Gentzen-type sequent calculus for modal logic S5

R. ALONDERIS*, *Institute of Data Science and Digital Technologies, Faculty of Mathematics and Informatics, Vilnius University, Akademijos str. 4, LT-08412 Vilnius, Lithuania.*

H. GIEDRA, *Institute of Computer Science, Faculty of Mathematics and Informatics, Vilnius University, Didlaukio str. 47, LT-08303 Vilnius, Lithuania.*

Abstract

We consider a Gentzen-type cut-free sequent calculus GS5 for the modal logic S5 with a restriction on backward applications of modal rule ($\Box \Rightarrow$). Using Schütte's method of reduction trees, we prove that the calculus is complete for S5. We also prove that all rules are invertible and the cut rule is admissible in the calculus. We show that any backward proof search terminates, obtaining a decision procedure for S5 using the introduced calculus GS5.

Keywords: modal logic S5; cut-free sequent calculus

1 Introduction

Cut-free Gentzen-type sequent calculi are rather convenient tools for proof search analysis and construction of decision procedures. Such calculi can be relatively easily constructed for logics weaker than **S5**, such as **T** or **S4**, see e.g. [22]. **S5** is more problematic in this aspect. The first sequent calculus for **S5** was presented in [14, 15]. Cut is not admissible in the calculus. The Gentzen-type sequent calculi in [19, 23] are rather extended sequent calculi; the calculus in [19] does not have the subformula property. A proof in the sequent calculus in [5] involves a constraint of dependency of occurrences of formulas in the proof. The use of this calculus for a decision procedure is problematic since the check for dependency is performed for ready proof trees. A cut-free Gentzen-type sequent calculus **G3S5** for **S5** is presented also in [1]. Cut admissibility in **G3S5** is proved syntactically by using its modified version **G3S5**$^{;}$, which is obtained by imposing partitions on sequents. The termination of proof search is not considered in [1].

There are also various proof systems for **S5** that are proposed in the literature as extensions of Gentzen-type sequent calculi. Such are e.g. labelled sequent calculi [12, 13]. Labels in such systems represent states in a model and are used along with formulas in derivation rules. Hypersequent calculi [3, 11, 16, 17], where some finite collections of sequents, called hypersequents, are used instead of ordinary sequents. Display calculi [4, 24], which include some structural connectives of fixed arity in addition to logical formulas. Deep inference systems [6, 21], using nested sequents that are slightly more complex than hypersequents. We also mention the double sequent calculus in [9], where two types of sequents instead of one are used. Extensions of Gentzen-type calculi are aimed mainly at

*E-mail: romas.alonderis@mif.vu.lt

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obtaining cut-free proof systems for **S5**. A rather thorough survey and analysis of **S5** proof systems based on sequent calculi can be found in the book [10]. They usually allow us to prove syntactically proof-theoretic properties, such as cut elimination.

Gentzen-type calculi have a simpler syntax compared to their extensions. For this reason, they are often more convenient to use for decision procedures. In the present work, we seek a proof search system with ordinary sequents and the termination of backward proof search rather than the syntactic proofs of proof-theoretic properties. Our goal is a sound and complete Gentzen-type sequent calculus that can be used for both a decision procedure of **S5** and in a construction of cut-free and decidable proof search systems of other logics involving **S5**, e.g. the logic of common knowledge over **S5** (LCKS5). Cut-elimination from sequent calculi of the logic of common knowledge (and of fixpoint logics in general) is problematic [2]. Loop-type (or cyclic) sequent calculi are used to get decidable proof-search systems in such cases. Ordinary (one- or two-sided) sequents are usually used in cyclic and analytic proof systems of fixpoint logics, e.g. the mentioned LCKS5 [18]. This has to do with the fact that the extension of syntax by e.g. labels or more complex sequents makes it more difficult to obtain and check loops in a proof search.

The Gentzen-type sequent calculus **GS5** with ordinary sequents is considered in the present paper. Each logical and modal connective in the antecedent (succedent) of a sequent is handled by a corresponding left-hand side (right-hand side, respectively) rule of **GS5**. The rules for propositional connectives are traditional. The rule ($\Box \Rightarrow$) is well known. For the sake of the backward proof search strategy, we introduce a restriction on backward applications of ($\Box \Rightarrow$). The rule ($\Rightarrow \Box$) for '$\Box$' in the succedent is constructed by making use of the rule ($R\Box$) of **G3S5** in [1]. We directly prove the completeness of **GS5** for **S5**, using a variant of Schütte's reduction tree method [20]. The admissibility of cut in **GS5** is proved using the soundness and completeness of **GS5**. We prove that any **GS5** backward proof search terminates, obtaining a decision procedure for **S5**. The procedure, together with the proof of Theorem 3, allows us to construct counter-models for non-valid sequents.

The layout of the present paper is as follows. In Section 2, we present the syntax used in the paper and recall the semantics of **S5**. The calculus **GS5** is defined in Section 3. In Section 4, we prove the height-preserving invertibility of all rules of **GS5**, except ($\Rightarrow \Box$), and soundness of **GS5**. In Section 5, we prove that **GS5** is complete for **S5**; making use of the soundness and completeness of **GS5**, we prove the invertibility of rule ($\Rightarrow \Box$), and the admissibility of the structural rules of weakening and contraction and cut rule in **GS5**. At the end of this section, we describe a decision procedure for **S5** using the calculus **GS5**, and show that the procedure together with the proof of Theorem 3 allow us to obtain counter-models of non-derivable sequents.

2 Syntax and semantics

2.1 Syntax

We use the following language: a set of propositional symbols $\{p, p_1, p_2, \ldots, q, q_1, q_2, \ldots\}$; the propositional connectives $\neg, \vee, \wedge, \rightarrow$; the modal operator $\Box$. The formulas ϕ are defined as usual by the following grammar:

$$\phi ::= p \mid \neg\phi \mid \phi \vee \psi \mid \phi \wedge \psi \mid \phi \rightarrow \psi \mid \Box\phi.$$

We use the letters ϕ and ψ, possibly with subscripts, to denote arbitrary formulas. The modal operator $\Diamond$ is not included in the language: any formula of the shape $\Diamond\phi$ can be expressed by $\neg\Box\neg\phi$.

The letter Σ (possibly subscripted) denotes a multiset of propositional symbols.

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Sequents are objects of the shape $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite, possibly empty, multisets of formulas. The letter S (possibly subscripted) denotes a sequent.

If $\Gamma = (\phi_1, \ldots, \phi_n)$, then $\theta\Gamma = (\theta\phi_1, \ldots, \theta\phi_n)$, where $n \geq 1$ and $\theta \in \{\neg, \Box\}$; if $\Gamma = \emptyset$, then $\theta\Gamma = \Gamma$. Also, $\eta\Gamma = (\phi_1\eta \ldots \eta\phi_n)$, where $n \geq 2$ and $\eta \in \{\vee, \wedge\}$; if $n = 1$ or $\Gamma = \emptyset$, then $\eta\Gamma = \Gamma$.

2.2 Semantics

In this section, we recall the semantics of **S5**. A Kripke model $\mathcal{M}$ for **S5** is a pair (W, V), where W is a set of states, and $V : W \mapsto 2^{\mathbb{P}}$ is a valuation function, where $\mathbb{P}$ is the set of propositional symbols and $2^{\mathbb{P}}$ is the set of subsets of $\mathbb{P}$. The semantics of formulas is defined using the satisfaction relation $\models$ between $(\mathcal{M}, w)$ and formulas, where $w \in W$.

1. $\mathcal{M}, w \models p$, iff $p \in V(w)$;
   $\mathcal{M}, w \models \neg\phi$, iff $\mathcal{M}, w \not\models \phi$;
   $\mathcal{M}, w \models \phi \vee \psi$, iff $\mathcal{M}, w \models \phi$ or $\mathcal{M}, w \models \psi$;
   $\mathcal{M}, w \models \phi \wedge \psi$, iff $\mathcal{M}, w \models \phi$ and $\mathcal{M}, w \models \psi$;
   $\mathcal{M}, w \models \phi \rightarrow \psi$, iff $\mathcal{M}, w \not\models \phi$ or $\mathcal{M}, w \models \psi$;
   $\mathcal{M}, w \models \Box\phi$, iff $\mathcal{M}, v \models \phi$ for all $v \in W$.

(Such semantics is enough for **S5** [7].)

$\mathcal{M}, w \models (\Gamma \Rightarrow \Delta)$, iff $\mathcal{M}, w \not\models \phi$, where $\phi \in \Gamma$, or $\mathcal{M}, w \models \psi$, where $\psi \in \Delta$. If $\mathcal{M}, w \not\models S$, then $\mathcal{M}$ is *a counter-model* for S.

A formula ϕ (sequent S) is called valid, $\models \phi$ ($\models S$) in notation, iff $\mathcal{M}, w \models \phi$ ($\mathcal{M}, w \models S$) for any $\mathcal{M}, w$.

3 Sequent calculus GS5

The multiset $|\Gamma|$ is obtained from Γ by contracting coincident members, e.g. if $\Gamma = (p, p, \phi, \phi, \psi)$, then $|\Gamma| = (p, \phi, \psi)$. Also, $|\Gamma \Rightarrow \Delta| \overset{\text{def}}{=} |\Gamma| \Rightarrow |\Delta|$.

The Gentzen type sequent calculus **GS5** for **S5** is defined as follows:

1. Axioms: $\Gamma, p \Rightarrow \Delta, p$.
2. Propositional rules:

$$\frac{\phi, \psi, \Gamma \Rightarrow \Delta}{\phi \wedge \psi, \Gamma \Rightarrow \Delta}\ (\wedge \Rightarrow), \qquad \frac{\Gamma \Rightarrow \phi, \Delta \quad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \wedge \psi, \Delta}\ (\Rightarrow \wedge),$$

$$\frac{\phi, \Gamma \Rightarrow \Delta \quad \psi, \Gamma \Rightarrow \Delta}{\phi \vee \psi, \Gamma \Rightarrow \Delta}\ (\vee \Rightarrow), \qquad \frac{\Gamma \Rightarrow \phi, \psi, \Delta}{\Gamma \Rightarrow \phi \vee \psi, \Delta}\ (\Rightarrow \vee),$$

$$\frac{\Gamma \Rightarrow \phi, \Delta}{\neg\phi, \Gamma \Rightarrow \Delta}\ (\neg \Rightarrow), \qquad \frac{\Gamma, \phi \Rightarrow \Delta}{\Gamma \Rightarrow \neg\phi, \Delta}\ (\Rightarrow \neg),$$

$$\frac{\Gamma \Rightarrow \phi, \Delta \quad \psi, \Gamma \Rightarrow \Delta}{\phi \rightarrow \psi, \Gamma \Rightarrow \Delta}\ (\rightarrow\Rightarrow), \qquad \frac{\Gamma, \phi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \rightarrow \psi, \Delta}\ (\Rightarrow\rightarrow).$$

3. Modal rules:

$$\frac{\phi, \Box^*\phi, \Gamma \Rightarrow \Delta}{\Box\phi, \Gamma \Rightarrow \Delta}\ (\Box \Rightarrow), \qquad \frac{\Box\Gamma \Rightarrow \Box \vee (\neg|\Sigma_a|, |\Sigma_s|), \Box\Delta, \phi}{\Box^*\Gamma, \Sigma_a \Rightarrow \Sigma_s, \Box\Delta, \Box\phi}\ (\Rightarrow \Box).$$

Here: Π, Γ, Δ are finite, possibly empty, multisets of formulas. It is required that conclusions of the rules are not axioms.

The rule $(\Rightarrow \Box)$ is called *special*. Any sequent of the shape of a premise (conclusion) of $(\Rightarrow \Box)$ is called *p-special* (*c-special*, respectively).

The explicit formula in the conclusion of a non-special rule is called the *principal* formula of that rule. For example, $\Box\phi$ is the principal formula of $(\Box \Rightarrow)$. The formula $\Box\phi$ is the principal formula of $(\Rightarrow \Box)$. The expression $(\Rightarrow \Box\phi)$ denotes an application of $(\Rightarrow \Box)$ with the principal formula $\Box\phi$.

REMARK 1

The calculus with the rule

$$\frac{\Box\Gamma \Rightarrow \Box \vee \neg|\Sigma_a|, \Box \vee |\Sigma_s|, \Box\Delta, \phi}{\Box^*\Gamma, \Sigma_a \Rightarrow \Sigma_s, \Box\Delta, \Box\phi}\ (\Rightarrow \Box)'$$

instead of $(\Rightarrow \Box)$ is not complete. Lemma 4 cannot be proved and, e.g. the valid in **S5** sequent

$$p \Rightarrow q, \Box\neg\Box(\neg p \vee q)$$

is not derivable in such a calculus.

The letter V denotes a proof search tree obtained by applying backward the derivation rules of **GS5**, starting from the root. The expression $V(S)$ denotes a proof search tree V the root of which is the sequent S.

A branch of a backward proof search tree is a path from the root to a leaf. The length of a path is the number of rule applications in it. *The height of a tree* V ($h(V)$ in notation) is the length of the longest branch of V.

A sequent S is called *derivable in* **GS5** ($GS5 \vdash S$ in notation), iff there is $V(S)$ each leaf of which is an axiom. Such $V(S)$ is a *derivation* of S.

The asterisk mark on the modal operator in the premise of $(\Box \Rightarrow)$ is introduced for the sake of the strategy of backward proof search; $\Box^*$ does not differ semantically from $\Box$. The following useless backward proof search is possible if the mark is not used (read bottom-up this and all proof search trees in the present paper):

$$\cfrac{\cfrac{\cfrac{\cdots}{p, p, \Box p, \Gamma \Rightarrow \Delta}\ (\Box \Rightarrow)}{p, \Box p, \Gamma \Rightarrow \Delta}\ (\Box \Rightarrow)}{\Box p, \Gamma \Rightarrow \Delta}\ (\Box \Rightarrow)$$

The mark is removed in the premise of $(\Rightarrow \Box)$. Hence $(\Box \Rightarrow)$ with the same principal formula can be applied backward for the second time on the same branch only if the second application is preceded by an application of $(\Rightarrow \Box)$. Let us consider the following derivation of the sequent $\Box(\neg\Box p \wedge p) \Rightarrow$:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\neg\Box p, p, \Box^*(\neg\Box p \wedge p) \Rightarrow p, \Box\neg p}{\neg\Box p \wedge p, \Box^*(\neg\Box p \wedge p) \Rightarrow p, \Box\neg p}\ (\wedge \Rightarrow)}{\Box(\neg\Box p \wedge p) \Rightarrow p, \Box\neg p}\ (\Box \Rightarrow)}{p, \Box^*(\neg\Box p \wedge p) \Rightarrow \Box p}\ (\Rightarrow \Box)}{\neg\Box p, p, \Box^*(\neg\Box p \wedge p) \Rightarrow}\ (\neg \Rightarrow)}{\neg\Box p \wedge p, \Box^*(\neg\Box p \wedge p) \Rightarrow}\ (\wedge \Rightarrow)}{\Box(\neg\Box p \wedge p) \Rightarrow}\ (\Box \Rightarrow)$$

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The root-sequent is derived using two applications of $(\Box \Rightarrow)$ with the principal formula $\Box(\neg\Box p \wedge p)$, where the second application of $(\Box \Rightarrow)$ is preceded by $(\Rightarrow \Box)$.

A sequent is called *closing* if no derivation rule can be applied backward to it. Such are axioms and sequents of the shape $\Sigma_a, \Box^* \Gamma \Rightarrow \Sigma_s$.

A path in a proof search tree that starts with S_1 and ends with S_2 is denoted by $S_1 \nearrow S_2$. The expression $S \nearrow$ denotes a path that starts with the sequent S.

We introduce the restriction that

$$\frac{S^\dagger}{S'}\ (\Rightarrow \Box\phi)$$

cannot be applied backward to S' in $S \nearrow S'$ if there is S_1 in $S \nearrow S'$ such that $|S_1|$ and $|S^\dagger|$ coincide. Such an occurrence of S_1 in $S \nearrow S'$ is called *a restrictor of* S'. If S' is c-special and $(\Rightarrow \Box)$ cannot be applied backward to it, then S' is called *terminating*. For example, S_1 is a restrictor of S' in

$$\cfrac{\cfrac{\cfrac{S' : \Rightarrow p, \Box\neg p}{p \Rightarrow \Box p}\ (\Rightarrow \Box)}{S_1 : \Rightarrow \Box p, \neg p}\ (\Rightarrow \neg)}{\Rightarrow \Box p, \Box\neg p}\ (\Rightarrow \Box)$$

because

$$\frac{S^\dagger : \Rightarrow \Box p, \neg p}{S'}\ (\Rightarrow \Box)$$

and $|S_1|$ and $|S^\dagger|$ coincide. S' is terminating because it is c-special and $(\Rightarrow \Box)$ cannot be applied backward to it.

4 Some properties of GS5

The size of a formula ϕ ($sz(\phi)$ in notation) is defined as follows: $sz(p) = 1$, $sz(\phi_1 \gamma \phi_2) = sz(\phi_1) + sz(\phi_2) + 1$, and $sz(\mu\phi) = sz(\phi) + 1$, where γ is a binary propositional connective and μ is $\neg$ or a (marked) modal operator. *The size of a sequent* is the sum of the sizes of all members of its antecedent and succedent.

A backward proof search tree $V(S)$ is called *a reduction tree* iff 1) each its leaf is a closing sequent or a conclusion of $(\Rightarrow \Box)$, and 2) there is no application of $(\Rightarrow \Box)$ in it.

LEMMA 1
Any sequent $\Gamma, \phi \Rightarrow \Delta, \phi'$, where 1) $\phi' = \phi$, or 2) $\phi = \Box^*\psi$ and $\phi' = \Box\psi$, is derivable in **GS5**.

PROOF. The proof is by induction on $sz(\phi)$. If $sz(\phi) = 1$, then the considered sequent is an axiom. Assume that $sz(\phi) > 1$. Let $\phi = \phi' = \phi_1 \vee \phi_2$. We have:

$$\cfrac{\cfrac{S_1 :\ \phi_1, \Gamma \Rightarrow \Delta, \phi_1, \phi_2 \qquad S_2 :\ \phi_2, \Gamma \Rightarrow \Delta, \phi_1, \phi_2}{\phi_1 \vee \phi_2, \Gamma \Rightarrow \Delta, \phi_1, \phi_2}\ (\vee \Rightarrow)}{S :\ \phi_1 \vee \phi_2, \Gamma \Rightarrow \Delta, \phi_1 \vee \phi_2}\ (\Rightarrow \vee)$$

S_1 and S_2 are derivable, according to inductive hypothesis. Hence, S is derivable. The cases when $\phi = \neg\phi_1$ or $\phi = \phi_1 \gamma \phi_2$, where γ stands for a binary propositional connective, are considered in the same way. Let $\phi \in \{\Box\psi, \Box^*\psi\}$ and $\phi' = \Box\psi$. Let us consider a reduction tree

$$V(\phi, \Gamma \Rightarrow \Delta, \Box\psi).$$

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Each leaf of the tree is an axiom or a c-special sequent of the shape

$$\Box^*\psi, \Gamma_1 \Rightarrow \Delta_1, \Box\psi.$$

We have:

$$\dfrac{\dfrac{S_2:\ \psi, \Box^*\psi, \Gamma_2 \Rightarrow \Delta_2', \psi}{\Box\psi, \Gamma_2 \Rightarrow \Delta_2, \psi}\,(\Box\Rightarrow)}{S_1:\ \Box^*\psi, \Gamma_1 \Rightarrow \Delta_1, \Box\psi}\,(\Rightarrow\Box)$$

S_2 is derivable according to inductive hypothesis. It follows that S_1 is derivable. Hence

$$\phi, \Gamma \Rightarrow \Delta, \Box\psi$$

is derivable. □

A derivation rule is called *height-preserving invertible* if it is invertible and the premise(s) is (are, respectively) derivable with no greater derivation height than the conclusion.

The expression $\vdash^V S$ denotes that V is a derivation of S.

LEMMA 2
All **GS5** non-special rules are height-preserving invertible.

PROOF. The lemma is proved by induction on the derivation height h of the conclusion of a rule. The proof is trivial if $h = 0$ because both the conclusion and premise(s) of any considered rule are axioms in this case. Assume that $h > 0$. Let us consider the rule

$$\dfrac{S':\ \phi, \Box^*\phi, \Gamma \Rightarrow \Delta}{S:\ \Box\phi, \Gamma \Rightarrow \Delta}\,(\Box\Rightarrow).$$

Let V:

$$\dfrac{\begin{array}{c}\ldots\\ \Gamma_1 \Rightarrow \Delta_1\end{array} \qquad \begin{array}{c}(\cdots)\\ (\Gamma_2 \Rightarrow \Delta_2)\end{array}}{S}\,r$$

be a derivation of S, where r is not $(\Rightarrow\Box)$ (r cannot be $(\Rightarrow\Box)$ because S is not a c-special sequent). We want to show that there is a derivation V' of S' such that $h(V') \leq h(V)$. If r is $(\Box\Rightarrow)$ with the principal formula $\Box\phi$, then the proof is obtained. Otherwise, $\Gamma_1 = \Box\phi, \Gamma_1'$ ($\Gamma_2 = \Box\phi, \Gamma_2'$) and we have

$$\vdash^{V_1} (\phi, \Box^*\phi, \Gamma_1' \Rightarrow \Delta_1) \quad \left(\vdash^{V_2} (\phi, \Box^*\phi, \Gamma_2' \Rightarrow \Delta_2)\right),$$

where $h(V_1) \leq h(V) - 1$ ($h(V_2) \leq h(V) - 1$), according to inductive hypothesis. Hence

$$\dfrac{V_1\left\{\begin{array}{c}\ldots\\ \phi, \Box^*\phi, \Gamma_1' \Rightarrow \Delta_1\end{array}\right. \qquad \left(V_2\left\{\begin{array}{c}\ldots\\ \phi, \Box^*\phi, \Gamma_2' \Rightarrow \Delta_2\end{array}\right.\right)}{S':\ \phi, \Box^*\phi, \Gamma \Rightarrow \Delta}\,r,$$

obtaining the derivation V' of S' such that $h(V') \leq h(V)$. The height-preserving invertibility of the remaining non-special rules is proved in the same way. □

THEOREM 1
The calculus **GS5** is sound for **S5**.

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PROOF. We prove by induction on $h(V)$ that $\vdash^V S$ implies $\models S$ for any S. The proof is obvious if $h(V) = 0$. Assume that $h(V) > 0$. The tree V has the following shape:

$$\frac{\dots}{S}\ (r).$$

Let $(r) = (\Rightarrow \Box)$:

$$\frac{\begin{array}{c}\dots\\ \Box\Gamma \Rightarrow \Box \vee (\neg|\Sigma_a|, |\Sigma_s|), \Box\Delta, \phi\end{array}}{S:\ \Box^*\Gamma, \Sigma_a \Rightarrow \Sigma_s, \Box\Delta, \Box\phi}\ (\Rightarrow \Box)$$

and

$$\mathcal{M}, w \models \wedge(\Box^*\Gamma, \Sigma_a).$$

The latter fact implies $\mathcal{M}, w \models \wedge\Sigma_a$. If $\mathcal{M}, w \models \vee\Sigma_s$, then $\mathcal{M}, w \models S$. The theorem is proved in this case. If $\mathcal{M}, w \not\models \vee\Sigma_s$, then $\mathcal{M}, w \not\models \Box \vee (\neg|\Sigma_a|, |\Sigma_s|)$. Hence $\mathcal{M}, w \models \vee(\Box\Delta, \phi)$, according to inductive hypothesis. If $\mathcal{M}, w \models \vee\Box\Delta$, then $\mathcal{M}, w \models S$. If $\mathcal{M}, w \not\models \vee\Box\Delta$, then $\mathcal{M}, w' \not\models \vee\Box\Delta$ for all $w' \in W$. Hence $\mathcal{M}, w' \models \phi$, by inductive hypothesis. Hence $\mathcal{M}, w \models \Box\phi$. We get $\mathcal{M}, w \models S$.

The remaining cases of (r) are traditional. $\Box$

5 Completeness of GS5

PROPOSITION 1
If there is a finite number $n \geq 0$ of applications of $(\Rightarrow \Box)$ in $S \nearrow$, then the length of $S \nearrow$ is finite.

PROOF. The proof follows from the shapes of **GS5** rules. The size of any premise of any propositional rule is less than the size of conclusion. Repeated backward applications of $(\Box \Rightarrow)$ with the same principal formulas are blocked by the asterisk mark. The mark is removed only in premises of $(\Rightarrow \Box)$. Hence, an infinite number of backward applications of $(\Rightarrow \Box)$ is needed to get an infinite path $S \nearrow$ $\Box$

A sequent S_1 *is modulo-contraction coincident with* S_2 if $|S_1|$ and $|S_2|$ coincide.

LEMMA 3
The length of any upward path $S \nearrow$ in any $V(\Gamma \Rightarrow \Delta)$ is finite.

PROOF. Let $\mathcal{P}$ be the set of all sub-formulas of formulas in $\Gamma \Rightarrow \Delta$. The set is finite because $\Gamma \Rightarrow \Delta$ is finite. Each sequent of in $V(\Gamma \Rightarrow \Delta)$ consists of formulas in $\mathcal{P}$, formulas in some finite set of formulas of the shape $\Box \vee (\neg|\Sigma_a|, |\Sigma_b|)$, and sub-formulas of these formulas. We have that there is only a finite number of sequents in $V(\Gamma \Rightarrow \Delta)$ that are not modulo-contraction coincident. In particular, there is a finite number of p-special sequents that are not modulo-contraction coincident. Hence, no path in $V(\Gamma \Rightarrow \Delta)$ has an infinite number of backward applications of $(\Rightarrow \Box)$. We get that the length of $S \nearrow$ is finite, based on Proposition 1. $\Box$

THEOREM 2
Any **GS5** backward proof search terminates.

PROOF. The proof follows from Lemma 3. $\Box$

Let $\Gamma = (\phi_1, \dots, \phi_n)$ $(n \geq 1)$. The expression $\mathcal{M}, w \models \Gamma$ denotes that $\mathcal{M}, w \models \phi_i$ $(1 \leq i \leq n)$ and $\mathcal{M}, w \not\models \Gamma$ denotes that $\mathcal{M}, w \not\models \phi_i$ $(1 \leq i \leq n)$.

LEMMA 4
Let $\mathcal{M} = (W, V)$ be any model, and S_p and S_c be a premise and conclusion, respectively, of any **GS5** rule. If $\mathcal{M}, w \not\models S_p$, then there is $w' \in W$ such that $\mathcal{M}, w' \not\models S_c$.

PROOF. Let us consider e.g. rule $(\vee \Rightarrow)$. If $\mathcal{M}, w \not\models (\phi, \Gamma \Rightarrow \Delta)$, then $\mathcal{M}, w \models (\phi, \Gamma)$ and $\mathcal{M}, w \not\models \Delta$. Hence $\mathcal{M}, w \models (\phi \vee \psi, \Gamma)$. We get $\mathcal{M}, w \not\models (\phi \vee \psi, \Gamma \Rightarrow \Delta)$. The remaining propositional and $(\Box \Rightarrow)$ rules are considered identically. Let us consider the special rule

$$\frac{\Box\Gamma \Rightarrow \Box \vee (\neg|\Sigma_a|, |\Sigma_s|), \Box\Delta, \phi}{\Box^*\Gamma, \Sigma_a \Rightarrow \Sigma_s, \Box\Delta, \Box\phi} \ (\Rightarrow \Box).$$

It is given that

$$\mathcal{M}, w \not\models \quad \Box\Gamma \Rightarrow \Box \vee (\neg|\Sigma_a|, |\Sigma_s|), \Box\Delta, \phi,$$

i.e. $\mathcal{M}, w \models \Box\Gamma$ and $\mathcal{M}, w \not\models (\Box \vee (\neg|\Sigma_a|, |\Sigma_s|), \Box\Delta, \phi)$. Hence $\mathcal{M}, w' \models \Box\Gamma$ and $\mathcal{M}, w' \not\models (\Box \vee (\neg|\Sigma_a|, |\Sigma_s|), \Box\Delta, \Box\phi)$ for any $w' \in W$. The fact $\mathcal{M}, w \not\models \Box \vee (\neg|\Sigma_a|, |\Sigma_s|)$ implies that $\mathcal{M}, w_1 \models \wedge(\Sigma_a, \neg\Sigma_s)$ for some w_1. Hence $\mathcal{M}, w_1 \models \Sigma_a$ and $\mathcal{M}, w_1 \not\models \Sigma_s$. We attain

$$\mathcal{M}, w_1 \not\models \quad \Box^*\Gamma, \Sigma_a \Rightarrow \Sigma_s, \Box\Delta, \Box\phi.$$

$\Box$

THEOREM 3 (Completeness of **GS5**).
If $\models S$, then $GS5 \vdash S$, where the asterisk mark does not occur in S.

PROOF. We prove that if $GS5 \nvdash S$, then $\not\models S$. The proof is by the Schütte's method of reduction trees. Let $\flat = (S \nearrow L)$ be any branch in some $V(S)$, where L is a non-axiom closing or terminating sequent. First, we assume that L is not terminating, e.g. $S = (\Rightarrow \Box p)$ or $(\Rightarrow \Box)$ is not applied in $\flat$. Let

$$L = \quad \Box^*\Gamma, \Sigma_a \Rightarrow \Sigma_s.$$

An **S5** Kripke model $\mathcal{M} = (W, V)$ is constructed as follows: $W = \{w\}$ and $V(w) = \Sigma_a$. Using induction on $sz(\phi)$, we prove that $\mathcal{M}, w \models \phi$ ($\mathcal{M}, w \not\models \phi$) if $\phi \in \Gamma'$ ($\phi \in \Delta'$, respectively), where

$$S' = \quad \Gamma' \Rightarrow \Delta'$$

is any sequent in $\flat$. The proof is obvious if $sz(\phi) = 1$.

Let $sz(\phi) > 1$, $\phi \in \Gamma'$, and $\phi = \Box\phi'$ ($\phi = \Box^*\phi'$); it is true that there is an application of $(\Box \Rightarrow)$ with the principal formula $\Box\phi'$ in $S' \nearrow L$ ($S \nearrow S'$, respectively). Hence $\mathcal{M}, w \models \phi'$ by inductive hypothesis. It follows that $\mathcal{M}, w \models \Box\phi'$.

Let $\phi \in \Delta'$ and $\phi = \Box\phi'$. It is true that there is $(\Rightarrow \Box\phi')$ in $S' \nearrow L$. Hence $\mathcal{M}, w \not\models \phi'$ by inductive hypothesis. It follows that $\mathcal{M}, w \not\models \Box\phi'$.

The outermost symbol of ϕ is a propositional connective in the remaining cases. We skip these cases since they are considered traditionally. We attain $M, w \not\models S$. Hence $\not\models S$.

Now we consider the case when L is terminating. Let $\Box\Gamma$ consist of all formulas of the shape $\Box\phi$ that occur in antecedents of sequents in $\flat$. Let $S^\dagger = (\Box\Pi \Rightarrow \Lambda)$ be the first from the bottom p-special sequent in $\flat$, where $|\Box\Pi| = |\Box\Gamma|$. All restrictors of L are within $\flat^\dagger = (S^\dagger \nearrow L)$, based on the fact that the antecedent of L contains $\Box^*\Gamma$. Hence L is terminating in $\flat^\dagger$.

Let

$$S^i = \quad \Box^*\Gamma^i, \Sigma_a^i \Rightarrow \Sigma_s^i, \Box\Delta^i$$

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$(1 \leq i \leq n)$ be the i-th from the bottom c-special sequent in $\flat^{\dagger}$. Let $\pi_i = (S_i \nearrow S^i)$, where S_i is a p-special and there are no applications of $(\Rightarrow \Box)$ in π_i for $1 \leq i \leq n$. (It is true, that $S_1 = S^{\dagger}$, $S^n = L$, and $\flat^{\dagger} = \pi_1 \ldots \pi_n$.) A Kripke model $\mathcal{M} = (W, V)$ is constructed as follows: $W = \{w_i \mid 1 \leq i \leq n\}$ and $V(w_i) = \{\Sigma_a^i\}$ $(1 \leq i \leq n)$. We prove by induction on $sz(\phi)$ that if $\Gamma' \Rightarrow \Delta'$ is in π_ι ($\iota \in \{1, \ldots, n\}$), then $\mathcal{M}, w_\iota \models \phi$ ($\mathcal{M}, w_\iota \not\models \psi$) for each $\phi \in \Gamma'$ (each $\psi \in \Delta'$, respectively). If $sz(\phi) = 1$, then ϕ is a propositional symbol. The proof follows from the fact that $\mathcal{M}, w_\iota \models \phi$ only if $\phi \in \Sigma_a^\iota$.

Let $\phi \in \Gamma'$ and $\phi = \Box\psi$ or $\phi = \Box^*\psi$. There is an application of $(\Box \Rightarrow)$ with the principal formula $\Box\psi$ in each π_j because $\Box\psi$ is a member of the antecedent of S_j $(1 \leq j \leq n)$, where $\pi_j = (S_j \nearrow S^j)$. Hence ψ is a member of the antecedent of a sequent in each π_j; it follows that $\mathcal{M}, w_j \models \psi$, according to inductive hypothesis. Hence $\mathcal{M}, w_\iota \models \phi$.

Let $\phi \in \Delta'$ and $\phi = \Box\psi$. It is true that $(\Rightarrow \Box)$ with the principal formula ϕ is applied in $\flat^{\dagger}$ or succedent of some restrictor of L has the member ψ. In both cases, $\mathcal{M}, w_j \not\models \psi$ ($j \in \{1 \ldots, n\}$), according to inductive hypothesis. Hence $\mathcal{M}, w_\iota \not\models \phi$.

The outermost symbol of ϕ is a propositional connective in the remaining cases, consideration of which is skipped. We obtain $\mathcal{M}, w_1 \not\models S^{\dagger}$. Hence $\mathcal{M}, w_j \not\models S$ ($j \in \{1 \ldots, n\}$), based on Lemma 4. It follows that $\not\models S$. □

LEMMA 5
Let S' and S be a premise and conclusion, respectively, of $(\Rightarrow \Box)$. It is true that $\models S$ implies $\models S'$.

PROOF. The proof follows from the fact that $\not\models S'$ implies $\not\models S$, according to Lemma 4. □

A derivation rule is called *invertible* if whenever its conclusion is derivable, the premise(s) is (are, respectively) derivable as well.

THEOREM 4
The rule $(\Rightarrow \Box)$ is invertible.

PROOF. Let

$$\frac{S'}{S}\ (\Rightarrow \Box)$$

be any instance of $(\Rightarrow \Box)$. We have that $GS5 \vdash S$ implies $\models S$, according to Theorem 1. Hence $\models S'$, based on Lemma 5. It follows that $GS5 \vdash S'$, according to Theorem 3. □

$(\Rightarrow \Box)$ is not height-preserving invertible, e.g. the derivation height of the c-special sequent $p \Rightarrow p, \Box q$ is 0, while the derivation height of the corresponding p-special sequent $\Rightarrow q, \Box(\neg p \vee p)$ is 3:

$$\cfrac{\cfrac{\cfrac{p \Rightarrow \Box q, p}{\Rightarrow \neg\Box q, \neg p, p}\ (\Rightarrow \neg)}{\Rightarrow \Box q, \neg p \vee p}\ (\Rightarrow \vee)}{\Rightarrow q, \Box(\neg p \vee p)}\ (\Rightarrow \Box)$$

A derivation rule is called *admissible in* **GS5** if the derivability of its premise(es) implies the derivability of the conclusion.

THEOREM 5
The structural rules of weakening

$$\frac{\Gamma \Rightarrow \Delta}{\Pi, \Gamma \Rightarrow \Delta, \Lambda}\ (W)$$

and contraction

$$\frac{\phi, \phi, \Gamma \Rightarrow \Delta}{\phi, \Gamma \Rightarrow \Delta}\ (C \Rightarrow), \qquad \frac{\Gamma \Rightarrow \Delta, \phi, \phi}{\Gamma \Rightarrow \Delta, \phi}\ (\Rightarrow C),$$

where the asterisk mark does not occur in the conclusions, are admissible in **GS5**.

PROOF. The admissibility of (W) is proved as follows. If $GS5 \vdash \Gamma \Rightarrow \Delta$, then $\models \Gamma \Rightarrow \Delta$, by Theorem 1. This fact implies $\models \Pi, \Gamma \Rightarrow \Delta, \Lambda$. Hence $GS5 \vdash \Pi, \Gamma \Rightarrow \Delta, \Lambda$, according to Theorem 3.

The admissibility of $(C \Rightarrow)$ and $(\Rightarrow C)$ is proved in the same way. □

THEOREM 6
The cut rule

$$\frac{\Gamma \Rightarrow \Delta, \phi \quad \phi, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}\ (cut),$$

where the asterisk mark does not occur in the conclusion, is admissible in **GS5**.

PROOF. If $GS5 \vdash \Gamma \Rightarrow \Delta, \phi$ and $GS5 \vdash \phi, \Pi \Rightarrow \Lambda$, then $\models \Gamma \Rightarrow \Delta, \phi$ and $\models \phi, \Pi \Rightarrow \Lambda$, according to Theorem 1. Hence $\models \Gamma, \Pi \Rightarrow \Delta, \Lambda$. It follows that $GS5 \vdash \Gamma, \Pi \Rightarrow \Delta, \Lambda$, based on Theorem 3. □

Let S be an arbitrary sequent. Any backward **GS5** proof search of S terminates, according to Theorem 2. If a tree $V(S)$ has a closing non-axiom or terminating leaf, then $GS5 \nvdash S$, by the fact that all rules of **GS5** are invertible. Hence $\not\models S$, according to Theorem 3. If all leaves of $V(S)$ are axioms, then $GS5 \vdash S$. It follows that $\models S$, according to Theorem 1. We attain a decision procedure for **S5**, based on the calculus **GS5**. If $\not\models S$, then the proof search tree generated by the procedure can be used to construct $\mathcal{M} = (W, V)$ such that $\mathcal{M}, w \not\models S$ for some $w \in W$, in the same way as in the proof of Theorem 3. Let us take e.g. the sequent

$$S = \quad \Box(p \vee q) \Rightarrow \Box p, \Box q$$

and consider the following branch of a backward proof search tree with S at the root:

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$$
\dfrac{\cdots \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{S_c^3 : \; p,\Box^*(p \vee q) \Rightarrow \Box(\neg q \vee p), q \qquad \cdots}{p \vee q, \Box^*(p \vee q) \Rightarrow \Box(\neg q \vee p), q}\,(\vee \Rightarrow)}{S_p^2 : \; \Box(p \vee q) \Rightarrow \Box(\neg q \vee p), q}\,(\Box \Rightarrow)}{S_c^2 : \; q, \Box^*(p \vee q) \Rightarrow p, \Box q}\,(\Rightarrow \Box)}{}}{p \vee q, \Box^*(p \vee q) \Rightarrow p, \Box q}\,(\vee \Rightarrow)
$$

$$
\dfrac{\dfrac{\dfrac{\cdots \qquad \dfrac{\dfrac{\dfrac{\dfrac{S_c^5 : \; p,p,\Box^*(p \vee q) \Rightarrow \Box(\neg q \vee p), q \qquad \cdots}{p \vee q, p, \Box^*(p \vee q) \Rightarrow \Box(\neg q \vee p), q}\,(\vee \Rightarrow)}{p, \Box(p \vee q) \Rightarrow \Box(\neg q \vee p), q}\,(\Box \Rightarrow)}{S_p^4 : \; \Box(p \vee q) \Rightarrow \Box(\neg q \vee p), \neg p \vee q}\,(\Rightarrow \vee), (\Rightarrow \neg)}{S_c^4 : \; q, q, \Box^*(p \vee q) \Rightarrow \Box(\neg p \vee q), p}\,(\Rightarrow \Box)}{p \vee q, q, \Box^*(p \vee q) \Rightarrow \Box(\neg p \vee q), p}\,(\vee \Rightarrow)}{q, \Box(p \vee q) \Rightarrow \Box(\neg p \vee q), p}\,(\Box \Rightarrow)
$$

$$
\dfrac{\dfrac{q, \Box(p \vee q) \Rightarrow \Box(\neg p \vee q), p}{S_p^3 : \; \Box(p \vee q) \Rightarrow \Box(\neg p \vee q), \neg q \vee p}\,(\Rightarrow \vee), (\Rightarrow \neg)}{S_c^3 : \; p, \Box^*(p \vee q) \Rightarrow \Box(\neg q \vee p), q}\,(\Rightarrow \Box)
$$

$$
\dfrac{\dfrac{p \vee q, \Box^*(p \vee q) \Rightarrow p, \Box q}{S_p^1 : \; \Box(p \vee q) \Rightarrow p, \Box q}\,(\Box \Rightarrow)}{\Box(p \vee q) \Rightarrow \Box p, \Box q}\,(\Rightarrow \Box)
$$

S_c^5 is terminating because S_p^3 is its restrictor. Using the proof of Theorem 3, we construct $\mathcal{M} = (W, V)$: a state w_i is added in W for each path $S_p^i \nearrow S_c^{i+1}$, where $1 \leq i \leq 4$; hence $W = \{w_1, w_2, w_3, w_4\}$; finally, $V(w_1) = \{q\}$, $V(w_2) = \{p\}$, $V(w_3) = \{q\}$, and $V(w_4) = \{p\}$. It is true that

$$\mathcal{M}, w_1 \not\models \quad \Box(p \vee q) \Rightarrow \Box p, \Box q.$$

6 Conclusion

The cut-free Gentzen-type sequent calculus **GS5** for **S5** with a restriction on backward applications of $(\Box \Rightarrow)$ has been considered in the paper. The termination of any backward proof search has been achieved by introducing the notion of a restrictor for sequents that have the shape of a conclusion of $(\Rightarrow \Box)$. Using the termination, we have proved that **GS5** is complete for **S5**. It has been proved that all rules of **GS5** are invertible. Using **GS5**, we have described a decision procedure for **S5** based on the soundness and completeness, termination of backward proof search, and invertibility of rules of **GS5**. If a sequent is not valid, then the decision procedure and proof of Theorem 3 allow us to get its counter-model.

Loops are easy to define and use in **GS5** backward proof-search trees. For example, a path $S \nearrow S'$ is a loop if the sequents $|S|$ and $|S'|$ coincide. Such loops were used in Section 3 to define restrictors. For future work, we plan to use the calculus **GS5** in the construction of cyclic, cut-free and decidable proof systems of logics involving **S5**, such as the logic of common knowledge over **S5** or the fixpoint logic for reasoning about knowledge and time [8].

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