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Production Factors' Heterogeneity, Substitution and Technological Shifts

Gamybos veiksnių heterogeniškumas, pakeičiamumas ir technologijos pobūdžio poslinkiai

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Abstract

In this thesis, a Time Varying Elasticity of Substitution (TVES) production function is constructed based on the hypothesis that elasticity of substitution is dependent on the distribution of public and private sectors in an economic system. The TVES function is written as a dynamic linear model and its estimation algorithm is specified using normalization, Kalman filtering and smoothing. Empirical estimation of elasticity of substitution and production growth is done using EU KLEMS data for 9 countries in a 28-year period. The CES model is rejected in favor of the TVES model for 8 countries, meaning that the elasticity of substitution is dependent on the aforementioned distribution, however, it does not explain all variation.

Key words: production function, time varying elasticity of substitution, public sector, private sector, dynamic linear model, Kalman filter.

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Santrauka

Šiame darbe sukonstruojama gamybos funkcija, kuroje gamybos veiksnių pakeičiamumas yra laike kintantis (TVES funkcija). Tai padaryta remiantis hipoteze, kad viešasojo ir privačiojo gamybos sektorių pasiskirstymas turi įtakos gamybos veiksnių pakeičiamumui. Darbe TVES funkcija yra aprašoma dinaminiu tiesiniu modeliu ir nuorodomas jos parametrų įvertinimo algoritmas naudojantis normalizavimu, Kalmano filtravimu ir glodinimu. Empirinės pakeičiamumo ir technologinio augimo vertės yra įvertinamos naudojant 9 šalių duomenis iš EU KLEMS duomenų bazės 28 metų laikotarpyje. 8 šalyse CES funkcija buvo atmesta lyginant su TVES funkcija, tai reiškia, kad minėtasis pasiskirstymas gali turėti įtakos gamybos veiksnių pakeičiamumui, tačiau jis nepaaiškina visos šio parametro variacijos.

Raktiniai žodžiai: gamybos funkcija, laike kintantis pakeičiamumas, viešasis sektorius, privatusis sektorius, dynaminis tiesinis model, Kalmano filtras

Contents

Introduction

Elasticity of substitution between production factors plays a crucial role in modeling macroeconomic data and is one of the determinants of economic growth [20]. When modeling such data, the choice of the production function is of high importance. In recent years, the Cobb-Douglas production function, which implies elasticity of substitution to be equal to one, has been rejected in favor of more general models – numerous papers have achieved similar results, leading to believe that a more general model is the way to go [10, 16, 31].

Most of the recent research is done using a Constant Elasticity of Substitution (CES) [3] or Variable Elasticity of Substitution (VES) production function, of which there are different forms, e.g. [35] assumes that elasticity is linearly dependent on the capital to labor ratio. This is a class of VES functions; however each class requires assumptions of dependence, whereas the CES model is derived directly from the definition of elasticity of substitution, but allows for other parameters to be time varying, e.g. mark-up or technology growth.

There are two different approaches to estimating elasticity of substitution: using time series variation for specific countries or cross-country regression using panel data [2, 15]. Neither has yet to prove more efficient than the other. In this thesis I hope to capture the elasticity of substitution with cross-country dynamic linear model regression, allowing elasticity of substitution to be dependent on other data.

When using macroeconomic data, a researcher often finds himself using aggregated data, which may lead to biased results [10, 40]. Allowing for unobserved heterogeneity between countries has shown to provide positive results [40], however each country is an aggregate itself. Further disaggregation to production sectors (like public and private, market and non-market, goods and services) within a country could lead to more consistent results. [7] shows this to be true for market and non-market sectors. Public and private sectors also have major differences, e.g. the public sector is more labor intensive with low elasticity of substitution, while the private sector may not necessarily have similar properties [34]. Modeling disaggregated data could give a better understanding of the growth of the whole economy as well.

The aim of this paper is to specify a Time Varying Elasticity of Substitution (TVES) production model, based on the hypothesis that the elasticity of substitution is dependent on the production distribution between private and public sectors. The first step is to perform a test, whether the aggregated elasticity in a country is a time varying parameter. A TVES model is constructed, it is normalized and adjusted for technical bias and estimation is proposed using Kalman filtering and smoothing. Lastly, the model is illustrated with data from EU KLEMS database.

1 Production Function in Economics

No model of a micro or macro economy is sufficient if it does not include a production function. However, when choosing one, the researcher also chooses economic assumptions, associated with the aforementioned function, e.g. the Leontief production function, which states that the production factors can only be input with a certain ratio. The results of a model are dependent on the chosen form of a production function, thus the researcher has to choose wisely. One of the most common production functions used in recent years is the CES model, as it is a more general form of a variety of cases, distinguished by the elasticity of substitution. With the CES model the researcher can evaluate two important parameters: elasticity of substitution and technological growth. These parameters not only affect the production output, but the distribution of input as well [22] and can even determine the growth of an economy [16, 31].

This section contains discussions about the economic significance of elasticity of substitution, how it is determined. Production functions of constant and variable elasticity of substitution are compared and lastly estimations of the parameters of interest from other works are shown.

1.1 Significance of Elasticity of Substitution

Introduced by Hicks in 1932 [13], the elasticity of substitution made a great impact on the modeling of economics. It was first introduced as a tool for analyzing capital and labor income shares in a growing economy with a constant returns-to-scale technology and neutral technology change. The mathematical definition of elasticity of substitution was written as:

$$\sigma = \frac{d\log\left(\frac{K}{L}\right)}{d\log\left(\frac{F_L}{F_K}\right)} \tag{1}$$

Where K and L are production factors: capital and labor, respectively; F is a production function and $F_i = \frac{\partial F}{\partial i}, i = K, L$, so $\frac{F_L}{F_K}$ is the rate of technical substitution. Given that production is a linear homogeneous function $Y/L = F(K/L)$, elasticity of substitution can also be defined as

$$\sigma = \frac{d\log\left(\frac{Y}{L}\right)}{d\log w} \tag{2}$$

Here w is the real wage rate, defined as the nominal wage rate adjusted with the GDP deflator.

σ can have values from 0 to ∞ , this value shows how easy it is to substitute one production factor with another. If the value is less than 1, the production factors are gross supplements and profit maximization is achieved with a combination of both capital and labor, thus an increase in one factor should be followed by an increase in the other. If the value is larger than 1, the factors are gross substitutes and profit maximization is achieved by using more of the cheaper factor and/or more effective factor.

Production factors input can also by changed technological growth. This is done by increasing the technical possibilities of production for one or both factors. If, for example, labor technology increases, one can achieve higher production values, with the same amount of labor. Even when the technological progress is clear, the distribution of production factors is still dependent on the elasticity of substitution: if the factors are gross supplements, the productivity growth of one factor leads to a case, where we need less of the more efficient factor than the less efficient one. On the other hand, if the factors are gross substitutes, productivity growth for one factor leads to a decreased input of the other. Knowledge of the true value of the elasticity of substitution would not only lead to knowledge of true technological growth, but also to an explanation of why in recent years the labor share of production is decreasing.

Another economic situation that needs elasticity of substitution to be fully explained is the growth of the economy itself. Different economic growth theories rely upon different assumptions for growth. The Solow-Swan growth model specifies the elasticity of substitution to be unitary [37]. On the other hand, by saying that elasticity of substitution is between 0 and 1, Acemoglu [1] explains the U shaped dynamics for technical progress. Another economic growth model - capital fundamentalism - states that capital accumulation is central to increasing the rate of economic growth [18] and the accumulation is only possible with elasticity of substitution value above 1.

1.2 CES vs. VES

The first widespread production function used for aggregated data was the Cobb-Douglas function:

$$Y = AL^{\alpha}K^{1-\alpha} \tag{3}$$

which, having a simple form, was easy to estimate and gave relatively precise estimates [9]. It was widely used for different estimations, however the function restricts the elasticity of substitution to be unitary. In 1956 Solow first introduced a constant elasticity of substitution (or CES) production function with elasticity of substitution to be 0.5, as Solow wrote, "just to add a bit of variety" [37].

The CES production function is stated as:

$$Y = \left[\delta(A_L L)^{\frac{\sigma-1}{\sigma}} + (1-\delta)(A_K K)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \tag{4}$$

Here A_L and A_K denotes the technological factors for labor and capital, respectively, and δ is a distribution parameter. This functional form is appreciated for being a general function of different specific cases, separated only by the value of σ: with σ equal to 1, the CES function becomes the Cobb-Douglas function; with σ equal to 0 – the Leontief function and with σ equal to infinity – a linear production function. There are also downsides to this function, one of them is, as McFadden and Uzawa showed, that for a number of production factors greater than 2 it is impossible to obtain a functional form for the CES production function [39, 29]. Another problem with CES is that the elasticity of substitution remains constant across and along the isoquants.

To solve these problems different researchers proposed different generalizations of the CES function, each of which led the elasticity of substitution to vary in different ways. These production functions are called variable elasticity of substitution (or VES) functions. Among notable examples of VES functions is the Revankar function, which solves the problem of constancy across isoquants [35]:

$$Y = AK^{a\nu}[L + baK]^{(1-a)\nu} \tag{5}$$

Here ν denotes the returns-to-scale, which is often assumed to be unitary. The idea behind the variation of elasticity of substitution in this function is that it is dependent on the current capital to labor ratio:

$$\sigma = 1 + \gamma\frac{K}{L} \tag{6}$$

Another generalized CES function introduced by Sato [36] allows to obtain a functional form for the production function for more than 2 production factors

$$Y^p = \delta(A_L L)^p + (1-\delta)(A_K K)^p. \tag{7}$$

There have been numerous papers published, that rejected the CES production function in favor of VES - Lovell [24, 25], Lu and Fletcher [27], just to name a few; on the other hand, there have been cases where CES could not be rejected, such as discussed in [26]. Even though the VES functions seem to be more favorable, the CES function is still often used in current-day research analysis, as the true source of time variation is unclear.

1.3 Determining Elasticity of Substitution

The real wage rate mentioned in (2) is calculated in a differential equation, that means that the elasticity of substitution is a 2nd order differential equation. Solving this equation leads to the CES production function, but solving a 2nd order differential equation requires two integration constants. Usually integration constants can be found using boundary conditions and normalization does exactly that [21].

A normalized CES function without technological growth has the functional form:

$$Y = Y_0\left[\delta_0\left(\frac{L}{L_0}\right)^{\frac{\sigma-1}{\sigma}} + (1-\delta_0)\left(\frac{K}{K_0}\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \tag{8}$$

Here $\delta_0 = \frac{w_0 L_0}{Y_0}$ and every variable with a subscript 0 is the value of the variable at the point of normalization.

As mentioned in the previous subsection, the CES production function estimates the same elasticity of substitution across all isoquants, however, when defined as in (1) and in (2), elasticity of substitution is calculated at a single point, this means that it is related to a specific isoquant. Normalization allows all the isoquants to be non intersecting and distinguished only by σ, thus allowing a more precise estimate of the parameter of interest.

Another important feature of normalization is that normalization fixes a benchmark value for factor income shares, so it is convenient when estimating biases in the direction of technical progress. This progress is often assumed to have this functional form:

$$A_i = A_{i,0}e^{\alpha_i(t-t_0)} \tag{9}$$

here $i = K, L$ and $A_{i,0}$ is the technological factor at the point of normalization. So the full normalized CES function with technological progress is written as:

$$Y = Y_0\left[\delta_0\left(\frac{L}{L_0}e^{\alpha_L(t-t_0)}\right)^{\frac{\sigma-1}{\sigma}} + (1-\delta_0)\left(\frac{K}{K_0}e^{\alpha_K(t-t_0)}\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \tag{10}$$

This function may seem complicated, however the only three parameters that are unknown in each system are σ, α_K and α_L.

Estimating the mentioned parameters from (10) directly is improbable, so a transformation is to be done. In a rational world profits are maximized:

$$\underset{L,K}{max} \; Y - wL - rK \tag{11}$$

Here r denotes the real interest rate. And thus the first order conditions for profit maximization are:

$$\frac{\partial Y}{\partial L} = w \tag{12}$$

$$\frac{\partial Y}{\partial K} = r \tag{13}$$

The solutions to these equations for a non-normalized CES function are:

$$\frac{wL}{Y} = \delta\Big(A_L\frac{L}{Y}\Big)^{\frac{\sigma-1}{\sigma}} \tag{14}$$

$$\frac{rK}{Y} = (1-\delta)\Big(A_K\frac{K}{Y}\Big)^{\frac{\sigma-1}{\sigma}} \tag{15}$$

And for a normalized CES function:

$$\frac{wL}{Y} = \delta_0\Big(\frac{Y_0L}{L_0Y}e^{\alpha_L(t-t_0)}\Big)^{\frac{\sigma-1}{\sigma}} \tag{16}$$

$$\frac{rK}{Y} = (1-\delta_0)\Big(\frac{Y_0K}{K_0Y}e^{\alpha_K(t-t_0)}\Big)^{\frac{\sigma-1}{\sigma}} \tag{17}$$

Taking the logarithms of each side we get a linear function which can be used in estimations. With some simplifications added, the estimation functions are:

$$\log(sh_L) = \log\delta_0 + \rho\log\left(\frac{Y_0L}{L_0Y}\right) + \rho\alpha_L(t-t_0) + \epsilon_L \tag{18}$$

$$\log(sh_K) = \log(1-\delta_0) + \rho\log\left(\frac{Y_0K}{K_0Y}\right) + \rho\alpha_K(t-t_0) + \epsilon_K \tag{19}$$

Here $sh_L = \frac{wL}{Y}$ and $sh_K = \frac{rK}{Y}$ are the production factors' shares and $\rho = \frac{\sigma-1}{\sigma}$. The error terms were added for estimation purposes. Note that data about one production factor is enough to estimate σ and its corresponding technological progress term.

1.4 Previous Estimations of Elasticity and Technological Growth

There have been a great number of researchers who have tried to estimate elasticity of substitution and technological growth in the global economy, different countries and economic sectors, and most estimates of σ have values less than 1, but the values are not consistent. For the U.S. economy alone, estimates vary from 0.11 [6] to 1.25 [4].

A more recent analysis was done by [40] who estimated elasticity of substitution and technological growth in 82 countries for different time periods. This paper estimated different elasticity of substitution values for different countries, some even being above 1.

Estimated technological growth often has values between 0 and 1, however there have been cases when estimated technological progress was negative, up to -8.1% for the capital-augmenting technological growth [40]. The author justifies these values by a calculating weighted productivity $\delta\alpha_L + (1-\delta)\alpha_K$, which in the case of highly negative technological growth for one factor, can still have positive (or at least not as negative) values.

2 Time Varying Elasticity of Substitution

As mentioned in 1.2 there have been different generalizations of the CES production function, each of them have a different hypothesis for the variations of the elasticity of substitution. Here a new hypothesis for differences in elasticity of substitution is proposed and an its estimation model is introduced.

This section is constructed as follows: first a test for structural change is performed to check if elasticity of substitution is constant, then a hypothesis for the variance in elasticity of substitution is formulated, a Time Varying Elasticity of Substitution (or TVES) model is constructed and lastly, a method for parameter estimation is discussed.

2.1 Testing for Structural Change

Before specifying a model with a varying elasticity of substitution, the CES model is tested for structural change. The idea behind this test is that if the test shows existence of a structural break in the model, elasticity of substitution is changed from being a constant to being a dependent variable in the hope that the variation will capture the change shown in the test. In other words, if structural change is observed, TVES is used instead of CES.

Chow test for structural change is performed for equation (18) with data discussed in 3.1 [8]. Although elasticity of substitution is not the only parameter in the mentioned equation, it is a factor in both estimation terms so there is a high probability that a TVES model will capture the variance, not captured in a CES model.

The existence of a structural break was tested for all values in an interval of time. The p value selected is the lowest from all values:

$$p = \inf_{1998 \le t \le 2001} p_t \tag{20}$$

Results from the Chow test are shown in Table 1. It is easy to see that structural change occurs in every country. Graphs, depicting F statistics at each point in time are shown in the Appendix (Figures 1 to 9).

Country	*p value*
Austria	<0.001
Australia	<0.001
Denmark	<0.001
Finland	<0.001
France	<0.001
Italy	<0.001
Japan	<0.001
Netherlands	<0.001
United Kingdom	<0.001

Table 1: p values for Chow test for structural change in different countries.

2.2 TVES model

Elasticity of substitution is a parameter which is affected by various economic factors, trying to control for some of them might lead to positive results. In order to do that the value added in a country's economy is divided into two sectors: the public and the private.

There are differences in these sectors concerning production that could capture the variance of elasticity of substitution. As all tax payments in the public sector cancel out, it will be more labor intensive than the private sector, when wage taxes are present [34]. As the real wage rate is different in the public sector, it can have an effect on the elasticity of substitution as shown in (2).

Another difference observed between these sectors is the rate of economic growth. [17] estimate that the private sector has a larger direct effect on growth than the public sector. As discussed in 1.1, different economic growth might be caused by a different value of elasticity of substitution.

As there are observed relevant differences between the public and private sectors, a hypothesis is formulated that the elasticity of substitution in an economic system is dependent on the economic structure of the system, namely the sectoral distribution. In order to see the effect of said distribution on the value of elasticity of substitution, a distribution parameter θ is added. The value of the distribution parameter is the part of total value added, which was created in the public sector:

$$\theta = \frac{Y_{public}}{Y} \tag{21}$$

Now a Time Varying Elasticity of Substitution (TVES) model can be formulated simply as a system of of a normalized CES function and the promoted hypothesis:

$$\begin{cases} Y_t = Y_0 \left[\delta_0 \left(\frac{L_t}{L_0} e^{\alpha_L (t-t_0)} \right)^{\frac{\sigma_t - 1}{\sigma_t}} + (1-\delta_0) \left(\frac{K_t}{K_0} e^{\alpha_K (t-t_0)} \right)^{\frac{\sigma_t - 1}{\sigma_t}} \right]^{\frac{\sigma_t}{\sigma_t - 1}} \\ \sigma_t = f(\theta_t) \end{cases} \tag{22}$$

Estimation of this model is discussed further.

2.3 TVES Estimation

As discussed in 1.3, the CES production function is not estimated directly, but after a transformation involving profit maximization. In the TVES model this too is done and the first order profit maximizing conditions are exactly the same as (18) and (19) only with time varying parameters. Since σ is not directly estimated but through a parameter ρ

$$\sigma = \frac{1}{1-\rho} \tag{23}$$

the second part of (22) is not exactly relevant, however if σ is a function of θ and ρ is a function of σ, it is clear that ρ is a function of θ as well:

$$\rho_t = f_\rho(\theta_t) \tag{24}$$

The true functional form of (24) is unclear so a first order Taylor series expansion around the first value of θ is done:

$$\rho_t = f_\rho(\theta_0) + f'_\rho(\theta_0)(\theta - \theta_0) \tag{25}$$

The first term of the right hand side equation is directly related to the value of elasticity of substitution in the first year

$$f_\rho(\theta_0) = \rho_0 = \frac{\sigma_0 - 1}{\sigma_0},$$

and the factor $f'_\rho(\theta_0)$ in the second term shows the change in ρ for a unitary change in θ. For equation simplicity, $f'_\rho(\theta_0)$ is further denoted as β.

In this paper estimation is done only for the labor share (the reasons are discussed in 3.1), so the estimation model is written as

$$\begin{cases} \log(sh_{L,t}) = \log\delta_0 + \rho_t \log\left(\frac{Y_0 L_t}{L_0 Y_t}\right) + \rho_t\alpha_L(t - t_0) + \epsilon_t \\ \rho_t = \rho_0 + \beta(\theta_t - \theta_0) + u_t \end{cases} \tag{26}$$

This dynamic linear model is estimated with Kalman filtering and smoothing [32]. If estimated β value is 0 the model becomes a standard CES model and if both β and ρ_0 have estimated values of zero, the model becomes the Cobb-Douglas model.

2.3.1 Kalman Filtering and Smoothing

The Kalman filter was first introduced by Kalman in 1960 [14]. It is an algorithm that solves state space models by minimizing the mean of square errors. In this subsection the filtering algorithm is reviewed. The iterative Kalman filtering algorithm written in this paper is discussed in more detail and with more extensive references in [41].

The Kalman filter addresses the problem of trying to estimate the state Γ_t of a discrete-time controlled process that is governed by the linear stochastic difference equation

$$\Gamma_t = G_t\Gamma_{t-1} + w_{t-1} \tag{27}$$

with a measurement

$$y_t = X_t\Gamma_t + v_t \tag{28}$$

Here y_t and X_t are the measured data, G_t is the data that has an effect on the state. w_t and v_t are random variables that represent the process and measurement noise, respectively. They are assumed to be independent and with normal probability distributions

$$p(w) \sim N(0, Q) \tag{29}$$

$$p(v) \sim N(0, R) \tag{30}$$

The covariance matrices Q and R are assumed to be constant and known.

An *a priori* state is defined as an estimate $\hat{\Gamma}_t^-$ at time t given information prior to that time. An *a posteriori* state is defined as an estimate $\hat{\Gamma}_t^+$ at time t given measurement y_t. The errors for both estimates are defined as

$$\begin{aligned} e_t^- &= \Gamma_t - \hat{\Gamma}_t^- \\ e_t^+ &= \Gamma_t - \hat{\Gamma}_t^+ \end{aligned} \tag{31}$$

And the estimate error covariance matrices are

$$\begin{aligned} P_t^- &= E\Big[e_t^- e_t^{-T}\Big] \\ P_t^+ &= E\Big[e_t^+ e_t^{+T}\Big] \end{aligned} \tag{32}$$

A linear function is constructed, which estimates the *a posteriori* state knowing the *a priori* one and the and the weighted difference between the actual measurement y_t and the measurement prediction $\Gamma_t X_t$. This function is shown in (33)

$$\hat{\Gamma}_t^+ = \hat{\Gamma}_t^- + K(y_t - X_t \Gamma_t) \tag{33}$$

Here K is the *gain* or *blending factor*, it minimizes *a posteriori* error covariance defined in (32). One form of K that does this is

$$K_t = P_t^- X_t^T (X_t P_t^- X_t^T + R)^{-1} \tag{34}$$

The *a posteriori* error covariance is calculated as

$$P_t^+ = (I - K_t X_t) P_t^- \tag{35}$$

Finally, the *a posteriori* values are used to calculate the *a priori* values of the next iteration.

$$\hat{\Gamma}_{t-1}^- = G_{t+1} \hat{\Gamma}_t^+ \tag{36}$$

$$P_{t+1}^- = G_{t+1} P_t^+ G_{t+1}^T + Q \tag{37}$$

The process is calculated iteratively, starting from t = 1, where the initial values for Γ_0^+ and P_0^+ are given. After a few iterations these initial values have no effect [30, 11]. It is important to set P_0^- to a relatively high value so that the parameter values are less likely to converge to a local minimum [38].

The Kalman filter calculates the values for all parameters for each point in time, however in our case some parameters need to have the same values throughout the whole period. Smoothing is done to get these values. It is also an iterative process, however it calculates the values starting from the last point in time and goes back to the first. Denoting the state value at time t after smoothing as Φ_t with variance Ψ_t, these values are calculated as

$$\Phi_t = \Gamma_t^+ + \hat{P}_t^+ G_{t+1}^T \Big(\hat{P}_{t+1}^-\Big)^{-1} (\Phi_{t+1} - \Gamma_t^-) \tag{38}$$

$$\Psi_t = \hat{P}_t^+ + \hat{P}_t^+ G_{t+1}^T \Big(\hat{P}_{t+1}^-\Big)^{-1} (\Psi_{t+1}) - \hat{P}_t^-) \Big(\hat{P}_{t+1}^-\Big)^{-1} G_{t+1} \hat{P}_t^+ \tag{39}$$

When estimating the TVES model the corresponding variables to the ones used in Kalman filtering are

$$\Psi_t = \begin{pmatrix} \rho_t \\ \alpha_L \rho_t \\ \rho_0 \\ \beta \end{pmatrix} \quad G_t = \begin{pmatrix} 0 & 0 & 1 & (\theta - \theta_0) \\ 0 & 0 & \alpha_L & \alpha_L(\theta - \theta_0) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad X_t = \begin{pmatrix} \log\left(\frac{Y_0 L_t}{L_0 Y_t}\right) & 0 & 0 & 0 \\ 0 & t - t_0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and lastly

$$y_t = \log(sh_{L,t}) - \delta_0$$

The initial values given to the dynamic linear model are that the all Γ_0^+ values are zero so that the initial model is Cobb-Douglas.

2.3.2 Estimating Technological Progress

A problem with estimating the technological progress parameter α_L is that it only affects the labor share through a product of itself and ρ_t. This means that when estimating the model specified in (26), the value of α_L is not estimated directly, instead the value of the product $\mu_t = \alpha_L \rho_t$ is estimated. The problem with this estimation is that numerically μ is not restricted to differ from ρ_t by a constant α_L.

In order to solve this problem, another iterative algorithm has been used. The product μ_t is estimated simultaneously to ρ so the model becomes

$$\begin{cases} \log(sh_{L,t}) = \log \delta_0 + \rho_t \log\left(\frac{Y_0 L_t}{L_0 Y_t}\right) + \mu_t (t - t_0) + \epsilon_t \\ \rho_t = \rho_0 + \beta(\theta_t - \theta_0) + u_t \\ \mu_t = \alpha_L \rho_0 + \alpha_L \beta(\theta_t - \theta_0) + v_t \end{cases} \tag{40}$$

An initial value of $\alpha_{L,0}$ is given to α_L. As the value of α_L is now biased, this increases the error term

$$\mu_t = \alpha_{L,0}\rho_0 + \alpha_{L,0}\beta(\theta_t - \theta_0) + \eta_t \tag{41}$$

where

$$\eta_t = (\alpha_L - \alpha_{L,0})\rho_t + v_t \tag{42}$$

When the initial value is specified, the system (40) is estimated using the Kalman filter. When ρ_t and η_t are estimated, their values are used in (42):

$$\hat{\eta}_t = (\alpha_L - \alpha_{L,0})\hat{\rho}_t + v_t \tag{43}$$

After an OLS regression, an estimate the bias of α_L is achieved. Together with the initial value, a new value of α_L is calculated

$$\hat{\alpha_{L,1}} = \alpha_{L,0} + \widehat{bias}(\alpha_L) \tag{44}$$

and is inserted into (41). The process is repeated until the values converge and an estimate of α_L is achieved. The variance of α_L is estimated with the bootstrap method.

3 Practical Estimation

3.1 Data Used

Practical estimation was done using data from the EU KLEMS database, collected by the Groningen Growth and Development Center [19]. This database contains yearly data for 25 European countries, Australia, Canada, Japan, Korea and U.S. This database provides carefully constructed data on labor compensation, which considers the labor income of the self employed which is important to study the evolution of labor share [12]. The EU KLEMS database uses standardized data that allows comparability between the set of countries.

The data used in this paper are (i) aggregate real value added, (ii) real value added in public sectors (education, health and social work, public administration and defense, compulsory social security, community, social and personal services, other community), (iii) number of employees, (iv) labor compensation for employees. Only 9 countries had data for all these variables for a moderate period of time - 28 years, so they were selected for estimation.

In this thesis only the labor share evolution is estimated, as it observable, it does not require explicit production function estimation and tends to allow researchers to abstract from capital accumulation. Also it tends to yield the correct slope sign [23]. The are some disadvantages of using labor share, one of which is that the use of labor share as a measure of real marginal cost implies that the number of workers or their utilization rate can be adjusted without extra costs.

Reasons not to use (19) are that there are difficulties in measuring the capital value and determining the real user costs of capital, as not all capital is well priced with interest rates [23]. Therefore stable empirical tests are obtained analyzing just (18).

3.2 Estimated Values

All countries that were investigated converge to the same long term economic growth [28, 33], so an additional restriction in estimation was given that technological growth in all countries is the same.

After evaluation using the TVES production function, 7 out of 9 countries have an estimated average elasticity of substitution above 1, meaning that in these coun-

Country	$\bar{\hat{\sigma}}$
Austria	1.08
Australia	1.24
Denmark	0.98
Finland	1.06
France	1.07
Italy	1.08
Japan	0.67
Netherlands	1.85
United Kingdom	1.04

Table 2: Mean values of estimated elasticity of substitution

tries the production factors are gross substitutes. The countries with an estimated average elasticity of substitution below 1 are Japan and Denmark. These results are shown in Table 2.

Although many countries have $\bar{\hat{\sigma}}$ values around 1, none of the values actually cross the unitary value. Compared to results published in other papers, the values of $\hat{\sigma}$ are quite different, as in most cases the value of σ is below 1 for the same countries or the global economy [40, 10, 31]. All estimated parameter values are shown in Table (3)

It is worth noting that the estimated initial value of elasticity of substitution in Denmark is (not different from) 1. In [40], where elasticity of substitution is estimated with a Bayesian approach using the same database, the estimated value for Denmark is also mentioned as exceptional, as it has a mean above 1 (1.09) and median below 1 (0.99).

8 out of 9 countries have statistically significant values of $\hat{\beta}$. Most of the values of $\hat{\beta}$ (especially the most statistically significant ones) are negative, meaning that as the percent of value added by the public sector increases, the elasticity of substitution decreases. This justifies the conclusion that heterogeneity exists in most countries and that it is easier to substitute production factors in the private sector than in the public.

At first look at some graphs of the labor share evolution, an example is given in Figure 4 in the Appendix, it is clear that while the CES model can grasp the trend

Country	*Parameter*	*Value*	*Variance*	*p value*
Austria	$\hat{\rho}_0$	0.09	$6{\cdot}10^{-5}$	<0.001
	$\hat{\beta}_0$	-0.67	0.097	0.03
Australia	$\hat{\rho}_0$	0.24	$2{\cdot}10^{-4}$	<0.001
	$\hat{\beta}_0$	-2.37	0.171	<0.001
Denmark	$\hat{\rho}_0$	0.02	$9{\cdot}10^{-5}$	0.06
	$\hat{\beta}_0$	-0.95	0.067	<0.001
Finland	$\hat{\rho}_0$	0.1	$5{\cdot}10^{-5}$	<0.001
	$\hat{\beta}_0$	-0.81	0.007	<0.001
France	$\hat{\rho}_0$	0.05	$7{\cdot}10^{-5}$	0.01
	$\hat{\beta}_0$	0.26	0.01	0.008
Italy	$\hat{\rho}_0$	0.08	$2{\cdot}10^{-5}$	<0.001
	$\hat{\beta}_0$	0.09	0.005	0.18
Japan	$\hat{\rho}_0$	-0.41	$9{\cdot}10^{-4}$	<0.001
	$\hat{\beta}_0$	-2.16	0.58	0.004
Netherlands	$\hat{\rho}_0$	0.36	0.01	<0.001
	$\hat{\beta}_0$	-1.92	0.271	0.002
United Kingdom	$\hat{\rho}_0$	0.06	$4{\cdot}10^{-5}$	<0.0001
	$\hat{\beta}_0$	-0.54	0.271	<0.001
All countries	$\hat{\alpha}_L$	0.0186	$9{\cdot}10^{-5}$	<0.001

Table 3: Estimated values using TVES production function

Country	R^2 TVES	R^2 CES
Austria	0.699	0.673
Australia	0.878	0.704
Denmark	-0.01	-0.11
Finland	0.398	0.207
France	0.6	0.564
Japan	0.208	-0.134
Netherlands	0.48	0.395
United Kingdom	0.404	0.317

Table 4: R^2 values for TVES and CES regressions for countries, that rejected CES production function

of the labor share evolution, TVES explains more of the short-term variance. This is also confirmed by the R^2 values shown in Table 4. Some R^2 values are negative, meaning that the mean of the data has more prediction power than the regressed variables. This is mostly due to the restriction of normalization ($\delta = \delta_0$) when the evolution of labor share has high unexplained variation, as in Denmark's case (Figure 10 in Appendix).

Since not all $\hat{\beta}$ values are significant, there is a country that could not reject the CES model in favor of the CES model. This is natural, as there are more factors that determine the elasticity of substitution and that are uncontrolled for in this paper.

More precise estimates of elasticity of substitution in turn led to less biased estimates of the labor technology augmenting parameter $\hat{\alpha}_L$. The estimated yearly labor technological growth is about 1.8% (standard error 0.77%), which is a feasible estimate. With CES the technological growth is estimated at 2.1% with a standard error of 0.96%. The hypothesis of all countries having the same technological growth was also tested by conducting the same regression without each of the countries and the test result is that values do not differ.

4 Conclusions

This thesis suggests that the elasticity of substitution for production factors is dependent on the sectoral distribution in an economic system. A dynamic linear model was constructed for evaluating the effects of this hypothesis by expanding the CES model. The dynamic linear model was evaluated for 9 countries in a 28-year period using the labor share evolution data provided in the EU KLEMS database.

The research concludes that the structure of an economic system can have an effect on the production factors' elasticity of substitution: when the public sector is larger, the value of elasticity of substitution drops. In the recent years, the labor share of value added is dropping, meaning that labor is being replaced by capital. As the public sector is more labor intensive, it is harder to replace labor by capital, meaning the elasticity of substitution is lower. This matches the results of this research.

It is worth noting that using Italy's data the CES production function could not be rejected in favor of the suggested TVES function. This is natural, as there are more economic factors responsible for changes in elasticity of substitution and all variance cannot be explained with one variable.

The dynamic linear model proved to be able to estimate the elasticity of substitution, at least to some accuracy. In 7 out of 9 countries, the estimated value of elasticity of substitution is above 1, meaning that the production factors are gross substitutes. Even though this is not the first time these kind of values were estimated, they are rare, as other researchers estimate values mostly to be below 1. This may be because other economic factors were not controlled for in this research, so the estimated values of this paper should be treated with caution.

Further research that could be done in this area is allowing for more flexible technological growth estimation using the Box and Cox transformation [5]. Also, a more precise functional form of elasticity of substitution dependence on the sectoral distribution could be found, and, of course, the data could be controlled for other economic factors to achieve more accurate results.

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Appendices

Chow test graphs

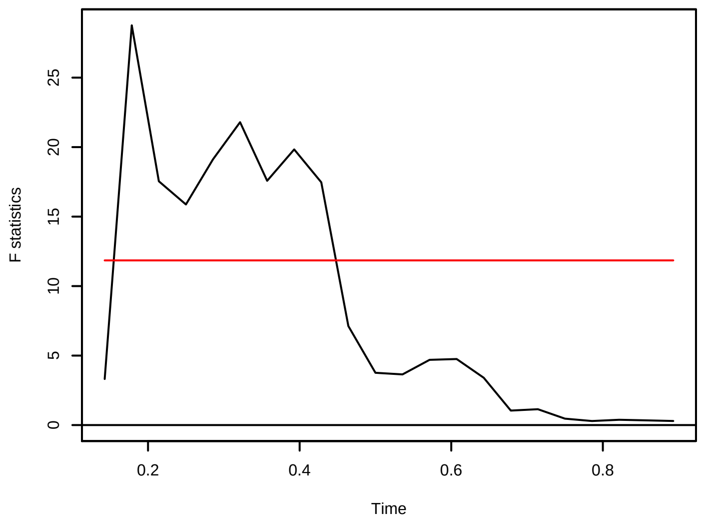

Figure 1: F statistics for Chow test for Austria at periods from 1982 to 2002. Red line indicates F statistics value, with which the confidence of rejecting H_0 is 95%.

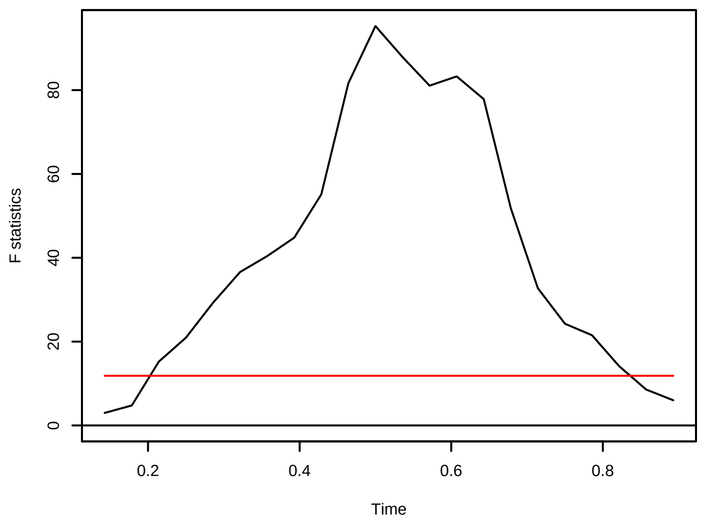

Figure 2: F statistics for Chow test for Australia at periods from 1982 to 2002. Red line indicates F statistics value, with which the confidence of rejecting H_0 is 95%.

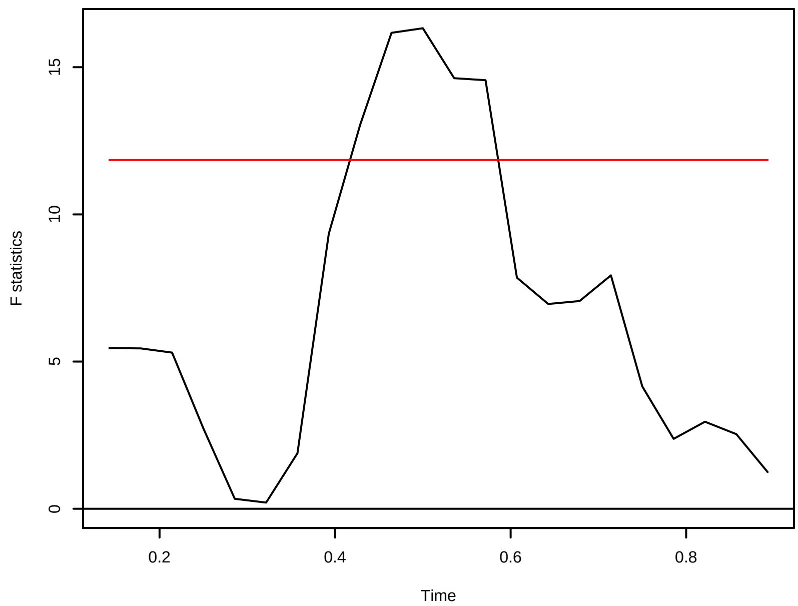

Figure 3: F statistics for Chow test for Denmark at periods from 1982 to 2002. Red line indicates F statistics value, with which the confidence of rejecting H_0 is 95%.

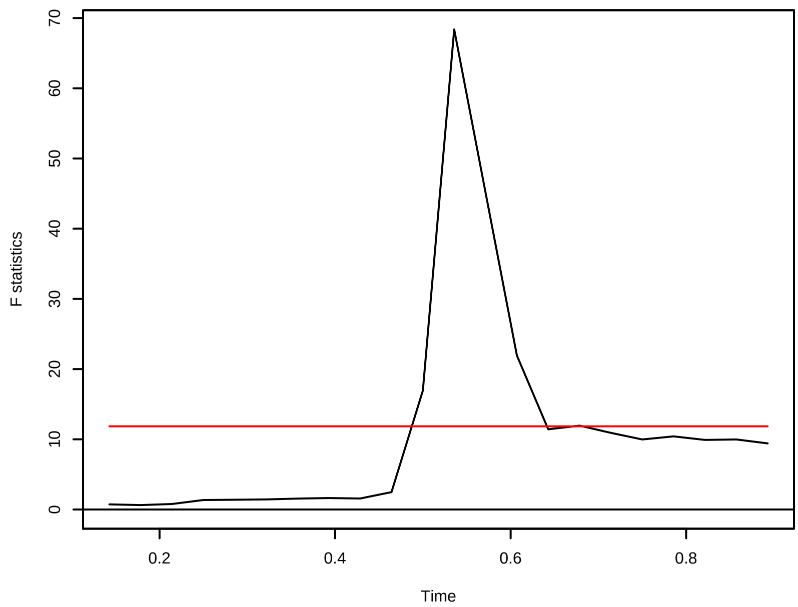

Figure 4: F statistics for Chow test for Finland at periods from 1982 to 2002. Red line indicates F statistics value, with which the confidence of rejecting H_0 is 95%.

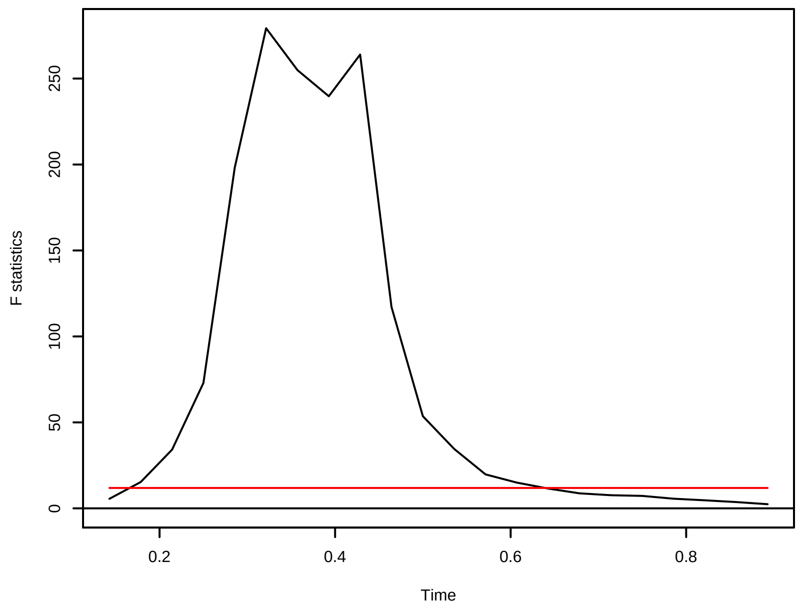

Figure 5: F statistics for Chow test for France at periods from 1982 to 2002. Red line indicates F statistics value, with which the confidence of rejecting H_0 is 95%.

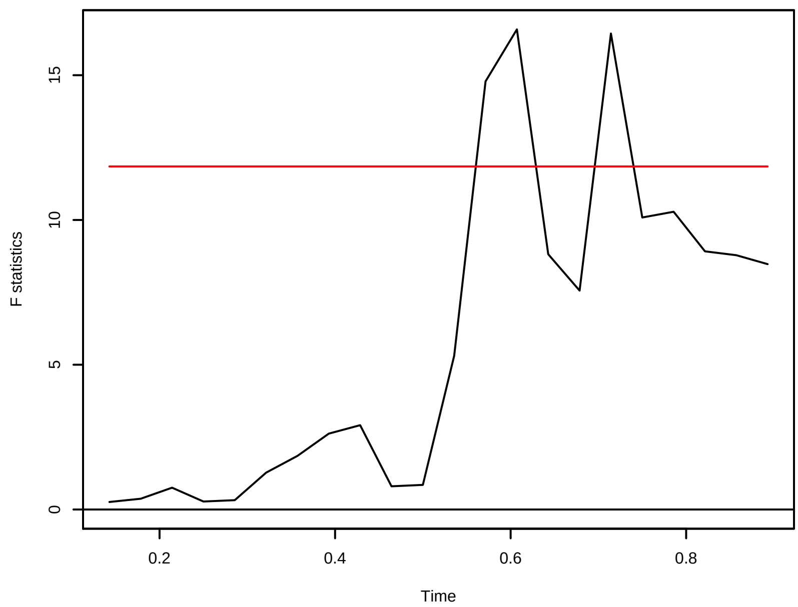

Figure 6: F statistics for Chow test for Italy at periods from 1982 to 2002. Red line indicates F statistics value, with which the confidence of rejecting H_0 is 95%.

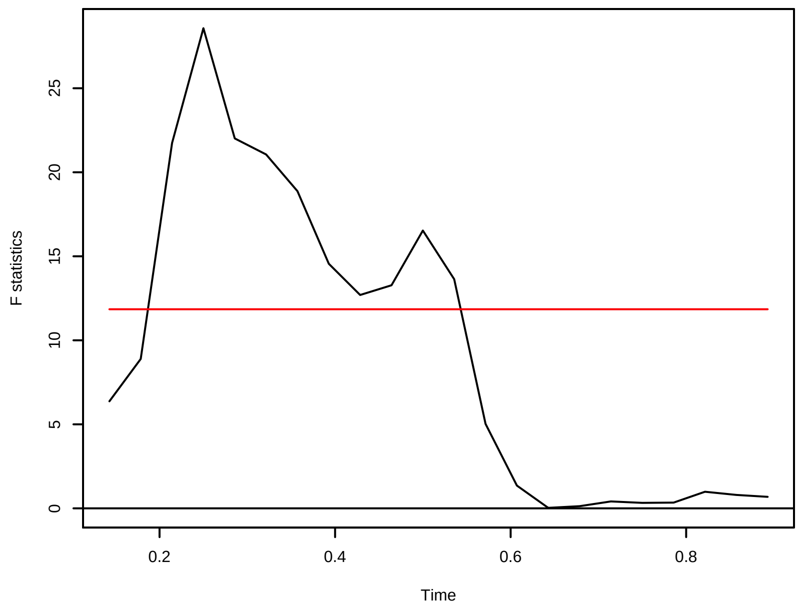

Figure 7: F statistics for Chow test for Japan at periods from 1982 to 2002. Red line indicates F statistics value, with which the confidence of rejecting H_0 is 95%.

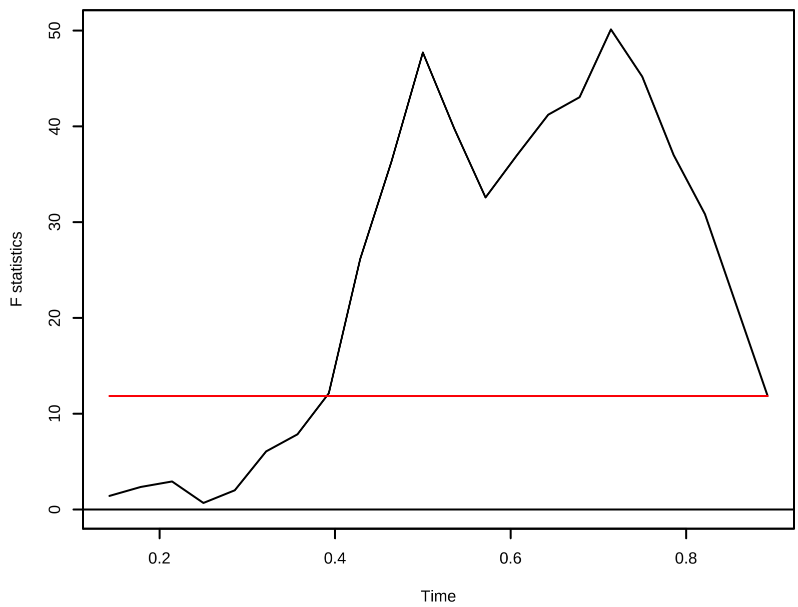

Figure 8: F statistics for Chow test for the Netherlands at periods from 1982 to 2002. Red line indicates F statistics value, with which the confidence of rejecting H_0 is 95%.

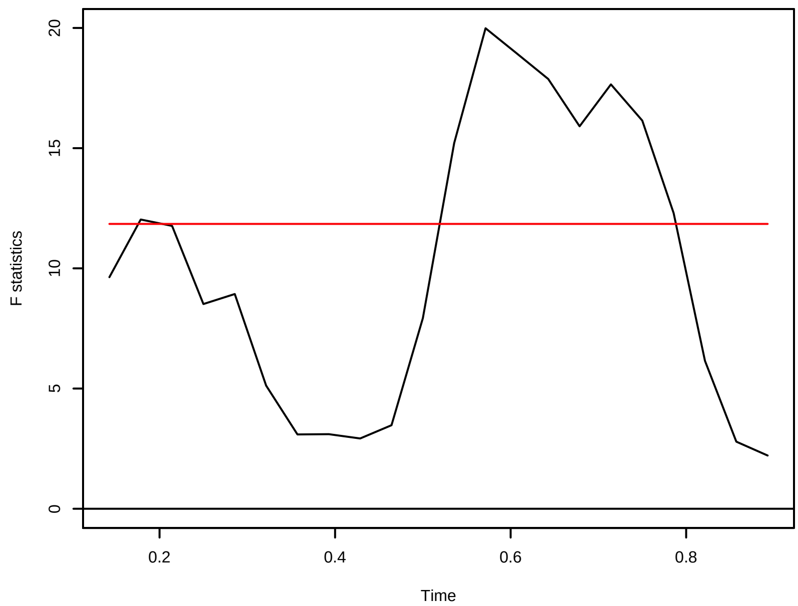

Figure 9: F statistics for Chow test for United Kingdom at periods from 1982 to 2002. Red line indicates F statistics value, with which the confidence of rejecting H_0 is 95%.

Labor share evolution graphs

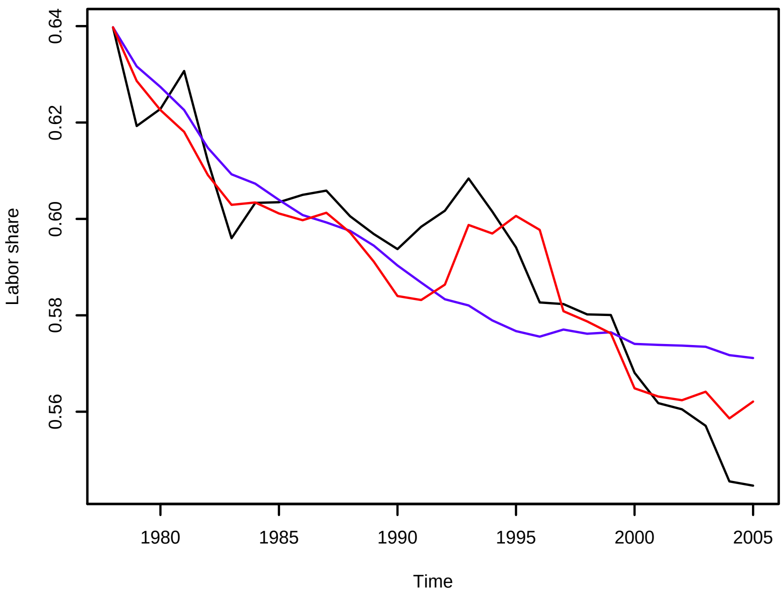

Figure 10: Labor share evolution in Australia. Black line denotes the actual data, blue line - CES fitted values, red line - TVES fitted values.

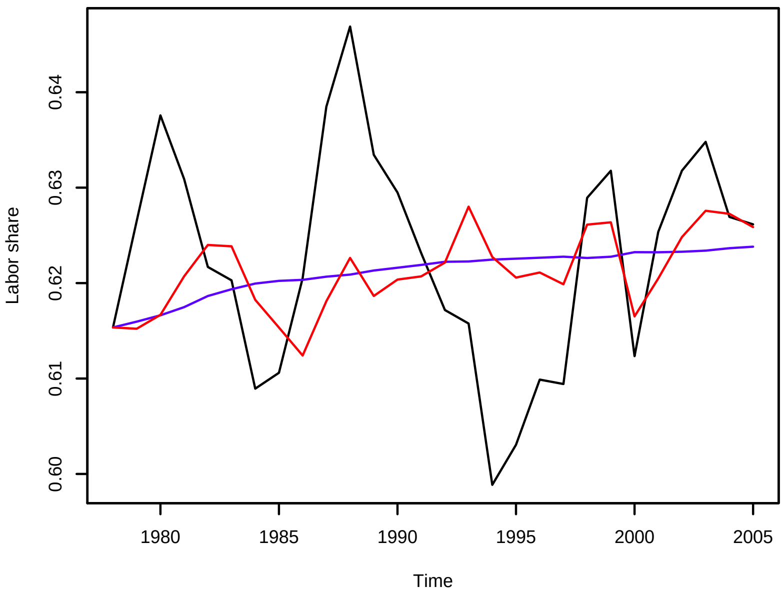

Figure 11: Labor share evolution in Denmark. Black line denotes the actual data, blue line - CES fitted values, red line - TVES fitted values.