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On Value Distribution for the Mellin Transform of the Fourth Power of the Riemann Zeta Function

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Abstract: In this paper, the asymptotic behavior of the modified Mellin transform $\mathcal{Z}_2(s)$, $s = \sigma + it$, of the fourth power of the Riemann zeta function is characterized by weak convergence of probability measures in the space of analytic functions. The main results are devoted to probability measures defined by generalized shifts $\mathcal{Z}_2(s + i\varphi(\tau))$ with a real increasing to $+\infty$ differentiable functions connected to the growth of the second moment of $\mathcal{Z}_2(s)$. It is proven that the mass of the limit measure is concentrated at the point expressed as $h(s) \equiv 0$. This is used for approximation of $h(s)$ by $\mathcal{Z}_2(s + i\varphi(\tau))$.

Keywords: limit theorem; Mellin transform; Riemann zeta function; space of analytic functions; weak convergence

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1. Introduction

Various transforms play an important role in the investigation of functions. Among them, Fourier and Mellin transforms occupy a central place. In analytic number theory, the Mellin transforms of powers of zeta-functions are useful for moment theory.

Let $s = \sigma + it$ be a complex variable and $x \in \mathbb{R}$. Suppose that the function $f(x)x^{\sigma-1}$ is integrable over $(0, \infty)$. The Mellin transform $M_f(s)$ of $f(x)$ is given by

$$M_f(s) = \int_0^\infty f(x)x^{s-1}\,\mathrm{d}x. \tag{1}$$

We observe that $M_f(s)$ is a partial case of Fourier transforms. Actually, after a change of variables $x = \mathrm{e}^y$ in (1), we find

$$M_f(s) = \int_{-\infty}^{\infty} \mathrm{e}^{iyt} f(\mathrm{e}^y)\mathrm{e}^{\sigma y}\,\mathrm{d}y.$$

This shows that $M_f(s)$ is the Fourier transform of the function $f(\mathrm{e}^x)\mathrm{e}^{\sigma x}$.

In (1), some convergence problems can arise at the point of $x = 0$. To avoid those problems, Y. Motohashi introduced [1] the modified Mellin transform $\widehat{M}_f(s)$ defined by

$$\widehat{M}_f(s) = \int_1^\infty f(x) x^{-s}\, \mathrm{d}x.$$

Thus, the modified Mellin transform is an integral form of Dirichlet series

$$\sum_{m=1}^\infty \frac{a(m)}{m^s}, \quad a(m) \in \mathbb{C},$$

which are widely used tools in analytic number theory.

There exists a close relation between $M_f(s)$ and $\widehat{M}_f(s)$. Let

$$\widehat{f}(x) = \begin{cases} f(x^{-1}) & \text{if } 0 < x \leqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then, in [2], it was found that

$$\widehat{M}_f(s) = M_{x^{-1}\widehat{f}(x)}(s).$$

Functions $f(x)$ and $\widehat{M}_f(s)$ are related by an inverse formula. Let $x^{-\sigma} f(x) \in L(1, \infty)$, where $f(x)$ is continuous for $x > 1$. Then, it is known [2] that

$$f(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \widehat{M}(s) x^{-s}\, \mathrm{d}s. \tag{2}$$

Sometimes, it is more convenient to consider $\widehat{M}(s)$, then, using (2), to investigate $f(x)$. This approach is confirmed in [1–11].

Modified Mellin transforms were introduced for the moment problem of the Riemann zeta function $\zeta(s)$. Recall that $\zeta(s)$, for $\sigma > 1$, is defined by

$$\zeta(s) = \sum_{m=1}^\infty \frac{1}{m^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

where the product is taken over all prime numbers and is analytically continuable to the whole complex plane, except a simple pole at the point $s = 1$ with residue 1.

In [1], the modified Mellin transform of $|\zeta(1/2 + it)|^4$

$$\mathcal{Z}_2(s) = \int_1^\infty \left|\zeta\left(\frac{1}{2} + ix\right)\right|^4 x^{-s}\, \mathrm{d}x, \quad \sigma > 1,$$

was introduced, studied, and applied for investigation of the fourth moment

$$m_2(T) = \int_0^T \left|\zeta\left(\frac{1}{2} + it\right)\right|^4 \mathrm{d}t, \quad T \to \infty.$$

Let

$$E_2(T) = m_2(T) - Tp(T),$$

where $p(\log T)$ is a polynomial of degree 4. Then, it was found in [1] that

$$E_2(T) \ll_\varepsilon T^{2/3+\varepsilon}$$

with arbitrary fixed $\varepsilon > 0$. Here, "$\ll_\varepsilon$" is equivalent to "$O(\dots)$" with an implied constant depending on ε. The latter estimate and other moment results show the importance of the transform $\mathcal{Z}_2(s)$ in the theory of the Riemann zeta function.

Modified Mellin transform of powers of the $\zeta(s)$ function were extensively studied in [3–15] in connection with the moment problem (meromorphic continuation, estimates, and mean square estimates). We recall some known results for $\mathcal{Z}_2(s)$. Motohashi meromorphically continued the $\mathcal{Z}_2(s)$ to the whole complex plane [1] (see also [8]). More precisely, let $\{\lambda_j = \kappa_j^2 + 1/4\} \cup \{0\}$ be the discrete spectrum of the non-Euclidean Laplacian acting on automorphic forms for the full modular group. Then, it was proven that $\mathcal{Z}_2(s)$ has a pole at $s = 1$ of order five, simple poles at $s = 1/2 + i\kappa_j$, and simple poles at $s = \rho/2$, where ρ are non-trivial zeros of $\zeta(s)$, i.e., zeros lying in the strip $\{s \in \mathbb{C} : 0 < \sigma < 1\}$. Moreover, in [8], the following estimate

$$\mathcal{Z}_2(\sigma + it) \ll t^{2-2\sigma}(\log t)^{18-14\sigma}, \quad \sigma \in \left(\frac{1}{2}, 1\right),\ t \geqslant t_0, \tag{3}$$

was given. Important results related to the mean square of $\mathcal{Z}_2(s)$

$$J_\sigma(T) \stackrel{\text{def}}{=} \int_0^T |\mathcal{Z}_2(\sigma + it)|^2 \,\mathrm{d}t$$

were obtained in [3–15]. In [5], for a fixed $\sigma \in (1/2, 1)$, the following estimate

$$J_\sigma(T) \ll T^{(10-8\sigma)/3} \log^c T, \quad c > 0,$$

was given. In [11], it was proven that, for a fixed $\sigma \in [5/6, 5/4]$, the following bound is valid:

$$J_\sigma(T) \ll_\varepsilon T^{(15-12\sigma)/5+\varepsilon}. \tag{4}$$

Earlier, it was conjectured in [2] that

$$J_\sigma(T) \ll_\varepsilon T^{2-2\sigma+\varepsilon}$$

for fixed a $\sigma \in (1/2, 1)$. Unfortunately, the asymptotics as $T \to \infty$ for the quantity $J_\sigma(T)$ is not known.

In this paper, we focus on probabilistic value distribution of the Mellin transform $\mathcal{Z}_2(s)$; therefore, we recall some probabilistic results in function theory. The application of the probabilistic approach in function theory was proposed by H. Bohr in [16] and realized in [17,18] for the Riemann zeta function. Let $\mathcal{R} \subset \mathbb{C}$ be a rectangle with edges parallel to the axis and $\mathfrak{M}_J$ denote the Jordan measure on $\mathbb{R}$. Then, roughly speaking, it was obtained in [17,18] that, for $\sigma > 1/2$, a limit

$$\lim_{T\to\infty} \frac{1}{T} \mathfrak{M}_J \{t \in [0, T] : \log \zeta(\sigma + it) \in \mathcal{R}\}$$

exists and depends only on $\mathcal{R}$ and σ. This result shows that the chaotic behavior of $\zeta(s)$ obeys statistical laws. In modern terminology, it is convenient to state Bohr–Jessen results in terms of weak convergence of probability measures. Let $\mathcal{B}(\mathbb{X})$ denote the Borel σ field of a topological space $\mathbb{X}$ and P_n, $n \in \mathbb{N}$, and P be probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. We recall that P_n converges weakly to P as $n \to \infty$ ($P_n \xrightarrow[n\to\infty]{\mathrm{w}} P$) if, for every real continuous bounded function x on $\mathbb{X}$,

$$\lim_{n\to\infty} \int_{\mathbb{X}} x \,\mathrm{d}P_n = \int_{\mathbb{X}} x \,\mathrm{d}P.$$

Then, the Bohr–Jessen theorem can be stated as follows: Suppose that $\sigma > 1/2$ is fixed; then, the probability measure expressed as

$$\frac{1}{T}\mathfrak{M}_L\{t \in [0,T] : \zeta(\sigma+it) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to a certain probability measure P_σ on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ as $T \to \infty$. Here, $\mathfrak{M}_L A$ stands for the Lebesgue measure of $A \in \mathbb{R}$.

More interesting are limit theorems for $\zeta(s)$ in functional spaces. Let $G = \{s \in \mathbb{C} : \sigma > 1/2\}$, and $\mathcal{H}(G)$ be the space of analytic functions on G endowed with the topology of uniform convergence on compacta. In this case, we can consider the weak convergence for

$$\frac{1}{T}\mathfrak{M}_L\{\tau \in [0,T] : \zeta(s+i\tau) \in A\}, \quad A \in \mathcal{B}(\mathcal{H}(G)).$$

Limit theorems for zeta functions in the space of analytic functions were proposed by B. Bagchi in [19] and are very useful for proof of universality theorems on approximation of analytic functions by shifts of zeta functions. Theorems of such a type have several theoretical and practical applications, including the functional independence of zeta functions [20–22] and description of the behavior of particles in quantum mechanics [23,24].

In [25–32], some probabilistic results were obtained on the value distribution of the modified Mellin transform defined, for $\sigma > 1$, by

$$\mathcal{Z}_1(s) = \int_1^\infty \left|\zeta\left(\frac{1}{2}+ix\right)\right|^2 x^{-s}\,\mathrm{d}x,$$

and by analytic continuation for $\sigma > -3/4$, except for a double pole at the point $s = 1$. Let $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$. Then, in the abovementioned works, the weak convergence for

$$\frac{1}{T}\mathfrak{M}_L\{t \in [0,T] : \mathcal{Z}_1(\sigma+it) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}),$$

and

$$\frac{1}{T}\mathfrak{M}_L\{\tau \in [0,T] : \mathcal{Z}_1(s+i\tau) \in A\}, \quad A \in \mathcal{B}(\mathcal{H}(D)),$$

as $T \to \infty$ was considered. Since the limit measure for the latter measures is degenerated at zero, the probability measures defined by generalized shifts $\mathcal{Z}_1(\sigma+i\varphi(t))$ and $\mathcal{Z}_1(s+i\varphi(\tau))$ were also studied [33,34]. In [33], it was required that $\varphi(t)$ be increasing to $+\infty$ differentiable function with a monotonically decreasing derivative $\varphi'(t)$ such that, for $\varepsilon > 0$,

$$\frac{I_{\sigma-\varepsilon}(\varphi(T))}{\varphi'(T)} \ll T, \quad T \to \infty. \tag{5}$$

Here,

$$I_\sigma(T) = \int_1^T |\mathcal{Z}_1(\sigma+it)|^2\,\mathrm{d}t.$$

In [34], hypothesis (5) was replaced by

$$\sup_{1/2<\sigma<1} \frac{I_\sigma(2\varphi(2T))}{T\varphi'(T)} \ll 1, \quad T \to \infty.$$

A natural problem arises to give to the probabilistic characterization of the transform $\mathcal{Z}_2$. Moreover, it is interesting to study approximation properties of $\mathcal{Z}_2(s)$. Since the function

$\zeta(s)$ with the Riemann conjecture is one of the important Millennium objects [35], all results on its value distribution have a significant value.

In this paper, we study the weak convergence of the following measures with some functions φ:

$$P_{T,\mathbb{C},\sigma,\varphi}(A) = \frac{1}{T}\mathfrak{M}_L\{t \in [T, 2T] : \mathcal{Z}_2(\sigma + i\varphi(t)) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}),$$

and

$$P_{T,\mathcal{H},\varphi}(A) = \frac{1}{T}\mathfrak{M}_L\{\tau \in [T, 2T] : \mathcal{Z}_2(s + i\varphi(\tau)) \in A\}, \quad A \in \mathcal{B}(\mathcal{H}),$$

where $\mathcal{H} = \mathcal{H}(\mathcal{D})$, $\mathcal{D} = \{s \in \mathbb{C} : 5/6 < \sigma < 1\}$.

2. Case of $\varphi(t) = t$

Let X be an $\mathbb{X}$-valued random element defined on a certain probability space $(\Omega, \mathfrak{B}, \nu)$. If the distribution of X is

$$\nu\{X \in A\} = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise,} \end{cases} \qquad A \in \mathcal{B}(\mathbb{X}),$$

then X is said to be degenerated at the point of $x \in \mathbb{X}$.

Suppose that X_n, $n \in \mathbb{N}$, are $\mathbb{X}$-valued random elements. If the distribution

$$\nu\{X_n \in A\}, \quad A \in \mathcal{B}(\mathbb{X}),$$

converges weakly to the distribution of X as $n \to \infty$, then we say that X_n converges to X in distribution ($X_n \xrightarrow[n\to\infty]{\mathcal{D}} X$).

Suppose that space $\mathbb{X}$ is metrisable and ρ is a metric inducing the topology of $\mathbb{X}$. If, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \nu\{\rho(X_n, x) \geqslant \varepsilon\} = 0,$$

then we say that X_n converges to $x \in X$ in probability ($X_n \xrightarrow[n\to\infty]{\mathrm{P}} x$).

It is known [36] that $X_n \xrightarrow[n\to\infty]{\mathrm{P}} x$ if and only if $X_n \xrightarrow[n\to\infty]{\mathcal{D}} X$, where X is degenerated at point x.

Proposition 1. *For $5/6 < \sigma < 1$, $P_{T,\mathbb{C},\sigma,t}$ converges weakly to the measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$, degenerated at the point $s = 0$ as $T \to \infty$.*

Proof. In view of the above remarks, it suffices to show that, for every $\varepsilon > 0$,

$$\lim_{T\to\infty} \nu\{d_{\mathbb{C}}(X_T, 0) \geqslant \varepsilon\} = 0, \tag{6}$$

where $X_T = \mathcal{Z}_2(\sigma + i\xi_T)$, $d_{\mathbb{C}}$ is the metric in $\mathbb{C}$ and ξ_T is a random variable on $(\Omega, \mathfrak{B}, \nu)$ uniformly distributed in $[T, 2T]$. According to (4), we have

$$\begin{aligned}
\frac{1}{T}\mathfrak{M}_L\{t \in [T, 2T] : d_{\mathbb{C}}(\mathcal{Z}_2(\sigma + it), 0) \geqslant \varepsilon\} &\leqslant \frac{1}{\varepsilon T}\int_T^{2T} |\mathcal{Z}_2(\sigma + it)|\, \mathrm{d}t \\
&\leqslant \left(\frac{1}{\varepsilon T}\int_T^{2T} |\mathcal{Z}_2(\sigma + it)|^2\, \mathrm{d}t\right)^{1/2} \\
&\ll_{\varepsilon,\varepsilon_1} T^{(10-12\sigma)/10+\varepsilon_1} = o(1), \quad T \to \infty,
\end{aligned}$$

and this proves (6). □

To obtain a similar result in the space $\mathcal{H} = \mathcal{H}(\mathcal{D})$, recall the metric in $\mathcal{H}$. Let $\{K_l : l \in \mathbb{N}\}$ be a sequence of compact embedded subsets of $\mathcal{D}$ such that $\mathcal{D}$ is the union of the sets K_l and every compact set $K \subset \mathcal{D}$ lies in some K_l. Then,

$$d_{\mathcal{H}}(h_1, h_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} |h_1(s) - h_2(s)|}{1 + \sup_{s \in K_l} |h_1(s) - h_2(s)|}, \quad h_1, h_2 \in \mathcal{H},$$

is the metric in $\mathcal{H}$ that induces its topology of uniform convergence on compacta.

Proposition 2. *$P_{T,\mathcal{H},\varphi}$ converges weakly to the measure on $(\mathcal{H}, \mathcal{B}(\mathcal{H}))$ degenerated at the point $h(s) \equiv 0$ as $T \to \infty$.*

Proof. Let $Y_T = \mathcal{Z}_2(s + i\xi_T)$. We have to prove that, for every $\varepsilon > 0$,

$$\lim_{T \to \infty} \nu\{d_{\mathcal{H}}(Y_T, 0) \geqslant \varepsilon\} = 0, \tag{7}$$

or

$$\lim_{T \to \infty} \frac{1}{T} \mathfrak{M}_L\{\tau \in [T, 2T] : d_{\mathcal{H}}(\mathcal{Z}_2(s + i\tau), 0) \geqslant \varepsilon\} = 0.$$

According to the Chebyshev-type inequality,

$$\frac{1}{T} \mathfrak{M}_L\{\tau \in [T, 2T] : d_{\mathcal{H}}(\mathcal{Z}_2(s + i\tau), 0) \geqslant \varepsilon\} \leqslant \frac{1}{T\varepsilon} \int_T^{2T} d_{\mathcal{H}}(\mathcal{Z}_2(s + i\tau), 0) \, \mathrm{d}\tau. \tag{8}$$

In view of the definition of the metric $d_{\mathcal{H}}$, it suffices to consider

$$\frac{1}{T} \int_T^{2T} \sup_{s \in K} |\mathcal{Z}_2(s + i\tau)| \, \mathrm{d}\tau$$

for compact subsets $K \subset \mathcal{D}$. Let $\mathcal{L}$ be a simple closed curve lying in $\mathcal{D}$ and enclosing set K such that

$$\inf_{s \in K} \inf_{z \in \mathcal{L}} |z - s| \gg_{\mathcal{L}} 1. \tag{9}$$

According to the Cauchy integral formula,

$$Z_2(s + i\tau) = \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\mathcal{Z}_2(z + i\tau)}{z - s} \, \mathrm{d}z.$$

Hence, (9) yields

$$\sup_{s \in K} |\mathcal{Z}_2(s + i\tau)| \ll_{\mathcal{L}} \int_{\mathcal{L}} |\mathcal{Z}_2(z + i\tau)| \, |\mathrm{d}z|.$$

Thus, according to the Cauchy–Schwarz inequality and (4),

$$\int_T^{2T} \sup_{s\in K}|\mathcal{Z}_2(s+i\tau)|\,\mathrm{d}\tau \ll_{\mathcal{L}} \int_{\mathcal{L}} \int_T^{2T} |\mathcal{Z}_2(z+i\tau)|\,\mathrm{d}\tau\,|\mathrm{d}z|$$
$$\ll_{\mathcal{L}} \int_{\mathcal{L}} \left(T \int_T^{2T} |\mathcal{Z}_2(z+i\tau)|^2\,\mathrm{d}\tau \right)^{1/2} |\mathrm{d}z|$$
$$\ll_{\mathcal{L}} \int_{\mathcal{L}} \left(T \int_{T-|\mathrm{Im}z|}^{2T+|\mathrm{Im}z|} |\mathcal{Z}_2(\mathrm{Re}z+i\tau)|^2\,\mathrm{d}\tau \right)^{1/2} |\mathrm{d}z|$$
$$\ll_{\mathcal{L}} \int_{\mathcal{L}} \left(T\left(2T+|\mathrm{Im}z|^{(15-12\mathrm{Re}z)/5}\right)\right)^{1/2} |\mathrm{d}z|$$
$$\ll_{\mathcal{L},\varepsilon} T^{(20-12\mathrm{Re}z)/10+\varepsilon_1} = o(T), \quad \varepsilon_1 > 0,$$

because $\mathrm{Re}z > 5/6$. This shows that quantity (8) is estimated as $o(1)$ and (7) holds. □

3. General Case

For brevity $P_{\mathbb{C},0}$ and $P_{\mathcal{H},0}$ denote the probability measures on $(\mathbb{C},\mathcal{B}(\mathbb{C}))$ and $(\mathcal{H},\mathcal{B}(\mathcal{H}))$ degenerated at $s=0$ and $g(s)\equiv 0$, respectively. We consider weak convergence for $P_{T,\mathbb{C},\sigma,\varphi}$ to $P_{\mathbb{C},0}$ and $P_{T,\mathcal{H},\varphi}$ to $P_{\mathcal{H},0}$, respectively, as $T\to\infty$. First, we observe that the case of $P_{T,\mathcal{H},\varphi}$ implies that of $P_{T,\mathbb{C},\sigma,\varphi}$. Actually, let $u:\mathcal{H}\to\mathbb{C}$ be given by

$$u(g(s)) = g(\sigma), \quad s=\sigma+it,\ g\in\mathcal{H}.$$

Since the topology in $\mathcal{H}$ is of uniform convergence on compacta, the mapping u is continuous. Moreover, for $A\in\mathcal{B}(\mathbb{C})$,

$$\begin{aligned} P_{T,\mathbb{C},\sigma,\varphi}(A) &= \frac{1}{T}\mathfrak{M}_L\{t\in[T,2T] : u(\mathcal{Z}_2(s+i\varphi(t)))\in A\} \\ &= \frac{1}{T}\mathfrak{M}_L\left\{\tau\in[T,2T] : \mathcal{Z}_2(s+i\varphi(\tau))\in u^{-1}A\right\} \\ &= P_{T,\mathcal{H},\varphi}(u^{-1}A). \end{aligned}$$

Hence,

$$P_{T,\mathbb{C},\sigma,\varphi} = P_{T,\mathcal{H},\varphi}u^{-1}, \tag{10}$$

where

$$P_{T,\mathcal{H},\varphi}u^{-1}(A) = P_{T,\mathcal{H},\varphi}(u^{-1}A), \quad A\in\mathcal{B}(\mathbb{C}).$$

Therefore, if $P_{T,\mathcal{H},\varphi} \xrightarrow[T\to\infty]{\mathrm{w}} P_{\mathcal{H},0}$, then the continuity of u, relation (10), and the well-known principle of preservation of weak convergence under continuous mappings (see [36], Theorem 5.1) imply that $P_{T,\mathbb{C},\sigma,\varphi} \xrightarrow[T\to\infty]{\mathrm{w}} P_{\mathcal{H},0}u^{-1}$. Since

$$P_{\mathcal{H},0}u^{-1}(A) = P_{\mathcal{H},0}(u^{-1}A) = \begin{cases} 1 & \text{if } (g(s)\equiv 0)\in u^{-1}A, \\ 0 & \text{otherwise,} \end{cases} = \begin{cases} 1 & \text{if } 0\in A, \\ 0 & \text{otherwise.} \end{cases}$$

The latter remark shows that the weak convergence of $P_{T,\mathbb{C},\sigma,\varphi}$ to $P_{\mathbb{C},0}$ is a necessary condition for that of $P_{T,\mathcal{H},\varphi}$ to $P_{\mathcal{H},0}$ as $T\to\infty$.

In this paper, we present some sufficient conditions of the weak convergence for $P_{T,\mathcal{H},\varphi}$ to $P_{\mathcal{H},0}$ as $T\to\infty$. These conditions are connected to $J_\sigma(T)$ and the derivative of the function $\varphi(\tau)$. We prove the following statement.

Theorem 1. *Suppose that $\varphi(\tau)$ is a differentiable function increasing to $+\infty$ with a decreasing derivative $\varphi'(\tau)$ on $[T_0,\infty)$, $T_0>1$ such that*

$$\sup_{\sigma\in(5/6,1)}\frac{J_\sigma(2\varphi(2\tau))}{\varphi'(2\tau)}\ll\tau,\quad \tau\to\infty.$$

Then, $P_{T,\mathcal{H},\varphi}\xrightarrow[T\to\infty]{\mathrm{w}}P_{\mathcal{H},0}$.

We notice that Theorem 1 does not follows directly from the estimate of the second moment for $\mathcal{Z}_2(s+i\varphi(\tau))$.

Actually, let $K\subset\mathcal{D}$ be a compact set. We then try to estimate

$$I\overset{\text{def}}{=}\int_T^{2T}\sup_{s\in K}|\mathcal{Z}_2(s+i\varphi(\tau))|\,\mathrm{d}\tau.$$

Let $\mathcal{L}$ be the same simple closed contour as in the proof of Proposition 2. Then, we have

$$\begin{aligned}I&\ll_{\mathcal{L}}\int_{\mathcal{L}}\left(T\int_T^{2T}|\mathcal{Z}_2(z+i\varphi(\tau))|^2\,\mathrm{d}\tau\right)^{1/2}|\mathrm{d}z|\\&=\int_{\mathcal{L}}\left(T\int_T^{2T}|\mathcal{Z}_2(\mathrm{Re}z+i\mathrm{Im}z+i\varphi(\tau))|^2\,\mathrm{d}\tau\right)^{1/2}|\mathrm{d}z|.\end{aligned}\tag{11}$$

Using properties of the function $\varphi(\tau)$ and the second mean value theorem, we find

$$\begin{aligned}\int_T^{2T}|\mathcal{Z}_2(\mathrm{Re}z+i\mathrm{Im}z+i\varphi(\tau))|^2\,\mathrm{d}\tau&=\int_T^{2T}|\mathcal{Z}_2(\mathrm{Re}z+i\mathrm{Im}z+i\varphi(\tau))|^2\,\frac{\mathrm{d}\varphi(\tau)}{\varphi'(\tau)}\\&\leqslant\frac{1}{\varphi'(2T)}\int_{\varphi(T)-|\mathrm{Im}z|}^{\varphi(2T)+|\mathrm{Im}z|}|\mathcal{Z}_2(\mathrm{Re}z+i\tau)|^2\,\mathrm{d}\tau\\&\leqslant\frac{1}{\varphi'(2T)}\int_{\varphi(T-|\mathrm{Im}z|)}^{2\varphi(2T)}|\mathcal{Z}_2(\mathrm{Re}z+i\tau)|^2\,\mathrm{d}\tau\\&\leqslant\frac{J_{\mathrm{Re}z}(\varphi(2T))}{\varphi'(2T)}\ll\sup_{\sigma\in(5/6,1)}\frac{J_\sigma(2\varphi(2T))}{\varphi'(2T)}\ll T\end{aligned}$$

for a large T. Thus, in view of (11),

$$I\ll_{\mathcal{L}}\int_{\mathcal{L}}T\,|\mathrm{d}z|\ll_{\mathcal{L}}T.$$

This shows only that

$$\frac{1}{T}\int_T^{2T}\sup_{s\in K}|\mathcal{Z}_2(s+i\varphi(\tau))|\,\mathrm{d}\tau$$

is bounded by a constant depending on K. Hence, we find that, in the notation of Proposition 2,

$$\begin{aligned}
\nu\{d_{\mathcal{H}}(\mathcal{Z}_2(s+i\varphi(\xi_T)),0)\geqslant\varepsilon\} &\leqslant \frac{1}{T\varepsilon}\int\limits_T^{2T} d_{\mathcal{H}}(\mathcal{Z}_2(s+i\varphi(\tau)),0)\,\mathrm{d}\tau \\
&= \frac{1}{T\varepsilon}\int\limits_T^{2T}\sum_{k=0}^{\infty}2^{-k}\frac{\sup_{s\in K_k}|\mathcal{Z}_2(s+i\varphi(\tau))|}{1+\sup_{s\in K_k}|\mathcal{Z}_2(s+i\varphi(\tau))|}\,\mathrm{d}\tau \\
&\leqslant \frac{1}{T\varepsilon}\sum_{k=0}^{\infty}2^{-k}\int\limits_T^{2T}\sup_{s\in K_k}|\mathcal{Z}_2(s+i\varphi(\tau))|\,\mathrm{d}\tau \\
&\ll \frac{1}{\varepsilon}\sum_{k=1}^{\infty}2^{-k}c_k,
\end{aligned}$$

and this does not mean that

$$\nu\{d_{\mathcal{H}}(\mathcal{Z}_2(s+i\varphi(\xi_T)),0)\}=0.$$

Thus, for the proof of Theorem 1, we need another approach.

4. Limit Lemmas

We start with a limit lemma for a certain integral over a finite interval. For brevity, we use the following notation. Let $\alpha>1/6$ be a fixed number; then, for $x,y\in[1,+\infty)$,

$$b(x,y)=\exp\left\{-\left(\frac{x}{y}\right)^{\alpha}\right\},$$

$$\zeta_2(x)=\left|\zeta\left(\frac{1}{2}+ix\right)\right|^4,$$

and, for $a>1$,

$$\mathcal{Z}_{2,a,y}(s)=\int\limits_1^a\zeta_2(x)b(x,y)x^{-s}\,\mathrm{d}x.$$

For $A\in\mathcal{B}(\mathcal{H})$, we define

$$P_{T,a,y}(A)=\frac{1}{T}\mathfrak{M}_L\{\tau\in[T,2T]:\mathcal{Z}_{2,a,y}(s+i\varphi(\tau))\in A\}.$$

Lemma 1. *Suppose that the function $\varphi(\tau)$ satisfies the hypotheses of Theorem 1. Then, for every fixed a and y, $P_{T,a,y}\xrightarrow[T\to\infty]{\mathrm{w}}P_{\mathcal{H},0}$.*

Proof. As we saw in Section 2, it suffices to find that, for each compact subset $K\subset\mathcal{D}$,

$$\lim_{T\to\infty}\frac{1}{T}\int\limits_T^{2T}\sup_{s\in K}|\mathcal{Z}_{2,a,y}(s+i\varphi(\tau))|\,\mathrm{d}\tau=0.$$

According to the Cauchy integral formula, the latter equality is implied by

$$\lim_{T\to\infty}\frac{1}{T}\int\limits_T^{2T}|\mathcal{Z}_{2,a,y}(\sigma+iu+i\varphi(\tau))|^2\,\mathrm{d}\tau=0 \tag{12}$$

with certain $5/6 < \sigma < 1$ and a bounded u. We have

$$
\begin{aligned}
&\left|\mathcal{Z}_{2,a,y}(\sigma + iu + i\varphi(\tau))\right|^2 = \mathcal{Z}_{2,a,y}(\sigma + iu + i\varphi(\tau))\overline{\mathcal{Z}_{2,a,y}(\sigma + iu + i\varphi(\tau))} \\
&= \int_1^a \zeta_2(x)b(x,y)x^{-\sigma-iu-i\varphi(\tau)}\,\mathrm{d}x \int_1^a \zeta_2(x)b(x,y)x^{-\sigma+iu+i\varphi(\tau)}\,\mathrm{d}x \\
&= \left(\underset{x_1=x_2}{\int_1^a\int_1^a} + \underset{x_1\neq x_2}{\int_1^a\int_1^a}\right)\zeta_2(x_1)\zeta_2(x_2)b(x_1,y)b(x_2,y)x_1^{-\sigma-iu-i\varphi(\tau)}x_2^{-\sigma+iu+i\varphi(\tau)}\,\mathrm{d}x_1\,\mathrm{d}x_2.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\frac{1}{T}\int_T^{2T}\left|\mathcal{Z}_{2,a,y}(\sigma + iu + i\varphi(\tau))\right|^2\mathrm{d}\tau \\
&= \frac{1}{T}\underset{x_1\neq x_2}{\int_1^a\int_1^a}\left(\zeta_2(x_1)\zeta_2(x_2)b(x_12,y)b(x_2,y)(x_1x_2)^{-\sigma}\left(\frac{x_2}{x_1}\right)^{iu}\int_T^{2T}\left(\frac{x_2}{x_1}\right)^{i\varphi(\tau)}\mathrm{d}\tau\right)\mathrm{d}x_1\,\mathrm{d}x_2.
\end{aligned}
\tag{13}
$$

The application of the second mean value theorem yields

$$
\begin{aligned}
\int_T^{2T}\cos\left(\varphi(\tau)\log\left(\frac{x_2}{x_1}\right)\right)\mathrm{d}\tau &= \left(\log\left(\frac{x_2}{x_1}\right)\right)^{-1}\int_T^{2T}\frac{1}{\varphi'(\tau)}\,\mathrm{d}\sin\left(\varphi(\tau)\log\left(\frac{x_2}{x_1}\right)\right) \\
&= \left(\log\left(\frac{x_2}{x_1}\right)\right)^{-1}\frac{1}{\varphi'(\tau)}\int_T^{\theta}\mathrm{d}\sin\left(\varphi(\tau)\log\left(\frac{x_2}{x_1}\right)\right) \\
&\ll \left|\log\left(\frac{x_2}{x_1}\right)\right|^{-1}\frac{1}{\varphi'(2T)},
\end{aligned}
$$

with $T \leqslant \theta \leqslant 2T$. Similarly,

$$
\int_T^{2T}\sin\left(\varphi(\tau)\log\left(\frac{x_2}{x_1}\right)\right)\mathrm{d}\tau \ll \left|\log\left(\frac{x_2}{x_1}\right)\right|^{-1}\frac{1}{\varphi'(2T)}.
$$

Therefore, in view of (13),

$$
\begin{aligned}
&\frac{1}{T}\int_T^{2T}\left|\mathcal{Z}_{2,a,y}(\sigma + iu + i\varphi(\tau))\right|^2\mathrm{d}\tau \\
&\ll \frac{1}{T\varphi'(2T)}\underset{x_1\neq x_2}{\int_1^a\int_1^a}\zeta_2(x_1)\zeta_2(x_2)b(x_1,y)b(x_2,y)(x_1x_2)^{-\sigma}\left|\log\left(\frac{x_2}{x_1}\right)\right|^{-1}\mathrm{d}x_1\,\mathrm{d}x_2.
\end{aligned}
\tag{14}
$$

According to Theorem 1 of [2], for $1/2 < \sigma < 1$,

$$
J_\sigma(T) \gg T^{2-2\sigma-\varepsilon}.
$$

Thus, $J_\sigma(T\varphi(2T)) \to \infty$ as $T \to \infty$. From this and

$$
\frac{J_\sigma(2\varphi(2T))}{T\varphi'(2T)} \ll 1,
$$

we find that
$$\frac{1}{T\varphi'(2T)} \to 0 \tag{15}$$
as $T \to \infty$. This and (14) prove (12). The Lemma is proven. □

Now, we will deal with
$$\mathcal{Z}_{2,y}(s) = \int_1^\infty \zeta_2(x) b(x,y) x^{-s}\,\mathrm{d}x.$$

For $A \in \mathcal{B}(\mathcal{H})$, we set
$$P_{T,y}(A) = \frac{1}{T}\mathfrak{M}_L\{\tau \in [T,2T] : \mathcal{Z}_{2,y}(s+i\varphi(\tau)) \in A\}.$$

In order to pass from $P_{T,a,y}$ to $P_{T,y}$, we apply the following general statement.

Lemma 2. *Let $(\mathcal{X}, d)$ be a separable metric space and X_{nk} and Y_n be $\mathcal{X}$-valued random elements in the same probability space with measure ν. Suppose that*
$$X_{nk} \xrightarrow[n\to\infty]{\mathcal{D}} X_k, \quad \forall k,$$
and
$$X_k \xrightarrow[k\to\infty]{\mathcal{D}} X.$$
If, for every $\varepsilon > 0$,
$$\lim_{k\to\infty} \limsup_{n\to\infty} \nu\{\rho(X_{nk}, Y_n) \geqslant \varepsilon\} = 0,$$
then
$$Y_n \xrightarrow[n\to\infty]{\mathcal{D}} X.$$

The proof of the lemma is given in Theorem 4.2 of [36].

Lemma 3. *Suppose that the function $\varphi(\tau)$ satisfies the hypotheses of Theorem 1. Then, for every $y > 1$, $P_{T,y} \xrightarrow[T\to\infty]{\mathrm{w}} P_{\mathcal{H},0}$.*

Proof. Let ξ_T be the same random variable as in the proof of Proposition 1. We define the $\mathcal{H}$-valued random element as
$$X_{T,a,y} = X_{T,a,y}(s) = \mathcal{Z}_{2,a,y}(s + i\varphi(\xi_T)),$$
and denote the $\mathcal{H}$-valued random element with distribution $P_{\mathcal{H},0}$ as $X_{a,y}$. Then, Lemma 1 implies the following relation:
$$X_{T,a,y} \xrightarrow[T\to\infty]{\mathcal{D}} X_{a,y}. \tag{16}$$
The distribution of $X_{a,y}$ is $P_{\mathcal{H},0}$ for all a and y. Thus,
$$X_{a,y} \xrightarrow[a\to\infty]{\mathcal{D}} P_{\mathcal{H},0}. \tag{17}$$

Since $b(x,y)$ decreases to zero exponentially, the following integral
$$\int_1^\infty \zeta_2(x) b(x,y) x^{-s}\,\mathrm{d}x$$

is absolutely convergent for $\sigma > \sigma_0$ with all finite σ_0. Hence,

$$\int_a^\infty \zeta_2(x)b(x,y)x^{-s}\,\mathrm{d}x = o_y(1)$$

for $\sigma > 5/6$ as $a \to \infty$. Let $K \subset \mathcal{D}$ be a compact set. Then, for $s \in K$,

$$\mathcal{Z}_{2,y}(s+i\varphi(\tau)) - \mathcal{Z}_{2,a,y}(s+i\varphi(\tau)) \ll \int_a^\infty \zeta_2(x)b(x,y)x^{-\mathrm{Re}s}\,\mathrm{d}x = o_y(1)$$

as $a \to \infty$. Therefore,

$$\lim_{a\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_T^{2T}\sup_{s\in K}\left|\mathcal{Z}_{2,y}(s+i\varphi(\tau)) - \mathcal{Z}_{2,a,y}(s+i\varphi(\tau))\right|\mathrm{d}\tau \ll \lim_{a\to\infty} o_y(1) = 0. \tag{18}$$

Let

$$X_{T,y} = X_{T,y}(s) = \mathcal{Z}_{2,y}(s+\varphi(\xi_T)).$$

Then, in view of (18), for every $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{a\to\infty}\limsup_{T\to\infty}\nu\{d_{\mathcal{H}}(X_{T,y},X_{T,a,y}) \geqslant \varepsilon\}\\
&\quad= \lim_{a\to\infty}\limsup_{T\to\infty}\frac{1}{T}\mathfrak{M}_L\{\tau\in[T,2T] : d_{\mathcal{H}}(\mathcal{Z}_{2,y}(s+i\varphi(\tau)),\mathcal{Z}_{2,a,y}(s+i\varphi(\tau))) \geqslant \varepsilon\}\\
&\quad\leqslant \lim_{a\to\infty}\limsup_{T\to\infty}\frac{1}{2T}\int_T^{2T} d_{\mathcal{H}}(\mathcal{Z}_{2,y}(s+i\varphi(\tau)),\mathcal{Z}_{2,a,y}(s+i\varphi(\tau)))\,\mathrm{d}\tau = 0
\end{aligned}$$

according to the definition of the metric $d_{\mathcal{H}}$. The latter remark, relations (16) and (17) and Lemma 2 lead to the relation

$$X_{T,y} \xrightarrow[T\to\infty]{\mathcal{D}} P_{\mathcal{H},0},$$

which is equivalent to the weak convergence of $P_{T,y}$ to $P_{\mathcal{H},0}$ as $T \to \infty$. $\square$

5. Integral Representation

In this section, we present the representation for $\mathcal{Z}_{2,y}(s)$ by a contour integral. Denote by $\Gamma(s)$ the Euler gamma function, and

$$l_y(s) = \frac{1}{\alpha}\Gamma\left(\frac{s}{\alpha}\right)y^s,$$

where α is from the definition of $b(x,y)$.

Lemma 4. *For $s \in \mathcal{D}$,*

$$\mathcal{Z}_{2,y}(s) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\mathcal{Z}_2(s+z)l_y(z)\,\mathrm{d}z.$$

Proof. We use the classical Mellin formula:

$$\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}\Gamma(z)\theta^{-z}\,\mathrm{d}z = \mathrm{e}^{-\theta}, \quad \beta,\theta > 0. \tag{19}$$

For brevity, let

$$g(x,\tau)=\frac{1}{2\pi i}l_y(\alpha+i\tau)\zeta_2(x)x^{-s-\alpha-i\tau}.$$

Then, for all $T\geqslant 1$ and $X>1$, we have

$$\int_1^X \mathrm{d}x\int_{-T}^T g(x,\tau)\,\mathrm{d}\tau=\int_{-T}^T \mathrm{d}\tau\int_1^X g(x,\tau)\,\mathrm{d}x. \tag{20}$$

The application of (19) and the definition of $l_y(s)$ yield

$$\begin{aligned}\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} l_y(z)x^{-z}\,\mathrm{d}z&=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{1}{\alpha}\Gamma\left(\frac{z}{\alpha}\right)\left(\frac{x}{y}\right)^{-z}\mathrm{d}z=\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\Gamma(z)\left(\frac{x}{y}\right)^{-\alpha z}\mathrm{d}z\\&=\exp\left\{-\left(\frac{x}{y}\right)^{\alpha}\right\}=b(x,y).\end{aligned} \tag{21}$$

It is well known that, uniformly in $\sigma\in[\sigma_1,\sigma_2]$,

$$\Gamma(\sigma+it)\ll\exp\{-c|t|\},\quad c>0. \tag{22}$$

Moreover,

$$\int_1^X\zeta_2(x)\,\mathrm{d}x\ll X\log^4 X.$$

The latter estimate, together with (22), shows that

$$\int_T^\infty\mathrm{d}\tau\int_1^X(|g(x,\tau)|+|g(x,-\tau)|)\,\mathrm{d}x\ll B(X,T)$$

and

$$\int_1^X\mathrm{d}x\int_T^\infty(|g(x,\tau)|+|g(x,-\tau)|)\,\mathrm{d}\tau\ll B(X,T),$$

where

$$\begin{aligned}B(X,T)&=y^\alpha\int_T^\infty\exp\left\{-\frac{c}{\alpha}\tau\right\}\mathrm{d}\tau\int_1^X\zeta_2(x)x^{-\sigma-\alpha}\,\mathrm{d}x\\&\leqslant y^\alpha\exp\{-c_1T\}\int_1^X x^{-\sigma-\alpha}\,\mathrm{d}\left(\int_1^X\zeta_2(u)\,\mathrm{d}u\right)\mathrm{d}x\\&\ll y^\alpha\exp\{-c_1T\}\left(X^{1-\sigma-\alpha}+1\right)\log^4 X.\end{aligned} \tag{23}$$

Therefore, by virtue of (20),

$$
\begin{aligned}
\int_{-\infty}^{\infty} \mathrm{d}\tau \int_{1}^{X} g(x,\tau)\,\mathrm{d}x &= \left(\int_{-T}^{T} + \int_{T}^{+\infty} + \int_{-\infty}^{-T} \right) \mathrm{d}\tau \int_{1}^{X} g(x,\tau)\,\mathrm{d}\tau \\
&= \int_{1}^{X} \mathrm{d}x \int_{-T}^{T} g(x,\tau)\,\mathrm{d}\tau + O(B(X,T)) \\
&= \int_{1}^{X} \mathrm{d}x \left(\int_{-\infty}^{\infty} - \int_{T}^{+\infty} - \int_{-\infty}^{-T} \right) g(x,\tau)\,\mathrm{d}\tau + O(B(X,T)) \\
&= \int_{1}^{X} \mathrm{d}x \int_{-\infty}^{\infty} g(x,\tau)\,\mathrm{d}\tau + O(B(X,T)).
\end{aligned}
$$

Thus, according to (23), as $T \to \infty$, for every $X > 1$, we obtain

$$
\int_{-\infty}^{\infty} \mathrm{d}\tau \int_{1}^{X} g(x,\tau)\,\mathrm{d}\tau = \int_{1}^{X} \mathrm{d}x \int_{-\infty}^{\infty} g(x,\tau)\,\mathrm{d}\tau.
$$

The latter equality, together with (21), yields

$$
\begin{aligned}
\frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \mathcal{Z}_2(s+z) l_y(z)\,\mathrm{d}z &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \zeta_2(x) l_y(\alpha+i\tau) x^{-s-\alpha-i\tau}\,\mathrm{d}x\,\mathrm{d}\tau = \int_{-\infty}^{\infty} \mathrm{d}\tau \int_{1}^{\infty} g(x,\tau)\,\mathrm{d}x \\
&= \int_{1}^{\infty} \zeta_2(x) x^{-s} \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} l_y(z) x^{-z}\,\mathrm{d}z\,\mathrm{d}x = \int_{1}^{\infty} \zeta_2(x) b(x,y) x^{-s}\,\mathrm{d}x \\
&= \mathcal{Z}_{2,y}(s).
\end{aligned}
$$

□

6. Difference Between $\mathcal{Z}_2(s)$ and $\mathcal{Z}_{2,y}(s)$

Recall that $d_{\mathcal{H}}$ is the metric in the space $\mathcal{H}$.

Lemma 5. *Suppose that the function $\varphi(\tau)$ satisfies the hypotheses of Theorem 1. Then,*

$$
\lim_{y\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_{T}^{2T} d_{\mathcal{H}}\big(\mathcal{Z}_2(s+i\varphi(\tau)), \mathcal{Z}_{2,y}(s+i\varphi(\tau))\big)\,\mathrm{d}\tau = 0.
$$

Proof. According to the definition of $d_{\mathcal{H}}$, it suffices to prove that, for every compact set $K \subset \mathcal{D}$,

$$
\lim_{y\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_{T}^{2T} \sup_{s\in K} \left| \mathcal{Z}_2(s+i\varphi(\tau)) - \mathcal{Z}_{2,y}(s+i\varphi(\tau)) \right| \mathrm{d}\tau = 0. \tag{24}
$$

Fix a compact set $K \subset \mathcal{D}$. Then, there exists $\delta > 0$ satisfying $5/6 + 2\delta \leqslant \sigma \leqslant 1 - \delta$ for all $s = \sigma + it \in K$. Take $\alpha = 1/6 + \delta$ in the definition of $b(x,y)$ and

$$
\alpha_1 = \sigma - \delta - \frac{5}{6}.
$$

Thus, $\alpha_1 > 0$. The point $z = 1 - s$ is a pole of order five, and $z = 0$ is a simple pole of the following function in the strip of $-\alpha_1 < \operatorname{Re} z < \alpha$:

$$\mathcal{Z}_2(s+z)\Gamma\left(\frac{s}{\alpha}\right)$$

because $(5/6 + \delta - \sigma)/(1/6 + \delta) > -1$. This, Lemma 4, and the residue theorem yield

$$\mathcal{Z}_{2,y}(s) - \mathcal{Z}_2(s) = \frac{1}{2\pi i} \int\limits_{-\alpha_1 - \infty}^{-\alpha_1 + i\infty} \mathcal{Z}_2(s+z) l_y(z)\,\mathrm{d}z + r(s),$$

where

$$r(s) = \operatorname*{Res}_{z=1-s} \mathcal{Z}_2(s+z) l_y(z).$$

Hence, for all $s \in K$,

$$\begin{aligned}
&\mathcal{Z}_{2,y}(s + i\varphi(\tau)) - \mathcal{Z}_2(s + i\varphi(\tau)) \\
&\quad = \frac{1}{2\pi i} \int\limits_{-\infty}^{\infty} \mathcal{Z}_2\left(\sigma - \sigma + \frac{5}{6} + \delta + it + iv + i\varphi(\tau)\right) l_y\left(\frac{5}{6} + \delta - \sigma + iv\right) \mathrm{d}v + r(s + i\varphi(\tau)) \\
&\quad = \frac{1}{2\pi i} \int\limits_{-\infty}^{\infty} \mathcal{Z}_2\left(\frac{5}{6} + \delta + iv + i\varphi(\tau)\right) l_y\left(\frac{5}{6} + \delta - s + iv\right) \mathrm{d}v + r(s + i\varphi(\tau)) \\
&\quad \leqslant \int\limits_{-\infty}^{\infty} \left|\mathcal{Z}_2\left(\sigma - \sigma + \frac{5}{6} + \delta + it + iv + i\varphi(\tau)\right)\right| \sup_{s\in K} \left| l_y\left(\frac{5}{6} + \delta - s + iv\right)\right| \mathrm{d}v \\
&\qquad + \sup_{s\in K} |r(s + i\varphi(\tau))|.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{1}{T} \int\limits_{-T}^{T} \sup_{s\in K} \left|\mathcal{Z}_2(s + i\varphi(\tau)) - \mathcal{Z}_{2,y}(s + i\varphi(\tau))\right| \mathrm{d}\tau \\
&\quad \ll \int\limits_{-\infty}^{\infty} \left(\frac{1}{T} \int\limits_{T}^{2T} \left|\mathcal{Z}_2\left(\frac{5}{6} + \delta + iv + i\varphi(\tau)\right)\right| \mathrm{d}\tau\right) \sup_{s\in K} \left| l_y\left(\frac{5}{6} + \delta - s + iv\right)\right| \mathrm{d}v \\
&\qquad + \frac{1}{T} \int\limits_{T}^{2T} \sup_{s\in K} |r(s + i\varphi(\tau))|\,\mathrm{d}\tau \overset{\text{def}}{=} I_1 + I_2.
\end{aligned} \tag{25}$$

Clearly, for all $v \in \mathbb{R}$,

$$a_T(v) \overset{\text{def}}{=} \frac{1}{T} \int\limits_{T}^{2T} \left|\mathcal{Z}_2\left(\frac{5}{6} + \delta + iv + i\varphi(\tau)\right)\right| \mathrm{d}\tau \leqslant \left(\frac{1}{T} \int\limits_{T}^{2T} \left|\mathcal{Z}_2\left(\frac{5}{6} + \delta + iv + i\varphi(\tau)\right)\right|^2 \mathrm{d}\tau\right)^{1/2}.$$

Properties of the function $\varphi(\tau)$ yield

$$
\begin{aligned}
a_T^2(v) &= \frac{1}{T}\int_T^{2T} \frac{1}{\varphi'(\tau)}\left|\mathcal{Z}_2\left(\frac{5}{6}+\delta+iv+i\varphi(\tau)\right)\right|^2 \mathrm{d}\varphi(\tau) \\
&\ll \frac{1}{T\varphi'(2T)}\int_T^{2T}\left|\mathcal{Z}_2\left(\frac{5}{6}+\delta+iv+i\varphi(\tau)\right)\right|^2 \mathrm{d}\varphi(\tau) \\
&\ll \frac{1}{T\varphi'(2T)}\int_{\varphi(T)-|v|}^{\varphi(2T)+|v|}\left|\mathcal{Z}_2\left(\frac{5}{6}+\delta+iu\right)\right|^2 \mathrm{d}u \ll_\delta \frac{1}{T\varphi'(2T)} J_{5/6+\delta}(\varphi(2T)+|v|). \quad (26)
\end{aligned}
$$

In the case of $|v| \leqslant \varphi(2T)$, we have

$$
a_T^2(v) \ll_\delta \frac{1}{T\varphi'(2T)} J_{5/6+\delta}(2\varphi(2T)) \ll_\delta \sup_{\sigma\in(5/6,1)} \frac{J_\sigma(2\varphi(2T))}{T\varphi'(2T)} \ll_\delta 1. \quad (27)
$$

In the case of $|v| > \varphi(2T)$, according to (4), we find

$$
\begin{aligned}
a_T^2(v) &\ll_\delta \frac{1}{T\varphi'(2T)} J_{5/6+\delta}(2|v|) \ll_{\delta,\varepsilon} \frac{1}{T\varphi'(2T)} |v|^{(15-12(5/6+\delta))/5+\varepsilon} \\
&\ll_{\delta,\varepsilon} \frac{1}{T\varphi'(2T)}|v| = o(|v|), \quad T\to\infty,
\end{aligned}
$$

because of (15). This, together with (26) and (27), shows that

$$
a_T(v) \ll_\delta (1+|v|)^{1/2}. \quad (28)
$$

Applying (22), we find that, for all $s \in K$,

$$
l_y\left(\frac{5}{6}+\delta-s+iv\right) \ll_\delta y^{5/6+\delta-\sigma}\exp\left\{-\frac{c}{\alpha}|v-t|\right\} \ll_{\delta,K} y^{-\delta}\exp\{-c_2|v|\}, \quad c_2>0.
$$

This, together with (28), yields

$$
I_1 \ll_{\delta,K} y^{-\delta}\int_{-\infty}^{\infty}(1+|v|)^{1/2}\exp\{c_2|v|\}\,\mathrm{d}v \ll_{\delta,K} y^{-\delta}. \quad (29)
$$

We have

$$
r(s+i\varphi(\tau)) = \left(\mathcal{Z}_2(s+i\varphi(\tau)+z)(z-s-i\varphi(\tau))^5 l_y(z)\right)^{(IV)}\Big|_{z=1-s-i\varphi(\tau)}. \quad (30)
$$

For brevity, let

$$
V(s,z,\varphi(\tau)) = \mathcal{Z}_2(s+i\varphi(\tau)+z)(z-s-i\varphi(\tau))^5.
$$

Then, according to the Leibnitz formula and (30),

$$
r(s+i\varphi(\tau)) = \sum_{k=1}^{5}\binom{k}{5} V^{5-k}(s,z,\varphi(\tau))\, l_y^{(k)}(z)\Big|_{z=1-s-i\varphi(\tau)}. \quad (31)
$$

The function $r(s+i\varphi(\tau))$ is analytic in strip $\mathcal{D}$. Therefore, according to the Cauchy integral formula,

$$
\sup_{s\in K}|r(s+i\varphi(\tau))| \ll_K \int_l |r(z+i\varphi(\tau))|\,|\mathrm{d}z|, \quad (32)
$$

where l is a suitable simple closed contour lying in $\mathcal{D}$ and enclosing set K. Using (3), (22), (31), and (32), we find

$$
\begin{aligned}
I_2 &\ll_{K,\delta} \int_l \left(\frac{1}{T} \int_T^{2T} (\varphi(\tau))^B y \log y \exp\{-c_3|t+\varphi(\tau)|\} \mathrm{d}\tau \right) |\mathrm{d}z| \\
&\ll_{K,\delta} \frac{y \log y}{T} \int_T^{2T} (\varphi(\tau))^B \exp\{-c_4 \varphi(\tau)\} \mathrm{d}\tau \\
&\ll_{K,\delta} \frac{y \log y \exp\{-(c_4/2)\varphi(T)\}}{T} \int_T^{2T} (\varphi(\tau))^B \exp\{-(c_4/2)\varphi(\tau)\} \mathrm{d}\tau,
\end{aligned}
$$

where c_3, c_4, and B are certain positive constants. This, (29), and (25) show

$$
\frac{1}{T} \int_T^{2T} \sup_{s \in K} \left| C Z_2(s + i\varphi(\tau)) - \mathcal{Z}_{2,y}(s + i\varphi(\tau)) \right| \mathrm{d}\tau \ll_{K,\delta} y^{-\delta} + y \log y \exp\{-(c_4/2)\varphi(T)\},
$$

and we have (24). $\square$

7. Proof of Theorem 1

We apply the same scheme as in the proof of Lemma 3. Let ξ_T be defined in the proof of Proposition 1.

Proof of Theorem 1. We define the following $\mathcal{H}$-valued random element:

$$
X_T = X_T(s) = \mathcal{Z}_2(s + i\varphi(\xi_T)).
$$

According to Lemma 3,

$$
X_{T,y} \xrightarrow[T\to\infty]{\mathcal{D}} X_y \tag{33}
$$

for all y, and the distribution of X_y is $P_{\mathcal{H},0}$ for all $y > 0$. Hence,

$$
X_y \xrightarrow[y\to\infty]{\mathcal{D}} P_{\mathcal{H},0}. \tag{34}
$$

Moreover, Lemma 5 implies that, for every $\varepsilon > 0$,

$$
\begin{aligned}
&\lim_{y\to\infty} \limsup_{T\to\infty} \nu\{d_{\mathcal{H}}(X_T, X_{T,y}) \geqslant \varepsilon\} \\
&\quad \leqslant \lim_{y\to\infty} \limsup_{T\to\infty} \frac{1}{2T} \int_T^{2T} d_{\mathcal{H}}\big(\mathcal{Z}_2(s+i\varphi(\tau)), \mathcal{Z}_{2,y}(s+i\varphi(\tau))\big) \, \mathrm{d}\tau = 0.
\end{aligned}
$$

This, together with (33), (34), and Lemma 2, yields the following relation:

$$
X_T \xrightarrow[T\to\infty]{\mathcal{D}} P_{\mathcal{H},0},
$$

which implies the assertion of the theorem. $\square$

8. Approximation by Shifts $\mathcal{Z}_2(s + i\varphi(\tau))$

It is well known that some zeta functions, including the Riemann zeta function, Dirichlet L functions, etc., are universal in the approximation sense, i.e., their shifts approximate a wide class of analytic functions (see [19,37–41]). A similar property remains valid for generalized shifts (see [42–60]). Approximation results for modified Mellin transforms of the Riemann zeta function were discussed in [30–32,34]. In this paper, we prove the following approximation statement of zero for the Mellin transform $\mathcal{Z}_2(s)$.

Theorem 2. *Suppose that $\varphi(\tau)$ is a differentiable function that increases to $+\infty$ with a decreasing derivative on $[T_0, \infty)$, $T_0 > 1$ such that*

$$\sup_{\sigma \in (5/6,1)} \frac{J_\sigma(2\varphi(2\tau))}{\varphi'(2\tau)} \ll \tau, \quad \tau \to \infty.$$

Then, for every compact set $K \subset \mathcal{D}$ and $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \mathfrak{M}_L \left\{ \tau \in [T, 2T] : \sup_{s\in K} |\mathcal{Z}_2(s + i\varphi(\tau))| < \varepsilon \right\} > 0.$$

Moreover, the limit

$$\lim_{T\to\infty} \frac{1}{T} \mathfrak{M}_L \left\{ \tau \in [T, 2T] : \sup_{s\in K} |\mathcal{Z}_2(s + i\varphi(\tau))| < \varepsilon \right\}$$

exists and is positive for all but, at most, countably many $\varepsilon > 0$.

Proof. The theorem is a simple consequence of Theorem 1 and the properties of weak convergence of probability measures.

According to Theorem 1, $P_{T,\mathcal{H},\varphi} \xrightarrow[T\to\infty]{\mathrm{w}} P_{\mathcal{H},0}$. According to this, denoting

$$\mathcal{G}_\varepsilon = \left\{ f(s) \in \mathcal{H} : \sup_{s\in K} |f(s)| < \varepsilon \right\},$$

we have

$$\liminf_{T\to\infty} P_{T,\mathcal{H},\varphi}(\mathcal{G}_\varepsilon) \geqslant P_{\mathcal{H},0}(\mathcal{G}_\varepsilon). \tag{35}$$

The support of $P_{\mathcal{H},0}$ is the set denoted as $\{h(s)\}$, $h(s) \equiv 0$. Therefore, $\mathcal{G}_\varepsilon$ is an open neighborhood of an element of the support of $P_{\mathcal{H},0}$. Hence,

$$P_{\mathcal{H},0}(\mathcal{G}_\varepsilon) > 0. \tag{36}$$

This, together with (35) and definitions of $P_{T,\mathcal{H},\varphi}$ and $\mathcal{G}_\varepsilon$, proves the first part of the theorem.

In order to prove the second part of the theorem, we apply the equivalent of weak convergence in terms of continuity sets, i.e., of sets $A \in \mathcal{B}(\mathbb{X})$ such that $P_{\mathcal{H},0}(\partial A) = 0$, where ∂A is the boundary of A. We observe that the boundaries of $\mathcal{G}_\varepsilon$ with different ε values do not intersect. Therefore, set $\mathcal{G}_\varepsilon$ is a continuity set of $P_{\mathcal{H},0}$ for all but, at most, countably many $\varepsilon > 0$. Thus, according to Theorem 1, the equivalence of weak convergence in terms of continuity sets, and (36), we have

$$\lim_{T\to\infty} P_{T,\mathcal{H},0}(\mathcal{G}_\varepsilon) = P_{\mathcal{H},0}(\mathcal{G}_\varepsilon) > 0$$

for all but, at most, countably many $\varepsilon > 0$, and the proof is complete. □

9. Conclusions

We considered the asymptotic behavior of the modified Mellin transform,

$$\mathcal{Z}_2(s) = \int_1^\infty \left|\zeta\left(\frac{1}{2}+ix\right)\right|^4 x^{-s}\,\mathrm{d}x,$$

where $\zeta(s)$ is the Riemann zeta function, by using a probabilistic approach. Let $\mathcal{D} = \{s \in \mathbb{C} : 5/6 < \sigma < 1\}$ and $\mathcal{H} = \mathcal{H}(\mathcal{D})$ denote the space of analytic on $\mathcal{D}$ functions with a topology of uniform convergence on compacta. We studied the weak convergence of the probability measure,

$$\frac{1}{T}\mathfrak{M}_L\{\tau \in [T,2T] : \mathcal{Z}_2(s+i\varphi(\tau)) \in A\}, \quad A \in \mathcal{B}(\mathcal{H}), \tag{37}$$

as $T \to \infty$, to the measure degenerated at the point of $h(s) \equiv 0$. We proved that this follows for a differentiable function increasing to $+\infty$ with a decreasing derivative on $[T_0,\infty)$ and $T_0 > 0$ such that

$$\sup_{\sigma\in(5/6,1)} \frac{1}{\varphi'(2\tau)} \int_0^{2\varphi(2\tau)} |\mathcal{Z}_2(\sigma+it)|^2\,\mathrm{d}t \ll \tau, \quad \tau \to \infty.$$

This shows that the asymptotic behavior of the $\mathcal{Z}_2(s)$ follows strong mathematical laws. From a limit theorem for (37), we derived that there are infinitely many shifts $\mathcal{Z}_2(s+i\varphi(\tau))$ that approximate the function expressed as $h(s) \equiv 0$.

An example is the function expressed as $\varphi(\tau) = \exp\{(\log\log\tau)^a\}$, $\tau \geqslant \mathrm{e}^2$, $a > 1$.

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