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A New Joint Limit Theorem of Bohr–Jessen Type for Epstein Zeta-Functions

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Abstract: For $j = 1,\dots,r$, let Q_j be a positive definite $n_j \times n_j$ matrix, and $\zeta(s_j;Q_j)$ denote the corresponding Epstein zeta-function. In this paper, assuming that $n_j \geqslant 4$ is even and $\underline{x}^{\mathrm{T}} Q_j \underline{x} \in \mathbb{Z}, \underline{x} \in \mathbb{Z}^r \setminus \{\underline{0}\}$, a joint limit theorem of Bohr–Jessen type for the functions $\zeta(s_1;Q_1),\dots,\zeta(s_r;Q_r)$, by using generalizing shifts $\zeta(\sigma_1 + i\varphi_1(t);Q_1),\dots,\zeta(\sigma_r + i\varphi_r(t);Q_r)$, is proved. Here, the functions $\varphi_1(t),\dots,\varphi_r(t)$ are increasing to $+\infty$, with monotonic derivatives $\varphi_j'(t)$ satisfying the asymptotic growth conditions: $\varphi_j(t) \ll t\varphi_j'(t)$, and $\varphi_j'(t) = o(\varphi_{j+1}'(t))$ as $t \to \infty$. An explicit form of the limit measure is given. This theorem extends and generalizes the previous result on the joint value-distribution of Epstein zeta-functions.

Keywords: Epstein zeta-function; Haar measure; probability measures; weak convergence

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1. Introduction

Let $\mathbb{P}, 2\mathbb{N}, \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ denote the sets of all prime, positive even integer, positive integer, integer, rational, real and complex numbers, respectively, and $s = \sigma + it \in \mathbb{C}$. Moreover, let Q be a positive definite $n \times n$, $n \in \mathbb{N}$, matrix and $Q[\underline{x}] = \underline{x}^{\mathrm{T}} Q \underline{x}$, $\underline{x} \in \mathbb{Z}^n = \underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{n}$. The Epstein zeta-function $\zeta(s;Q)$ is defined, for $\sigma > \frac{n}{2}$, by the series

$$\zeta(s;Q) = \sum_{\underline{x} \in \mathbb{Z}^n \setminus \{\underline{0}\}} (Q[\underline{x}])^{-s},$$

and has the analytic continuation to the whole complex plane, except for the point $s = \frac{n}{2}$ which is a simple pole with residue $\pi^{n/2}(\Gamma(\frac{n}{2})\sqrt{\det Q})^{-1}$, where $\Gamma(s)$ is the Euler gamma-function. The function $\zeta(s;Q)$ was introduced by P. Epstein [1] with the aim of generalizing the Riemann zeta-function

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \sigma > 1,$$

and its functional equation

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s). \tag{1}$$

Clearly, for $n = 1$ and $Q = (1)$, we have $\zeta(s; Q) = 2\zeta(2s)$. Epstein's attempt was successful, and he obtained the functional equation for $\zeta(s; Q)$:

$$\pi^{-s}\Gamma(s)\zeta(s;Q) = \sqrt{\det Q}\pi^{s-\frac{n}{2}}\Gamma\left(\frac{n}{2} - s\right)\zeta\left(\frac{n}{2} - s; Q^{-1}\right),$$

which, as in (1), is valid for all $s \in \mathbb{C}$, and Q^{-1} denotes the inverse matrix of Q. This and (1) show that $\zeta(s)$ has the symmetric functional equation, while in the functional equation for $\zeta(s; Q)$, a new function $\zeta(s; Q^{-1})$ appears, but symmetry with respect to s is preserved. Although the functions $\zeta(s)$ and $\zeta(s; Q)$ have functional equations of the same Riemann type, their properties are quite different. For example, the function $\zeta(s) \neq 0$ in the half-plane of absolute convergence $\sigma > 1$, while there exist matrices Q such that $\zeta(s; Q)$ has infinitely many zeros in the half plane $\sigma > \frac{n}{2}$. Zero distribution of $\zeta(s; Q)$ is also a significant problem, comparable to that of $\zeta(s)$, and has been studied by numerous authors. We mention some results here. It is known that, for certain matrices, the Riemann hypothesis for $\zeta(s; Q)$ does not hold; there exist zeros of $\zeta(s; Q)$ off the critical line $\sigma = \frac{n}{4}$ [2]. Moreover, it was shown in [3] that, differently from the case of $\zeta(s)$, the zeros of $\zeta(s; Q)$ are generally not symmetric with respect to the line $\sigma = \frac{n}{4}$. Estimates for the number of zeros in the strips have been studied by E. Bobmbieri and J. Mueller [4], Y Lee [5], and others. Also, it is known [6] that imaginary parts of the zeros of Epstein zeta-functions are uniformly distributed modulo 1. Recently, an interesting formula for the sum of values of $\zeta(s; Q)$ over the nontrivial zeros of $\zeta(s)$ was proved in [7]. Thus, Epstein provided mathematicians with a novel object of algebraic and analytic nature, which has stimulated extensive research in number theory and related fields.

The function $\zeta(s; Q)$ is an automorphic form with respect to an unimodular group; it appears in the problems of algebraic number theory. It also has a range of practical applications, including crystallography [8], quantum field theory [9,10] and temperature and energy problems [11–14]. In general, the Epstein zeta-function is an attractive analytical object and is widely studied.

Unfortunately, we do not know any monograph devoted to classical results on the function $\zeta(s; Q)$. Some desired results can be found in the works on automorphic forms; see, for example, [15,16].

In [17], we began to characterize the asymptotic behaviour of the function $\zeta(s; Q)$ by using the Bohr–Jessen method [18,19], and techniques developed in [20]. Note that H. Bohr and B. Jessen considered only the existence of density on certain sets (rectangles) for the Riemann zeta-function, without giving an explicit form. Denote by $\mathcal{B}(\mathbb{X})$ the Borel σ-field of the space $\mathbb{X}$, and by meas$A$ the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then, the asymptotic behaviour of $\zeta(s; Q)$ can be described by the asymptotics of

$$\frac{1}{T}\text{meas}\{t \in [0, T] : \zeta(\sigma + it; Q) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}),$$

as $T \to \infty$. For this, it is convenient to use the weak convergence of probability measures.

Really, the function $\zeta(s; Q)$ is a class of Dirichlet series depending on the matrix Q. This class is rather general in obtaining results that are full of sense. In order for the function $\zeta(s; Q)$ to be close to number-theoretical objects, it is sufficient to limit ourself by matrices Q for which $Q[\underline{x}] \in \mathbb{Z}$ for all $\underline{x} \in \mathbb{Z}^n \setminus \{\underline{0}\}$. In this case, the function $\zeta(s; Q)$, for $\sigma > \frac{n}{2}$, can be expressed in the following form [21]:

$$\zeta(s; Q) = \zeta(s; E_Q) + \zeta(s; F_Q),$$

where $\zeta(s; E_Q)$ and $\zeta(s; F_Q)$ are corresponding zeta-functions of a certain Eisenstein series and modular forms of weight $\frac{n}{2}$, respectively. Moreover, it is convenient to additionally require that $n \in 2\mathbb{N}$ and $n \geqslant 4$. Then, $\zeta(s;Q)$ is a combination of products of Dirichlet L-functions and an absolutely convergent Dirichlet series [15,16]. More precisely, let $q \in \mathbb{N}$ be such that $q(2Q)^{-1}$ is an integral matrix, k and l positive divisors of q, χ_k and χ_l Dirichlet characters modulo q/k and q/l, respectively, and

$$L(s,\chi_k) = \sum_{m=1}^{\infty} \frac{\chi_k(m)}{m^s}, \quad L(s,\psi_l) = \sum_{m=1}^{\infty} \frac{\psi_l(m)}{m^s}, \quad \sigma > 1,$$

the corresponding Dirichlet L-functions. Then, for $\sigma > \frac{n-1}{2}$,

$$\zeta(s;Q) = \sum_{k=1}^{K} \sum_{l=1}^{L} \frac{a_{kl}}{k^s l^s} L(s,\chi_k) L\Big(s - \frac{n}{2} + 1, \psi_l\Big) + \sum_{m=1}^{\infty} \frac{f_Q(m)}{m^s}, \tag{2}$$

where $a_{kl} \in \mathbb{C}$ are certain numbers, the characters χ_k, $1 \le k \le K$, are pairwise nonequivalent, and χ_l, $1 \le l \le L$, are pairwise nonequivalent too, and the Dirichlet series is absolutely convergent in the half-plane $\sigma > \frac{n-1}{2}$. In view of equality (2), the investigations of the function $\zeta(s;Q)$, under the above hypotheses, reduce to those of Dirichlet L-functions.

For the definition of the limit measure in a limit theorem for $\zeta(s;Q)$, the set

$$\Omega = \prod_{p \in \mathbb{P}} \gamma_p,$$

where $\gamma_p = \{s \in \mathbb{C} : |s| = 1\}$, for all $p \in \mathbb{P}$, plays a crucial role. The set Ω consists of all functions from $\mathbb{P}$ into the unit circle. With the product topology and pointwise multiplication, the torus Ω is a compact topological group; therefore, on $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measure m_{1H} exists, and we arrive at the probability space $(\Omega, \mathcal{B}(\Omega), m_{1H})$. Let $\omega(p)$ be the projection of $\omega \in \Omega$ to the coordinate space γ_p, $p \in \mathbb{P}$. Extend the function $\omega(p)$, $p \in \mathbb{P}$, to the set $\mathbb{N}$ by using the formula

$$\omega(m) = \prod_{p^\alpha \| m} \omega^\alpha(p), \quad m \in \mathbb{N},$$

where $p^\alpha \parallel m$ means that $p^\alpha \mid m$, but $p^{\alpha+1} \nmid m$. For an arbitrary Dirichlet L-function $L(s,\chi)$, define

$$L(s,\omega,\chi) = \sum_{m=1}^{\infty} \frac{\chi(m)\omega(m)}{m^s}.$$

The latter series, for almost all $\omega \in \Omega$, converges in the half-pane $\sigma > \frac{1}{2}$, and is a complex-valued random element on $(\Omega, \mathcal{B}(\Omega), m_{1H})$. Moreover, for almost all $\omega \in \Omega$, the equality

$$L(s,\omega,\chi) = \prod_{p \in \mathbb{P}} \left(1 - \frac{\chi(p)\omega(p)}{p^s}\right)^{-1}, \quad \sigma > \frac{1}{2},$$

is valid. For $\sigma > \frac{n-1}{2}$, set

$$\begin{aligned} \zeta(\sigma,\omega;Q) \quad &= \sum_{k=1}^{K} \sum_{l=1}^{L} \frac{a_{kl}\omega(k)\omega(l)}{k^\sigma l^\sigma} L(\sigma,\omega,\chi_k) L\Big(\sigma - \frac{n}{2} + 1, \omega, \psi_l\Big) \\ &+ \sum_{m=1}^{\infty} \frac{f_Q(m)\omega(m)}{m^\sigma}. \end{aligned}$$

Then, $\zeta(\sigma,\omega;Q)$ is a complex-valued random element on $(\Omega, \mathcal{B}(\Omega), m_{1H})$, and let

$$P_{\sigma;Q}(A) = m_{1H}\{\omega \in \Omega : \zeta(\sigma,\omega;Q) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}),$$

be its distribution. The main result of [17] is the following theorem on the weak convergence for

$$P_{T,\sigma;Q}(A) = \frac{1}{T}\text{meas}\{t \in [0,T] : \zeta(\sigma+it;Q) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}).$$

Theorem 1. *Suppose that $\sigma > \frac{n-1}{2}$ is fixed. Then, $P_{T,\sigma;Q}$ converges weakly to the measure $P_{\sigma;Q}$ as $T \to \infty$.*

In [22], a joint version of Theorem 1 has been obtained. For $j = 1,\dots,r$, let Q_j be a positive definite quadratic $n_j \times n_j$ matrix, and $\zeta(s_j;Q_j)$ be the corresponding Epstein zeta-function. Denote $\underline{s} = (s_1,\dots,s_r)$, $\underline{Q} = (Q_1,\dots,Q_r)$ and $\underline{\zeta}(\underline{s};\underline{Q}) = (\zeta(s_1;Q_1),\dots,\zeta(s_r;Q_r))$. On the probability space $(\Omega, \mathcal{B}(\Omega), m_{1H})$, define the $\mathbb{C}^r$-valued ($\mathbb{C}^r = \underbrace{\mathbb{C}\times\cdots\times\mathbb{C}}_{r}$) random element

$$\underline{\zeta}(\underline{\sigma},\omega;\underline{Q}) = (\zeta(\sigma_1,\omega;Q_1),\dots,\zeta(\sigma_r,\omega;Q_r)),$$

where $\sigma_j > \frac{n_j-1}{2}$, and

$$\begin{aligned}\zeta(\sigma_j,\omega;Q_j) \quad = \quad & \sum_{k=1}^{K_j}\sum_{l=1}^{L_j}\frac{a_{klj}\omega(k)\omega(l)}{k^{\sigma_j}l^{\sigma_j}}L(\sigma_j,\omega,\chi_{kj})L\left(\sigma_j-\frac{n_j}{2}+1,\omega,\psi_{lj}\right)\\ & + \sum_{m=1}^{\infty}\frac{f_{Q_j}(m)\omega(m)}{m^{\sigma_j}},\end{aligned}$$

with corresponding $a_{klj} \in \mathbb{C}$, $K_j \in \mathbb{N}$, $L_j \in \mathbb{N}$, and Dirichlet characters χ_{kj} and ψ_{lj}, $j = 1,\dots,r$.

For $A \in \mathcal{B}(\mathbb{C}^r)$, define

$$\hat{P}_{\underline{\zeta}}(A) = \hat{P}_{\underline{\zeta},\underline{\sigma};\underline{Q}}(A) = m_{1H}\Big\{\omega \in \Omega : \underline{\zeta}(\underline{\sigma},\omega;\underline{Q}) \in A\Big\}$$

and

$$\hat{P}_T(A) = \hat{P}_{T,\underline{\sigma};\underline{Q}}(A) = \frac{1}{T}\text{meas}\Big\{t \in [0,T] : \underline{\zeta}(\underline{\sigma}+it;\underline{Q}) \in A\Big\}.$$

Then, in [22], the following limit theorem has been given.

Theorem 2. *Suppose that $\sigma_j > \frac{n_j-1}{2}$ is fixed, where $j = 1,\dots,r$. Then, $\hat{P}_T$ converges weakly to the measure $\hat{P}_{\underline{\zeta}}$ as $T \to \infty$.*

In [23], a generalization of Theorem 1 has been given, i. e., the weak convergence for

$$\frac{1}{T}\text{meas}\{t \in [T,2T] : \zeta(\sigma+i\varphi(t);Q) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}),$$

with a certain differentiable function $\varphi(t)$ has been obtained as $T \to \infty$. The aim of this paper is to prove a joint version of the above-mentioned theorem from [23]. We note that using generalized shifts $\zeta(\sigma+i\varphi(t);Q)$ allows for the more complete characterization of the asymptotic behaviour of the function $\zeta(s;Q)$.

Let $T_0 > 0$ be a fixed sufficiently large number. We say that a collection of real-valued functions $(\varphi_1(t),\dots,\varphi_r(t))$ defined for $t \geqslant T_0$ belongs to the class $U_r(T_0)$ if the following conditions are satisfied:

1° for every $j = 1,\dots,r$, $\varphi_j(t)$ is an increasing to $+\infty$ function;

2° for every $j = 1,\dots,r$, $\varphi_j(t)$ has a monotonic derivative $\varphi_j'(t)$ such that

$$\varphi_j(t) \ll t\varphi_j'(t);$$

3° for every $j = 2, \dots, r$ and $k \le j$, $\varphi_k'(t) = o\Big(\varphi_j'(t)\Big)$ as $t \to \infty$.

For example, we can take $\varphi_j(t) = t^{\alpha+j}$ with fixed $\alpha > 0$. We recall that $a \ll_\theta b$, $b > 0$, means that there exists a constant $c = c(\theta) > 0$ such that $|a| \le cb$.

For the statement of a joint limit theorem with shifts $\zeta(\sigma + i\varphi_j(t); Q)$, we need a new probability space. Let Ω be the same group as above. Define

$$\Omega^r = \Omega_1 \times \cdots \times \Omega_r,$$

where $\Omega_j = \Omega$ for all $j = 1, \dots, r$. Then, by the classical Tikhonov theorem, Ω^r is a compact topological group. Therefore, on $(\Omega^r, \mathcal{B}(\Omega^r))$, the probability Haar measure m_H can be defined. Note that the measure m_H is the product of the Haar measures m_{jH} on $(\Omega_j, \mathcal{B}(\Omega_j))$, $j = 1, \dots, r$. Thus, we have the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H)$. Denote by ω_j the elements of Ω_j and by $\omega = (\omega_1, \dots, \omega_r)$ the elements of Ω^r. Now, on the probability space $(\Omega^r, \mathcal{B}(\Omega^r), m_H)$, the $\mathbb{C}^r$-valued random element

$$\underline{\zeta}(\underline{\sigma}, \omega; \underline{Q}) = \big(\zeta(\sigma_1, \omega_1; Q_1), \dots, \zeta(\sigma_r, \omega_r; Q_r)\big),$$

is defined, where $\sigma_j > \frac{n_j - 1}{2}$, and

$$\begin{aligned} \zeta(\sigma_j, \omega_j; Q_j) &= \sum_{k=1}^{K_j} \sum_{l=1}^{L_j} \frac{a_{klj}\omega_j(k)\omega_j(l)}{k^{\sigma_j} l^{\sigma_j}} L(\sigma_j, \omega_j, \chi_{kj}) L\left(\sigma_j - \frac{n_j}{2} + 1, \omega_j, \psi_{lj}\right) \\ &+ \sum_{m=1}^{\infty} \frac{f_{Q_j}(m)\omega_j(m)}{m^{\sigma_j}}, \quad j = 1, \dots, r. \end{aligned}$$

Let $P_{\underline{\zeta}}$ be the distribution of the random element $\underline{\zeta}(\underline{\sigma}, \omega; \underline{Q})$, i.e.,

$$P_{\underline{\zeta}}(A) = P_{\underline{\zeta};\underline{\sigma},\underline{Q}}(A) = m_H\Big\{\omega \in \Omega^r : \underline{\zeta}(\underline{\sigma}, \omega; \underline{Q}) \in A\Big\}, \quad A \in \mathcal{B}(\mathbb{C}^r).$$

Define

$$P_T(A) = P_{T,\underline{\sigma};\underline{Q}}(A) = \frac{1}{T}\text{meas}\Big\{t \in [T, 2T] : \underline{\zeta}(\underline{\sigma} + i\underline{\varphi}(t); \underline{Q}) \in A\Big\}, \quad A \in \mathcal{B}(\mathbb{C}^r),$$

where $\underline{\varphi}(t) = (\varphi_1(t), \dots, \varphi_r(t))$, and

$$\underline{\zeta}(\underline{\sigma} + i\underline{\varphi}(t); \underline{Q}) = (\zeta(\sigma_1 + i\varphi_1(t); Q_1), \dots, \zeta(\sigma_r + i\varphi_r(t); Q_r))$$

with

$$\begin{aligned} \zeta(\sigma_j + i\varphi_j(t); Q_j) &= \sum_{k=1}^{K_j} \sum_{l=1}^{L_j} \frac{a_{klj}}{k^{\sigma_j + i\varphi_j(t)} l^{\sigma_j + i\varphi_j(t)}} L(\sigma_j + i\varphi_j(t), \chi_{kj}) \\ &\quad \cdot L\left(\sigma_j + i\varphi_j(t) - \frac{n_j}{2} + 1, \psi_{lj}\right) \\ &+ \sum_{m=1}^{\infty} \frac{f_{Q_j}(m)}{m^{\sigma_j + i\varphi_j(t)}}. \end{aligned}$$

Theorem 3. *Suppose that $(\varphi_1(t), \dots, \varphi_r(t)) \in U_r(T_0)$, and $\sigma_j > \frac{n_j - 1}{2}$ is fixed, where $j = 1, \dots, r$. Then, P_T converges weakly to the measure $P_{\underline{\zeta}}$ as $T \to \infty$.*

Thus, Theorem 3 provides a joint extension of Theorem 2. We emphasize the importance of condition 3° in the definition of the class $\mathcal{U}_r(T_0)$. It is important to mention that probabilistic limit theorems accurately reflect the chaotic behaviour of the functions

$\zeta(s_1; Q_1), \ldots, \zeta(s_r; Q_r)$, and can be applied to further investigations related to approximation problems.

The proof of Theorem 3 is divided into parts. First, the weak convergence on Ω^r is established. Next, some absolutely convergent Dirichlet series are considered, and finally, the assertion of Theorem 3 is proved.

2. Case of Ω^r

For $A \in \mathcal{B}(\Omega^r)$, define

$$\begin{aligned} R_T(A) &= R_{T,\Omega^r,\underline{\varphi}}(A) \\ &= \frac{1}{T}\mathrm{meas}\Big\{t \in [T, 2T] : \Big(\Big(p^{-i\varphi_1(t)} : p \in \mathbb{P}\Big), \ldots, \Big(p^{-i\varphi_r(t)} : p \in \mathbb{P}\Big)\Big) \in A\Big\}. \end{aligned}$$

Lemma 1. *Suppose that $(\varphi_1(t), \ldots, \varphi_r(t)) \in U_r(T_0)$. Then, R_T converges weakly to the Haar measure m_H as $T \to \infty$.*

Proof. We have to prove that the Fourier transform $f_T(\underline{k}_1, \ldots, \underline{k}_r) = f_{T,\Omega^r,\underline{\varphi}}(\underline{k}_1, \ldots, \underline{k}_r)$ of R_T (where $\underline{k}_j = \big(k_{jp} : k_{jp} \in \mathbb{Z},\ p \in \mathbb{P}\big)$, $j = 1, ..., r$) converges to the Fourier transform

$$f(\underline{k}_1, \ldots, \underline{k}_r) = \begin{cases} 1 & \text{if} \quad (\underline{k}_1, \ldots, \underline{k}_r) = (\underline{0}, \ldots, \underline{0}), \\ 0 & \text{if} \quad (\underline{k}_1, \ldots, \underline{k}_r) \neq (\underline{0}, \ldots, \underline{0}), \end{cases}$$

of the Haar measure m_H as $T \to \infty$. Here, $\underline{0}$ denotes the collection of zeros.

By the definition of R_T, we have

$$\begin{aligned} f_T(\underline{k}_1, \ldots, \underline{k}_r) &= \int\limits_{\Omega^r} \left(\prod_{j=1}^{r} \prod_{p \in \mathbb{P}}{}^{*} \omega_j^{k_{jp}}(p)\right) \mathrm{d}R_T = \frac{1}{T}\int\limits_{T}^{2T} \left(\prod_{j=1}^{r} \prod_{p \in \mathbb{P}}{}^{*} p^{-ik_{jp}\varphi_j(t)}\right) \mathrm{d}t \\ &= \frac{1}{T}\int\limits_{T}^{2T} \exp\left\{-i\sum_{j=1}^{r} \varphi_j(t) \sum_{p \in \mathbb{P}}{}^{*} k_{jp} \log p\right\} \mathrm{d}t, \end{aligned} \tag{3}$$

where the star "*" shows that only a finite number of integers k_{jp} are distinct from zero. Obviously,

$$f_T(\underline{0}, \ldots, \underline{0}) = 1. \tag{4}$$

Thus, it remains to consider the case $(\underline{k}_1, \ldots, \underline{k}_r) \neq (\underline{0}, \ldots, \underline{0})$. Set

$$A(t) = \sum_{j=1}^{r} \varphi_j(t) \sum_{p \in \mathbb{P}}{}^{*} k_{jp} \log p = \sum_{j=1}^{r} \kappa_j \varphi_j(t),$$

where

$$\kappa_j = \sum_{p \in \mathbb{P}}{}^{*} k_{jp} \log p.$$

It is well known that the set of logarithms of prime numbers is linearly independent over $\mathbb{Q}$. Therefore, there exist $j \in \{1, \ldots, r\}$ such that $\kappa_j \neq 0$. Let $j_0 = \max(j : \kappa_j \neq 0)$. Then, by the definition of the class $U_r(T_0)$, we have, for $j \leqslant j_0$,

$$A'(t) = \sum_{j \leqslant j_0} \kappa_j \varphi_j'(t) = \kappa_{j_0} \varphi_{j_0}'(t)(1 + o(1)), \quad t \to \infty.$$

Hence, by the identity

$$\frac{1}{1+a} = 1 - \frac{a}{1+a}, \quad a \neq -1,$$

we find

$$\frac{1}{A'(t)} = \frac{1}{\kappa_{j_0}\varphi'_{j_0}(t)}(1+o(1)), \quad t \to \infty.$$

Therefore,

$$\begin{aligned}
\int_T^{2T} \cos(A(t))\mathrm{d}t &= \int_T^{2T} \frac{1}{A'(t)} \cos(A(t))\mathrm{d}(A(t)) \\
&= \frac{1}{\kappa_{j_0}} \int_T^{2T} \frac{1}{\varphi'_{j_0}(t)} \cos(A(t))\mathrm{d}(A(t)) + \frac{1}{\kappa_{j_0}} \int_T^{2T} \frac{o(1)}{\varphi'_{j_0}(t)} \cos(A(t))\mathrm{d}(A(t)) \\
&= \frac{1}{\kappa_{j_0}} \int_T^{2T} \frac{1}{\varphi'_{j_0}(t)} \mathrm{d}(\sin(A(t))) + \int_T^{2T} \frac{o(1)(1+o(1))}{A'(t)} \mathrm{d}(\sin(A(t))) \\
&= \frac{1}{\kappa_{j_0}} \int_T^{2T} \frac{1}{\varphi'_{j_0}(t)} \mathrm{d}(\sin(A(t))) + \int_T^{2T} o(1) \cos(A(t))\mathrm{d}t. \qquad (5)
\end{aligned}$$

The function $\varphi'_{j_0}(t)$, by 2° of the class $U_r(T_0)$, is monotonic and non-negative. Therefore, by the second mean value theorem,

$$\int_T^{2T} \frac{1}{\varphi'_{j_0}(t)} \mathrm{d}(\sin(A(t))) = \begin{cases} \frac{1}{\varphi'_{j_0}(T)} \int_T^{\xi} \mathrm{d}(\sin(A(t))) & \text{if } \varphi'_{j_0}(t) \text{ is increasing,} \\ \frac{1}{\varphi'_{j_0}(2T)} \int_\xi^{2T} \mathrm{d}(\sin(A(t))) & \text{if } \varphi'_{j_0}(t) \text{ is decreasing,} \end{cases}$$

$T \leqslant \xi \leqslant 2T$. Since $\varphi'_{j_0}(T) \geqslant \frac{\varphi_{j_0}(T)}{T}$ and $\varphi_{j_0}(T) \to \infty$ as $T \to \infty$, we have that

$$\int_T^{2T} \frac{1}{\varphi'_{j_0}(t)} \mathrm{d}(\sin(A(t))) = o(T), \quad T \to \infty.$$

This and (5) show that

$$\int_T^{2T} \cos(A(t))\mathrm{d}t = o(T), \quad T \to \infty.$$

Similarly, it follows that

$$\int_T^{2T} \sin(A(t))\mathrm{d}t = o(T), \quad T \to \infty.$$

Thus, in view of (3), in the case $(\underline{k}_1, \dots, \underline{k}_r) = (\underline{0}, \dots, \underline{0})$,

$$\lim_{T\to\infty} f_T(\underline{k}_1, \dots, \underline{k}_r) = 0,$$

and this together with (4) proves the lemma. □

3. Case of Absolute Convergence

Lemma 1 and the properties of weak convergence make it possible to obtain a limit lemma for $\underline{\zeta}_N(\underline{\sigma}; \underline{Q})$ involving certain absolutely convergent Dirichlet series. Let $\theta > 0$ be a fixed number, and

$$v_N(m) = \exp\left\{-\left(\frac{m}{N}\right)^\theta\right\}, \quad m, N \in \mathbb{N}.$$

Define

$$L_N\left(\sigma_j-\frac{n_j}{2}+1,\psi_{lj}\right)=\sum_{m=1}^{\infty}\frac{\psi_{lj}(m)v_N(m)}{m^{\sigma_j-\frac{n_j}{2}+1}},\quad l=1,\dots,L_j,\quad j=1,\dots,r.$$

Since $v_N(m)$ with respect to m decreases exponentially, the latter series are absolutely convergent for all finite σ_j. Moreover, as $n_j\geqslant 4$, we have that $\sigma_j>\frac{n_j-1}{2}\geqslant\frac{3}{2}>1$. Hence, the series for $L(\sigma_j,\chi_{kj})$, $k=1,\dots,K_j$, $j=1,\dots,r$, are absolutely convergent. Therefore,

$$\zeta_N(\sigma_j;Q_j)=\sum_{k=1}^{K_j}\sum_{l=1}^{L_j}\frac{a_{klj}}{k^{\sigma_j}l^{\sigma_j}}L(\sigma_j,\chi_{kj})L_N\left(\sigma_j-\frac{n_j}{2}+1,\psi_{lj}\right)+\sum_{m=1}^{\infty}\frac{f_{Q_j}(m)}{m^{\sigma_j}},\tag{6}$$

where $j=1,\dots,r$, is a combination of absolutely convergent Dirichlet series. Let

$$\underline{\zeta}_N(\underline{\sigma};\underline{Q})=\big(\zeta_N(\sigma_1;Q_1),\dots,\zeta_N(\sigma_r;Q_r)\big),$$

and

$$\underline{\zeta}_N(\underline{\sigma},\omega;\underline{Q})=\big(\zeta_N(\sigma_1,\omega_1;Q_1),\dots,\zeta_N(\sigma_r,\omega_r;Q_r)\big),$$

where $\zeta_N(\sigma_j,\omega_j;Q_j)$ is obtained from $\zeta(\sigma_j,\omega_j;Q_j)$ by putting $L_N\left(\sigma_j-\frac{n_j}{2}+1,\psi_{lj}\right)$ in the place of $L\left(\sigma_j-\frac{n_j}{2}+1,\psi_{lj}\right)$, where $j=1,\dots,r$. Then, $\zeta_N(\sigma_j,\omega_j;Q_j)$, where $j=1,\dots,r$, is also a combination of absolutely convergent Dirichlet series. Let the function $u_{N,\underline{\sigma};\underline{Q}}:\Omega^r\to\mathbb{C}^r$ be given by

$$u_N(\omega)=u_{N,\underline{\sigma};\underline{Q}}(\omega)=\underline{\zeta}_N(\underline{\sigma},\omega;\underline{Q}),\quad \sigma_j>\frac{n_j-1}{2},\quad j=1,\dots,r.$$

In virtue of the absolute convergence of the series in $\underline{\zeta}_N(\underline{\sigma},\omega;\underline{Q})$, the function $u_N(\omega)$ is continuous, and hence $(\mathcal{B}(\Omega^r),\mathcal{B}(\mathbb{C}^r))$-measurable. Then, the measure $W_N=W_{N,\underline{\sigma};\underline{Q}}=m_Hu_N^{-1}$, where

$$m_Hu_N^{-1}(A)=m_H\left(u_N^{-1}A\right),\quad A\in\mathcal{B}(\mathbb{C}^r),$$

can be defined. For $A\in\mathcal{B}(\mathbb{C}^r)$, define

$$P_{T,N}(A)=P_{T,N,\underline{\sigma};\underline{Q}}(A)=\frac{1}{T}\mathrm{meas}\left\{t\in[T,2T]:\underline{\zeta}_N(\underline{\sigma}+i\underline{\varphi}(t);\underline{Q})\in A\right\}.$$

Lemma 2. *Suppose that* $(\varphi_1(t),\dots,\varphi_r(t))\in U_r(T_0)$*. Then,* $P_{T,N}$ *converges weakly to* W_N *as* $T\to\infty$*.*

Proof. From the definitions of R_T, $P_{T,N}$ and u_N, we have

$$\begin{aligned}P_{T,N}(A)&=\frac{1}{T}\mathrm{meas}\left\{t\in[T,2T]:u_N\left(\left(p^{-i\varphi_1(t)}:p\in\mathbb{P}\right),\dots,\left(p^{-i\varphi_r(t)}:p\in\mathbb{P}\right)\right)\in A\right\}\\&=\frac{1}{T}\mathrm{meas}\left\{t\in[T,2T]:\left(\left(p^{-i\varphi_1(t)}:p\in\mathbb{P}\right),\dots,\left(p^{-i\varphi_r(t)}:p\in\mathbb{P}\right)\right)\in u_N^{-1}A\right\}\\&=R_T\left(u_N^{-1}A\right)\end{aligned}$$

for all $A\in\mathcal{B}(\mathbb{C}^r)$. Therefore, $P_{T,N}=R_Tu_N^{-1}$. Now, Lemma 1 and the preservation of weak convergence under continuous mappings, see, for example, Theorem 5.1 of [24], show that $P_{T,N}$ converges weakly to the probability measure $m_Hu_N^{-1}$ as $T\to\infty$. $\square$

The measure W_N is an important ingredient of the proof of Theorem 3. We see that W_N is independent on the functions $\varphi_1(t),\dots,\varphi_r(t)$. Therefore, we can use some statements from [22] to prove the following lemma.

Lemma 3. *The probability measure W_N converges weakly to $P_{\underline{\zeta}}$ as $N \to \infty$.*

Proof. By Lemma 8 from [22], the sequence of probability measures $\{W_N : N \in \mathbb{N}\}$ is tight, i.e., for every $\epsilon > 0$, there exists a compact set $K = K(\epsilon) \in \mathbb{C}^r$ such that

$$W_N(K) > 1 - \epsilon$$

for all $N \in \mathbb{N}$. Hence, by the Prokhorov theorem, see, for example, Theorem 6.1 of [24], the above sequence is relatively compact. This means that every subsequence of $\{W_N\}$ has a subsequence weakly convergent to a certain probability measure on $(\mathbb{C}^r, \mathcal{B}(\mathbb{C}^r))$. Thus, there exists a sequence $\{W_{N_l}\} \subset \{W_N\}$ such that W_{N_l} converges weakly to the measure $P_{\underline{\sigma};\underline{Q}}$ as $l \to \infty$. In the proof of Theorem 2 from [22], it is obtained that $P_{\underline{\sigma};\underline{Q}}$ coincides with $P_{\underline{\zeta}}$. Since the sequence $\{W_N\}$ is relatively compact, from this we have that W_N converges weakly to $P_{\underline{\zeta}}$ as $N \to \infty$. □

4. Estimate in the Mean

To derive Theorem 3 from Lemma 2, we have to show the nearest $\underline{\zeta}_N(\underline{\sigma} + i\underline{\varphi}(t); \underline{Q})$ to $\underline{\zeta}(\underline{\sigma} + i\underline{\varphi}(t); \underline{Q})$. Let, for $\underline{s}_j = (s_{j1}, \dots, s_{jr})$, $j = 1, 2$,

$$d_r(\underline{s}_1, \underline{s}_2) = \left(\sum_{j=1}^{r} |s_{j1} - s_{j2}| \right)^{1/2}.$$

Lemma 4. *Suppose that $(\varphi_1(t), \dots, \varphi_r(t)) \in U_r(T_0)$ and $\sigma_j > \frac{n_j - 1}{2}$, $j = 1, \dots, r$. Then,*

$$\lim_{N\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_T^{2T} d_r\Big(\underline{\zeta}(\underline{\sigma} + i\underline{\varphi}(t); \underline{Q}), \underline{\zeta}_N(\underline{\sigma} + i\underline{\varphi}(t); \underline{Q})\Big) \mathrm{d}t = 0.$$

Proof. Clearly,

$$\begin{aligned} &\int_T^{2T} d_r\Big(\underline{\zeta}(\underline{\sigma} + i\underline{\varphi}(t); \underline{Q}), \underline{\zeta}_N(\underline{\sigma} + i\underline{\varphi}(t); \underline{Q})\Big) \mathrm{d}t \\ \ll \quad &\sum_{j=1}^{r} \int_T^{2T} \big|\zeta(\sigma_j + i\varphi_j(t); Q_j) - \zeta_N(\sigma_j + i\varphi_j(t); Q_j)\big| \mathrm{d}t. \end{aligned}$$

Therefore, it suffices to show that

$$\lim_{N\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_T^{2T} |\zeta(\sigma + i\varphi(t); Q) - \zeta_N(\sigma + i\varphi(t); Q)| \mathrm{d}t = 0$$

for Q and σ satisfying the hypotheses of Theorem 1, and $\varphi(t)$ satisfying 1° and 2° of the class $U_r(T_0)$. However, the latter equality was proved in [23] and Lemma 2. We only mention that, in view of (6), it is suficient to prove that

$$\lim_{N\to\infty} \limsup_{T\to\infty} \frac{1}{T} \int_T^{2T} |L(\sigma + i\varphi(t), \chi) - L_N(\sigma + i\varphi(t), \chi)| \mathrm{d}t = 0 \tag{7}$$

for $\sigma > \frac{1}{2}$. For this, the representation

$$L_N(s,\chi) = \frac{1}{2\pi i} \int_{\theta-i\infty}^{\theta+i\infty} L(s+z,\chi) l_N(z) \frac{\mathrm{d}z}{z}$$

with $l_N(z) = \frac{z}{\theta}\Gamma\left(\frac{z}{\theta}\right)N^z$, is applied. Thus, the latter representation, the mean square estimate

$$\int_T^{2T} |L(\sigma + i\varphi(t) + i\tau, \chi)|^2 \mathrm{d}t \ll_{\sigma,\chi,\varphi} T(1 + |\tau|)$$

for all real τ, and the classical estimate

$$\Gamma(\sigma + it) \ll \exp\{-c|t|\}, \quad c > 0,$$

lead to equality (7). □

5. Proof of Theorem 3

Theorem 3 follows from Lemmas 2–4 and the following statement on convergence in distribution ($\xrightarrow{\mathcal{D}}$); see, for example, Theorem 4.2 from [24].

Lemma 5. *Suppose that the space $(\mathbb{X}, \rho)$ is separable, and the $\mathbb{X}$-valued random elements X_{nk} and Y_n, $n, k \in \mathbb{N}$, are defined on the same probability space with measure P. Let*

$$X_{nk} \xrightarrow[n\to\infty]{\mathcal{D}} X_k \quad \textit{and} \quad X_k \xrightarrow[k\to\infty]{\mathcal{D}} X,$$

and, for every $\epsilon > 0$,

$$\lim_{k\to\infty} \limsup_{n\to\infty} P\{\rho(X_{nk}, Y_n) \geqslant \epsilon\} = 0.$$

Then, $Y_n \xrightarrow[n\to\infty]{\mathcal{D}} X$.

Proof of Theorem 3. Let ξ_T be a random variable defined on a certain probability space with the measure P and uniformly distributed in the interval $[T, 2T]$. Define the $\mathbb{C}^r$-valued random element

$$X_{T,N} = X_{T,N,\underline{\sigma};\underline{Q}} = \underline{\zeta}_N(\underline{\sigma} + i\underline{\varphi}(\xi_T); \underline{Q})$$

and

$$X_T = X_{T,\underline{\sigma};\underline{Q}} = \underline{\zeta}(\underline{\sigma} + i\underline{\varphi}(\xi_T); \underline{Q}),$$

and denote by $Y_N = Y_{N,\underline{\sigma};\underline{Q}}$ the $\mathbb{C}^r$-valued random element having the distribution W_N. Then, in view of Lemma 2, we have

$$X_{T,N} \xrightarrow[T\to\infty]{\mathcal{D}} X_N, \tag{8}$$

while Lemma 3 implies that

$$Y_N \xrightarrow[N\to\infty]{\mathcal{D}} P_{\underline{\zeta}}. \tag{9}$$

Moreover, by Lemma 4, for every $\epsilon > 0$,

$$\begin{aligned} & \lim_{N\to\infty} \limsup_{T\to\infty} P\{d_r(X_{T,N}, X_T) \geqslant \epsilon\} \\ = & \lim_{N\to\infty} \limsup_{T\to\infty} \frac{1}{T} \mathrm{meas}\Big\{t \in [T, 2T] : d_r\Big(\underline{\zeta}(\underline{\sigma} + i\underline{\varphi}(t); \underline{Q}), \underline{\zeta}_N(\underline{\sigma} + i\underline{\varphi}(t); \underline{Q})\Big) \geqslant \epsilon\Big\} \\ \leqslant & \lim_{N\to\infty} \limsup_{T\to\infty} \frac{1}{\epsilon T} \int_T^{2T} d_r\Big(\underline{\zeta}(\underline{\sigma} + i\underline{\varphi}(t); \underline{Q}), \underline{\zeta}_N(\underline{\sigma} + i\underline{\varphi}(t); \underline{Q})\Big) \mathrm{d}t = 0. \end{aligned}$$

This, together with (8) and (9), shows that the random elements $X_{T,N}$, X_T and Y_N satisfy the hypotheses of Lemma 5. Therefore,

$$X_T \xrightarrow[T\to\infty]{\mathcal{D}} P_{\underline{\zeta}},$$

and this is equivalent to the assertion of the theorem. □

6. Conclusions

For $j = 1, \dots, r$, let Q_j be a positive definite $n_j \times n_j$ matrix, such that $\underline{x}^{\mathrm{T}} Q_j \underline{x} \in \mathbb{Z}$ for all $\underline{x} \in \mathbb{Z}^r \setminus \{\underline{0}\}$, $n_j \in 2\mathbb{N}$ and $n_j \geqslant 4$. In this paper, it is obtained that, for a collection of Epstein zeta-functions $\underline{\zeta}(\underline{s}; \underline{Q}) = \big(\zeta(s_1; Q_1), \dots, \zeta(s_r; Q_r)\big)$, a limit theorem on weakly convergent probability measures with generalized shifts $\underline{\zeta}(\underline{\sigma} + i\underline{\varphi}(t); \underline{Q})$ is valid, where $\varphi_j(t)$ are certain differentiable functions. The proven theorem generalizes the main result of [22] obtained for $\varphi_j(t) = t$. Note that the main theorem remains valid even if the functions $\varphi_j(t)$ coincide for some j. For example, one may consider $\varphi_j(t) = t \log^{\alpha_j} t$ with different $\alpha_j > 0$. Also, polynomials $\varphi_j(t) = a_j t^{\alpha_j} + \dots + a_0$, with $a_j > 0$ and different $\alpha_j > 0$, can be used.

As shown in the proof of Lemma 1, using generalized shifts $\zeta(\sigma_j + i\varphi_j(t); Q_j)$ makes it possible to obtain a desired rate of convergence for R_T to m_H. We conjecture that this phenomenon is also preserved for the measures P_T and $P_{\underline{\zeta}}$.

The next paper will be devoted to a joint generalized discrete version, i.e., for weak convergence of

$$\frac{1}{N+1} \#\Big\{N \leqslant k \leqslant 2N : \underline{\zeta}(\underline{\sigma} + i\underline{\varphi}(k); \underline{Q}) \in A\Big\}, \quad A \in \mathcal{B}(\mathbb{C}^r),$$

as $N \to \infty$. Here, $\#A$ denotes the number of elements of the set A.

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