \mathrm{d}\tau.
\end{aligned}$$

Therefore, Lemma 7 implies that

$$\lim_{n\to\infty} \limsup_{T\to\infty} \mu\left\{\rho_2\left(\underline{\zeta}_{T,H,\alpha}, \underline{\zeta}_{T,H,n,\alpha}\right) \geqslant \varepsilon\right\} = 0.$$

This equality and relations (12) and (13) show that all hypotheses of Theorem 4.2 of [38] are fulfilled because the space $H^2(D)$ is separable. In consequence,

$$\underline{\zeta}_{T,H,\alpha} \xrightarrow[T\to\infty]{\mathcal{D}} \underline{\zeta}(s,\alpha),$$

and this relation is equivalent to the weak convergence of $P_{T,H,\alpha}$ to $P_{\underline{\zeta},\alpha}$ as $T \to \infty$. □

## 4. Proof of the Main Theorem

Theorem 9 is the main ingredient of the proof of Theorem 8. However, the support of the limit measure $P_{\underline{\zeta},\alpha}$ is also needed. We recall that the support of $P_{\underline{\zeta},\alpha}$ is a minimal closed set $S_{\underline{\zeta},\alpha} \subset H^2(D)$ such that $P_{\underline{\zeta},\alpha}(S_{\underline{\zeta},\alpha}) = 1$. The elements of $S_{\underline{\zeta},\alpha}$ have a property that, for every open neighborhood $G$ of $\underline{g}$, the inequality $P_{\underline{\zeta},\alpha}(G) > 0$ is satisfied.

Since the space $H^2(D)$ is separable, we have [38]

$$\mathcal{B}(H^2(D)) = \mathcal{B}(H(D)) \times \mathcal{B}(H(D)).$$

Therefore, it suffices to deal with sets of the form

$$A = A_1 \times A_2, \quad A_1, A_2 \in \mathcal{B}(H(D)).$$

It is well known [10] that

$$L_T(\zeta(s + i\tau) \in A), \quad A \in \mathcal{B}(H(D)),$$

converges weakly to the measure $P_\zeta$ as $T \to \infty$, where $P_\zeta$ is the distribution of the random element $\zeta(s, \omega_1)$.

$$L_T(\zeta(s + i\tau, \alpha) \in A), \quad A \in \mathcal{B}(H(D)),$$

with transcendental $\alpha$ converges weakly to the measure $P_{\zeta,\alpha}$ as $T \to \infty$, where $P_{\zeta,\alpha}$ is the distribution of the random element $\zeta(s, \omega_2, \alpha)$ [15]. Moreover, the support of $P_\zeta$ is the set

$$S \stackrel{\text{def}}{=} \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\},$$

while the support of $P_{\zeta,\alpha}$ is the whole $H(D)$ [10,15].

**Lemma 11.** *The support of the measure $P_{\underline{\zeta},\alpha}$ is the set $S \times H(D)$.*

**Proof.** By a property of the Haar measures $m_{1H}$, $m_{2H}$, and $m_H$, and the above remark, we have

$$m_H(S \times H(D)) = m_{1H}(S) \cdot m_{2H}(H(D)).$$

This and the minimality of the sets $S$ and $H(D)$ such that $m_{1H}(S) = 1$ and $m_{2H}(H(D)) = 1$ show that $S \times H(D)$ is a minimal set satisfying $m_H(S \times H(D)) = 1$. □

**Proof of Theorem 8.** By the Mergelyan theorem on the approximation of analytic functions by polynomials [39] (see also [40]), we have the existence of polynomials $p(s)$ and $q(s)$ such that

$$\sup_{s \in K_1} \left| f_1(s) - e^{p(s)} \right| < \frac{\varepsilon}{2} \tag{14}$$

and

$$\sup_{s \in K_2} |f_2(s) - q(s)| < \frac{\varepsilon}{2}. \tag{15}$$

We stress that the Mergelyan theorem can be applied because $K_1, K_2 \in \mathcal{K}$.

Define the set

$$G_\varepsilon = \left\{ (g_1, g_2) \in H^2(D) : \sup_{s \in K_1} \left| g_1(s) - e^{p(s)} \right| < \frac{\varepsilon}{2}, \sup_{s \in K_2} |g_2(s) - q(s)| < \frac{\varepsilon}{2} \right\}.$$

Then, $G_\varepsilon$ is an open neighborhood of an element $(e^{p(s)}, q(s)) \in S \times H(D)$. By Lemma 11 and properties of the support, we have

$$P_{\underline{\zeta},\alpha}(G_\varepsilon) > 0. \tag{16}$$

Let

$$\mathcal{G}_\varepsilon = \left\{ (g_1, g_2) \in H^2(D) : \sup_{s \in K_1} |g_1(s) - f_1(s)| < \varepsilon, \sup_{s \in K_2} |g_2(s) - f_2(s)| < \varepsilon \right\}.$$

The inequalities (14)–(15) imply the inclusion of $G_\varepsilon \subset \mathcal{G}_\varepsilon$. Therefore, in view of (16),

$$P_{\underline{\zeta},\alpha}(\mathcal{G}_\varepsilon) > 0.$$

The set $\mathcal{G}_\varepsilon$ is open in $H^2(D)$. Therefore, Theorem 9 with the equivalent of weak convergence in terms of open sets (see Theorem 2.1 of [38]) gives

$$\liminf_{T\to\infty} P_{T,H,\alpha}(\mathcal{G}_\varepsilon) \geqslant P_{\underline{\zeta},\alpha}(\mathcal{G}_\varepsilon) > 0.$$

This and the definitions of $\mathcal{G}_\varepsilon$ and $P_{T,H,\alpha}$ imply the first assertion of the theorem.

The boundary of the set $\mathcal{G}_\varepsilon$ is denoted by $\partial\mathcal{G}_\varepsilon$. Then, we have that $\partial\mathcal{G}_{\varepsilon_1} \cap \partial\mathcal{G}_{\varepsilon_2} = \varnothing$ for different positives $\varepsilon_1$ and $\varepsilon_2$. The set $\mathcal{G}_\varepsilon$ is a continuity set of $P_{\underline{\zeta},\alpha}$ if $P_{\underline{\zeta},\alpha}(\partial\mathcal{G}_\varepsilon) = 0$. From the above remark, it follows $P_{\underline{\zeta},\alpha}(\partial\mathcal{G}_\varepsilon) \neq 0$ for at most countably many $\varepsilon > 0$. Applying Theorem 9 again in terms of continuity sets (see Theorem 2.1 of [38]), we obtain that

$$\lim_{T\to\infty} P_{T,H,\alpha}(\mathcal{G}_\varepsilon) = P_{\underline{\zeta},\alpha}(\mathcal{G}_\varepsilon) > 0$$

for all but at most countably many $\varepsilon > 0$. This proves the second statement of the theorem. □

## 5. Conclusions

Let $\zeta(s)$ and $\zeta(s,\alpha)$ denote the Riemann and Hurwitz zeta-functions, respectively, and the parameter $\alpha$ is transcendental. We obtained the set of shifts $(\zeta(s+i\tau), \zeta(s+i\tau,\alpha))$, $\tau \in \mathbb{R}$, that approximate a given pair of analytic functions defined on the strip $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$, has a positive lower density in the interval $[T, T+H]$, $T \to \infty$. Here, $T^{27/82} \leqslant H \leqslant T^{1/2}$. More precisely, the following result is proven. Let $K_1$ and $K_2$ be compact subsets of the strip $D$ with connected complements, and $f_1(s) \neq 0$ and $f_2(s)$ continuous functions on $K_1$ and $K_2$ that are analytic inside of $K_1$ and $K_2$, respectively. Then, for every $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{H} \mathrm{meas}\left\{\tau \in [T, T+H] : \sup_{s\in K_1} |f_1(s) - \zeta(s+i\tau)| < \varepsilon \right.$$
$$\left. \sup_{s\in K_2} |f_2(s) - \zeta(s+i\tau,\alpha)| < \varepsilon \right\} > 0.$$

Moreover, except for at most countably many values of $\varepsilon > 0$, "lim inf" can be replaced by "lim". This result extends that of H. Mishou [23].

We are planing to consider similar problems for discrete shifts and generalized shifts $(\zeta(s+i\varphi(\tau)), \zeta(s+i\varphi(\tau),\alpha))$ with a certain function $\varphi(\tau)$.

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