



# Article Belyi Maps from Zeroes of Hypergeometric Polynomials

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**Abstract:** The evaluation of low-degree hypergeometric polynomials to zero defines algebraic hypersurfaces in the affine space of the free parameters and the argument of the hypergeometric function. This article investigates the algebraic surfaces defined by the hypergeometric equation  $_2F_1(-N, b; c; z) = 0$  with N = 3 or N = 4. As a captivating application, these surfaces parametrize certain families of genus 0 Belyi maps. Thereby, this article contributes to the systematic enumeration of Belyi maps.

Keywords: Gauss hypergeometric function; Belyi map (of genus 0); elliptic surfaces

MSC: 33C05; 57M12; 14J27

# 1. Introduction

The systematic cataloging of Belyi maps is an active research undertaking [1–3]. Contributing to this venture, this article considers comprehensively Belyi maps  $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$  of the form

$$\varphi(x) = (1-x)^p (1-\lambda x)^q G_m(x)^r \tag{1}$$

or

 $\varphi(x) = (1 + \alpha x + \beta x^2)^p G_m(x)^r, \qquad (2)$ 

where  $G_m(x)$  is a polynomial of degree *m* without multiple roots, and

$$\varphi(x) = 1 + O(x^{m+2})$$
 (3)

asymptotically. This implies  $G_m(0) = 1$ . The powers p, q, r are allowed to be positive or negative integers, and they are assumed to be different. If the powers are positive integers, then  $\varphi(x)$  is a polynomial. Recall that polynomial Belyi maps are called *Shabat polynomials* ([4] §2.2) or *generalized Chebyshev polynomials* [5]. We are particularly interested in Belyi maps defined over  $\mathbb{Q}$ .

Assuming that the point  $x = \infty$  is above  $\varphi \in \{0, \infty\}$ , the distinct points in the three canonical fibers  $\varphi \in \{0, 1, \infty\}$  are as follows:

- Two roots of  $(1 x)(1 \lambda x)$  or  $1 + \alpha x + \beta x^2$ , of branching order |p| or |q|, above  $\varphi = 0$  or  $\varphi = \infty$  depending on the signs of p, q;
- The roots of  $G_m(x)$ , of branching order |r|, above  $\varphi \in \{0, \infty\}$ ;
- The point  $x = \infty$ , of branching order  $|p + q + mr| \neq 0$  or  $|2p + mr| \neq 0$ , respectively, for the two forms, above  $\varphi \in \{0, \infty\}$ ;
- The point x = 0, of branching order m + 2, above  $\varphi = 1$ ;
- d (m+2) non-branching points above  $\varphi = 1$ , where *d* is the degree of  $\varphi$ .

In total, we count

$$2 + m + 1 + 1 + (d - m - 2) = d + 2 \tag{4}$$



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Copyright: © 2025 by the author. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/ licenses/by/4.0/). distinct points in the three fibers, which is the minimal number of points in three fibers for a covering  $\mathbb{P}^1 \to \mathbb{P}^1$  by the Riemann–Hurwitz formula ([6] Theorem 5.9), and the number of points that Belyi maps of genus 0 must have in the three fibers. This will be recapped in Lemma 3.

As it turns out, the considered families of Belyi maps are parametrized by the zero sets of hypergeometric polynomial equations, namely

$${}_{2}F_{1}\left(\begin{array}{c} -m-1, \frac{q}{r} \\ -m-\frac{p}{r} \end{array} \middle| \lambda\right) = 0$$
(5)

for the Belyi maps of the form (1), and

$${}_{2}F_{1}\left(\begin{array}{c}-\frac{m+1}{2},-\frac{m}{2}\\-m-\frac{p}{r}\end{array}\right|\frac{4\beta}{\alpha^{2}}\right)=0$$
(6)

for the form (2). These hypergeometric equations were derived earlier by Adrianov ([5] Propositions 3.5, 3.8) in the context of Shabat polynomials. The equations give the generic number of distinct Belyi maps (up to Möbius transformations) of the considered branching patterns, namely, m + 1 for fixed p, r, q, m in the form (1), or  $\lceil m/2 \rceil$  for fixed p, q, m in the form (2). As we show in Lemmas 1 and 2, this number is smaller when one of the quotients p/r, q/r is a larger negative integer. It appears that the hypergeometric polynomials in (5), (6) are irreducible over  $\mathbb{Q}$  typically, giving the maximal Galois orbits of Belyi maps with the considered branching patterns.

The considered Belyi maps constitute a borderline easy case that continues extensively the examples in ([4] §2.2) and [7]. The new examples and methodology will not be radically novel for active readers of [4,7]. In particular, we encounter new families of Belyi maps defined over  $\mathbb{Q}$  that are parametrized by the following:

- Rational points on an elliptic curve, as in ([4] §2.2.4.3); see Examples 8, 13 and 14 here.
- Integer solutions of Pell's equation, as in ([7] Ch. 10); see Example 18.
- Rational points on a cubic surface or a cubic pencil; see Examples 4, 5.
- Integral points on elliptic curves or elliptic surfaces; see Example 9.

The investigation of this article starts with recalling the relevant properties of hypergeometric functions. Section 2 introduces the hypergeometric  $_2F_1$ -function, the Pochhammer symbol ( $\alpha$ )<sub>*m*</sub>, and the key properties of hypergeometric polynomials that we use. Sections 3 and 4 consider the algebraic surfaces defined by the equation

$${}_{2}\mathrm{F}_{1}\left(\begin{array}{c}-N,\ b\\c\end{array}\middle|z\right) = 0\tag{7}$$

with N = 3 or N = 4. We modify the lower parameter to -c + 1 - N, so that after clearing the denominators, we have the symmetric identity

$$(c)_{m 2} F_1 \left( \begin{array}{c} -N, b \\ -c+1-N \end{array} \middle| z \right) = (b)_m z^N {}_2 F_1 \left( \begin{array}{c} -N, c \\ -b+1-N \end{array} \middle| \frac{1}{z} \right)$$
(8)

of the same hypergeometric summation in the opposite directions. The parameters b, c are, thereby, interchangeable in combination with the Möbius transformation  $z \mapsto 1/z$ . These equations with  $N \in \mathbb{Z}$  define algebraic surfaces in the affine space defined by the parameters b, c, z as coordinates. Further symmetries of these hypergeometric polynomials are presented in (11)–(14).

The technical contribution of this article is the careful consideration of the hypergeometric polynomial equations, including their degenerations, and analysis of their algebraic structure and elliptic fibrations. The algebraic surfaces defined by the cubic or quartic Equation (7) are investigated in Sections 3 and 4, respectively. As Belyi maps are defined for  $b, c \in \mathbb{Q}$  and  $z \in \overline{\mathbb{Q}}$  (but preferably for us,  $z \in \mathbb{Q}$ ), we are particularly interested in the rational points and elliptic fibrations on these algebraic surfaces.

The later sections describe the Belyi maps of the forms (1) and (2). Section 5 presents the easy examples of Belyi maps that the considered two forms generalize. These easy examples are largely covered in ([4] §2.2). Section 6 considers Belyi maps of the form (1) using rational parametrizations of the examined surfaces (7). The cases with fewer than N Belyi maps in Examples 8 and 9 are parametrized by particular elliptic curves on the same surfaces. Section 7 considers Belyi maps of the form (2). This leads to considering rational or integral points on some other elliptic curves on the same surfaces. Our thorough investigation of algebraic surfaces (7) allows to refine the examples in ([4] §2.2.4.3) and ([7] Ch. 10), and to explore beyond them.

## 2. Hypergeometric Details

The Gauss hypergeometric function ([8] Ch. 2) is defined by the series

$${}_{2}\mathrm{F}_{1}\left(\begin{array}{c|c}a, b\\c\end{array}\middle| z\right) = \sum_{k=0}^{\infty} \frac{(a)_{k}(b)_{k}}{(c)_{k}k!} z^{k}.$$
(9)

Here,  $(\alpha)_k = \alpha (\alpha + 1)(\alpha + 2) \dots (\alpha + k - 1)$  denotes the *Pochhammer symbol*, or the "raising" variant of the factorial. Using the  $\Gamma$ -function ([8] Ch. 1) we can write

$$(\alpha)_k = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}.$$
(10)

The standard analytic continuation of the  $_2F_1$ -function is onto  $\mathbb{C} \setminus [1, \infty)$ . This function is undefined when *c* equals zero or a negative integer.

## 2.1. Hypergeometric Polynomials

If well defined, the hypergeometric series (9) is a polynomial in z when a or b are non-positive integers. In addition to (8), we have these symmetries of hypergeometric polynomials:

$$(c)_{N} {}_{2}F_{1} \left( \begin{array}{c} -N, b \\ -c+1-N \end{array} \middle| z \right) = (-1)^{N} (b+c)_{N} {}_{2}F_{1} \left( \begin{array}{c} -N, b \\ b+c \end{array} \middle| 1-z \right)$$
(11)

$$= (b+c)_N (-z)^N {}_2F_1 \left( \begin{array}{c} -N, c \\ b+c \end{array} \right| 1 - \frac{1}{z} \right)$$
(12)

$$= (c)_{N} (1-z)^{N} {}_{2}F_{1} \left( \begin{array}{c} -N, e \\ -c+1-N \\ \end{array} \right) \left( \begin{array}{c} z \\ z-1 \\ \end{array} \right)$$
(13)

$$= (b)_N (z-1)^N {}_2F_1 \left( \begin{array}{c} -N, e \\ -b+1-N \end{array} \middle| \frac{1}{1-z} \right).$$
(14)

where e = -b - c + 1 - N. The following expression defines a polynomial of degree  $\lfloor N/2 \rfloor$  for any integer  $N \ge 0$ :

$${}_{2}F_{1}\left(\begin{array}{c}-\frac{N}{2},-\frac{N-1}{2}\\c\end{array}\middle|z\right).$$
(15)

If *c* is a non-negative integer but *a* or *b* is a non-negative integer  $\geq c$ , the hypergeometric summation (9) can be considered a polynomial of degree -a or -b. This interpretation is often used in theory of orthogonal polynomials [9]. We adopt this interpretation of hypergeometric functions like

$${}_{2}F_{1}\left(\begin{array}{c}-n, \ b\\-n-m\end{array}\middle|z\right) \tag{16}$$

with integer  $n, m \ge 0$ . The well-known identities

$${}_{2}\mathrm{F}_{1}\left(\begin{array}{c}a,b\\c\end{array}\right|z\right) = (1-z)^{-a} {}_{2}\mathrm{F}_{1}\left(\begin{array}{c}a,c-b\\c\end{array}\right|\frac{z}{z-1}\right) \tag{17}$$

$$= (1-z)^{c-a-b} {}_{2}F_{1} \begin{pmatrix} c-a, c-b \\ c \\ \end{pmatrix}$$
(18)

of Pfaff and Euler ([8] Th. 2.2.5) should not be automatically applied then; see ([10] Lemma 3.1). Yet these identities are valid when all three numbers *a*, *b*, *c* are non-negative integers, and  $-c \ge \max(-a, -b)$ ; see ([10] §9).

### 2.2. Distinctive Roots of Hypergeometric Polynomials

The distinctiveness of the roots of hypergeometric polynomials is easily proved by using *contiguous relations* ([8] §2.5, §3.7) of  $_2F_1$ -functions. In particular, we use the observations in the following two lemmas. Complementarily, one may consider the relation of hypergeometric polynomials to classical orthogonal polynomials [9] and the interlacing properties of the zeroes of orthogonal polynomials ([8] §5.4).

**Lemma 1.** Consider the sequence of hypergeometric polynomials

$$P(k) = {}_{2}\mathbf{F}_{1}\left(\begin{array}{c} -k, \ b \\ 1-k-c \end{array} \middle| z\right), \tag{19}$$

with some  $b, c \in \mathbb{C}$ . Suppose that for a positive integer k, we have  $(b)_k \neq 0$ ,  $(c)_k \neq 0$ , and  $(b+c)_k \neq 0$ . Then, we have the following:

- (*i*) The polynomials P(k) and P(k-1) do not have common roots.
- (*ii*) The polynomial P(k) has k distinct roots.
- (iii) z = 1 is not a root of P(k).

**Proof.** The sequence P(k) satisfies the recurrence relation

$$\frac{k(k-1+b+c)z}{k-1+c} P(k-1) = (kz+k+bz+c) P(k) - (k+c) P(k+1).$$
(20)

This follows from the contiguous relations [11], or by applying Zeilberger's algorithm ([8] §3.11), or by checking the series expansion. We apply the recurrence with decreasing k, and use the assumption that  $(b + c)_k \neq 0$ . The recurrence implies that a common root of P(k) and P(k - 1) is either z = 0, which conflicts with the hypergeometric values 1 at z = 0, or it would be a common root of  $P(k - 2), \ldots$ , and of P(0) = 1 contradictorily.

The second statement follows from the degree of *k* of P(k) thanks to  $(b)_k \neq 0$ , and the fact that another contiguous relation

$$\frac{1-z}{k}\frac{dP(k)}{dz} + P(k) = \frac{k-1+b+c}{k-1+c}P(k-1)$$
(21)

implies that a multiple root of P(k) would be also a root of P(k-1), leading to the established first statement.

The last statement follows from the Chu-Vandermonde identity ([8] p. 67)

$${}_{2}\mathrm{F}_{1}\left(\begin{array}{c}-k,B\\C\end{array}\middle|1\right) = \frac{(C-B)_{k}}{(C)_{k}},$$
(22)

giving the value  $(b + c)_k / (c)_k$  for P(k) evaluated at z = 1.  $\Box$ 

Lemma 2. Consider the sequence of hypergeometric polynomials

$$P(k) = {}_{2}F_{1} \left( \begin{array}{c} -\frac{k}{2}, -\frac{k-1}{2} \\ 1-k-c \end{array} \middle| z \right),$$
(23)

with some  $c \in \mathbb{C}$ . Suppose that for a positive integer k we have  $(c)_k \neq 0$  and  $(2c)_k \neq 0$ . Then, we have the following:

- (*i*) The polynomials P(k) and P(k-1) do not have common roots.
- (*ii*) The polynomial P(k) has  $\lfloor k/2 \rfloor$  distinct roots.
- (iii) z = 1 is not a root of P(k).

**Proof.** The sequence P(k) satisfies the recurrence relation

$$P(k) - P(k+1) = \frac{kz \left(2c + k - 1\right)}{4(c+k)(c+k-1)} P(k-1).$$
(24)

Under the assumption that  $(b + c)_k \neq 0$ , the recurrence shows that a common root of P(k) and P(k + 1) is either the rejectable z = 0, or it would be a common root of P(k - 1),  $P(k - 2), \ldots$ , and of P(0) = 1 contradictorily.

The second statement follows from the degree of *k* of  $c_k$ , thanks to  $(a)_k \neq 0$ , and the fact that another contiguous relation

$$\frac{2(1-z)}{k}\frac{dP(k)}{dz} + P(k) = \frac{2c+k-1}{2(k+c-1)}P(k-1)$$
(25)

implies that a multiple root of P(k) would be also a root of P(k+1), leading to the established first statement.

The last statement follows from applying the Chu–Vandermonde identity (22). The evaluation at z = 1 is

$$\left(c+\frac{1}{2}\right)_{\lfloor\frac{k}{2}\rfloor} \left/ \left(c+\left\lceil\frac{k}{2}\right\rceil\right)_{\lfloor\frac{k}{2}\rfloor} \right. \tag{26}$$

for both even and odd k.  $\Box$ 

# 3. Cubic Hypergeometric Polynomials

After clearing the denominators in the equation  $_{2}F_{1}\begin{pmatrix} -3, b \\ -c-2 \end{pmatrix} z = 0$ , we obtain

$$b(b+1)(b+2)z^{3} + 3bc(b+1)z^{2} + 3bc(c+1)z + c(c+1)(c+2) = 0,$$
(27)

or rather more compactly,

$$(bz+c)^3 + 3(bz+c)(bz^2+c) + 2(bz^3+c) = 0.$$
(28)

Let  $S_3$  denote the algebraic surface defined by (27), of degree 6.

Fixing *z* leads to a cubic equation in *b*, *c*. For  $z \in \{0, 1\}$ , the equation factorizes into three linear factors. For  $z \in \{-1, 2, \frac{1}{2}\}$ , it factorizes into quadratic and linear parts. For other fixed *z*, we obtain an irreducible cubic curve of genus 0. Here is a parametrization of those cubic curves by e = bz + c:

$$b = \frac{e(e+1)(e+2)}{z(1-z)(3e+2z+2)}, \qquad c = \frac{e(e+z)(e+2z)}{(z-1)(3e+2z+2)}.$$
(29)

This also gives a birational parametrization of  $S_3$  by z and e. If 3e + 2z + 2 = 0 in the denominators, then the polynomial in (27) can be factored as

$$-\frac{4}{27}(z+1)(z-2)(2z-1).$$
(30)

Both numerators in (29) equal 0 as well at the exceptional three points  $(e, z) \in \{(0, -1), (-2, 2), (-1, \frac{1}{2})\}$ . The three points are blown up to these lines on  $S_3$ :

$$b = c, z = -1;$$
  $2b + c + 2 = 0, z = 2;$   $b + 2c + 2 = 0, z = \frac{1}{2}.$  (31)

We are interested in the points on  $S_3$  with both b, c being positive, while  $z \notin \{0, 1\}$ and  $b \neq c$ . The dependence of the signs of b, c on the parameters e, z by the parametrization (29) is depicted in Figure 1a. There are two regions marked by + where b and c are positive; they are delineated by

$$z < 0$$
 and  $e(3e+2z+2) < 0.$  (32)

The exceptional lines (30) do not apply to this interest.



**Figure 1.** The signs of *b* and *c* depending on (**a**) *e*, *z* in formula (29), and (**b**) *t*, *y* in formula (40). The upper sign in  $\pm$ ,  $\mp$ , = is for *b*, while the lower one is for *c*. There sign + indicates that both *b*, *c* are positive.

### 3.1. Complete Factorization for Some b, c

Suppose that the parameters b, c have the form (29) for some  $z = s \in \mathbb{Q}$  and  $e \in \mathbb{Q}$ . Considering (27) as a cubic polynomial in z, it then has the root z = s. For generic values of e, s, the other two roots solve the quadratic equation

$$\frac{e^2 - 2es - 2s^2 + 3e + 2}{e + 2s - 2}z^2 - (2e + s + 1)sz + \frac{e^2 + 3es + 2s^2 - 2e - 2}{e - 2s + 2}s^2 = 0.$$
 (33)

We are interested when those other roots are in  $\mathbb{Q}$  as well. For that, the discriminant

$$\frac{3s^2(s-1)^2(e+2s+2)(3e+2s+2)}{(e+2s-2)(e-2s+2)}$$
(34)

must be a full square. We apply the birational transformation

$$e = 2 \frac{\xi - 1}{\eta - \xi}, \qquad s = \frac{\eta + 1}{\eta - \xi}, \tag{35}$$

and the discriminant becomes

$$\frac{3\eta(1-\xi-\eta)(\xi+1)^2(\eta+1)^2}{\xi(\eta-\xi)^4}.$$
(36)

The values  $\xi$ ,  $\eta$  should therefore determine a Q-rational point on the cubic surface

$$3\varrho^2 = \xi \eta (1 - \xi - \eta). \tag{37}$$

This cubic surface has three singularities  $(\xi, \eta, \varrho) \in \{(0, 0, 0), (0, 1, 0), (0, 0, 1)\}$ . By intersecting this surface with the pencil of lines  $\xi = t\varrho$ ,  $\eta = y\varrho$  through the singularity (0, 0, 0), we obtain this birational parametrization by  $(t, y) \in \mathbb{P}^2$ :

$$\xi = \frac{ty-3}{y(y+t)}, \qquad \eta = \frac{ty-3}{t(y+t)}, \qquad \varrho = \frac{ty-3}{ty(y+t)}.$$
 (38)

This translates to

$$e = \frac{2t(y^2 + 3)}{(t - y)(ty - 3)}, \qquad s = \frac{y(t^2 + 2ty - 3)}{(y - t)(ty - 3)},$$
(39)

and ultimately to

$$b = -\frac{(t^2+3)(y^2+3)(ty+3)}{3(t^2+2ty-3)(y^2+2ty-3)}, \qquad c = \frac{(y^2+3)(t^2y-3y-6t)}{3(y-t)(y^2+2ty-3)}.$$
 (40)

This parametrizes the cases when the cubic polynomial (27) factors completely. The special case y = -t gives the following neat family:

$$b = \frac{1}{3}t^2 - 1, \qquad c = -\frac{1}{6}t^2 - \frac{1}{2}.$$
 (41)

The algebraic map (39) is 2-to-1 generically, as the central symmetry  $(y, t) \mapsto (-y, -t)$  keeps *e*, *s* invariant and permutes the roots of (33). Up to this symmetry, the inverse map to (39) is

$$y = \sqrt{\frac{3(e+2s+2)(e-2s+2)}{(e+2s-2)(3e+2s+2)}}, \qquad t = \sqrt{\frac{3(e+2s-2)(e-2s+2)}{(e+2s+2)(3e+2s+2)}}.$$
 (42)

There is no point  $(y, t) \in \mathbb{R}^2$  where both *b* and *c* are positive by the parametrization (40); see Figure 1b, and note that the only points  $(y, t) \in \{(3, -1), (-3, 1)\}$  where both *b*, *c* are undetermined by (40) give (e, s) = (-1, -3) in (39) and hence are rejectable (b, c) = (0, -1) in (29).

The following map moves z = s to another root of (27):

$$(y,t) \mapsto \left(\frac{y-3}{y+1}, \frac{t-3}{t+1}\right).$$

$$(43)$$

By composition with the central symmetry (on either or both sides), one realizes all six permutations of the roots of (27) by fractional linear transformations of the parameters y, t.

## 3.2. Elliptic Fibration by b or c

For fixed  $b \notin \{0, -1, -2\}$ , Equation (27) defines a cubic curve of genus 1. Its *j*-invariant equals

$$-\frac{27 b^2 (9b^2 + 32b + 32)^3}{64 (b+1)^3 (b+2)^2}.$$
(44)

We can consider  $S_3$  as an elliptic surface ([12] III.11) over  $\mathbb{Q}(b)$ . We are interested in its basic  $\mathbb{Q}$ -arithmetic properties [6,12]. The following propositions give an isomorphic Weierstrass form for the elliptic surface, and a characterization of rational curves on it in terms of the Mordell–Weil group.

**Proposition 1.** For  $b \notin \{0, -1, -2\}$ , the cubic relation (27) between *c* and *z* defines a curve of genus 1, isomorphic to the elliptic curve

$$\mathcal{E}_3: \quad v^2 = u^3 + b^2 \left(3u - 16b - 16\right)^2 \tag{45}$$

by the isomorphism

$$c = -(b+2)\frac{v+3bu-16b(b+1)}{2v}, \qquad z = \frac{v+(3b+4)u-16b(b+1)}{2v}.$$
 (46)

The Mordell–Weil group for (45) of rational points over  $\mathbb{Q}(b)$  is isomorphic to  $(\mathbb{Z}/3\mathbb{Z}) \times \mathbb{Z}$ . It is generated the torsion point (0, 16b(b+1)) and the point (8b, 8b(b+2)).

**Proof.** It is straightforward to check the isomorphism (46). The inverse map is given as

$$u = \frac{16b(b+1)(bz+2z+c)}{bc+(b+2)(3bz+2c+2)}, \qquad v = \frac{32b(b+1)(b+2)}{bc+(b+2)(3bz+2c+2)}.$$
(47)

As a side note, the elliptic involution  $v \mapsto -v$  corresponds to the hypergeometric symmetry (11) under this isomorphism.

The canonical Weierstrass form  $v^2 = u^3 + Au + B$  can be obtained by applying the shift  $u \mapsto u - 3b^2$  in (45). The discriminant ([12] III.1) of the elliptic surface equals  $\Delta = 2^{18}3^3b^4(b+1)^3(b+2)^2$ . By considering  $A, B, \Delta$  and the *j*-invariant in (44), we can use ([13] Table 5.1) and conclude that the singular fibers  $b \in \{0, -1, -2, \infty\}$  have the Kodaira types IV, I<sub>3</sub>, I<sub>2</sub>, I<sub>3</sub>, respectively. By [13] (Fig. 5.1–5.2), they have, respectively,  $m_v = 3, 3, 2, 3$ irreducible components in the Kodaira–Neron model. By the Shioda–Tate formula ([13] Corollary 6.7), the Mordell–Weil rank equals 10 - 2 - 7 = 1, as the Neron–Severi rank  $\rho$  is 10 for rational elliptic surfaces ([13] §7.2). By [13] (Theorem 8.33), the Mordell–Weil group is generated by polynomial points (u, v) with deg  $u \leq 2$ , deg  $v \leq 3$ . Direct computation with undetermined coefficients gives eight possibilities for such u:

$$u \in \{0, 8b, -16b, 4(b+1), 16(b+1), -8b(b+1), 16b(b+1), \frac{16}{9}(1-b)(1+2b)\}.$$
 (48)

The points with u = 0 are flex points with the tangents  $\pm v = 3bu - 16b(b+1)$ ; hence, they are 3-torsion points. An investigation of the group relations between all candidate points shows that we can take a point with  $u \in \{8b, 16(b+1), -8b(b+1)\}$  for a free generator.  $\Box$ 

Taking  $b \in \mathbb{Q} \setminus \{0, -1, -2\}$  in (45) gives an elliptic curve over  $\mathbb{Q}$ , with the same Mordell–Weil group typically ([12] III.11). But the Mordell–Weil group group over  $\mathbb{Q}$  could be larger than the projected set of points from the elliptic surface. For example, b = -7 gives the Mordell–Weil group  $(\mathbb{Z}/3\mathbb{Z}) \times \mathbb{Z}^2$ , as we will consider in Example 8.

It is instructive to consider several of the simplest sections (u, v) on  $\mathcal{E}_3$ , parametrized by b, and map them to the original surface  $\mathcal{S}_3$  defined by (27). Here are a few obtained non-degenerate c-values of low degree in b:

$$\frac{b+2}{4b-1}, -\frac{4b(b+2)}{5b+1}, -\frac{27b(b+1)}{16b^2+19b+1}, -\frac{(2b+1)^3}{7b^2+b+1}, -\frac{2(b-1)^3}{2b^2+17b-1}.$$
 (49)

After substituting these values into (27), there is a linear factor in *z*. That parametrizes fully the corresponding section in (c, z) on  $S_3$ .

If we have a root described by (46) of the cubic hypergeometric polynomial with fixed *c* (and possibly *b*), the other two roots are equal to

$$z_{2,3} = \frac{1}{2v} \left( v + (3b - 2)u - 16b(b + 1) \pm u \sqrt{\frac{3u}{b + 1} - 12} \right).$$
(50)

To seek the rationality of these roots, we may replace  $u = (b + 1)(w^2 + 12)/3$  and obtain a family of hyperelliptic curves (w, v) of genus 2. One may go through a limited list of Mordell–Weil points on  $S_3$  (be it specialized, with b = -5/2 for example) and check that apparently, the complete factorization of the cubic polynomial happens generically only in degenerate cases such as  $c \in \{-1, -b - 1\}$ . We need *b* to be expressible as in (40). The family (41) corresponds to the infinite point on the elliptic surface.

For fixed  $c \notin \{0, -1, -2\}$ , Equation (27) has degree six, but it defines an isomorphic elliptic surface by the symmetry (8) of hypergeometric polynomials. One can take  $b \leftrightarrow c$ ,  $z \mapsto 1/z$  in Proposition 1.

## 4. Quartic Hypergeometric Polynomials

A compact polynomial form of the equation  $_{2}F_{1}\begin{pmatrix} -4, b \\ -c-3 \end{pmatrix} z = 0$  is

$$\left((bz+c)^2+3bz^2+3c\right)^2+2(bz^2+c)^2+8bcz(z-1)^2+6(bz^4+c)=0.$$
(51)

Let us denote this surface by  $S_4$ . We will associate two elliptic surfaces  $\mathcal{E}_4$  and  $\mathcal{E}_4^*$  to it.

#### 4.1. Elliptic Fibration by z

For fixed *z*, the curve has genus 1 generically. We identify  $S_4$  directly as an elliptic surface. As such, it is isomorphic to

$$\mathcal{E}_4: \quad v^2 = u^3 - 20Z \, u^2 + 108Z^2 \, u - 648(Z-1)^2, \tag{52}$$

where  $Z = z^2 - z + 1$ . The *j*-invariant is rather untidy in the denominator:

$$-\frac{2^{8} 19^{3} Z^{6}}{3^{4} (72 Z^{6} - 860 Z^{5} + 3907 Z^{4} - 9608 Z^{3} + 13122 Z^{2} - 8748 Z + 2187)}.$$
(53)

An isomorphism is given by

$$b = -1 + \varphi_4(z, u, v), \qquad c = -1 + \varphi_4\left(\frac{1}{z}, \frac{u}{z^2}, \frac{v}{z^3}\right)$$
(54)

with

$$\varphi_4(z,u,v) = \frac{2(u^2 - 4Z^2 u + 8Z^3 - 12Z^2)(1 - 2z)v + 2W_1}{z(1-z)(u^2 - 12Zu + 12Z^2)^2},$$
(55)

where

$$W_1 = (5Z - 3)u^3 + 6(2Z^3 - 24Z^2 + 27Z - 9)u^2$$

$$- 4Z(20Z^3 - 159Z^2 + 171Z - 5)u + 72Z^2(Z - 1)(2Z^2 - 12Z + 9).$$
(56)

The infinite point on (52) is mapped to the central point (b, c) = (-1, -1) among the degenerations. The parametric expression for bz + c simplifies to

$$bz + c = \frac{2Zv - (z+1)\left(u^2 - 2(8Z - 3z)u + 36(z-1)^2Z\right)}{u^2 - 12Zu + 12Z^2}.$$
(57)

As we consider in the next subsection,  $\mathcal{E}_4$  is a rational elliptic surface. Its Mordell–Weil group has no torsion, and the maximal rank 8 for elliptic surfaces over  $\mathbb{CP}^1$ , as is evident ([13] §7.3) from the irreducible degree 12 denominator of the *j*-invariant (53), meaning that the surface  $\mathcal{E}_4$  has 12 singular fibers of Kodaira type  $I_1$ . By [13] (Theorem 7.12 (*i*)), there are 240 candidate points with a polynomial coordinate  $u(z) \in \mathbb{C}[z]$  of degree  $\leq 2$  to generate the Mordell–Weil group. Computations show that 60 of them are defined over  $\mathbb{Q}(z)$ . They have

$$u \in \{6, 9, 18, 54, 9z^2 - 6z + 9, 18z^2 + 12z + 18, 2z^2 + 4z + 6, 6z^2 + 12\},$$
(58)

or can be obtained by further applying the hypergeometric symmetries (8), (11)–(14), generated by  $u(z) \mapsto z^2 u(1/z)$  and  $u(z) \mapsto u(1-z)$ . Further, the two points with  $u = 8z^2 - 8z + 8$  are defined over  $\mathbb{Q}(\sqrt{6}, z)$ , and the points with u = 0 are defined over  $\mathbb{Q}(\sqrt{-2}, z)$ . Additionally, 64 points have  $u \in \mathbb{Q}(\sqrt{6})[z]$ , 64 points have  $u \in \mathbb{Q}(\sqrt{-2})[z]$ , and 48 points have  $u \in \mathbb{Q}(\sqrt{6}, \sqrt{-2})[z]$ . The Mordell–Weil group over  $\mathbb{Q}(z)$  has rank 6. It is generated by points with  $u \in \{6, 6z^2, 6(z-1)^2, 9, 9z^2, 9(z-1)^2\}$ . Examples of simpler rational sections on the original surface (51) have these non-degenerate values of *b*:

$$\frac{(z+1)(z+2)}{3z(1-z)}, \quad \frac{3(z+1)(3z+2)}{z(1-z)}, \quad \frac{6(z+1)(3z-2)}{25z(1-z)}, \quad \frac{2(2z+1)(4z-3)}{z(1-z)(2z^2-3)^2}, \\ \frac{(2-z)(z^2-4z+1)(z^2-6z+3)}{z(5z^2-2z-1)^2}, \quad \frac{2z(2z+1)(6z^3+z^2-2)}{(1-z)(3z^2-2)^2}.$$
(59)

The corresponding *c*-coordinate is obtained from a linear factor of (51) that arises after substituting *b*. Due to the hypergeometric symmetry (8), some possible values for *c* can be obtained by substituting  $z \mapsto 1/z$  in (59).

#### 4.2. The Rational Surface

The discriminant of the elliptic surface (52) is of degree 12 in *z*; see the denominator of (53). Therefore, it is a rational surface ([13] §7.5) that can be obtained from a pencil of cubic curves in  $\mathbb{P}^2$  (linearly parametrized by *z*) by blowing up the nine intersection points of the pencil. Rather equivalently [14], it is a del Pezzo surface in the weighted projective space with weights(z : 1 : u : v) = (1 : 1 : 2 : 3), of degree 1.

The surface  $S_4$  is birational to the elliptic surface (52) in the Weierstrass form; hence, it can also be obtained from the pencil of cubic curves in  $\mathbb{P}^2$ . To obtain a rational parametrization of  $S_4$ , we first notice the simpler defining equation

$$f(3f + 6(z^2 + z + 1) + 8ez + 6e^2 + 14e) + e(e+1)(e+2)(e+3) = 0$$
(60)

in the coordinates

$$e = bz + c, \qquad f = bz(z - 1).$$
 (61)

The new equation defines a del Pezzo surface in the weighted projective space with weights (z : e : 1 : f) = (1 : 1 : 1 : 2), of degree 2. It contains four lines in the hyperplane f = 0. Choosing the line f = e = 0 to blow down, we apply a standard step (called *unprojection* in ([14] §5)) in resolving del Pezzo surfaces by introducing the coordinate

$$w = \frac{f + 2(z^2 + z + 1)}{e}.$$
 (62)

We can eliminate *f* straightaway and obtain the following non-singular cubic surface in  $\mathbb{P}^3$ :

$$(ew - 2z^2 - 2z - 2)(3w + 8z + 6e + 14) + (e + 1)(e + 2)(e + 3) = 0.$$
 (63)

Subsequently, we can blow down one of the lines on the plane 3w + 8z + 6e + 14 = 0, or apply a classical parametrization recipe ([13] §10.5.3) using two skew lines on the cubic surface, say, e = 3w + 8z = -2 and w = e + 1 = -z - 1. Eventually, a suitable simplified cubic pencil is defined by U + zV = 0, where

$$U = 2t^2y - 6t^2 + 4ty - 3y^2 + 3y,$$
(64)

$$V = ty^2 + 4t^2 - 2ty + 3t - 6y.$$
(65)

A parametrization of  $S_4$  is given by z = -U/V and

$$b = \frac{3(y^2 - 2t + y)(ty^2 - 8t^2 + 4ty - 3y^2 + 3t + 3y)}{U(U + V)},$$
(66)

$$c = -\frac{6(4t^2 + 2t - 3y - 3)(t^2y - t^2 + ty - y^2)}{V(U + V)}.$$
(67)

The inverse map is given by

$$t = \frac{3(b+1)}{2} - \frac{3(c+1)(c+2)}{2bz} + \frac{3(b+c+1)(b+c+2)}{2b(z-1)},$$
(68)

$$y = -c + \frac{(b+2)(b+3)z}{c} + \frac{(b+c+2)(b+c+3)z}{c(z-1)}.$$
(69)

Computations with two versions of (t, y) in (64)–(66) show that the hypergeometric symmetries (8), (11)–(14) are realized by non-linear Cremona [15] transformations of the parameters t, y. For example, (8) is realized by

$$(t,y) \mapsto \left(\frac{t(y^2 - ty^2 - 3t + 3y)}{2(t^2y - t^2 + ty - y^2)}, \frac{(3 - 2t)(ty^2 - y^2 + 3t - 3y)}{ty^2 - 8t^2 + 4ty - 3y^2 + 3t + 3y}\right),\tag{70}$$

while (11) is realized by

$$(t,y) \mapsto \left(t, \frac{ty - 4t^2 - 3t + 3y}{ty - t - 3}\right).$$
 (71)

#### 4.3. The Fibration by b

Equation (51) with fixed b defines a quartic curve of the generic genus 3. We can reduce the genus to 1 by factoring out the hypergeometric symmetry (11). This gives an elliptic surface isomorphic to

$$\mathcal{E}_4^*: \quad v^2 = u \left( u^2 - 4b(5b+9)u + 108 \, b(b+1)^2(b+2) \right). \tag{72}$$

The symmetry invariants are parametrized as follows:

$$z(1-z) = -6 \frac{v + 2(2b+3)u - 36b(b+1)(b+2)}{W_2},$$
(73)

$$c(b+c+3) = -216 \frac{b(b+1)(b+2)(b+3)^2}{W_2},$$
(74)

$$cz + (b+c+3)(z-1) = -(b+3) \frac{u^2 + 2(4b+3)v - 4b(b+3)u + 108b(b+1)(b+2)(b+5)}{W_2},$$
 (75)

where  $W_2 = 8bv + u^2 - 4b(b+9)u + 108b(b+1)(b+2)(b+9)$ . The inverse projection is given by

$$u = -6b(b+1)(b+2)\frac{(bz+c+3z)^2}{c(b+c+3)},$$
(76)

$$v = 12b(b+1)(b+2)(b+3)\frac{(bz+c+3z)(2bz+2c+3)}{c(b+c+3)}.$$
(77)

Rational points on the quartic curves can be found by trying to lift from the rational fibers on the elliptic surface. Its Mordell–Weil group can be determined similarly to the proof of Proposition 1. The discriminant of the elliptic surface equals  $2^{13}3^6b^3(b+1)^4(b+2)^2(b+3)^3$ , and the *j*-invariant equals

$$-\frac{32\left(19b^3 + 36b^2 - 81b - 162\right)^3}{729\left(b+1\right)^4\left(b+2\right)^2\left(b+3\right)^3}.$$
(78)

Using ([13] Table 5.1), we conclude that the singular fibers  $b \in \{0, -1, -2, -3\}$  have the Kodaira types III, I<sub>4</sub>, I<sub>2</sub>, I<sub>3</sub>, respectively. They have, respectively,  $m_v = 2, 4, 2, 3$  irreducible components. The Mordell–Weil rank equals 10 - 2 - 7 = 1. The candidates for the generators have

$$u \in \{0, 9(b+1)^2, \frac{9}{4}(b-1)^2, 6b(b+1), 54b(b+1), 12b(b+2), (79) 2(b+1)(b+2), 18(b+1)(b+2)\}.$$

The Mordell–Weil group is isomorphic to  $(\mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}$ . The group is generated by the 2-torsion point (0,0) and a point having  $u \in \{6b(b+1), 18(b+1)(b+2)\}$ . The point

$$(6b(b+1), 12b(b+1)(b+3))$$
(80)

can be taken as a free generator.

The discriminant of the quadratic Equation (73) for z equals

$$\frac{(v+4bu+12u)^2}{(v+4bu)^2+864b(b+1)(b+2)u}$$
(81)

on  $\mathcal{E}_4^*$ . To find rational sections on the genus 3 curve, we need the denominator

$$(v+4bu)^2 + 864b(b+1)(b+2)u$$
(82)

to be a full square on  $\mathcal{E}_4^*$ . The point (u, v) = 0 leads to

$$c = -\frac{b+3}{2} \left( 1 \pm \sqrt{\frac{b+1}{b+9}} \right), \qquad z = \frac{1}{2} \pm \frac{1}{2} \sqrt{\frac{b+1}{b+9}}.$$
(83)

A rational section is obtained after the base change  $b = (9\zeta^2 - 1)/(1 - \zeta^2)$ .

## 5. Belyi Maps

Recall that Belyi maps are algebraic coverings  $\varphi : C \to \mathbb{P}^1$  that branch in the three fibers  $\varphi \in \{0, 1, \infty\}$ . We consider Belyi of genus 0 only. The following characteristic property of Belyi maps of genus 0 follows from the Riemann–Hurwitz formula.

**Lemma 3.** A Belyi map of genus 0 and degree d has exactly d + 2 distinct points in the 3 fibers  $\{0, 1, \infty\}$ .

As summed up in (4), the considered maps (1) and (2) satisfy the condition of this lemma. The degree of (1) equals

$$d = \max(|p|, |q|, |p+q|, m|r|, |p+mr|, |q+mr|, |p+q+mr|),$$

while the degree of (2) equals  $\max(2|p|, m|r|, |2p + mr|)$ . Without loss of generality, we may assume  $\max(|p|, |q|) \le |p + q + mr|$ .

## Easy Maps

The simplest Belyi maps are the power functions  $\varphi(x) = x^p$ . They have just two points in the two fibers  $\varphi = 0$  and  $\varphi = \infty$ . Our considered rational maps can be viewed as a close neighborhood of this exemplar in the landscape of Belyi maps.

**Example 1.** The Belyi maps with exactly three points in two fibers are easy to find; see [7] §6.6.2 with k = 2. These maps necessarily have two points with some branching orders p, q in one fiber (say,  $\varphi = 0$ ) and one point (say,  $x = \infty$ ) of branching order p + q in the other fiber (say,  $\varphi = \infty$ ). There must be then p + q - 1 distinct points in the third fiber  $\varphi = 1$  by Lemma 3; hence, there, we have exactly one branching point, of order two. The most compact expression for the Belyi map is obtained after choosing this branching point to be x = 0:

$$\varphi(x) = (1 - qx)^p (1 + px)^q \tag{84}$$

$$= 1 + O(x^2). (85)$$

*This form matches the case* m = 0,  $\lambda = p/q$  *of* (1), *where we take* p,q *to be positive, and rescale*  $x \mapsto x/q$ .

**Example 2.** Consider Belyi maps of the more general form

$$\varphi(x) = (1 - qx)^p F_m(x)^q \tag{86}$$

$$= 1 + O(x^{m+1}), (87)$$

where  $F_m(x)$  is a polynomial or degree m, as in [7] §6.6.2. If p, q are positive integers, then these maps have one point of order p, m points of order q above  $\varphi = 0$ , a single point of order p + mqabove  $\varphi = \infty$ , and a branching point of order m + 1 above  $\varphi = 1$ . The other points above  $\varphi = 1$  are non-branching by Lemma 3. By the equality of (86) and (87), we have

$$F_m(x) = (1 - qx)^{-p/q} \mod x^{m+1}.$$
(88)

This means that the polynomial  $F_m(x)$  equals the truncated Taylor series of  $(1-qx)^{-p/q}$  at x = 0. Explicitly,

$$F_m(x) = \sum_{k=0}^m \frac{q^k (p/q)_k}{k!} x^k.$$
(89)

Following our interpretation of hypergeometric polynomials (16), we can write

$$F_m(x) = {}_2F_1 \left( \begin{array}{c} -m, \frac{p}{q} \\ -m \end{array} \middle| qx \right).$$
(90)

**Remark 1.** We may allow p or q to be negative integers in (86). That just means that the corresponding point(s) are above  $\varphi = \infty$  rather than above  $\varphi = 0$ . Formula (88) still holds then. The point  $x = \infty$  remains above  $\varphi = \infty$  if p + mq > 0, and it moves to the fiber  $\varphi = 0$  if p + mq < 0.

**Remark 2.** If we take p = 0 or p + mq = 0, then we have m + 1 (rather than m + 2) points in the two fibers  $\varphi \in \{0, \infty\}$ . The branching order of x = 0 can be at most m then, and (88) should be modified to  $F_m(x) = 1 + O(x^m)$ . We obtain the Belyi maps  $\varphi(x) = (1 - x^m)^q$ , after scaling x additionally.

**Example 3.** Now we look at the Belyi maps of the form

$$\varphi(x) = (1 - x)^p (1 - \lambda x)^q (1 - \mu x)^r, \tag{91}$$

with  $\varphi(x) = 1 + O(x^3)$ . This is the case m = 1 of (1). The series expansion of (91) starts with

$$1 - (p + \lambda q + \mu r) x + \frac{1}{2} \Big( (p + \lambda q + \mu r)^2 - p - \lambda^2 q - \mu^2 r \Big) x^2 + \dots$$
(92)

The coefficients to  $x, x^2$  must vanish. Eliminating  $\lambda$ , we obtain a quadratic equation for  $\mu$ . Its discriminant equals -pqr(p+q+r); hence, the Belyi maps are expressed with the radical  $\sigma = \pm \sqrt{-pqr(p+q+r)}$ . Explicitly,

$$\varphi(x) = \left(1 - x\right)^p \left(1 + \frac{pq + \sigma}{q(q+r)}x\right)^q \left(1 + \frac{pr - \sigma}{r(q+r)}x\right)^r.$$
(93)

These Belyi maps with the definition field  $\mathbb{Q}(\sigma)$  are obtained in [4] (Example 2.2.25). If p, q, r are positive integers, the definition field is an imaginary quadratic extension of  $\mathbb{Q}$ . But the Belyi maps could be defined over  $\mathbb{Q}$  if both positive and negative powers are prescribed. For example, here is a nice family of paired Belyi maps with  $p = \binom{n+1}{2}, q = \binom{n}{2}, r = -1$ :

$$\frac{(1+x)^p (1-x)^q}{1+nx}, \qquad \frac{\left(1+(n-2)x\right)^p \left(1-(n+2)x\right)^q}{1-n^2x}.$$
(94)

A general rational parametrization of the triples (p,q,r) with  $\rho \in \mathbb{Q}$  is obtained by parametrizing the singular cubic surface  $y^2 + st(s + t + 1) = 0$ , identifying s = p/r, t = q/r. Here is such a parametrization by  $u = \sigma/qr$ ,  $v = \sigma/pr$ , up to the simultaneous scaling of p,q,r:

$$p = u(uv + 1), \quad q = v(uv + 1), \quad r = -u - v.$$
 (95)

Then,  $\sigma = -uv(u+v)(uv+1)$ . After rescaling x in the two forthcoming Belyi maps, we obtain the expressions

$$\left(1 - (v+1)x\right)^{u} \left(1 + (u-1)x\right)^{v} \left(1 - (uv+1)x\right)^{-(u+v)/(uv+1)},\tag{96}$$

$$(1+(v-1)x)^{u}\left(1-(u+1)x\right)^{v}\left(1-(uv+1)x\right)^{-(u+v)/(uv+1)},$$
(97)

which are to be raised to a common power so as to make the three powers integral. The cases with  $\sigma = 0$  fall outside the considered shape (91), and the cases  $(u + v)(u^2 - 1)(v^2 - 1) = 0$ 

with coalescing branching points require special attention. In the latter situation, we have (p+q)(q+r)(q+r) = 0.

**Remark 3.** If p + q = 0 in the last example, then  $\sigma = \pm pr$ . Taking  $\sigma = pr$  gives the trivial function  $\varphi(x) = 1$ . Taking  $\sigma = -pr$  gives

$$\varphi(x) = \frac{\left(1 - (p - r)x\right)^p \left(1 - 2px\right)^r}{\left(1 - (p + r)x\right)^p}.$$
(98)

In the cases p + r = 0 or q + r = 0, we also have just one Belyi map analogously. If simultaneously p + q = p + r = 0, both candidate functions collapse to  $\varphi(x) = 1$ .

## 6. Belyi Maps of the Form (1)

Belyi maps of the shape (1) generalize Example 3 from m = 1. We assume that the numbers p, q, r, p + q + mr are non-zero such that there are indeed m + 3 distinct points above  $\varphi = 0$  and  $\varphi = \infty$ . The points  $x = 1, x = 1/\lambda, x = \infty$  and their branching orders p, q, -p - q - mr are permutable. We also assume  $p \neq q, p \neq q + mr, q \neq p + mr$ , as these cases are considered in Section 7 with the smallest field of definition.

The condition (3) translates into the power series relation

$$G_m(x) = (1-x)^{-p/r} (1-\lambda x)^{-q/r} \mod x^{m+2}.$$
(99)

The power series term with  $x^{m+1}$  has to equal 0 then. The polynomial  $G_m(x)$  is determined uniquely by the power series

$$(1-x)^{-p/r}(1-\lambda x)^{-q/r} = \sum_{k=0}^{\infty} h_k x^k$$
(100)

truncated at the (m + 1)-th term.

**Lemma 4.** No Belyi map of the form (1), (3) exist when  $p = -\ell_1 r$  and  $q = -\ell_2 r$  for some positive integers  $\ell_1, \ell_2$  satisfying  $\ell_1 + \ell_2 \le m$ .

**Proof.** The left-hand side of (100) is the polynomial  $(1 - x)^{\ell_1}(1 - ax)^{\ell_2}$  under the stated conditions. When  $\ell_1 + \ell_2 < m$ , the polynomial  $G_m(x)$  is forced to be of a smaller degree than m. If  $\ell_1 + \ell_2 = m$ , then  $G_m(x)$  has the undue roots x = 1,  $x = 1/\lambda$ .  $\Box$ 

The coefficients  $h_k$  in (100) can be expressed as follows:

$$h_k = \sum_{j=0}^k \frac{(p/r)_{k-j} (q/r)_j \lambda^j}{(k-j)! \, j!},\tag{101}$$

or

$$h_k = \frac{(p/r)_k}{k!} \, {}_2\mathrm{F}_1 \left( \begin{array}{c} -k, \frac{q}{r} \\ 1-k-\frac{p}{r} \end{array} \right| \lambda \right) \tag{102}$$

$$= \frac{(q/r)_k \lambda^k}{k!} \, {}_2\mathrm{F}_1 \left( \begin{array}{c} -k, \frac{p}{r} \\ 1-k-\frac{q}{r} \end{array} \middle| \frac{1}{\lambda} \right). \tag{103}$$

We must have  $h_{m+1} = 0$ , which leads to Equation (5), generically.

## 6.1. Full Sets of Belyi Maps

Here we consider examples of the generic case (104) with m + 1 Belyi maps. We are interested in the cases  $\lambda \notin \{0, 1\}$  and  $p \neq q$ . The degenerate value  $\lambda = 1$  would coalesce the points of branching order p and q, but this point is excluded by Lemma 1 (*iii*). By Lemma 1 (*ii*), the hypergeometric polynomial  $h_{m+1}(\lambda)$  has m + 1 distinct roots when

$$\left(\frac{p}{r}\right)_{m+1} \neq 0, \quad \left(\frac{q}{r}\right)_{m+1} \neq 0, \quad \left(\frac{p+q}{r}\right)_{m+1} \neq 0.$$
 (104)

The condition  $((p+q)/r)_{m+1} \neq 0$  is equivalent to

$$\left(-\frac{p+q+mr}{r}\right)_{m+1} \neq 0, \tag{105}$$

underscoring the symmetry of the branching orders p, q, and -p - q - mr.

The following examples investigate the Belyi maps with m = 2 and m = 3. The case m = 1 is considered in Example 3.

**Example 4.** In the case m = 2, we consider Equation (27) specialized by b = q/r, c = p/r,  $z = \lambda$ :

$$q(q+r)(q+2r)\lambda^3 + 3pq(q+r)\lambda^2 + 3pq(p+r)\lambda + p(p+r)(p+2r) = 0.$$
 (106)

The discriminant

$$-108p^2q^2r^3(p+r)(q+r)(p+q+r)(p+q+2r)^2$$

*is proportional to*  $(p/r)_3(q/r)_3((p+q)/r)_3$ *, and it indeed vanishes only when we have fewer than three distinct roots, and those roots*  $\lambda \notin \{0,1\}$  *as in Lemma* 1 (*ii*).

Belyi maps defined over  $\mathbb{Q}$  can be found using the parametrization (29), identifying q = br, p = cr, and  $\lambda = z$ . Polynomial Belyi maps defined over  $\mathbb{Q}$  are obtained from the + regions (32) in Figure 1a. For example,  $e = -z = \frac{1}{5}$  and  $e = -z = \frac{1}{6}$  produce these Shabat polynomials, respectively:

$$(1-5x)^2(1+x)^{20}(1-2x+9x^2)^5,$$
 (107)

$$(1-6x)(1+x)^{20}(1-2x+6x^2)^7.$$
 (108)

Their companions with the same p, q, r are defined, respectively, over  $\mathbb{Q}(\sqrt{-35})$  and  $\mathbb{Q}(\sqrt{-2})$ . There are no triples of Shabat polynomials defined over  $\mathbb{Q}$ , as there is no point  $(t, y) \in \mathbb{R}^2$  giving positive values for b = q/r and c = p/r by Figure 1b. The family (41) with  $t \in \mathbb{Q}$  provides most of the triples of Belyi maps defined over  $\mathbb{Q}$  with small absolute values of p, q, r. For example, t = 2 gives the case (p, q, r) = (2, -7, 6) with these three Belyi maps defined over  $\mathbb{Q}$ :

$$\frac{(1-x)^2(1-2x-\frac{1}{6}x^2)^6}{(1-2x)^7}, \ \frac{(1-x)^2(1+5x+\frac{10}{3}x^2)^6}{(1+4x)^7}, \ \frac{(1-5x)^2(1-3x-\frac{2}{3}x^2)^6}{(1-4x)^7}.$$
 (109)

*Here are a few projective ratios* (p : q : r) *giving three Belyi maps defined over*  $\mathbb{Q}$  *outside the family* (41):

$$(1:-\frac{7}{8}:\frac{1}{19}), (2:-\frac{1}{5}:\frac{1}{37}), (1:-\frac{26}{31}:\frac{1}{37}), (1:-\frac{11}{41}:\frac{1}{43}), (1:-\frac{14}{17}:\frac{1}{61}), (7:-\frac{5}{2}:\frac{4}{13}), (7:-\frac{65}{19}:3), (19:-\frac{35}{8}:1), (-19:-\frac{37}{7}:\frac{1}{2}).$$
(110)

Families of Belyi maps can be obtained from sections on the elliptic surface  $\mathcal{E}_3$ , particularly from the c-values in (48). Taking c = (b+2)/(4b-1) with the reparametrization t = 4b-1 and independent rescaling of x and of the powers p, q, r gives

$$(1-2tx)^{t+9} \left(1+6x\right)^{t(t+1)} \left(1-(t-3)x-(t+3)^2 x^2\right)^{4t}.$$
(111)

*Similarly, taking* c = -4b(b+2)/(5b+1) *and* t = (5b+1)/(3b) *gives* 

$$(1-tx)^{12-8t} \left(1-2(t-1)x\right)^t \left(1-2x-\frac{2(t-2)^2}{3t-5}x^2\right)^{t(3t-5)}.$$
(112)

*Or one may employ the whole two-parameter uniformizations* (29) *or* (41).

**Example 5.** Similarly, for m = 3, we obtain the surface  $S_4$  in (51) with b = q/r, c = p/r,  $z = \lambda$ . We can use the full parametrization (66) and (67), or a section of the elliptic surface  $\mathcal{E}_4$  in (52). For example, the first b-value from (59) gives the family

$$(1-x)^{2t(t-2)(t+2)} (1-tx)^{(t+1)(t+2)} \times \left(1-(t+2)x - \frac{(t+1)(t+2)}{3}x^2 - \frac{(t+1)(t-2)(t+2)}{9}x^3\right)^{3t(1-t)},$$
(113)

with t = z. Here are some values of b, c, with one of them being a positive integer, found by an extensive search through the parametrization (66):

$$(5, -\frac{7}{2}), (5, -\frac{7}{11}), (5, -\frac{14}{13}), (13, -\frac{13}{3}), (13, -\frac{26}{5}), (14, -\frac{7}{2}), (15, -\frac{45}{43}), (19, -\frac{28}{3}), (19, -\frac{19}{7}), (20, -\frac{92}{11}), (22, -\frac{40}{7}), (23, -\frac{25}{2}), (24, -\frac{117}{47}), (45, -\frac{256}{13}), (45, -\frac{207}{19}), (45, -\frac{47}{51}), (51, -\frac{442}{71}), (54, -\frac{135}{7}), (55, -\frac{35}{2}), (55, -\frac{99}{4}), (56, -\frac{29}{5}), (56, -\frac{133}{5}), (68, -\frac{92}{7}), (68, -\frac{353}{33}), (68, -\frac{85}{33}), (77, -\frac{79}{2}), (84, -\frac{192}{5}), (91, -\frac{188}{41}).$$

The negative integer values of b, c are considered in Example 9. Here are some other found values of b, c, excluding the values  $\{-1/2, -3/2, -5/2, -7/2, -9/2\}$  with reference to Examples 15 and 16:

$$\begin{pmatrix} \frac{1}{2}, -\frac{8}{3} \end{pmatrix}, \begin{pmatrix} \frac{1}{2}, -\frac{7}{5} \end{pmatrix}, \begin{pmatrix} \frac{7}{2}, -\frac{26}{5} \end{pmatrix}, \begin{pmatrix} \frac{17}{2}, -\frac{3}{10} \end{pmatrix}, \begin{pmatrix} \frac{4}{3}, -\frac{27}{7} \end{pmatrix}, \begin{pmatrix} \frac{7}{3}, -\frac{13}{9} \end{pmatrix}, \begin{pmatrix} \frac{7}{3}, -\frac{26}{5} \end{pmatrix}, \begin{pmatrix} \frac{14}{3}, -\frac{34}{9} \end{pmatrix}, \\ \begin{pmatrix} \frac{20}{3}, -\frac{23}{9} \end{pmatrix}, \begin{pmatrix} \frac{29}{3}, -\frac{63}{5} \end{pmatrix}, \begin{pmatrix} \frac{74}{3}, -\frac{37}{2} \end{pmatrix}, \begin{pmatrix} \frac{11}{4}, -\frac{40}{7} \end{pmatrix}, \begin{pmatrix} \frac{15}{4}, -\frac{45}{7} \end{pmatrix}, \begin{pmatrix} \frac{15}{4}, -\frac{46}{7} \end{pmatrix}, \begin{pmatrix} \frac{4}{5}, -\frac{19}{7} \end{pmatrix}, \\ \begin{pmatrix} \frac{7}{6}, -\frac{35}{9} \end{pmatrix}, \begin{pmatrix} \frac{8}{5}, -\frac{13}{4} \end{pmatrix}, \begin{pmatrix} \frac{1}{6}, -\frac{9}{22} \end{pmatrix}, \begin{pmatrix} \frac{5}{6}, -\frac{2}{9} \end{pmatrix}, \begin{pmatrix} \frac{1}{7}, -\frac{1}{4} \end{pmatrix}, \begin{pmatrix} \frac{1}{7}, -\frac{11}{5} \end{pmatrix}, \begin{pmatrix} \frac{3}{7}, -\frac{17}{5} \end{pmatrix}, \begin{pmatrix} \frac{6}{7}, -\frac{18}{5} \end{pmatrix}, \\ \begin{pmatrix} -\frac{2}{3}, -\frac{7}{5} \end{pmatrix}, \begin{pmatrix} -\frac{5}{3}, -\frac{2}{9} \end{pmatrix}, \begin{pmatrix} -\frac{8}{3}, -\frac{3}{8} \end{pmatrix}, \begin{pmatrix} -\frac{1}{4}, -\frac{7}{5} \end{pmatrix}, \begin{pmatrix} -\frac{5}{4}, -\frac{8}{11} \end{pmatrix}, \begin{pmatrix} -\frac{3}{5}, -\frac{4}{7} \end{pmatrix}, \begin{pmatrix} -\frac{7}{6}, -\frac{9}{10} \end{pmatrix}.$$

#### 6.2. Cases with Fewer Belyi Maps

The number of Belyi maps (1) with  $\lambda \notin \{0,1\}$  is smaller than m + 1 when at least one of the conditions (104) is not satisfied. Indeed, if  $q = -\ell r$  for a positive integer  $\ell \leq m$ , the polynomial  $h_{m+1}(\lambda)$  has the degree  $\ell$  rather than m + 1, as the terms with  $j > \ell$  in (101) vanish then. If  $p = -\ell r$  for a positive integer  $\ell \leq m$ , then the first  $m + 1 - \ell$  terms in (101) vanish, and

$$h_{m+1} = \frac{(-1)^{\ell} (q/r)_{m+1-\ell}}{(m+1-\ell)!} \lambda^{m+1-\ell} {}_{2}F_{1} \left( \begin{array}{c} -\ell, \frac{q}{r} + m + 1 - \ell \\ m + 2 - \ell \end{array} \right| \lambda \right).$$
(114)

The factor  $\lambda^{m+1-\ell}$  reduces the number of relevant roots to  $\ell$ . If  $p + q = -\ell r$  for a positive integer  $\ell \leq m$  but  $(p/r)_m \neq 0$  and  $(q/r)_m \neq 0$ , then we apply Euler's transformation (18) to the hypergeometric polynomial  $h_{m+1}(\lambda)$  and obtain

$$h_{m+1} = \frac{(p/r)_{m+1}}{(m+1)!} (1-\lambda)^{\ell+1} {}_{2}F_{1} \left( \begin{array}{c} -m+\ell, 1+\frac{p}{r} \\ -m-\frac{p}{r} \end{array} \right| \lambda \right).$$
(115)

The factor  $(1 - \lambda)^{\ell+1}$  reduces the number of relevant roots to  $m - \ell$ . If  $p/r \in \mathbb{Z}$  and  $q/r \in \mathbb{Z}$ , then these possibilities may combine, giving a lower degree of  $h_{m+1}(\lambda)$  and multiple undue roots  $\lambda = 0$  and  $\lambda = 1$ . Figure 2 depicts the regions and line segments or rays on the integer lattice for the values of p/r and q/r where the number of Belyi maps (1) is fixed  $\leq m + 1$ . Lemma 4 applies to the visible triangle inside the dark middle region.



**Figure 2.** The number of Belyi maps (1) for integer values of p/r and q/r. The dark region in the middle, and the lines emanating from it, represent the cases with no Belyi maps. The light grey regions represent m + 1 Belyi maps; the dashed lines—unique Belyi maps; the dense dotted lines—pairs of Belyi maps; the sparser dotted lines—triples of maps; the two kinds of dashed–dotted lines: (m - 1) or m maps.

Let us consider the cases of reduced sets of distinct Belyi maps of size at most 4. Let us use  $b^* = q/r$  and  $c^* = p/r$  for shorthand.

**Example 6.** The cases of single Belyi maps are represented by some lattice points on the lines p = -r, q = -r, p + q = (1 - m)r or p = 2r, q = 2r, p + q = -(m + 2)r in Figure 2. It is enough to consider two cases, that is, one from both displayed triples of lines, due to the hypergeometric symmetries (8), (11)–(14). When q = -r and  $(c^* - 1)_m \neq 0$ , the hypergeometric polynomial  $h_{m+1}$  is linear in  $\lambda$ . It gives  $\lambda = (m + c^*)/(m + 1)$ . Following (99), consider the power series

$$(1-x)^{-c^*}\left(1-\frac{m+c^*}{m+1}x\right) = 1 + \frac{1}{m+1}\sum_{k=1}^{\infty}\frac{(c^*-1)_k}{k!}(m+1-k)x^k.$$
 (116)

The term with  $x^{m+1}$  indeed vanishes, and the earlier terms define  $Q_m(x)$ . When p = 2r and  $(b^*)_{m+1} = 0$ , then (115) reads

$${}_{2}F_{1}\left(\begin{array}{c} -m-1, \ b^{*}\\ -m-2 \end{array} \middle| \lambda\right) = (1-\lambda)^{-b^{*}-1} {}_{2}F_{1}\left(\begin{array}{c} -1, -b^{*}-m-2\\ -m-2 \end{array} \middle| \lambda\right).$$
(117)

*The hypergeometric polynomial on the right-hand side is linear, and gives*  $\lambda = (m+2)/(b^*+m+2)$ *. The implication for the power series of* 

$$(1-x)^{-2} \left(1 - \frac{m+2}{b^* + m + 2}x\right)^{-b^*}$$
(118)

is remarkable: the k-th term is divisible by  $(b^*+1)_k$  when m = k - 1. Here is a revealing form of the starting terms:

$$1 + \frac{m(b^*+2) + 4(b^*+1)}{b^*+m+2} + \frac{(m-1)(b^*+3)(m(b^*+2) + 9b^*+10) + 27(b^*+1)(b^*+2)}{2(b^*+m+2)^2} + \frac{(m-2)(b^*+4)(m^2(b^*+2)(b^*+3) + \dots) + 256(b^*+1)(b^*+2)(b^*+3)}{6(b^*+m+2)^3} + \dots$$
(119)

**Example 7.** Here, we consider the cases with two Belyi maps. They are represented by some lattice points on the lines p = -2r, q = -2r, and p + q = (2 - m)r or p = 3r, q = 3r, and p + q = -(m + 3)r in Figure 2. It is enough to consider two cases. When q = -2r and  $(c^* - 2)_m \neq 0$ , the hypergeometric polynomial  $h_{m+1}$  is quadratic in  $\lambda$ :

$$m(m+1)\lambda^2 - 2(m+1)(m+c^*-1)\lambda + (m+c^*)(m+c^*-1).$$
(120)

The two solutions are

$$\lambda = \frac{m + c^* - 1}{r} \pm \frac{\sqrt{(m+1)(c^* - 1)(m + c^* - 1)}}{m(m+1)}.$$
(121)

We have two Belyi maps that are defined over  $\mathbb{Q}$  when

$$c^* = \frac{m+1+(m-1)u^2}{m+1-u^2}.$$
(122)

for some  $u \in \mathbb{Q}$ . Then, the  $\lambda$ -values are  $(m + 1 \pm u)/(m + 1 - u^2)$ . The power series then looks like this:

$$(1-x)^{-c^*} \left(1 - \frac{m+1+u}{m+1-u^2}x\right) = 1 + \left(c^* - 1 - \frac{m+(u+1)^2}{m+1-u^2}\right)x + \frac{1}{m}\sum_{k=2}^{\infty}\frac{(c^*-1)_{k-1}}{k!}\left(c^* - 1 - \frac{m+k(u+1)^2}{m+1-u^2}\right)(m+1-k)x^k.$$
 (123)

*Note that if*  $c^* = 2$  *then*  $u = \pm 1$ *, and we have just one Belyi map with* u = 1*, which is a special case of* (118).

*Like in the previous example, we apply* (115) *to the case* p = 3r *and*  $(q/r)_{m+1} = 0$ :

$${}_{2}F_{1}\left(\begin{array}{c} -m-1, b^{*}\\ -m-3 \end{array} \middle| \lambda\right) = (1-\lambda)^{-b^{*}-2} {}_{2}F_{1}\left(\begin{array}{c} -2, -b^{*}-m-3\\ -m-3 \end{array} \middle| \lambda\right).$$
(124)

*We obtain a quadratic equation analogous to* (120)*. It has solutions*  $\lambda \in \mathbb{Q}$  *when* 

$$b^* = -\frac{(m+3)u^2}{m+2+u^2} \tag{125}$$

for some  $u \in \mathbb{Q}$ . Then, the  $\lambda$ -values are  $(m + 2 + u^2)/(m + 2 \pm u)$ .

**Example 8.** Now we consider the cases with larger *m* that have three Belyi maps. Suppose that q = -3r. Then, the hypergeometric polynomial (114) for  $h_{m+1}$  defines a cubic relation between  $\lambda$  and p/r. This cubic relation defines a curve of genus 1 as stated in Proposition 1. The elliptic curve

has the Mordell–Weil group isomorphic to  $(\mathbb{Z}/3\mathbb{Z}) \times \mathbb{Z}$  for general *m*. The isomorphism (46) has to be adjusted with

$$b = -m - 1, \qquad c = 3 - m - \frac{p}{r}.$$
 (126)

Analysis with the database [16] found the examples with  $m \in \{6, 11, 13, 17, 23, 25\}$  giving the Mordell–Weil group  $(\mathbb{Z}/3\mathbb{Z}) \times \mathbb{Z}^2$ , though the cases  $m \in \{14, 20, 21, 22\}$  are not in the database yet. In particular, the elliptic curve for the case m = 6 has the label 39690.bj2 in [16]. The equation specialized from (45) is

$$y^2 = x^3 + 49(3x + 96)^2. (127)$$

Besides the specialized generators (0, 672) and (-56, 280) of Proposition 1, its Mordell–Weil group has also the generator (-48, 48) over  $\mathbb{Q}$ . This extra generator corresponds to the hypergeometric evaluation

$$_{2}F_{1}\left(\begin{array}{c} -7, -3\\ 13 \end{array} \middle| -1 \right) = 0,$$
 (128)

and the Belyi map

$$\frac{1 - 16x + 117x^2 - 512x^3 + 1463x^4 - 2736x^5 + 2907x^6}{(1 - x)^{19}(1 + x)^3}.$$
 (129)

It is interesting to observe that the expanded polynomial  $(1 - x)^{19}(1 + x)^3$  does not have the terms with  $x^7$  (as we just used) and with  $x^{11}$ ,  $x^{15}$ . Therefore, we can obtain two more Belyi maps by extending the numerator of (129) to polynomials of degree m = 10 or m = 14. This phenomenon is typical for integer points on our hypergeometric surfaces as the next example suggests.

**Example 9.** Here we look at the cases with larger *m* that have four Belyi maps. For q = -4r, the parametrization (66) and (67) has to be adjusted by (126). We seek negative values of *b* or c, -b - c - m due to the hypergeometric symmetries (8), (11)–(14). Here are some cases for (b, c) being found:

$$(-5, -\frac{4}{7}), (-7, -\frac{7}{5}), (-9, -\frac{9}{2}), (-14, -\frac{26}{5}), (-14, -\frac{22}{19}), (-19, -\frac{3}{20}), (-19, -\frac{17}{31}), (-22, -\frac{5}{31}), (-22, -\frac{57}{107}), (-24, -\frac{77}{4}), (-24, -\frac{33}{10}), (-33, -\frac{33}{7}), (-36, -\frac{20}{3}), (-43, -\frac{43}{2}), (-43, -\frac{168}{127}), (-54, -\frac{39}{85}), (-54, -\frac{51}{233}), (-59, -\frac{19}{85}), (-59, -\frac{63}{139}), (-65, -\frac{52}{7}), (-68, -\frac{871}{19}), (-71, -\frac{180}{19}), (-78, -\frac{22}{3}), (-78, -\frac{190}{7}), (-79, -\frac{247}{5}), (-80, -\frac{231}{23}).$$

Of particular interest are the points with both b, c negative integers. Found instances are

$$(-7, -10), (-14, -66), (-28, -65), (-14, -52), (-30, -36), (-22, -35), (-63, -64), (-41, -247), (-78, -210), (-115, -210), (-247, -780), (-341, -715), (-901, -1856), (-1730, -8478), (-2795, -16512).$$
 (131)

They give two Belyi maps though for different cases of m, p/r. For example, the first instance gives the hypergeometric evaluations

$$_{2}F_{1}\left(\begin{array}{c}-7,-4\\7\end{array}\right|-1\right)=0, \quad _{2}F_{1}\left(\begin{array}{c}-10,-4\\4\end{array}\right|-1\right)=0.$$
 (132)

They both lead to consider the polynomial

$$(1-x)^4(1+x)^{13} = 1 + 9x + 32x^2 + 48x^3 - 12x^4 - 156x^5 - 208x^6$$
(133)  
+ 286x<sup>8</sup> + 286x<sup>9</sup> - 208x<sup>11</sup> - 156x<sup>12</sup> - 12x<sup>13</sup> + 32x<sup>15</sup> + 9x<sup>16</sup> + x<sup>17</sup>

as a power series without the terms  $x^7$  or  $x^{10}$ .

**Remark 4.** *The cases like* (128), (132) *of hypergeometric equations with integer parameters can be expressed in terms of* Krawtchouk polynomials ([9] §9.11):

$$K_n(x;p,N) = {}_2\mathbf{F}_1\left(\begin{array}{c} -n,-x \\ -N \end{array} \middle| \begin{array}{c} 1 \\ p \end{array}\right).$$
(134)

Following (11), hypergeometric polynomials in (128), (132) are identified by

$${}_{2}F_{1}\left(\begin{array}{c}-m,-n\\M\end{array}\middle|1-\frac{1}{p}\right) = \frac{(M+m)_{n}}{(M)_{n}}K_{n}(m;p,M+m+n-1).$$
(135)

Finding the integer roots of Krawtchouk polynomials is an active field of research ([17,18] §7.2) with special implications for graph and coding theories [19], algebraic geometry [20], Padé approximations ([21] §2.2), and quantum entanglement [22].

## 7. Belyi Maps of the Form (2)

The form (2) of Belyi maps is the compacted case q = p of the form (1), where  $(1-x)^p(1-\lambda x)^p$  is replaced by the aggregate power  $(1 + \alpha x + \beta x^2)^p$ . This grouping of points with the same branching order is routinely used to maximally simplify the *field of definition* of Belyi maps [4]. Similarly, here we are not interested in the case r = p. These maps correspond to the cases  $(1 - x^{m+2})^p$  of Remark 2, with a quadratic factor of  $1 - x^{m+2}$  separated.

Let us denote  $H_2(x) = 1 + \alpha x + \beta x^2$ . The condition (3) translates into the power series relation

$$G_m(x) = H_2(x)^{-p/r} \mod x^{m+2}.$$
 (136)

The power series term with  $x^{m+1}$  has to equal 0. The polynomial  $G_m(x)$  is determined uniquely by the power series

$$H_2(x)^{-p/r} = \sum_{k=0}^{\infty} g_k x^k \tag{137}$$

truncated at the (m + 1)-the term. The power series of can be computed by expanding  $(-\alpha - \beta x)^j$  in

$$H_2(x)^{-p/r} = \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{p}{r}\right)_j (-\alpha - \beta x)^j x^j.$$
 (138)

Explicitly,

$$g_k = \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{(-1)^{k-j} (p/r)_{k-j} \alpha^{k-2j} \beta^j}{(k-2j)! \, j!}.$$
(139)

If  $(p/r)_k \neq 0$  and  $\alpha \neq 0$ , we have the hypergeometric expression

$$g_k = \frac{(-\alpha)^k}{k!} \left(\frac{p}{r}\right)_k {}_2F_1 \left(\begin{array}{c} -\frac{k}{2}, -\frac{k-1}{2} \\ 1-k-\frac{p}{r} \end{array}\right) \left(\frac{4\beta}{\alpha^2}\right). \tag{140}$$

Distinct Belyi maps correspond to the roots of  $g_{m+1}(\alpha:\beta)$ , identified up to the (1:2)weighted homogeneous action of scaling x. There is a Belyi map with  $\alpha = 0$  when m is even. It is obtained from Example 2 after the substitution  $x \mapsto x^2$ , q = r (with q, m over there equal to the current r, m/2).

The following degeneratated cases are encountered:

- There are no Belyi maps when  $p = -\ell r$  for a positive integer  $\ell \le \lfloor m/2 \rfloor$  because then  $G_m(x) = H_2(x)^j$ , and possibly of lesser degree than m.
- If  $p = -\ell r$  for a positive integer satisfying  $\lfloor m/2 \rfloor < \ell \le m$ , then the first  $m + 1 \ell$  terms for the sum (139) for  $g_{m+1}$  are zero, and  $g_{m+1}$  has the degenerate root  $\beta = 0$  of that multiplicity. After shifting the summation index by  $k \ell$ ,

$$g_{k} = \frac{\alpha^{2\ell-k}\beta^{k-\ell}\,\ell!}{(k-\ell)!\,(2\ell-k)!}\,{}_{2}F_{1}\left(\begin{array}{c}\frac{k}{2}-\ell,\frac{k+1}{2}-\ell\\1+k-\ell\end{array}\right|\frac{4\beta}{\alpha^{2}}\right).$$
(141)

There are  $\ell - \lceil m/2 \rceil$  proper Belyi maps then, including the  $\alpha = 0$  case for even *m*. If  $p = -\ell r/2$  for a positive odd integer  $\ell \le m$ , then

$$g_{m+1} = \frac{(-\alpha)^{m+1} \left(-\frac{\ell}{2}\right)_{m+1}}{(m+1)!} \left(1 - \frac{4\beta}{\alpha^2}\right)^{\frac{\ell+1}{2}} {}_2F_1 \left(\begin{array}{c} \frac{\ell-m}{2}, \frac{\ell+1-m}{2} \\ \frac{\ell}{2} - m \end{array} \middle| \frac{4\beta}{\alpha^2} \right).$$
(142)

by Euler's formula (18). The root  $\beta = \alpha^2/4$  corresponds to the degeneration of  $H_2(x)$  to a full square. The transformed hypergeometric sum can be identified as  $g_{m-\ell}$  with the substituted  $p/r = 1 + \ell/2$ . There are then  $\lceil (m - \ell)/2 \rceil$  Belyi maps, including the  $\alpha = 0$  case for even m. In particular, there are no Belyi maps when the odd  $\ell = m$ , and there is only the map with  $\alpha = 0$  when the odd  $\ell = m - 1$ .

Otherwise, that is when  $p \neq -\ell r$  for all positive  $\ell \leq m$  and  $p \neq -\ell r/2$  for all positive odd  $\ell \leq m$ , the <sub>2</sub>F<sub>1</sub>-factor in (140) for  $g_{m+1}$  gives  $\lceil m/2 \rceil$  distinct values with  $\beta \notin \{0, \alpha^2/4\}$  by Lemma 2. Together with  $\alpha = 0$ , in total, we then have  $\lceil (m+1)/2 \rceil$  Belyi maps.

## 7.1. Full Sets of Belyi Maps

Let us explore Belyi maps of the form (2) with  $m \le 8$ . We are principally interested in Belyi maps defined over  $\mathbb{Q}$ .

**Example 10.** For m = 2, there is a Belyi map defined with  $\alpha = 0$ . When  $p \neq -r/2$ , there is another map. We can scale x to obtain

$$H_2(x) = 1 + rx + \frac{1}{6}r(p+2r)x^2,$$

$$G_2(x) = 1 - px + \frac{1}{4}p(2p+r)x^2.$$
(143)

**Example 11.** For m = 3, the Belyi maps are defined by

$$H_{2}(x) = 1 + rx + \frac{1}{6}r(3p + 6r + R_{3})x^{2},$$

$$G_{3}(x) = 1 - px + \frac{1}{6}p(3r + R_{3})x^{2} + \frac{1}{6}p(p + r)(2p + 4r + R_{3})x^{3},$$
(144)

where  $R_3 = \pm \sqrt{3(p+2r)(2p+3r)}$ . There are no Belyi maps for  $p/r \in \{-1, -2, -\frac{3}{2}\}$ . For p = -r/2 or p = -3r, we have single Belyi maps, respectively:

$$\frac{(1+2x+5x^2)^p}{(1+x+2x^2-2x^3)^{2p}}, \qquad \frac{(1+3x-5x^3)^r}{(1+x-x^2)^{3r}}.$$
(145)

*The cases when*  $R_3 \in \mathbb{Q}$  *are parametrized as* 

$$\frac{p}{r} = \frac{9-2s^2}{s^2-6}, \qquad s \in \mathbb{Q}.$$
 (146)

**Example 12.** *For* m = 4*, the Belyi maps with*  $\alpha \neq 0$  *are defined by* 

$$H_{2}(x) = 1 + rx + \frac{1}{30}r(5p + 15r + R_{4})x^{2},$$

$$G_{4}(x) = 1 - px + \frac{1}{30}p(10p - R_{4})x^{2} + \frac{1}{30}p(p + r)(5r + R_{4})x^{3}$$

$$- \frac{1}{180}p(p + r)(2p + 3r)(2p + 6r + R_{4})x^{4},$$
(147)

where  $R_4 = \pm \sqrt{5(p+3r)(2p+3r)}$ . There are no Belyi maps for  $p/r \in \{-1, -2, -3, -\frac{3}{2}\}$ . For p = -r/2 or p = -4r, we have single Belyi maps, respectively:

$$\frac{(1+6x+21x^2)^p}{(1+3x+6x^2-18x^3+36x^4)^{2p}}, \quad \frac{(1+12x+42x^2-189x^4)^r}{(1+3x-3x^2)^{4q}}.$$
 (148)

*The cases when*  $R_4 \in \mathbb{Q}$  *are parametrized as* 

$$\frac{p}{r} = \frac{15 - 3s^2}{2s^2 - 5}, \qquad s \in \mathbb{Q}.$$
(149)

**Example 13.** The case m = 5 is considered in ([4] §2.2.4.3). The cubic hypergeometric relation

$${}_{2}F_{1}\left(\begin{array}{c}-3,-\frac{5}{2}\\-\frac{p}{r}-5\end{array}\middle|\frac{4\beta}{\alpha^{2}}\right)=0$$
(150)

can be analyzed within the context of Section 3.2 with b = -5/2, c = 3 + p/r, and  $z = 4\beta/\alpha^2$ . We combine Proposition 1 with a simple transformation

$$(u,v)\mapsto \left(\frac{u-75}{4},\frac{v}{8}\right).$$
(151)

*The obtained elliptic curve is the same as in* [4]:

$$\mathcal{E}_5: \quad v^2 = u^3 - 2475u - 5850. \tag{152}$$

*The isomorphism obtained from* (46) *is simpler than that in* ([4] pg. 105):

$$\frac{p}{q} = \frac{645 - 15u - 11v}{4v}, \qquad z = \frac{45 - 7u + v}{2v}.$$
(153)

The database of elliptic curves [16] identifies this curve by the label 4050.y2, and confirms that the Mordell–Weil group of  $E_1$  is isomorphic to  $\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})$ . Any  $\mathbb{Q}$ -rational point on  $E_1$  can be expressed using the addition law on the elliptic curve as

$$n(-5,80) + \varepsilon(75,480), \quad with \ n \in \mathbb{Z}, \ \varepsilon \in \{0,1,-1\}.$$
 (154)

Several of the rational points correspond to the degenerate cases  $p/r \in \{0, -3, -5/2\}$  of no Belyi maps,  $p/r \in \{-4, -3/2\}$  of single maps, and  $p/r \in \{-5, -1/2\}$  of coupled maps. The elliptic involution represents the hypergeometric identity (11), and acts as

$$\left(\frac{p}{r},z\right)\mapsto\left(-\frac{p}{r}-\frac{11}{2},1-z\right).$$

*There are rational points with*  $p/r \in \{-11/2, -11/4\}$ *. Here are some other values of* p/r *that give Belyi maps defined over*  $\mathbb{Q}$ *:* 

$$-\frac{28}{11}, -\frac{65}{22}; -\frac{59}{23}, -\frac{135}{46}; -\frac{65}{107}, -\frac{1047}{214}; \frac{33}{124}, -\frac{715}{124}; -\frac{251}{169}, -\frac{1357}{338}.$$
 (155)

This list has only one positive value, representing a Shabat polynomial defined over  $\mathbb{Q}$ . A rarity of positive values for p/r is observed in [4]. Here are the next few:

#### $\frac{1317156026567}{649800344821}, \frac{487834953714776556005}{104793834699948131134}, \frac{1999297558019926898176821908516}{208841813498019535906845150263}.$ $\frac{2008145}{1653242}$

Following (153), the positivity region in Figure 3a is cut out by the lines v = 0 and 15u + 11v = 645. We have three separate small regions on  $\mathcal{E}_5$  between these lines, the one with u > 0 being especially tiny. The rational points are distributed ergodically on  $\mathcal{E}_5$ , with the density proportional to invariant holomorphic differential du / v. The whole measure is the real period over the finite oval or the infinite piece. It can be computed avoiding numeric issues near v = 0 or  $u = \infty$ by integrating du/v between points that differ by a three-torsion point, and then multiplying by three. For example,

 $a_5 = 3 \int_{-5}^{-5} \frac{du}{du} = 3 \int_{-5}^{315} \frac{du}{du} \approx 0.732116211$ 

$$\varrho_{5} = 3 \int_{-45}^{5} \frac{du}{v} = 3 \int_{51}^{51} \frac{du}{v} \approx 0.732116211.$$
(156)

Figure 3. (a) The elliptic curves (152), with the numerator (153) line that together with the horizontal line v = 0 determines the sign of p/r. (b) The elliptic curves (159), with the numerator (160) line that together with the horizontal line v = 0 determines the sign of p/r.

The integrals over the three small regions could be stably computed by shifting the integration range by a three-torsion point. The two integrals on the finite oval evaluate to  $\approx 0.0564864103 +$ 0.0524120276. After division by  $\varrho_5$ , we can compute the percentage and conclude that positive values of p/q asymptotically occur about 5.72 less frequently than the negative ones on the finite oval. The points (154) with |n| < 200 give 46 and 43 positive values in the two regions. This is within the rounding error from the asymptotic prediction. The tiny integral on the infinite branch is  $\approx 0.00407438266$ . This gives the odds ratio  $\approx 179$  for positive values. The points (154) with |n| < 200 give 3 positive values of p/q, with  $n \in \{54, 162, -188\}$ . Their denominators have 736, 6640 or 8944 decimal digits, respectively. The same search looks for complete factorizations of the cubic polynomial in z, but only predictable degenerations are found. By (50) and (151), we need  $\sqrt{102-2u} \in \mathbb{Q}$  for those factorizations.

**Example 14.** Similarly, for m = 6, we can investigate the cubic relation

$${}_{2}F_{1}\left(\begin{array}{c|c} -3, -\frac{7}{2} \\ -\frac{p}{r} - 6 \end{array} \middle| z^{*}\right) = 0$$
(157)

between p/r and  $z^* = 4\beta/\alpha^2$  by applying Proposition 1 with b = -7/2, c = 4 + p/r. Additionally, we apply the simple transformation

$$(u,v) \mapsto \left(\frac{u-147}{4}, \frac{v}{8}\right) \tag{158}$$

and derive the elliptic curve

$$\mathcal{E}_6: \quad v^2 = u^3 - 17787u + 692566 \tag{159}$$

with the isomorphism

$$\frac{p}{r} = \frac{5901 - 63u - 13v}{4v}, \qquad z^* = \frac{791 - 13u + v}{8v}.$$
(160)

The database of elliptic curves ([16] 13230.dp1) tells that the Mordell–Weil group is  $\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})$  here as well. The generators of  $\mathbb{Q}$ -rational points are (u, v) = (35, 336) and a torsion point (147, 1120). The elliptic involution acts as

$$\left(\frac{p}{r}, z^*\right) \mapsto \left(-\frac{p}{r} - \frac{13}{2}, 1 - z^*\right)$$

*Here are some values of* p/r *that give Belyi maps defined over*  $\mathbb{Q}$ *:* 

$$-\frac{13}{4}, 1, -\frac{15}{2}, -\frac{13}{5}, -\frac{39}{10}, -\frac{30}{11}, -\frac{83}{22}, -\frac{52}{37}, -\frac{377}{74}, -\frac{20}{29}, -\frac{337}{58}, \frac{41}{16}, -\frac{145}{16}.$$
 (161)

Positive values appear more frequently than in the m = 2 case. Besides the discardable p/r = 1 representing a quadratic factor of  $1 - 16x^8$  in  $\mathbb{Q}[x]$ , we see  $\frac{41}{16}$  as well. Here are the next few positive values:

 $\frac{83031}{5198}, \frac{572982199}{35574034}, \frac{66779978696204}{16641846371989}, \frac{77057440650930450189}{148795415472621031586}, \frac{559984659717697802475460}{36618976557904027122191}, \frac{100}{100}$ 

Following (160), the positivity region in Figure 3b is cut out by the lines v = 0 and 63u + 13v = 5901. Again, we have three separate small regions on  $\mathcal{E}_6$  between these lines. The whole real period can be computed to be

$$\varrho_6 = 3 \int_{-133}^{35} \frac{du}{v} = 3 \int_{107}^{707} \frac{du}{v} \approx 0.541858251.$$
 (162)

The two integrals on the finite oval evaluate to  $\approx 0.0507070923 + 0.0430448636$ . After division by  $\varrho_6$ , we can compute the odds ratio for positive p/r to be  $\approx 4.78$  for a rational point with u < 50. The small integral on the infinite branch is  $\approx 0.00766222865$ , giving the odds ratio  $\approx 69.7$ . For a complete factorization of the cubic polynomial we need  $\sqrt{30(107 - u)} \in \mathbb{Q}$ , but only a few predictable degenerations are found.

**Example 15.** The case m = 7 gives a genus 3 polynomial relation

$${}_{2}F_{1}\left(\begin{array}{c}-4, -\frac{7}{2}\\-\frac{p}{r}-7\end{array}\middle| z^{*}\right) = 0$$
(163)

between p/r and  $z^* = 4\beta/\alpha^2$ . The Faltings theorem [23] implies that the genus 3 curve has only finitely many  $\mathbb{Q}$ -rational points. Section 4.3 implies that it could be projected to an elliptic curve (72) with b = -7/2, which is

$$\mathcal{E}_7: \quad v^2 = u \left( u^2 - 119u + \frac{14175}{4} \right). \tag{164}$$

The database of elliptic curves ([16] 94080.el2) tells that the Mordell–Weil group of this curve has rank two and is isomorphic to  $\mathbb{Z}^2 \times (\mathbb{Z}/2\mathbb{Z})$ . Besides the 2-torsion point (u, v) = (0, 0) and

the generator  $(\frac{105}{2}, -\frac{105}{2})$  specialized from (80), another free generator is (60, 15). To obtain rational points on the genus 3 curve, we need rational points (u, v) on  $\mathcal{E}_7$  giving a full square in (82). That means  $(v - 14u)^2 - 11340u$  must be a full square. An extensive search through  $2 \cdot 61 \cdot 51 > 6000$  points on  $\mathcal{E}_7$  gives the following 12 values of p/r = c - 4, paired by (74):

$$1, -\frac{17}{2}; 10, -\frac{35}{2}; -\frac{7}{3}, -\frac{31}{6}; -\frac{14}{5}, -\frac{47}{10}; -\frac{79}{11}, -\frac{7}{22}; -\frac{34}{11}, -\frac{97}{22}.$$
 (165)

*The value* 1 *should be discarded along with the degenerate*  $p/r \in \{-5, -6, -7, -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}\}$ . *The positive value* 10 *gives the Shabat polynomial* 

$$(1+2x+4x^2)^{10}(1-20x+180x^2-880x^3+1760x^4+6336x^5-59840x^6+183040x^7).$$
 (166)

**Example 16.** Similarly, the case m = 8 gives a genus 3 polynomial relation

$${}_{2}F_{1}\left(\begin{array}{c}-4,-\frac{9}{2}\\-\frac{p}{r}-8\end{array}\middle|z^{*}\right)=0$$
(167)

between p/r and  $z^* = 4\beta/\alpha^2$ . Section 4.3 implies that this curve could be projected to an elliptic curve (72) with b = -9/2, which is

$$\mathcal{E}_8: \quad v^2 = u \left( u^2 - 243u + \frac{59535}{4} \right). \tag{168}$$

The database of elliptic curves ([16] 40320.bf2) tells that the Mordell–Weil group is isomorphic to  $\mathbb{Z}^2 \times (\mathbb{Z}/2\mathbb{Z})$ . Besides the generator  $(\frac{189}{2}, -\frac{567}{2})$  specialized from (80), another free generator is  $(\frac{945}{4}, \frac{14175}{8})$ . We need rational points (u, v) on  $\mathcal{E}_8$  giving a full square in (82). That means  $(v - 18u)^2 - 34020u$  must be a full square. An extensive search through  $2 \cdot 61 \cdot 41 > 5000$  points on  $\mathcal{E}_8$  gives these 13 pertinent values of p/r = c - 5 from (74):

$$-\frac{17}{2}; -14, \frac{11}{2}; -\frac{8}{3}, -\frac{35}{6}; -\frac{16}{5}, -\frac{53}{10}; -\frac{17}{5}, -\frac{51}{10}; -\frac{17}{7}, -\frac{85}{14}; -\frac{371}{151}, -\frac{1825}{302}.$$
 (169)

*The positive value*  $p/r = \frac{11}{2}$  *gives the Shabat polynomial* 

$$(1+2x+5x^2)^{11}(1-11x+44x^2-715x^4+2717x^5-572x^6-29172x^7+97240x^8)^2.$$
(170)

#### 7.2. Cases with Fewer Belyi Maps

Let us explore reduced sets of Belyi maps (2). We look at the cases when the hypergeometric polynomials in (141) or (142) have degree at most 4 in  $z^* = 4\beta/\alpha^2$ .

**Example 17.** We start with sampling linear hypergeometric polynomials in (141) or (142). For odd m > 2, we obtain a single Belyi map when, respectively,

$$\ell = -\frac{p}{r} \in \left\{\frac{m+3}{2}, m-2\right\}.$$
(171)

Indeed, setting  $\ell = (m+3)/2$  in (141) gives  $z^* = -m - 1$ . Then,  $H_2(x) = 1 + 2x - (m+1)x^2$ up to x-scaling. According to (136), the terms to  $x^{2k}$  in the power series

$$\left(1+2x-(m+1)x^2\right)^{\frac{m+3}{2}} = 1+(m+3)x-\frac{(m+1)(m+2)(m+3)}{3}x^3 - \frac{(m+1)(m+2)(m+3)}{12}(m-3)x^4 + \frac{(m+1)(m+2)(m+3)(m-1)(m+5)}{20}x^5 + \frac{(m+1)(m+2)(m+3)^2(m-1)}{45}(m-5)x^6 - \frac{(m+1)(m+2)(m+3)(m-1)(m-3)(m^2+19m+42)}{252}x^7 - \frac{(m+1)(m+2)(m+3)(m-1)(m-3)(m+5)(11m+26)}{3360}(m-7)x^8 + \dots$$
(172)

must be divisible by (m - 2k + 1). Setting  $\ell = m - 2$  in (142) gives  $z^* = m + 2$ . Here as well, the terms to  $x^{2k}$  in the power series

$$(1 + 2x + (m+2)x)^{\frac{m-2}{2}} = 1 + (m-2)x + (m-2)(m-1)x^2 + \frac{2m(m-2)(m-4)}{3}x^3 + \frac{(m-2)(m-4)(5m+2)}{12}(m-3)x^4 + \frac{(m-2)(m-4)(m-6)(13m^2 - 9m - 10)}{60}x^5 + \frac{(m-2)(m-4)(m-6)(19m^2 + 5m - 6)}{180}(m-5)x^6 + \dots$$

$$(173)$$

must be hypnotically divisible by (m - 2k + 1). For even m > 2, we obtain isolated Belyi maps when

$$\ell = -\frac{p}{r} \in \left\{\frac{m+4}{2}, m-3\right\}.$$
(174)

We obtain then  $z^* = -m/3$  or  $z^* = m/3 + 1$  from (141) or (142), respectively. The terms to  $x^{2k+1}$  in the following power series must be divisible by (m - 2k):

$$\left(1+2x-\frac{m}{3}x^2\right)^{\frac{m+4}{2}} = 1+(m+4)x+\frac{(m+3)(m+4)}{3}x^2-\frac{m(m+2)(m+3)(m+4)}{36}x^4 - \frac{m(m+2)(m+3)(m+4)}{180}(m-4)x^5+\frac{m(m+2)(m+3)(m+4)(m^2+15m-36)}{1620}x^6 + \frac{m(m-2)(m+2)(m+3)(m+4)(m+6)}{2835}(m-6)x^7+\dots,$$
(175)  
$$\left(1+2x+\frac{m+3}{3}x^2\right)^{\frac{m-3}{2}} = 1+(m-3)x+\frac{2(m-3)^2}{3}x^2+\frac{(m-3)(m-5)}{3}(m-2)x^3 + \frac{(m-3)(m-5)(5m^2-33m+36)}{36}x^4+\frac{(m-3)(m-5)(m-7)(m-1)}{20}(m-4)x^5 + \frac{(m-3)^2(m-5)(m-7)(13m^2-105m+72)}{810}x^6 + \frac{(m-3)(m-5)(m-7)(m-9)(53m^2-153m+72)}{11340}(m-6)x^7+\dots.$$
(176)

**Example 18.** *Here, we look at the cases when the hypergeometric polynomials in* (141) *or* (142) *are quadratic in*  $z^*$ *. For odd m, we then have* 

$$\ell = -\frac{p}{r} \in \left\{\frac{m+5}{2}, m-4\right\}.$$
(177)

Setting  $\ell = (m+5)/2$  in (141) gives this equation for  $z^*$ :

$$3(z^*)^2 + 6(m+1)z^* + m^2 - 1 = 0.$$
(178)

It has roots in  $\mathbb{Q}$  when the discriminant 24(m+1)(m+2) is a square. An equivalent condition is the existence of integer solutions of the Pell equation

$$(2m+3)^2 - 6\,d^2 = 1,\tag{179}$$

as observed in ([7] Ch. 10). These solutions correspond to the units in the field  $\mathbb{Q}(\sqrt{6})$ , which are generated by  $5 + 2\sqrt{6}$ . We should express them as

$$\left(5+2\sqrt{6}\right)^n = (2m+3) \pm \sqrt{6} \left(z^*+m+1\right),\tag{180}$$

*leading to two integer values of*  $z^*$  *for suitable m. For*  $n \in \{2, 3, 4\}$ *, we obtain the suitable values*  $m \in \{23, 241, 2399\}$ *. The case* m = 23 *gives two Belyi maps with* p/r = -14 *defined*  $\mathbb{Q}$ *, with*  $H_2(x) = 1 + x - x^2$  or  $H_2(x) = 1 + x - 11x^2$ .

Setting  $\ell = m - 4$  in (142) gives this equation for  $z^*$ :

$$3(z^*)^2 - 6(m+2)z^* + (m+2)(m+4) = 0.$$
(181)

It has the same discriminant as (178), leading to the same Pell Equation (179). The case m = 23 gives two Belyi maps with p/r = -19/2 defined  $\mathbb{Q}$ , with  $H_2 = 1 + 2x + 5x^2$  or  $H_2 = 1 + 2x + 45x^2$ .

For even m, we have

$$\ell = -\frac{p}{r} \in \left\{\frac{m+6}{2}, \frac{m-5}{2}\right\}.$$
(182)

Setting  $\ell = (m + 6)/2$  in (141) gives this equation for  $z^*$ :

$$15(z^*)^2 - 10mz^* + m(m-2) = 0.$$
(183)

It has roots in  $\mathbb{Q}$  when the discriminant 40m(m+3) is a square. An equivalent condition is the existence of integer solutions of the equation

$$(2m+3)^2 - 10\,d^2 = 9. (184)$$

If m is divisible by 3 (and thus by 6), we have a reduction to Pell's equation

$$\left(\frac{2}{3}m+1\right)^2 - 10 \, (d^*)^2 = 1. \tag{185}$$

The solutions correspond to units in the field  $\mathbb{Q}(\sqrt{10})$ , which are generated by  $3 + \sqrt{10}$ . We should express the solutions as

$$\left(3 + \sqrt{10}\right)^n = \left(\frac{2}{3}m + 1\right) \pm \sqrt{10}\left(z^* + \frac{1}{3}m\right),\tag{186}$$

leading to two integer values of  $z^*$  for suitable m. But the norm of  $3 + \sqrt{10}$  in  $\mathbb{Q}(\sqrt{10})$  is -1 rather than 1; thus n should be even. Further, m is prescribed to be even (while n = 2 gives m = 27, for example). For that, we need n to be divisible by 4. The smallest possibility n = 4 gives m = 1080.

*If m is not divisible by* 3*, Equation* (184) *is solved by considering the numbers of the norm* 9 *in*  $\mathbb{Q}(\sqrt{10})$ *:* 

$$(1+\sqrt{10})(3+\sqrt{10})^n, \qquad (1-\sqrt{10})(3+\sqrt{10})^n.$$
 (187)

The two options correspond to the fact that  $\mathbb{Q}(\sqrt{10})$  is not a unique factorization domain; its class number equals 2. In the first case, we need  $n = 3 \mod 4$  for even m; the smallest n = 3 gives m = 242. In the second case, we need  $n = 1 \mod 4$ ; the smallest interesting n = 5 gives m = 4802. Setting  $\ell = m - 5$  in (142) gives this equation for  $z^*$ :

$$15(z^*)^2 - 10(m+3)z^* + (m+3)(m+5) = 0.$$
(188)

*It has the same discriminant as* (183), *leading to the same Equation* (184).

**Example 19.** Let us consider hypergeometric polynomials (141) of degree 3 or 4. Note that the lower parameter  $1 + k - \ell$  is supposed to be a positive integer for k = m + 1. With odd m and  $\ell = (m + 7)/2$ , we are considering

$$_{2}F_{1}\left(\begin{array}{c|c} -3, -\frac{5}{2} \\ \ell - 5 \end{array} \middle| z^{*}\right) = 0.$$
 (189)

Comparing with (150), we conclude that we need integer values  $p/r = -\ell$  of Example 13 satisfying p/r < -5. That excludes the discarded  $p/r \in \{0, -1, -2, -3, -4, -5\}$  again. The number of points on  $\mathcal{E}_5$  that give integer values of p/r (or any other regular function) is finite by Siegel's theorem ([6] IX.3). In this case, there appear to be no relevant integer values.

Similarly, the consideration of integer  $\ell \in \{\frac{m+8}{2}, \frac{m+9}{2}, \frac{m+10}{2}\}$  in (141) leads to the hypergeometric polynomials

$${}_{2}F_{1}\left(\begin{array}{c}-3, -\frac{7}{2}\\\ell-6\end{array}\middle|z^{*}\right), \qquad {}_{2}F_{1}\left(\begin{array}{c}-4, -\frac{7}{2}\\\ell-7\end{array}\middle|z^{*}\right), \qquad {}_{2}F_{1}\left(\begin{array}{c}-4, -\frac{9}{2}\\\ell-8\end{array}\middle|z^{*}\right), \qquad (190)$$

comparable to (157), (163), and (167), respectively. We need sufficiently negative integer values of  $p/r = -\ell$  from Examples 14–16. Only the value p/r = -14 of Example 16 suits us. The pivotal hypergeometric evaluation is

$$F_1 \begin{pmatrix} -4, -\frac{9}{2} \\ 6 \\ \end{pmatrix} = 0.$$
(191)

This gives  $\ell = 14$ , m = 18 and  $H_2(x) = 1 + x - x^2$ . The expansion of  $H_2(x)^{14}$  misses the term with  $x^{19}$  accordingly. The polynomial  $G_{18}(x)$  is defined by the lower-degree terms:

2

$$1 + 14x + 77x^{2} + 182x^{3} - 910x^{5} - 1365x^{6} + 1430x^{7} + 5005x^{8} - 10010x^{10} - 3640x^{11} + 14105x^{12} + 6930x^{13} - 15625x^{14} - 6930x^{15} + 14105x^{16} + 3640x^{17} - 10010x^{18}.$$
 (192)

The terms with  $x^4$ ,  $x^9$  are missed as well. The broken symmetry around the term with  $x^{14}$  is notable. The Belyi map is  $H_2(x)^{14}/G_{18}(x)$ .

**Example 20.** Hypergeometric polynomials (142) of degree 3 or 4 are considered similarly. The consideration of odd  $\ell \in \{m - 6, m - 7, m - 8, m - 9\}$  in (142) leads to the hypergeometric polynomials

$${}_{2}F_{1}\left(\begin{array}{c}-3,-\frac{5}{2}\\-\frac{\ell}{2}-6\end{array}\right|z^{*}\right), \quad {}_{2}F_{1}\left(\begin{array}{c}-3,-\frac{7}{2}\\-\frac{\ell}{2}-7\end{array}\right|z^{*}\right), \quad {}_{2}F_{1}\left(\begin{array}{c}-4,-\frac{7}{2}\\-\frac{\ell}{2}-8\end{array}\right|z^{*}\right), \quad {}_{2}F_{1}\left(\begin{array}{c}-4,-\frac{9}{2}\\-\frac{\ell}{2}-9\end{array}\right|z^{*}\right),$$

comparable to (150), (157), (163), and (167), respectively. The relation with p/r of Examples 13–16 is always  $p/r = 1 + \ell/2$ . We look for positive half-integer values p/r. Again, only Example 16 provides an instance: p/r = 11/2. This gives  $\ell = 9$ , m = 18, and also  $H_2(x) = 1 + 2x + 5x^2$ . The expansion of  $H_2(x)^{9/2}$  misses the term with  $x^{19}$  indeed. We have

$$G_{18}(x) = 1 + 9x + 54x^{2} + 210x^{3} + 630x^{4} + 1386x^{5} + 2394x^{6} + 2394x^{7} + 2655x^{8}$$
(193)  
+ 1195x^{9} + 252x^{10} - 252x^{11} + 168x^{12} - 180x^{14} + 228x^{15} - 18x^{16} - 378x^{17} + 560x^{18}.

For some reason, the coefficients 2394 and  $\pm 252$  repeat twice consequently. The Belyi map is  $G_{18}(x)^2/H_2(x)^9$ .

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