mathematics

Article

Dual Connectivity in Graphs

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Abstract: An edge-coloring σ of a connected graph G is called rainbow if there exists a rainbow path connecting any pair of vertices. In contrast, σ is monochromatic if there is a monochromatic path between any two vertices. Some graphs can admit a coloring which is simultaneously rainbow and monochromatic; for instance, any coloring of K_n is rainbow and monochromatic. This paper refers to such a coloring as dual coloring. We investigate dual coloring on various graphs and raise some questions about the sufficient conditions for connected graphs to be dual connected.

Keywords: edge coloring; rainbow coloring; monochromatic coloring; dual connected graphs

MSC: 05C15

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Academic Editor: Janez Žerovnik

Received: 29 November 2024
Revised: 9 January 2025
Accepted: 10 January 2025
Published: 11 January 2025

Citation: Mutar, M.A.; Otera, D.E.; Khawwan, H.A. Dual Connectivity in Graphs. *Mathematics* **2025**, *13*, 229. https://doi.org/10.3390/math13020229

1. Introduction

The edge coloring of graphs is a fundamental topic of graph theory that includes various interesting concepts. Recent studies have concentrated on modifying or extending the concept of coloring to special kinds of graphs. For instance, the proper coloring of graphs is a well-studied concept referring to the assignment of colors to the edges such that no two adjacent edges share the same color. Recently, Richard Behr [1] and Zhang et al. [2] independently extended the proper coloring to signed graphs. Similarly, the Ramsey number $R(s,t)$ is an extensively studied concept of the well-known Ramsey theory, which asks about the smallest positive integer r such that edge 2-colored K_r contains either a monochromatic K_s of color 0 or K_t of color 1. Recently, Mutar et al. [3] modified this concept to signed graphs in which $r_{\pm}(s,t)$ is the smallest positive integer r, such that signed K_r contains $+K_s$ or $-K_t$.

Let G be a simple undirected connected graph of order n and size m. Let $\mathbb{N}_k$ denote the set of non-negative integers up to $k-1$. An edge k-coloring is a function $\sigma : E(G) \longrightarrow \mathbb{N}_k$ that assigns colors to the edges of G with no restrictions in the way adjacent edges are allowed to have the same color. The coloring σ is called rainbow if, for any pair of vertices, there exists a rainbow path joining that pair (that is, a path in which all the edges are colored with distinct colors). The concept of edge rainbow coloring was introduced by Chartrand et al. [4]. Later, Caro et al. in [5] introduced monochromatic coloring as the opposite concept of rainbow coloring. That is, the coloring σ is monochromatic if, for any pair of vertices, there exists a monochromatic path joining the pair. By simply assigning distinct colors or exactly one color to the edges of any connected graph, one always ends up with a rainbow or monochromatic coloring. Although such colorings might seem trivial for some graphs, like trees, these assignments are the only possible option. The minimum number of colors required for ensuring rainbow coloring is called the *rainbow connection number* and is denoted by $rc(G)$. Correspondingly, $mc(G)$ denotes the *monochromatic connection number*, which is the maximum number of colors required for obtaining a monochromatic coloring. We refer to [6–9] for their significant contributions to both concepts, as well as to the well-established surveys [10,11].

Both concepts have been extensively studied over the years. A total coloring refers to assigning colors to both the vertices and edges of a graph. In 2015, Yuefang Sun [12] introduced total rainbow coloring, which requires the existence of a path between every pair of vertices such that the edges and the internal vertices of the path all have distinct colors. The *total rainbow connection number* of G is then automatically defined and denoted as $trc(G)$. Recently, Zhang et al. [13] studied this concept on families of connected graphs. In contrast, Hui Jiang et al. [14] introduced the so-called total monochromatic coloring where every pair of vertices is connected by a path whose vertices and edges are assigned the same color. The *total monochromatic connection number* is denoted by $tmc(G)$.

Some graphs can admit a coloring that is simultaneously rainbow and monochromatic. This observation motivated us to investigate connected graphs admitting such a coloring. In this paper, we call this coloring *dual*. That is, for any pair of vertices, there exist both rainbow and monochromatic paths connecting this pair. If G admits dual coloring, then the minimum number of colors needed to obtain a dual coloring will be denoted $ldc(G)$ and called the *lower dual connection number*. Moreover, the inequality $rc(G) \leq ldc(G) \leq mc(G)$ holds. Note that any coloring of complete graphs is simultaneously rainbow and monochromatic. However, it turns out that a tree T of order $n \geq 3$ does not admit dual coloring, because of the next two results.

Theorem 1 ([5])**.** *If G is K_3-free, then $mc(G) = m - n + 2$.*

Proposition 1 ([4])**.** *Let G be a connected graph.*

1. *If G is a cycle on $n \geq 4$ vertices, then $rc(G) = \lceil \frac{n}{2} \rceil$.*
2. *If G is a tree, then $rc(G) = m$.*

2. A Discussion on Necessary and Sufficient Conditions

An obvious necessary condition for G to admit dual coloring is that $rc(G) \leq mc(G)$. For instance, if C_n is a cycle with $n \geq 5$ vertices, then C_n does not admit dual coloring. Theorem 1 shows that $mc(C_n) = 2$, while Proposition 1 assures that $rc(C_n) = \lceil \frac{n}{2} \rceil > mc(C_n)$.

However, this condition is not enough, as we will see shortly. Let us consider the generalized Petersen graph, denoted by $G_{n,k}$, which is a class of graphs characterized by the positive integers $n \geq 3$ and $k \geq 1$. This graph is a 3-regular connected graph on $2n$ vertices, which are divided into two sets: an outer set $V = \{v_0, v_1 \ldots, v_{n-1}\}$ and an inner set $U = \{u_0, v_1 \ldots, u_{n-1}\}$. The construction of this graph is as follows: each vertex v_i in the outer set is adjacent to v_{i+1} and v_{i-1} (with indices taken modulo n). In other words, the outer vertices induce a cycle. In addition, every vertex u_i in the inner set is adjacent to the corresponding vertex v_i in the outer set as well as to the vertex u_{i+k} in the inner set (with indices taken modulo n). The graph $G_{n,1}$ is called a prism; it consists of an outer cycle and an inner cycle connected by the edges (v_i, u_i).

Figure 1 exhibits three examples of dual colorings of prisms with $n = 3, 4, 5$. For $n \geq 6$, the graph $G_{n,1}$ is Hamiltonian and K_3-free with $3n$ edges. Thus, $rc(G_{n,1}) \leq n < n + 2 = mc(G_{n,1})$. However, Theorem 2 shows that $G_{n,1}$ is not dual connected.

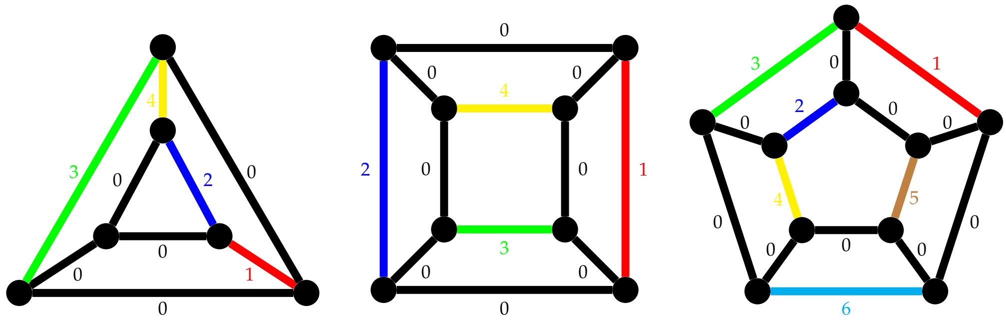

Figure 1. Dual coloring of prisms with $n = 3, 4, 5$. The colors black, red, blue, green, yellow, brown, and cyan correspond the numbers 0, 1, 2, 3, 4, 5, and 6, respectively.

Theorem 2. *For $n \geq 6$, the graph $G_{n,1}$ is not dual connected.*

The proof of Theorem 2 will follow from the following several intermediate claims.

Lemma 1. *The vertices of $G_{n,1}$ can be partitioned into n pairs such that the distance between the vertices in each pair is $\lfloor \frac{n}{2} \rfloor + 1$.*

Proof. Let $V = \{v_0, v_1 \ldots, v_{n-1}\}$ and $U = \{u_0, v_1 \ldots, u_{n-1}\}$ be the sets of vertices in $G_{n,1}$ that form the outer and inner cycles, respectively. For $i = 0, 1, \ldots, n-1$, define $f : V \longrightarrow U$ as $f(v_i) = u_j$ where $j = i + \lfloor \frac{n}{2} \rfloor \pmod{n}$. This is a bijection where the distance between v_i and u_j is $\lfloor \frac{n}{2} \rfloor + 1$ for all $i = 0, 1, \ldots, n-1$. $\square$

Claim 1. *For a fixed $n \geq 6$, a monochromatically colored $G_{n,1}$ has a monochromatic spanning tree.*

Proof. If the incident edges at some vertex v are assigned same color, then monochromatic connectivity implies that the graph $G_{n,1}$ must have a monochromatic spanning tree rooted at v. Thus, assume that a monochromatically colored $G_{n,1}$ has no vertex v whose incident edges are identically colored. We will show that $G_{n,1}$ has a monochromatic path of length $2n - 1$.

Let P be the longest monochromatic path with length l. First, we claim that $l \geq \lfloor \frac{n}{n} \rfloor + 2$. Otherwise, the longest monochromatic path will be of length $\lfloor \frac{n}{2} \rfloor + 1$. By Lemma 1, let P_i denote the monochromatic path connecting v_i to u_j for all $i = 0, 1, \ldots, n-1$ and $j = i + \lfloor \frac{n}{2} \rfloor \pmod{n}$. Clearly, for any $i \neq j$, the paths P_i and P_j cannot share edges. If they do, it would either result in a vertex with its incident edges assigned the same color, or one of the paths would not remain monochromatic. Hence, they are pairwise edge-disjoint. This yields that the total edges are at least $n(\lfloor \frac{n}{2} \rfloor + 1)$, contradicting the fact that $G_{n,1}$ has only $3n$ edges. Therefore, $l \geq \lfloor \frac{n}{2} \rfloor + 2$.

Now, we will prove that $l = 2n - 1$. Suppose, by contradiction, that $l < 2n - 1$. Then, there exists a vertex x_0 that is not on the path P, which contains at least four internal vertices x_1, x_2, x_3, x_4. Let P_1, P_2, P_3, and P_4 be four monochromatic paths connecting x_1, x_2, x_3 and x_4 to x_0, respectively. Since $G_{n,1}$ has no vertex with its incident edges identically colored, each path is colored differently from P. Moreover, since each vertex has degree 3, the paths P_1, P_2, P_3, and P_4 must emanate from x_1, x_2, x_3, and x_4, respectively. This implies that either some of the paths P_1, P_2, P_3, and P_4 share edges, or x_0 has degree 4, which is a contradiction. $\square$

Claim 2. *A coloring σ of $G_{n,1}$ is not rainbow if the colored $G_{n,1}$ contains a monochromatic spanning tree having degree 3.*

Proof. Let σ be a coloring of $G_{n,1}$. Suppose that H is a monochromatic spanning tree of the colored $G_{n,1}$. That is, at least $2n-1$ edges of $G_{n,1}$ are assigned a color i. Let v be a vertex which has degree 3 in H, or equivalently, the incident edges to v in $G_{n,1}$ have the same color i. Since v is not adjacent to $2n-4$ vertices in $G_{n,1}$, it follows that, in order for v to be rainbow connected to all other vertices, there must be at least $2n-4$ edges of colors different to i. This is impossible because the size of $G_{n,1}$ is $3n$. Thus, σ is not rainbow. □

Claim 3. *A coloring σ of $G_{n,1}$ is not rainbow if the colored $G_{n,1}$ contains a monochromatic path of length $2n-1$, where $n \geq 6$.*

Proof. Let P be a monochromatic path of length $2n-1$ given as $P : x_1, x_2, \ldots, x_{2n-1}, x_{2n}$, and $H = G_{n,1} - P$, a spanning subgraph obtained by removing from $G_{n,1}$ the edges of P. Clearly, the subgraph H has $2n$ vertices, each of degree 1 except x_1 and x_{2n}, which are of degree 2. Moreover, H consists of $n-1$ components because it has only $n+1$ edges.

If x_1 and x_2 are not adjacent, then for some distinct vertices x_r, x_s and x_t, the paths $C_1 : x_r, x_1, x_s$ and $C_2 : x_t, x_{2n}, x_u$ are the only components consisting of more than one edge in H; see Figure 2. Recall from Lemma 1 that the vertices can be partitioned into n pairs of vertices, such that the distance between the vertices of each pair is $\lfloor \frac{n}{2} \rfloor + 1$. That is, there are at least 6 such pairs, each with a distance of at least 4. Therefore, either x_1 is at a distance at least 4 from a vertex in C_2, and consequently, there exists a pair of distinct vertices x_i and x_j at a distance of at least 4, belonging to two distinct single-edge components C_i and C_j, respectively. As a result, any path connecting x_i to x_j cannot be rainbow, as it must traverse at least two edges of P. Or x_1 is at a distance of at least 4 from the vertex x_w in a single-edge component C_w. Now, since any path connecting x_1 to x_w will use, at most, either the edge x_1x_r or x_1x_t, such a path cannot be rainbow as well.

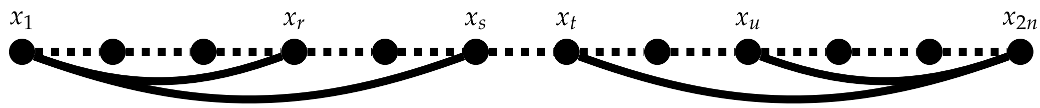

Figure 2. Dashed edges represent the edges of P while the solid ones represent those of H.

Otherwise, x_1 and x_{2n} are adjacent, see Figure 3. Then for some distinct vertices x_i and x_j, the path $C_3 : x_i, x_1, x_{2n}, x_j$ is the only component with more than one edge in H. This leaves a pair of vertices in two distinct single-edge components, where the distance between the vertices of that pair is at least 4. Therefore, they cannot be rainbow connected, as we wanted to prove. □

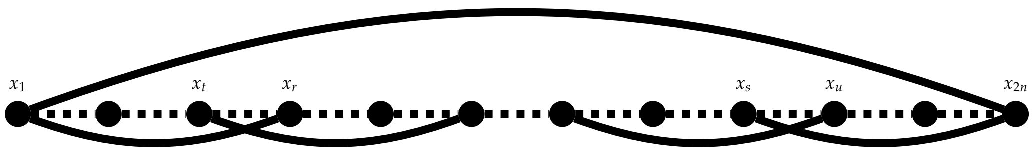

Figure 3. Dashed edges represent the edges of P while the solid ones represent those of H.

Claim 1 shows that any monochromatic coloring of $G_{n,1}$ will always result in a monochromatic spanning tree. Claims 2 and 3 together show the coloring will not be rainbow as long it has a monochromatic spanning tree. Thus, no coloring of $G_{n,1}$ can be monochromatic and rainbow at the same time, which proves the statement of Theorem 2. This raises the following natural questions.

Question 1. *Let G be a connected K_3-free graph. Does every monochromatic coloring of G induce a monochromatic spanning tree?*

Question 2. *Does every connected graph G with $rc(G) \leq mc(G)$ and $diam(G) \leq 3$ admit dual coloring?*

Question 3. *Is $G_{n,k}$ dual connected when $k = \lceil \frac{n}{2} \rceil - 1$?*

3. Lower Dual Connection Number of Some Graphs

In this section, we will focus on the lower dual connection number of wheels and complete bipartite graphs.

Theorem 3. *Let W_n be a wheel where $n \geq 3$; then,*

$$ldc(W_n) = \begin{cases} 1 & \text{if } n = 3, \\ 2 & \text{if } 4 \leq n \leq 6, \\ 3 & \text{if } n \geq 7. \end{cases}$$

Proof. For $n \geq 3$, let $\{v_0, v_1, \ldots, v_n\}$ be the vertices of the wheel W_n, where the vertices $\{v_1, \ldots, v_n\}$ induce a cycle C and v_0 is adjacent to every vertex in C. If $n = 3$, then $W_3 = K_3$ and a single color is sufficient to ensure dual connectivity. Thus, $\mathrm{ldc}(W_3) = 1$.

For $4 \leq n \leq 6$, W_n is not a complete graph. In this case, there exists a pair of vertices at distance 2. Therefore, $\mathrm{ldc}(W_n) \geq 2$. For $i \geq 1$, consider the coloring below:

$$\sigma(v_i, v_{i+1}) = \sigma(v_i, v_0) = \begin{cases} 0 & \text{if } i \text{ is even}, \\ 1 & \text{if } i \text{ is odd}. \end{cases}$$

This coloring is monochromatic since the subgraph induced by the edges of color 1 is a connected spanning subgraph. Consider two non-adjacent vertices v_i and v_j on the cycle C. If i and j have a different parity, then they are connected by a rainbow path through the vertex v_0 because $\sigma(v_i, v_0) \neq \sigma(v_j, v_0)$. If i and j have the same parity, then, since the cycle C has length $l \leq 6$, v_i and v_j must be at a distance of 2 on the cycle. Thus, there exists a vertex v_k in C between v_i and v_j such that k has a different parity from i and j. Consequently, the path v_i, v_k, v_j is rainbow. Therefore, the coloring is rainbow and, consequently, is dual.

For $n \geq 7$, we claim that $ldc(W_n) \geq 3$. Suppose that σ is any 2-coloring of W_n. Then, at least $\lceil \frac{n}{2} \rceil$ of the edges incident to v_0 will receive the color x for some $x \in \mathbb{N}_2$. Equivalently, at least $\lceil \frac{n}{2} \rceil$ vertices of the cycle C will be the endpoints of edges colored by x. With respect to C, two of these $\lceil \frac{n}{2} \rceil$ vertices, say, u and v, will be at distance greater than 2 because $\lceil \frac{n}{2} \rceil \geq 4$ for $n \geq 7$. Therefore, there is no rainbow path connecting u to v which either goes through v_0 or is completely contained within the cycle C. Hence, $ldc(W_n) \geq 3$.

It remains to be shown that $ldc(W_n) \leq 3$. To this end, consider the 3-coloring σ, defined as follows:

$$\sigma(v_i, v_j) = \begin{cases} 0 & \text{if } i \text{ even and } j = 0, \\ 1 & \text{if } i \text{ odd and } j = 0, \\ 2 & \text{otherwise}. \end{cases}$$

This coloring is monochromatic because all edges of C are assigned color 2 and all vertices of C are adjacent to v_0. Additionally, let v_i and v_j be a pair of non-adjacent vertices on the cycle C. If i and j have a different parity, then v_i and v_j are connected by a rainbow path through the vertex v_0 because $\sigma(v_i, v_0) \neq \sigma(v_j, v_0)$. Otherwise, i and j have the same parity. Without the loss of generality, suppose $i < j$. Then, the path v_i, v_{i+1}, v_0, v_j is rainbow. Therefore, σ is dual, which finishes the proof. □

Example 1. *Figure 4 illustrates the coloring used in Theorem 3.*

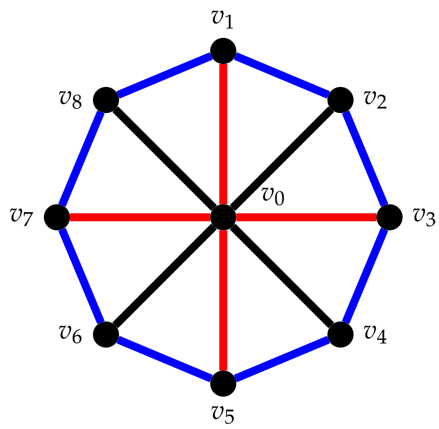

Figure 4. Dual coloring of wheel graph W_8, where black, red, and blue represent 0, 1, and 2, respectively.

Theorem 4. *Let $K_{r,s}$ be a complete bipartite graph with $2 \leq r \leq s$ and $t = \sqrt[r]{s+1}$ and $u = \sqrt[r]{s+3}$. If $r = 2$, then $ldc(K_{r,s}) = \lceil \frac{s+1}{2} \rceil$. Otherwise,*

$$ldc(K_{r,s}) = \begin{cases} 2 & \text{if } t \leq 2, \\ 3 & \text{if } t > 2 \text{ and } u \leq 3, \\ 4 & \text{if } u > 3. \end{cases}$$

The proof of Theorem 4 will be presented later, after the establishment of several necessary concepts and results.

Definition 1. *An $r \times s$ matrix Σ over $\mathbb{N}_k$ is called transitive if the following conditions hold:*

1. *The columns of Σ are distinct.*
2. *For any pair of columns v and u, there exists an element $x \in \mathbb{N}_k$ and a sequence of vectors $v = w_1, w_2, \ldots, w_l = u$, such that for each consecutive pair w_i and w_{i+1}, there exists at least one position where w_i and w_{i+1} have the value x.*

Example 2. *For $r \geq 3$, the identity matrix I_r over $\mathbb{N}_2$ is transitive.*

Lemma 2. *For $r \leq s \leq 2^r - 1$ and $r \geq 3$, there exists an $r \times s$ matrix Σ over $\mathbb{N}_2$ such that both Σ and Σ^T are transitive.*

Proof. Let Σ be an $r \times s$ matrix consisting of all r-dimensional column vectors over $\mathbb{N}_2$, where each vector contains at least one 0 entry. Consequently, Σ has $2^r - 1$ columns, including the identity matrix I_r as a block. Suppose u and v are two distinct columns of Σ with 0 entries in non-corresponding positions i and j, respectively. Then, there exists a standard basis vector w in I_r that has 0 at positions i and j, allowing for a sequence u, w, v for every such pair of columns. Therefore, Σ is transitive, and so is its transpose Σ^T, thanks to the existence of the block I_r in Σ^T. Furthermore, any submatrix obtained by removing columns from Σ other than those in I_r, along with its transpose, will also be transitive. □

Lemma 3. *For $2^r \leq s \leq 3^r - 3$ and $r \geq 3$, there exists a transitive $r \times s$ matrix over $\mathbb{N}_3$.*

Proof. Aside from the all-zeros column vector, let Σ_0 be an $r \times s$ matrix consisting of all r-dimensional column vectors over $\mathbb{N}_3$ where each vector contains at least one 0 entry. That is, Σ_0 consists of $3^r - (2^r + 1)$ distinct columns, including the identity matrix I_r as a block. In a manner similar to the argument made in the proof of Lemma 2, for any $r \leq s \leq 3^r - 2^r - 1$, there exists an $r \times s$ matrix that is transitive, and its transpose is also transitive.

For $3^r - 2^r \leq s \leq 3^r - 3$, let Σ be a matrix consisting of distinct column vectors from $\mathbb{N}_3^r$ such that Σ includes Σ_0 as a block but no vector like $\begin{bmatrix} x & x & \dots & x \end{bmatrix}^T$ for all $x \in \mathbb{N}_3$. Thus, the transpose Σ^T is transitive because Σ includes I_r. Now, let u and v be two columns. Then, u and v contain a same entry y (not necessarily in corresponding positions, say, in position i and j, respectively). This implies that there exists a standard vector w having y as entry at both positions i and j. Therefore, Σ is transitive. $\square$

Proof of Theorem 4. Let the vertices of $K_{r,s}$ be partitioned into two independent sets $A = \{a_1, \dots, a_r\}$ and $B = \{b_1, \dots, b_s\}$. For a k-coloring σ, let $\Sigma = [\sigma_{ij}]$ be an $r \times s$ matrix and Σ^T denote the transpose of Σ, where $\sigma_{ij} = \sigma(a_i, b_j)$.

First of all, let us note that σ is dual if Σ and Σ^T are transitive. To verify this statement, suppose that v and u are two column vectors of Σ. Then, v and u must differ in at least one coordinate, say, at coordinate c. This implies that the corresponding vertices b_v and b_u are connected by a rainbow path of length 2 through the vertex a_c. Additionally, the transitivity guarantees the existence of a vector sequence $v = w_1, w_2, \dots, w_l = u$ such that for some $x \in \mathbb{N}_k$, two consecutive vectors w_i and w_{i+1} agree in at least one coordinate by the element x. In other words, for $i = 1, \dots, l-1$, both w_i and w_{i+1} have x at a coordinate c_i. Consequently, there is a monochromatic path of color x connecting b_v to b_u, given as $b_v = b_{w_1}, a_{c_1}, b_{w_2}, \dots, b_{w_{l-1}}, a_{c_{l-1}}, b_{w_l} = b_u$. A similar argument applies to Σ^T, leading to the same conclusion. That is, there exist rainbow and monochromatic paths between any pair of vertices in the same partition. Hence, σ is a dual coloring.

If $r = 2$, then σ is a dual coloring if and only if Σ is transitive and contains a column vector $v = \begin{bmatrix} x & x \end{bmatrix}^T$ for some $x \in \mathbb{N}_k$. To prove this claim, suppose that σ is a dual coloring. Since all paths between a_1 and a_2 are of lengths 2, there is a monochromatic path joining a_1 to a_2 through a vertex b_v. Hence, $\sigma(a_1, b_v) = \sigma(a_2, b_v) = x$ and such a vector v exists. Now suppose that there is a column vector u which does not coincide with the vector v at any coordinate. Then, there exists no monochromatic path joining b_v to b_u since the two incident edges at b_u share no common color with the incident edges at b_v. This is a contradiction. Therefore, all column vectors must coincide in at least a coordinate with v. Furthermore, if $u = w$ for some column vectors u and w, then there is no rainbow path joining b_u to b_w. To verify this point, suppose without the loss of generality that u and w coincide with v in their first position. That is, $\sigma(a_1, b_u) = \sigma(a_1, b_w) = x$ and $\sigma(a_2, b_u) = \sigma(a_2, b_w)$. One can see immediately that there is no rainbow path of length 2 passing through a_1 or a_2. Moreover, assuming that there is a rainbow path of length at least 4, yielding that this path must start with color $\sigma(a_1, b_u)$ and end with color $\sigma(a_2, a_w)$. Therefore, there must be a vertex b_l on the path with a column vector l not coinciding with v in any of its coordinates. This is a contradiction.

Conversely, suppose that Σ is transitive and contains a column vector $v = \begin{bmatrix} x & x \end{bmatrix}^T$ for some $x \in \mathbb{N}_k$. Since the columns of Σ are distinct and have x at one of their two coordinates, the two columns of Σ^T are also distinct but coincide with x at one coordinate. Thus, the matrix Σ^T is also transitive.

Now, the maximum number of two-dimensional columns that a transitive matrix Σ over $\mathbb{N}_k$, containing a vector like $v = \begin{bmatrix} x & x \end{bmatrix}^T$, can have is given by $k^2 - (k-1)^2$. Therefore, the lower dual connection number of $K_{2,s}$ must be the smallest positive integer k satisfying the following inequality.

$$(k-1)^2 - (k-2)^2 < s \leq k^2 - (k-1)^2. \tag{1}$$

Basic simplifications of inequality (2) yields

$$\frac{s+1}{2} \leq k < \frac{s+1}{2} + 1. \tag{2}$$

Therefore, $ldc(K_{2,s}) = \lceil \frac{s+1}{2} \rceil$, the smallest positive integer greater or equal $\frac{s+1}{2}$. Lastly, we will address the cases when $r \geq 3$, as follows:

Case 1 Suppose that $t \leq 2$. This implies that $s \leq 2^r - 1$. By Lemma 2, there exists an $r \times s$ matrix Σ over $\mathbb{N}_2^r$, such that both Σ and Σ^T are transitive. The 2-coloring induced by Σ is dual; thus, $ldc(K_{r,s}) \leq 2$. In general, $ldc(K_{r,s}) \geq 2$ because the distance between any two vertices in the same partition is 2. Therefore, $ldc(K_{r,s}) = 2$.

Case 2 If $t > 2$ and $u \leq 3$, then $2^r - 1 < s \leq 3^r - 3$. Lemma 3 guarantees the existence of an $r \times s$ matrix Σ over $\mathbb{N}_3$, such that both Σ and Σ^T are transitive. Thus, $ldc(K_{r,s}) \leq 3$. It remains to be shown that $ldc(K_{r,s}) \geq 3$. Since $s \geq 2^r$, using the elements of $\mathbb{N}_2$ to color the edges of $K_{r,s}$ results in Σ having either two identical columns or the columns share no entry. Assume that v_i and v_j are two identical columns in Σ. Then, the corresponding vertices b_i and b_j cannot be connected by a rainbow path of length 2 because every path of length 2 will be monochromatic. In addition, if v_i and v_j share no entry, then the corresponding vertices b_i and b_j will not be connected by any monochromatic path. Thus, $ldc(K_{r,s}) \geq 3$.

Case 3 If $u > 3$, then $s > 3^r - 3$. In a manner similar to the argument presented in Case 2, using the elements of $\mathbb{N}_3$ would result in two identical columns or columns sharing no common entry. Thus, $ldc(K_{r,s}) \geq 4$.
Consider the 4-coloring σ given as follows: for $i = 1$ or $j = 1$, $\sigma(a_i, a_j) = 0$, and for $i \geq 2$ and $j \geq 2$,

$$\sigma(a_i, b_j) = \begin{cases} 0 & \text{if i is odd and j is even,} \\ 1 & \text{if i is even and j is even,} \\ 2 & \text{if i is even and j is odd,} \\ 3 & \text{if i is odd and j is odd.} \end{cases}$$

Note that the incident edges at a_1 and b_1 are colored with 0, making σ monochromatic. Furthermore, for $i \geq 2$ and $j \geq 2$, the vertices a_1 and b_1 are connected to a_i and b_j through the rainbow paths a_1, b_3, a_i and b_1, a_2, b_j, respectively. Additionally, let $a_\alpha, a_\beta \in A$ and $b_\lambda, b_\sigma \in B$, where $1 < \alpha < \beta$ and $1 < \lambda < \sigma$. If α and β have different a parity, and similarly, λ and σ have different parity, then the paths a_α, b_2, a_β and b_λ, a_2, b_σ are also rainbow. Otherwise, the paths $a_\alpha, b_2, a_{\alpha+1}, b_3, a_\beta$ and $b_\lambda, a_2, b_{\lambda+1}, a_3, b_\sigma$ are rainbow. Therefore, the coloring σ is dual, and $ldc(K_{r,s}) = 4$.

□

Example 3. *Figure 5 illustrates the dual coloring proposed in case 3 of Theorem 4.*

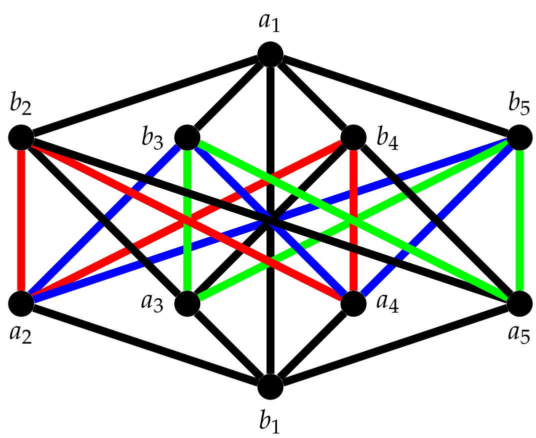

Figure 5. Dual coloring of $K_{5,5}$, where black, red, blue, and green represent 0, 1, 2, and 3, respectively.

4. Conclusions

An edge coloring of a complete graph will always be rainbow and monochromatic, while a tree does not enjoy this property. This observation motivated us to study dual connectivity on some graphs. Clearly, if a connected graph can be decomposed into two disjoint spanning trees, it is dual connected. This condition is sufficient but not necessary. For instance, the prism on 10 vertices is dual connected, but it cannot be decomposed into two spanning trees. Furthermore, one might conjecture that dense graphs are more likely to admit dual coloring. However, this claim is not always true. Consider the graph formed by the union of a path with vertices $\{v_1, v_2, v_3\}$ and a complete graph with vertices $\{v_3, \ldots, v_{n+2}\}$. This graph G is not dual connected because the segment v_1, v_2, v_3 exists in every path connecting v_1 to the vertices of K_n; see Figure 6. Moreover, we showed that the prism $G_{n,1}$, where $n \geq 6$ is not dual connected neither. This naturally poses questions mentioned in Section 2.

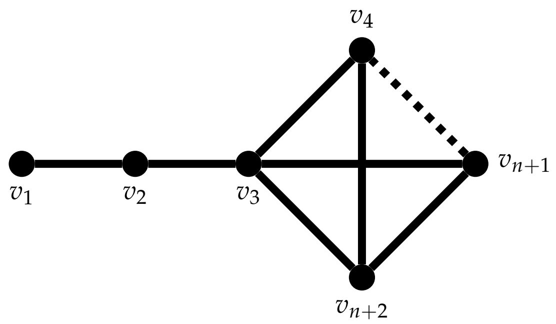

Figure 6. An example of a dense graph that is not dual connected.

As a correspondence to the rainbow connection number, we defined the *lower dual connection number* of a dual connected graph G, denoted by $ldc(G)$, as the minimum number of colors needed to obtain a dual coloring. We then investigated this number in some graphs that admit dual colorings. We can also define the *upper dual connection number* of a dual connected graph G, denoted as udc, as the maximum number of colors required for G to obtain a dual coloring. Clearly, the inequality $rc(G) \leq ldc(G) \leq udc(G) \leq mc(G)$. For future work, two questions need to be investigated:

1. What is the upper dual connection number of a complete graph $K_{r,s}$, i.e, $udc(K_{r,s})$?
2. Are there graphs where $ldc(G) = udc(G)$?

A biorientation of G is the process of replacing each undirected edge e connecting vertices u to v with the two directed edges (u, v) and (v, u). Wang et al. most recently investigated the rainbow connection number of bioriented graphs; see [15]. In contrast, a bidirection of graph refers to assigning two directional arrows to every edge in the graph. This concept was introduced by Edmonds and Johnson [16] and recently studied by Busch et al. [17]. Directed, bioriented, and bidirected graphs offer a framework for investigating dual connectivity in future research.

Author Contributions: Conceptualization, M.A.M.; Methodology, M.A.M., D.E.O. and H.A.K.; Validation, D.E.O.; Investigation, M.A.M. and H.A.K.; Supervision, D.E.O. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Acknowledgments: The authors thank the referees for the comments and the suggestions provided.

Conflicts of Interest: The authors declare no conflicts of interest.

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