

Nonstationary heat equation with nonlinear side condition

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Abstract. The initial boundary value problem for the nonstationary heat equation is studied in a bounded domain with the specific over-determination condition. This condition is nonlinear and can be interpreted as the energy functional. In present paper we construct the class of solutions to this problem.

Keywords: nonstationary heat equation; inverse problem; very weak solution; nonlinear side condition.

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1 Introduction

Let us start with the nonstationary boundary value problem to the heat equation

$$\begin{cases} u_t(x, t) - \Delta u(x, t) = f(x, t), & (x, t) \in \Omega \times (0, T), \\ u(x, t)|_{\partial\Omega \times [0, T]} = 0, \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases} \quad (1.1)$$

where $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, is a simply connected bounded domain, the boundary $\partial\Omega$ is C^2 smooth, f is the internal source that heats or cools the system, u is the temperature and u_0 is the initial temperature. For the case when functions f , u_0 are prescribed and u is the unknown function, we have the classical initial boundary value problem for heat equation. The unique solvability of this problem is standard and well-established (see, for example, [8]).

There is an amount of papers where some additional integral condition

$$\int_{\Omega} u(x, t) dx = F(t), \quad F(0) = \int_{\Omega} u_0(x) dx \quad (1.2)$$

is prescribed (see, e.g., [2, 3, 4, 6, 9, 10, 11, 12, 13, 14]). Then, the solution of problem (1.1)–(1.2) is a pair of functions u and f . In other words, problem

(1.1), (1.2) can be seen as an inverse problem where we prescribe the time dependent function F . Inverse problems were studied by many mathematicians, starting with the works of J. R. Cannon (see [3, 4]) and then by the others (see [6, 11, 12, 13, 16]). However, in all mentioned papers, problem (1.1)–(1.2) was considered under assumption that function $F(t)$ is sufficiently smooth, for example, assuming that the derivative $F'(t)$ exists. Nevertheless, in recent papers (see [7, 15]) problem (1.1)–(1.2) was studied under the minimal regularity of function $F(t)$, i.e., assuming that $F \in L^2(0, T)$.

In some papers, for example in [3, 4], integral (1.2) is called an energy¹. However, in this paper instead of the linear side condition (1.2) we consider the nonlinear side condition:

$$\int_{\Omega} u^2(x, t) dx = E^2(t), \quad E(0) = \|u_0\|_{L^2(\Omega)}.$$

It can be interpreted as the energy functional for the heat equation and it measures the distance (in L^2 -norm) from the trivial equilibrium solution $u = 0$. This expression also reminds the elastic potential energy of a spring.

2 Notation and auxiliary results

In this paper, we will use the following notation. If G is the domain in $\mathbb{R}^n$, $C^\infty(G)$ means, as usual, the set of all infinitely differentiable functions in G and $C_0^\infty(G)$ is the subset of functions from $C^\infty(G)$ with compact supports in G . The space $C^m(\overline{G})$ (m is a nonnegative integer number) consists of m times continuously differentiable functions in $\overline{G}$ with the norm

$$\|u\|_{C^m(\overline{G})} = \sum_{|\alpha|=0}^m \sup_{x \in \overline{G}} |D^\alpha u(x)|.$$

For nonnegative integer l and $q > 0$ we use the usual notation for Lebesgue $L^q(G)$ and Sobolev $W^{l,q}(G)$ spaces with the norms

$$\|u\|_{L^q(G)} = \left(\int_G |u(x)|^q dx \right)^{1/q}, \quad \|u\|_{W^{l,q}(G)} = \left(\sum_{|\alpha|=0}^l \int_G |D^\alpha u(x)|^q dx \right)^{1/q}.$$

$W^{l-1/q,q}(\partial G)$ is the trace space on ∂G of functions from $W^{l,q}(G)$. The space $\dot{W}^{1,2}(G)$ is the closure of $C_0^\infty(G)$ in the norm of $W^{1,2}(G)$ (see [1, 8]).

The space $W^{-2,2}(G)$ denotes the dual space of $W^{2,2}(G) \cap \dot{W}^{1,2}(G)$ with pairing $\langle h, \zeta \rangle_G$ for any functional $h \in W^{-2,2}(G)$ and test function $\zeta \in W^{2,2}(G) \cap \dot{W}^{1,2}(G)$. The norm in $W^{-2,2}(G)$ is defined in a usual way:

$$\|h\|_{W^{-2,2}(G)} = \sup_{\zeta \in W^{2,2}(G) \cap \dot{W}^{1,2}(G)} \frac{|\langle h, \zeta \rangle|}{\|\zeta\|_{W^{2,2}(G)}}.$$

The space $W^{-1/2,2}(\partial G)$ denotes the dual space of $W^{1/2,2}(\partial G)$ with pairing $\langle g, \xi \rangle_{\partial G}$ for any functional $g \in W^{-1/2,2}(\partial G)$ and function $\xi \in W^{1/2,2}(\partial G)$.

¹ Notice that integral (1.2) does not actually describe "energy" in the physical sense.

The norm of an element u in the function space V is denoted by $\|u\|_V$. Then, $L^2(0, T; V)$ is the space of functions u , depending on the space variable x and time variable t , such that $u(\cdot, t) \in V$ for almost all $t \in [0, T]$ and the norm

$$\|u\|_{L^2(0, T; V)} = \left(\int_0^T \|u(\cdot, t)\|_V^2 dt \right)^{1/2}$$

is finite.

Lemma 1. *Let G is a bounded domain in $\mathbb{R}^n$ and ∂G is C^2 -smooth. Let $v_k(x) \in W^{2,2}(G) \cap \dot{W}^{1,2}(G)$ and numbers λ_k are eigenfunctions and eigenvalues of the Laplace operator:*

$$\begin{cases} -\Delta v_k(x) = \lambda_k v_k(x), & x \in G, \\ v_k(x)|_{\partial G} = 0. \end{cases}$$

Then, $\lambda_k > 0$ and $\lim_{k \rightarrow \infty} \lambda_k = \infty$. The eigenfunctions $v_k(x)$ are orthogonal in $L^2(G)$ and we assume that $v_k(x)$ are normalized in $L^2(G)$, i.e.,

$$\int_G v_k(x) v_l(x) dx = \delta_{lk} = \begin{cases} 1, & l = k, \\ 0, & l \neq k. \end{cases}$$

Moreover,

$$\int_G \nabla v_k(x) \cdot \nabla v_l(x) dx = \lambda_k \delta_{lk} = \begin{cases} \lambda_k, & l = k, \\ 0, & l \neq k. \end{cases}$$

For the details see [8].

Lemma 2. *Let G be a bounded domain in $\mathbb{R}^n$, $n \geq 1$, $\{v_k(x)\}$ be a basis in Hilbert space $\dot{W}^{1,2}(G)$ and $h(x) = \sum_{k=1}^{\infty} h_k v_k(x)$.*

- 1. If $\sum_{k=1}^{\infty} \frac{h_k^2}{1 + \lambda_k^2} < \infty$, where λ_k is an eigenvalue corresponding eigenfunction $v_k(x)$, then $\int_G h(x) \eta(x) dx$ is a bounded functional in $W^{2,2}(G)$.*
- 2. If $H(\eta)$ be a bounded functional in $W^{2,2}(G)$, i.e. $H(\eta) \in W^{-2,2}(G)$, then $H(\eta) = \int_G h(x) \eta(x) dx$ and $\sum_{k=1}^{\infty} \frac{h_k^2}{1 + \lambda_k^2} < \infty$ for any $\eta \in W^{2,2}(G)$.*

Proof.

1. Using the properties of the eigenfunctions and the Cauchy–Schwarz inequality we get

$$\begin{aligned} \int_G h(x) \eta(x) dx &= \lim_{N \rightarrow \infty} \int_G \sum_{k=1}^N h_k v_k(x) \sum_{k=1}^N \eta_k v_k(x) dx = \lim_{N \rightarrow \infty} \sum_{k=1}^N h_k \eta_k \\ &\leq \lim_{N \rightarrow \infty} \left(\sum_{k=1}^N \frac{h_k^2}{1 + \lambda_k^2} \right)^{1/2} \left(\sum_{k=1}^N \eta_k^2 (1 + \lambda_k^2) \right)^{1/2} \leq c \left(\sum_{k=1}^{\infty} \eta_k^2 (1 + \lambda_k^2) \right)^{1/2} \\ &= c \|\eta(x)\|_{W^{2,2}(G)}. \end{aligned}$$

2. Let us denote $h_k = H(v_k(x))$. Then,

$$\begin{aligned} H(\eta(x)) &= \lim_{N \rightarrow \infty} H\left(\sum_{k=1}^N \eta_k v_k(x)\right) = \lim_{N \rightarrow \infty} \sum_{k=1}^N \eta_k H(v_k(x)) = \lim_{N \rightarrow \infty} \sum_{k=1}^N \eta_k h_k \\ &= \sum_{k=1}^{\infty} \eta_k h_k = \int_G h(x) \eta(x) dx, \end{aligned}$$

where $h(x) = \sum_{k=1}^{\infty} h_k v_k(x)$.

Next, we prove that $\sum_{k=1}^{\infty} \frac{h_k^2}{1 + \lambda_k^2} < \infty$. Since $H(\eta)$ is a functional in $W^{2,2}(G)$, we have that

$$|H(\eta)| \leq c \|\eta\|_{W^{2,2}(G)},$$

i.e.,

$$\sum_{k=1}^{\infty} h_k \eta_k \leq c \|\eta\|_{W^{2,2}(G)} \leq c \left(\sum_{k=1}^{\infty} \eta_k^2 (1 + \lambda_k^2) \right)^{1/2}. \quad (2.1)$$

Let us take

$$\eta_k = \begin{cases} h_k / (1 + \lambda_k^2), & k \leq N, \\ 0, & k > N. \end{cases} \quad (2.2)$$

Substituting (2.2) into (2.1) we obtain

$$\begin{aligned} \left| \sum_{k=1}^{\infty} h_k \eta_k \right| &= \left| \sum_{k=1}^N \frac{h_k^2}{1 + \lambda_k^2} \right| \leq c \left(\sum_{k=1}^N \frac{h_k^2}{(1 + \lambda_k^2)^2} (1 + \lambda_k^2) \right)^{1/2} \\ &\leq c \left(\sum_{k=1}^N \frac{h_k^2}{1 + \lambda_k^2} \right)^{1/2}, \end{aligned}$$

$$\text{i.e., } \left| \sum_{k=1}^N \frac{h_k^2}{1 + \lambda_k^2} \right| \leq c \left(\sum_{k=1}^N \frac{h_k^2}{1 + \lambda_k^2} \right)^{1/2}.$$

Dividing both sides by $\left(\sum_{k=1}^N \frac{h_k^2}{1 + \lambda_k^2} \right)^{1/2}$ we get $\left(\sum_{k=1}^N \frac{h_k^2}{1 + \lambda_k^2} \right)^{1/2} \leq c$.

Since constant c in the last estimate does not depend on N , we can pass to a limit as $N \rightarrow \infty$ and we obtain:

$$\sum_{k=1}^{\infty} \frac{h_k^2}{1 + \lambda_k^2} < \infty.$$

□

Remark 1. If function h depends on time variable t and space variable x , Lemma 2 remains valid with only difference that h_k depends on t .

Remark 2. Notice that $\|H\|_{W^{-2,2}(G)} \sim \sum_{k=1}^{\infty} \frac{h_k^2}{1 + \lambda_k^2}$.

3 Formulation of problem and main result

In a bounded simply connected domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, with C^2 smooth boundary $\partial\Omega$ we consider

$$\begin{cases} u_t(x, t) - \Delta u(x, t) = f(x, t), & (x, t) \in \Omega \times (0, T), \\ u(x, t)|_{\partial\Omega \times [0, T]} = 0, \\ u(x, 0) = u_0(x), \end{cases} \quad (3.1)$$

with additionally prescribed nonlinear side condition

$$\int_{\Omega} u^2(x, t) dx = E^2(t), \quad E(0) = \|u_0\|_{L^2(\Omega)}, \quad (3.2)$$

where u and f are unknown functions while E and u_0 are given functions.

DEFINITION 1. The pair $(u(x, t), f(x, t))$ with functions $u \in L^2(0, T; L^2(\Omega))$, $u_t \in L^2(0, T; L^2(\Omega))$ and $f \in L^2(0, T; W^{-2,2}(\Omega))$ is called a very weak solution of problem (3.1)–(3.2) if the function u satisfies the initial condition $u(x, 0) = u_0(x)$, the pair $(u(x, t), f(x, t))$ satisfies the integral identity for any $\eta \in L^2(0, T; W^{2,2}(\Omega) \cap \dot{W}^{1,2}(\Omega))$

$$\int_0^T \int_{\Omega} u_t(x, t) \eta(x, t) dx - \int_0^T \int_{\Omega} u(x, t) \Delta \eta(x, t) dx = \int_0^T \int_{\Omega} f(x, t) \eta(x, t) dx$$

and u satisfies the nonlinear side condition (3.2).

Deriving the definition of a very weak solution, we multiplied the heat equation (3.1)₁ by the test function $\eta \in L^2(0, T; W^{2,2}(\Omega) \cap \dot{W}^{1,2}(\Omega))$ and then we integrated twice by parts over Ω the second term on the left-hand side. Doing this we got two integrals over the boundary $\partial\Omega$:

$$\int_{\partial\Omega} (\nabla u \cdot \mathbf{n}) \eta dS, \quad \int_{\partial\Omega} u (\nabla \eta \cdot \mathbf{n}) dS,$$

where $\mathbf{n}$ is a unit vector of the outward normal to $\partial\Omega$.

Since $\eta \in L^2(0, T; W^{2,2}(\Omega) \cap \dot{W}^{1,2}(\Omega))$, i.e., $\eta=0$ on the boundary $\partial\Omega$ in the trace sense, the integral $\int_{\partial\Omega} (\nabla u \cdot \mathbf{n}) \eta dS$ is equal to zero. The integral $\int_{\partial\Omega} u (\nabla \eta \cdot \mathbf{n}) dS$ must be understood as the functional $u \in W^{-1/2,2}(\partial\Omega)$ applied to the test function $\nabla \eta \in W^{1/2,2}(\partial\Omega)$. Indeed, since $\eta \in W^{2,2}(\Omega) \cap \dot{W}^{1,2}(\Omega)$, we have $\nabla \eta \in W^{1,2}(\Omega)$ and $\nabla \eta \in W^{1/2,2}(\partial\Omega)$ (see [1]). This implies that the boundary condition (3.1)₂ yields $\int_{\partial\Omega} u (\nabla \eta \cdot \mathbf{n}) dS = 0$.

The main result of this paper is formulated in the following theorem.

Theorem 1. Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a bounded simply connected domain, the boundary $\partial\Omega$ is C^2 smooth, initial function $u_0 \in L^2(\Omega)$ and function $E \in W^{1,2}(0, T)$, $E(0) = \|u_0\|_{L^2(\Omega)}$. Then, there exists at least one very weak solution of problem (3.1)–(3.2).

4 Proof of the main result

We look for the approximate solution in the form:

$$u^{(N)}(x, t) = \sum_{k=1}^N w_k^{(N)}(t) v_k(x), \quad f^{(N)}(x, t) = \sum_{k=1}^N q_k^{(N)}(t) v_k(x), \quad (4.1)$$

where $v_k(x)$ are eigenfunctions of the Laplace operator.

Functions $w_k^{(N)}(t)$ and $q_k^{(N)}(t)$ can be found from the following system:

$$\begin{cases} \int_{\Omega} u_t^{(N)}(x, t) v_k(x) dx - \int_{\Omega} u^{(N)}(x, t) \Delta v_k(x) dx = \int_{\Omega} f^{(N)}(x, t) v_k(x) dx, \\ u^{(N)}(x, 0) = \sum_{k=1}^N \beta_k v_k(x), \\ \int_{\Omega} |u^{(N)}(x, t)|^2 dx = \frac{1}{\|u_0\|_{L^2(\Omega)}^2} \sum_{k=1}^N \beta_k^2 E^2(t), \end{cases} \quad (4.2)$$

where β_k , $k = 1, \dots, N$, are the Fourier coefficients of $u_0(x)$.

Equality (4.2)₁ and initial condition (4.2)₂ yields the following problem:

$$\begin{cases} \left(w_l^{(N)}(t) \right)' + \lambda_l w_l^{(N)}(t) = q_l^{(N)}(t), \\ w_l^{(N)}(0) = \beta_l. \end{cases} \quad (4.3)$$

For all $l = 1, 2, \dots, N$ the solution of (4.3) is:

$$w_l^{(N)}(t) = \int_0^t e^{-\lambda_l(t-\tau)} q_l^{(N)}(\tau) d\tau + \beta_k. \quad (4.4)$$

Substituting (4.1) into the nonlinear condition (4.2)₃ and using the orthogonality properties of the eigenfunctions v_l (see Lemma 1) we get

$$\begin{aligned} \int_{\Omega} |u^{(N)}(x, t)|^2 dx &= \int_{\Omega} \left| \sum_{l=1}^N w_l^{(N)}(t) v_l(x) \right|^2 dx \\ &= \sum_{l=1}^N \left(w_l^{(N)}(t) \right)^2 \int_{\Omega} v_l^2(x) dx = \sum_{l=1}^N \left(w_l^{(N)}(t) \right)^2 = \sum_{l=1}^N \gamma_l^2 E^2(t), \end{aligned} \quad (4.5)$$

where $\sum_{l=1}^{\infty} \gamma_l^2 = 1$.

In order to satisfy condition (4.5) we choose that

$$\left(w_l^{(N)}(t) \right)^2 = \gamma_l^2 E^2(t), \quad \text{i.e., } w_l^{(N)}(t) = \gamma_l E(t), \quad (4.6)$$

where we take $\gamma_l = \beta_l / \|u_0\|_{L^2(\Omega)}$.

Let us calculate the norms of $u^{(N)}$ and $u_t^{(N)}$ in $L^2(0, T; L^2(\Omega))$:

$$\begin{aligned} \|u^{(N)}\|_{L^2(0, T; L^2(\Omega))}^2 &= \int_0^T \int_{\Omega} \sum_{k=1}^N |w_k^{(N)}(t)|^2 v_k^2(x) dx dt \\ &= \int_0^T \sum_{k=1}^N |w_k^{(N)}(t)|^2 dt = \int_0^T \sum_{k=1}^N \left| \frac{\beta_k E(t)}{\|u_0\|_{L^2(\Omega)}} \right|^2 dt \\ &= \frac{1}{\|u_0\|_{L^2(\Omega)}^2} \sum_{k=1}^N \beta_k^2 \int_0^T E^2(t) dt \end{aligned} \quad (4.7)$$

and

$$\begin{aligned} \|u_t^{(N)}\|_{L^2(0, T; L^2(\Omega))}^2 &= \int_0^T \int_{\Omega} \sum_{k=1}^N |(w_k^{(N)}(t))'_t|^2 v_k^2(x) dx dt \\ &= \int_0^T \sum_{k=1}^N |(w_k^{(N)}(t))'_t|^2 dt = \int_0^T \sum_{k=1}^N \left| \frac{\beta_k E'(t)}{\|u_0\|_{L^2(\Omega)}} \right|^2 dt \\ &= \frac{1}{\|u_0\|_{L^2(\Omega)}^2} \sum_{k=1}^N \beta_k^2 \int_0^T |E'(t)|^2 dt. \end{aligned} \quad (4.8)$$

Next, we get the estimate for $q_l^{(N)}$ which leads to the estimate of the unknown function $f^{(N)}$. Notice that from (4.3)₁ and (4.6) we have

$$q_l^{(N)}(t) = \frac{\beta_l}{\|u_0\|_{L^2(\Omega)}} \left(\lambda_l E(t) + E'(t) \right), \quad \forall l = 1, \dots, N. \quad (4.9)$$

Let us square both sides of (4.9) and then divide by $1 + \lambda_k^2$:

$$\begin{aligned} \frac{|q_k^{(N)}(t)|^2}{1 + \lambda_k^2} &= \frac{\beta_k^2}{\|u_0\|_{L^2(\Omega)}^2} \frac{\left(\lambda_k E(t) + E'(t) \right)^2}{1 + \lambda_k^2} \leq \frac{c \beta_k^2}{\|u_0\|_{L^2(\Omega)}^2} \left(E^2(t) + \frac{|E'(t)|^2}{1 + \lambda_k^2} \right) \\ &\leq \frac{c \beta_k^2}{\|u_0\|_{L^2(\Omega)}^2} \left(E^2(t) + |E'(t)|^2 \right). \end{aligned}$$

Summing up from 1 to N , we derive:

$$\sum_{k=1}^N \frac{|q_k^{(N)}(t)|^2}{1 + \lambda_k^2} \leq \frac{c}{\|u_0\|_{L^2(\Omega)}^2} \left(E^2(t) + |E'(t)|^2 \right) \sum_{k=1}^N \beta_k^2, \quad (4.10)$$

i.e., due to Lemma 2 we have

$$\begin{aligned} \|f^{(N)}\|_{L^2(0, T; W^{-2,2}(\Omega))}^2 &\leq \sum_{k=1}^N \frac{|q_k^{(N)}(t)|^2}{1 + \lambda_k^2} \\ &\leq \frac{c}{\|u_0\|_{L^2(\Omega)}^2} \sum_{k=1}^N \beta_k^2 \int_0^T \left(|E'(t)|^2 + E^2(t) \right) dx. \end{aligned} \quad (4.11)$$

Therefore, estimates (4.7), (4.8) and (4.11) show that sequences $\{u^{(N)}\}$, $\{u_t^{(N)}\}$ are bounded in the space $L^2(0, T; L^2(\Omega))$ and the sequence $\{f^{(N)}\}$ is bounded in the space $L^2(0, T; W^{-2,2}(\Omega))$. Thus, we can choose subsequences $\{u^{(N_j)}\}$, $\{u_t^{(N_j)}\}$ and $\{f^{(N_j)}\}$ weakly converging in the spaces $L^2(0, T; L^2(\Omega))$ and $L^2(0, T; W^{-2,2}(\Omega))$, respectively.

Let us take integral identity (4.2)₁ for $N = N_j$:

$$\int_{\Omega} u_t^{(N_j)}(x, t) v_k(x) dx - \int_{\Omega} u^{(N_j)}(x, t) \Delta v_k(x) dx = \int_{\Omega} f^{(N_j)}(x, t) v_k(x) dx. \quad (4.12)$$

We multiply (4.12) by $d_k(t) \in L^2(0, T)$, then sum up from 1 to M , $M \leq N_j$ and integrate with respect to t from 0 to T :

$$\begin{aligned} & \int_0^T \int_{\Omega} u_t^{(N_j)}(x, t) \sum_{k=1}^M v_k(x) d_k(t) dx dt - \int_0^T \int_{\Omega} u^{(N_j)}(x, t) \Delta \left(\sum_{k=1}^M v_k(x) d_k(t) \right) dx dt \\ &= \int_0^T \int_{\Omega} f^{(N_j)}(x, t) \sum_{k=1}^M v_k(x) d_k(t) dx dt. \end{aligned}$$

Denote $\sum_{k=1}^M v_k(x) d_k(t) = \eta(x, t)$. Then, we have

$$\begin{aligned} & \int_0^T \int_{\Omega} u_t^{(N_j)}(x, t) \eta(x, t) dx dt - \int_0^T \int_{\Omega} u^{(N_j)}(x, t) \Delta \eta(x, t) dx dt \\ &= \int_0^T \int_{\Omega} f^{(N_j)}(x, t) \eta(x, t) dx dt, \end{aligned} \quad (4.13)$$

where $\eta(x, t) \in L^2(0, T; W^{2,2}(\Omega) \cap \dot{W}^{1,2}(\Omega))$. Since $\{u^{(N_j)}\}$ and $\{u_t^{(N_j)}\}$ weakly converge in $L^2(0, T; L^2(\Omega))$, and $\{f^{(N_j)}\}$ weakly converges in $L^2(0, T; W^{-2,2}(\Omega))$, we can pass to a limit as $N_j \rightarrow \infty$ in equality (4.13):

$$\begin{aligned} & \int_0^T \int_{\Omega} u_t(x, t) \eta(x, t) dx dt - \int_0^T \int_{\Omega} u(x, t) \Delta \eta(x, t) dx dt \\ &= \int_0^T \int_{\Omega} f(x, t) \eta(x, t) dx dt. \end{aligned} \quad (4.14)$$

Note that (4.14) is now proved for $\eta(x, t) = \sum_{k=1}^M v_k(x) d_k(t) \in L^2(0, T; W^{2,2}(\Omega) \cap \dot{W}^{1,2}(\Omega))$, and M is an arbitrary natural number. Since the set of all linear combinations $\sum_{k=1}^M v_k(x) d_k(t)$ is dense in the space $L^2(0, T; W^{2,2}(\Omega) \cap \dot{W}^{1,2}(\Omega))$, for every $\eta(x, t) \in L^2(0, T; W^{2,2}(\Omega) \cap \dot{W}^{1,2}(\Omega))$ there exists a subsequence $\{\eta_l\}$ such that

$$\|\eta_l - \eta\|_{L^2(0, T; W^{2,2}(\Omega) \cap \dot{W}^{1,2}(\Omega))} \rightarrow 0 \quad \text{as } l \rightarrow \infty.$$

So, for every η_l the equality (4.14) is valid, i.e.,

$$\begin{aligned} & \int_0^T \int_{\Omega} u_t(x, t) \eta_l(x, t) dx dt - \int_0^T \int_{\Omega} u(x, t) \Delta \eta_l(x, t) dx dt \\ &= \int_0^T \int_{\Omega} f(x, t) \eta_l(x, t) dx dt. \end{aligned} \quad (4.15)$$

Then, we can pass to a limits as $l \rightarrow \infty$ in (4.15):

$$\int_0^T \int_{\Omega} u_t(x, t) \eta(x, t) dx dt - \int_0^T \int_{\Omega} u(x, t) \Delta \eta(x, t) dx dt = \int_0^T \int_{\Omega} f(x, t) \eta(x, t) dx dt,$$

for arbitrary function $\eta(x, t)$ from the space $L^2(0, T; W^{2,2}(\Omega) \cap \dot{W}^{1,2}(\Omega))$.

Next, we need to prove that

$$\int_{\Omega} |u^{(N)}(x, t)|^2 dx \rightarrow E^2(t).$$

Let us denote

$$\varphi^{(N)}(t) = \|u^{(N)}(\cdot, t)\|_{L^2(\Omega)} = \left(\int_{\Omega} |u^{(N)}|^2 dx \right)^{1/2}. \quad (4.16)$$

Then, $(\varphi^{(N)}(t))'$ is:

$$\begin{aligned} (\varphi^{(N)}(t))' &= \left(\left(\int_{\Omega} |u^{(N)}|^2 dx \right)^{1/2} \right)'_t = \frac{\int_{\Omega} u^{(N)} u_t^{(N)} dx}{\|u^{(N)}(\cdot, t)\|_{L^2(\Omega)}} \\ &\leq \frac{\left(\int_{\Omega} |u^{(N)}|^2 dx \right)^{1/2} \left(\int_{\Omega} |u_t^{(N)}|^2 dx \right)^{1/2}}{\|u^{(N)}(\cdot, t)\|_{L^2(\Omega)}} = \|u_t^{(N)}\|_{L^2(\Omega)}. \end{aligned} \quad (4.17)$$

So, applying estimates (4.16) and (4.17) we get

$$\int_0^T |\varphi^{(N)}(t)|^2 dt = \int_0^T \|u^{(N)}\|_{L^2(\Omega)}^2 dt = \|u^{(N)}\|_{L^2(0,T;L^2(\Omega))}^2 \quad (4.18)$$

and

$$\int_0^T |(\varphi^{(N)}(t))'|^2 dt \leq \int_0^T \|u_t^{(N)}\|_{L^2(\Omega)}^2 dt = \|u_t^{(N)}\|_{L^2(0,T;L^2(\Omega))}^2. \quad (4.19)$$

Since the $L^2(0, T; L^2(\Omega))$ - norms of functions $u^{(N)}$ and $u_t^{(N)}$ are finite (see (4.7) and (4.8)) and the estimates (4.18), (4.19) are valid, we conclude $\varphi(t) \in W^{1,2}(0, T)$. The embedding $W^{1,2}(0, T) \hookrightarrow C([0, T])$ is completely continuous (see [5]). Therefore, from $\varphi^{(N)} \rightarrow \varphi$ in $W^{1,2}(0, T)$, follows that $\varphi^{(N)} \rightarrow \varphi$ in $C([0, T])$, i.e.,

$$\int_{\Omega} |u^{(N)}(\cdot, t)|^2 dx \rightarrow \int_{\Omega} |u(\cdot, t)|^2 dx. \quad (4.20)$$

Since $\frac{1}{\|u_0\|_{L^2(\Omega)}^2} \sum_{k=1}^N \beta_k^2 E^2(t) \rightarrow E^2(t)$ as $N \rightarrow \infty$, we can conclude that $\int_{\Omega} |u(x, t)|^2 dx = E^2(t)$. Due to the condition $E(0) = \|u_0\|_{L^2(\Omega)}$, we obtain $\int_{\Omega} |u(x, 0)|^2 dx = \int_{\Omega} |u_0(x)|^2 dx$.

In order to prove $u(x, 0) = u_0(x)$ we need to get that

$$\lim_{t \rightarrow 0} \|u(\cdot, t) - u_0\|_{L^2(\Omega)} = 0. \quad (4.21)$$

Let us estimate the norm $\|u(\cdot, t) - u_0\|_{L^2(\Omega)}$:

$$\begin{aligned} \|u(\cdot, t) - u_0\|_{L^2(\Omega)} &\leq \|u(\cdot, t) - u^{(N_l)}(\cdot, t)\|_{L^2(\Omega)} \\ &\quad + \|u^{(N_l)}(\cdot, t) - u_0^{(N_l)}\|_{L^2(\Omega)} + \|u_0^{(N_l)} - u_0\|_{L^2(\Omega)}. \end{aligned} \quad (4.22)$$

Let us choose an arbitrary small $\varepsilon > 0$. From the weak convergence $u^{(N_l)}$ to u and (4.20) we conclude that $u^{(N_l)}$ converges strongly. So, there exists N_* such that for every $N_l > N_*$

$$\|u(\cdot, t) - u^{(N_l)}(\cdot, t)\|_{L^2(\Omega)} \leq \varepsilon/3. \quad (4.23)$$

Since $u_0^{(N_l)}$ is the partial sum of the Fourier series of the initial function u_0 , the number N_* can be find such that for every $N_l > N_*$

$$\|u_0^{(N_l)} - u_0\|_{L^2(\Omega)} \leq \varepsilon/3. \quad (4.24)$$

Let us fix N_l . Then,

$$\begin{aligned} \|u^{(N_l)}(\cdot, t) - u_0^{(N_l)}\|_{L^2(\Omega)} &= \left(\int_{\Omega} \left| \sum_{k=1}^{N_l} w_k^{(N_l)}(t) v_k(x) - \sum_{k=1}^{N_l} \beta_k v_k(x) \right|^2 dx \right)^{1/2} \\ &= \left(\sum_{k=1}^{N_l} (w_k^{(N_l)}(t) - \beta_k)^2 \right)^{1/2} = \left(\sum_{k=1}^{N_l} (w_k^{(N_l)}(t) - w_k^{(N_l)}(0))^2 \right)^{1/2} \leq \varepsilon/3 \end{aligned} \quad (4.25)$$

for any $t \leq \delta(\varepsilon)$. Here we used the fact that functions $w_k^{(N_l)}$ are continuous. Substituting (4.23), (4.24) and (4.25) into inequality (4.22) we arrive at (4.21). This implies that the obtained solution u satisfies the initial condition $u(x, 0) = u_0(x)$.

Remark 3. The constructed solutions of problem (3.1)–(3.2) depend on the Fourier coefficients β_k of the initial function u_0 . In the case when $u_0 = 0$ the function $E(t)$ has to satisfy condition $E(0) = 0$. Then instead of the coefficients γ_k we can take in (4.6) arbitrary coefficients α_k such that $\sum_{k=1}^{\infty} \alpha_k = 1$. So, there is no uniqueness of the solution if $u_0 = 0$.

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