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On Equivalents of the Riemann Hypothesis Connected to the Approximation Properties of the Zeta Function

Antanas Laurinčikas

Institute of Mathematics, Faculty of Mathematics and Informatics, Vilnius University, Naugarduko Str. 24, LT-03225 Vilnius, Lithuania; antanas.laurincikas@mif.vu.lt

Abstract: The famous Riemann hypothesis (RH) asserts that all non-trivial zeros of the Riemann zeta function $\zeta(s)$ (zeros different from $s = -2m$, $m \in \mathbb{N}$) lie on the critical line $\sigma = 1/2$. In this paper, combining the universality property of $\zeta(s)$ with probabilistic limit theorems, we prove that the RH is equivalent to the positivity of the density of the set of shifts $\zeta(s + it_\tau)$ approximating the function $\zeta(s)$. Here, t_τ denotes the Gram function, which is a continuous extension of the Gram points.

Keywords: Gram function; Riemann hypothesis; Riemann zeta function; limit theorem; universality; weak convergence of probability measures

MSC: 11M06

1. Introduction

Let $s = \sigma + it$, $\sigma \in \mathbb{R}$, $t \in \mathbb{R}$, be a complex variable. The main object of analytic number theory—the Riemann zeta function $\zeta(s)$—is given by the Dirichlet series

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}, \quad \sigma > 1,$$

or, equivalently, by the Euler product

$$\zeta(s) = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \sigma > 1, \tag{1}$$

where $\mathbb{P}$ denotes the set of all prime numbers. L. Euler was the first who began to study the function $\zeta(s)$; however, he did so with $s \in \mathbb{R}$. In this way, he obtained identity (1).

B. Riemann, differently from Euler, began to consider $\zeta(s)$ as a function of a complex variable [1]; he extended analytically the function $\zeta(s)$ to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1, proving the functional equation

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s), \tag{2}$$

where $\Gamma(s)$ denotes the Euler gamma function. He proposed a method to apply $\zeta(s)$ for the investigation of the distribution of prime numbers in the set $\mathbb{Z}^+$. This method is connected to the zero-distribution of $\zeta(s)$ in the strip $1/2 \leqslant \sigma \leqslant 1$. Set

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$$\pi(x) = \sum_{\substack{p \leqslant x \\ p \in \mathbb{P}}} 1.$$

Riemann's ideas have been realized in works by J. Hadamard [2] and C.J. de la Vallée Poussin [3–5]. They have proven independently the asymptotic distribution law of prime numbers:

$$\pi(x) = \int_2^x \frac{\mathrm{d}u}{\log u} + O\left(x\mathrm{e}^{-c\sqrt{\log x}}\right), \quad x \to \infty, \ c > 0.$$

The proof is based on the non-vanishing of the function $\zeta(s)$ in the region

$$\sigma > 1 - \frac{c_1}{\log(|t| + 2)}, \quad c_1 > 0.$$

Riemann also stated some conjectures. The most important of these, now called the Riemann hypothesis (RH), concerns the zeros of the function $\zeta(s)$. From the functional Equation (2) and the properties of the function $\Gamma(s)$, we have $\zeta(-2m) = 0$ for all $m \in \mathbb{N}$. The zeros $s = -2m$, $m \in \mathbb{N}$, are well known and called trivial. Moreover, the function $\zeta(s)$ has infinitely many of the so-called nontrivial complex zeros lying in the strip $\{s \in \mathbb{C} : 0 < \sigma < 1\}$. The RH states that all nontrivial zeros are on the line $\sigma = 1/2$ or, equivalently, $\zeta(s) \neq 0$ for $\sigma > 1/2$.

The first result regarding the number of zeros of $\zeta(s)$ on the line $\sigma = 1/2$ belongs to A. Selberg. Let

$$N(T) = \#\{\rho = \sigma + i\gamma : \zeta(\rho) = 0, 0 < \gamma < T\}$$

and

$$N_0(T) = \#\left\{\rho = \frac{1}{2} + i\gamma : \zeta(\rho) = 0, 0 < \gamma < T\right\}.$$

In [6], Selberg obtained that

$$\liminf_{T \to \infty} \frac{N_0(T)}{N(T)} > 0.$$

A famous result in this direction was given by N. Levinson in [7], namely

$$\liminf_{T \to \infty} \frac{N_0(T)}{N(T)} \geqslant 0.3474 > \frac{1}{3}.$$

The latter bound was improved in [8] to obtain

$$\liminf_{T \to \infty} \frac{N_0(T)}{N(T)} \geqslant 0.4105 > \frac{2}{5}.$$

The best known result,

$$\liminf_{T \to \infty} \frac{N_0(T)}{N(T)} > \frac{5}{12},$$

was given in [9].

All large-scale computer calculations support the RH. For example, it was obtained in [10] that the first 10^{13} nontrivial zeros of $\zeta(s)$ are on the line $\sigma = 1/2$.

The Riemann hypothesis is among the most important seven Millennium Prize problems of mathematics [11]; for its proof or disproof, a large sum is promised.

Many equivalents of the RH in various terms are known. We mention some of them below.

1. The RH and the estimate

$$\pi(x) = \int_2^x \frac{\mathrm{d}u}{\log u} + O(\sqrt{x}\log x), \quad x \to \infty,$$

are equivalent [12].

2. Let $\Lambda(m)$, $m \in \mathbb{N}$, be the von Mangoldt function, i.e.,

$$\Lambda(m) = \begin{cases} \log p & \text{if } m = p^k,\, p \in \mathbb{P},\, k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

The RH is equivalent to the estimate [13]

$$\sum_{m \leqslant x} \Lambda(m) = x + O\left(x^{1/2} \log^2 x\right), \quad x \to \infty.$$

The equality

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{m=1}^{\infty} \frac{\Lambda(m)}{m^s}, \quad \sigma > 1,$$

is valid. This indicates the importance of the functions $\Lambda(m)$ and $\zeta(s)$ in the theory of distribution of prime numbers.

3. Let $\mu(m)$ denote the Möbius function, i.e.,

$$\mu(m) = \begin{cases} (-1)^r & \text{if } m = p_1 \cdots p_r,\, p_1, \dots, p_r \in \mathbb{P}, \\ 0 & \text{if } p^2 \mid m,\, p \in \mathbb{P}. \end{cases}$$

The RH is equivalent to the estimate [14]

$$\sum_{m \leqslant x} \mu(m) = O\left(x^{1/2+\varepsilon}\right), \quad x \to \infty,$$

for every $\varepsilon > 0$.

The function $\mu(m)$ is connected to $\zeta(s)$ by the equality

$$\frac{1}{\zeta(s)} = \sum_{m=1}^{\infty} \frac{\mu(m)}{m^s}, \quad \sigma > 1.$$

The convergence of the latter series for $\sigma > 1/2$ also is one of the criteria for the RH [15].

4. The Bombieri–Weil positivity criterion. Let

$$G(s) = \int_0^{\infty} g(x) x^{s-1}\, \mathrm{d}x.$$

The RH is equivalent to

$$\sum_{\rho} G(\rho)\overline{G}(1-\rho) > 0$$

for every $g \in C_0^{\infty}(0, \infty)$, $g(x) \neq 0$, where summing runs over zeros $\rho = 1/2 + i\gamma$ of $\zeta(s)$ [16].

5. Define

$$\xi(s) = \frac{1}{2} s(s-1)\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

The RH is equivalent to the inequality [17]

$$\operatorname{Re}\frac{\xi'(s)}{\xi(s)} > 0, \quad \sigma > \frac{1}{2}.$$

6. The RH is equivalent to the estimate

$$\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{(m-1)!\zeta(2k)}x^m = O\Big(x^{1/2+\varepsilon}\Big)$$

with every $\varepsilon > 0$ [18].

More equivalents of the RH can be found in [19,20].

In this paper, we are interested in equivalents of the RH connected to the universality of $\zeta(s)$. Set $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$. Denote by $\mathcal{K}$ the class of compact subsets of the strip D having connected complements, and by $H_0(K)$ with $K \in \mathcal{K}$ a class of non-vanishing continuous functions on K that are analytic inside K. Moreover, let meas$A$ stand for the Lebesgue measure of a measurable set A on the real line. Then, the universality of $\zeta(s)$ is described by the following statement.

Proposition 1 ([21], Corollary 5.3.6; see also [22–25])**.** *Suppose that $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty}\frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K}|f(s) - \zeta(s+i\tau)| < \varepsilon\right\} > 0.$$

We note that the universality property of $\zeta(s)$ in the approximation of non-vanishing analytic functions on discs of the strip D was discovered by S.M. Voronin [26]. Let $0 < d < 1/4$ be a fixed number. Then, Voronin's theorem states [26] that, for every non-vanishing continuous function $f(s)$ on the disc $|s| \leqslant d$ and analytic inside this disc, and $\varepsilon > 0$, there is a real number $\tau = \tau(\varepsilon, f)$ satisfying the inequality

$$\max_{|s|\leqslant d}\left|\zeta\left(s + \frac{3}{4} + i\tau\right) - f(s)\right| < \varepsilon.$$

In Proposition 1, the universality of $\zeta(s)$ is stated in terms of a lower density for the set of shifts $\zeta(s+i\tau)$ approximating a given analytic function. Namely, Proposition 1 asserts that the latter set has a positive lower density. Moreover, a version of universality for $\zeta(s)$ in terms of a positive density is known [27,28].

Proposition 2. *Suppose that $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, the limit*

$$\lim_{T\to\infty}\frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K}|f(s) - \zeta(s+i\tau)| < \varepsilon\right\}$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

From a mathematical point of view, a set is more precisely characterized by its density. Thus, Proposition 2 has a certain advantage against Proposition 1. On the other hand, the exceptional set of values of $\varepsilon > 0$ is not given effectively. Therefore, Proposition 2 has only certain theoretical value. Moreover, we conjecture that Proposition 2 remains valid for all $\varepsilon > 0$.

B. Bagchi obtained [29] the equivalence of the RH in terms of the lower density of the set of shifts $\zeta(s+i\tau)$ approximating $\zeta(s)$. More precisely, the Bagchi theorem is of the following form.

Theorem 1 ([29])**.** *The RH is true if and only if, for every $K \in \mathcal{K}$ and $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\mathrm{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s) - \zeta(s+i\tau)| < \varepsilon\right\} > 0.$$

In [30], the equivalence of the RH was described by self-approximation in the spirit of Proposition 2.

Theorem 2 ([30])**.** *The RH is true if and only if, for every $K \in \mathcal{K}$, the limit*

$$\lim_{T\to\infty} \frac{1}{T}\mathrm{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau) - \zeta(s)| < \varepsilon\right\}$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

There are several works on the positivity of a lower density

$$\liminf_{T\to\infty} \frac{1}{T}\mathrm{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau d) - \zeta(s+i\tau)| < \varepsilon\right\}$$

for all sets $K \in \mathcal{K}$, $\varepsilon > 0$ with a real d; see [31–33]. By Theorem 1, this, with $d = 0$, implies the RH.

It is well known that Proposition 1 remains valid for more general shifts $\zeta(s + i\varphi(\tau))$ with a certain class of real functions $\varphi(\tau)$. In [34], the Gram function t_τ has been used, which is defined as follows. Denote by $g(s)$ the ingredient $\pi^{-s/2}\Gamma(s/2)$ of the functional Equation (2), and by $\theta(t)$ the increment in the argument of the function $g(s)$ along the segments connecting the points $s = 1/2$ and $s = 1/2 + it$. It is known [35] that the function $\theta(t)$ is increasing for $t \geqslant t^* = 6.289835\ldots$; therefore, the equation

$$\theta(t) = \pi(\tau - 1), \quad \tau \geqslant 0,$$

has the unique solution t_τ, which is called the Gram function. The following results regarding the asymptotics of the function t_τ as $\tau \to \infty$

$$t_\tau = \frac{2\pi\tau}{\log\tau}\left(1 + \frac{\log\log\tau}{\log\tau}\right)(1 + o(1))$$

and

$$t'_\tau = \frac{2\pi}{\log\tau}\left(1 + \frac{\log\log\tau}{\log\tau}\right)(1 + o(1))$$

are known; see [36]. The points t_n, $n \in \mathbb{N}$, were studied in [37] by J.-P. Gram in connection with the imaginary parts of nontrivial zeros of $\zeta(s)$. He observed that the interval $(t_{n-1}, t_n]$ with $1 \leqslant n \leqslant 15$ contains one zero of the function

$$e^{i\theta(t)}\zeta\left(\frac{1}{2} + it\right),$$

and conjectured that this is not true if $n > 15$. Later, this conjecture was confirmed by various authors. It is well known that

$$t_n \sim \gamma_n, \quad n \to \infty,$$

where $\{\gamma_n : n \in \mathbb{N}\}$ is the sequence of positive imaginary parts of nontrivial zeros of $\zeta(s)$.

For the first time, the function t_τ in the approximation of analytic functions by generalized shifts was applied in [34]. For $j = 1, \dots, r$, let $\chi_j(m)$ denote a Dirichlet character and

$$L(s, \chi_j) = \sum_{m=1}^{\infty} \frac{\chi_j(m)}{m^s}, \quad \sigma > 1,$$

the corresponding Dirichlet L-function. Moreover, let $a_1, \dots, a_r$ be real algebraic numbers that are linearly independent over the field of rational numbers $\mathbb{Q}$. Suppose that $K_j \in \mathcal{K}$ and $f_j(s) \in H_0(K_j)$, $j = 1, \dots, r$. Then, it was obtained in [34] that, for every $\varepsilon > 0$,

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas}\left\{ \tau \in [0, T] : \sup_{1 \leqslant j \leqslant r} \sup_{s \in K_j} \left| f(s) - L(s + i a_j t_\tau, \chi_j) \right| < \varepsilon \right\} > 0.$$

Moreover, "lim inf" can be replaced with "lim" for all but at most countably many $\varepsilon > 0$.

From the latter result with $r = 1$ and a character modulo 1, we have the following statement.

Theorem 3. *Suppose that $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas}\left\{ \tau \in [0, T] : \sup_{s \in K} |f(s) - \zeta(s + i t_\tau)| < \varepsilon \right\} > 0.$$

Moreover, the limit

$$\lim_{T \to \infty} \frac{1}{T} \text{meas}\left\{ \tau \in [0, T] : \sup_{s \in K} |f(s) - \zeta(s + i t_\tau)| < \varepsilon \right\}$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

The aim of this paper is to extend Theorems 1 and 2 by using generalized shifts $\zeta(s + i t_\tau)$. We will prove the following equivalents of the RH.

Theorem 4. *The RH holds if and only if, for every $K \in \mathcal{K}$ and $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas}\left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s) - \zeta(s + i t_\tau)| < \varepsilon \right\} > 0.$$

Theorem 5. *The RH holds if and only if, for every $K \in \mathcal{K}$, the limit*

$$\lim_{T \to \infty} \frac{1}{T} \text{meas}\left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s) - \zeta(s + i t_\tau)| < \varepsilon \right\}$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

We notice that the shift $\zeta(s+it_\tau)$ is only an example of a possible shift. Theorems 4 and 5 remain true also for other shifts satisfying statements of the type of Theorem 3. The choice of the function t_τ is only due to the illustration of its importance in the theory of $\zeta(s)$.

Considering the rapid progress of the approximation theory by shifts of zeta functions, we expect that Theorems 4 and 5 will have a certain influence when considering the RH.

2. Limit Lemmas

For proofs of universality theorems in the approximation of analytic functions by shifts of zeta functions, B. Bagchi proposed [21] to apply limit probabilistic theorems in the space of analytic functions; see also [23,24]. We recall some facts connected to the latter approach.

Let $\mathcal{X}$ be a topological space with the Borel σ-field $\mathcal{B}(\mathcal{X})$, and let P and P_n, $n \in \mathbb{N}$, be probability measures on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$. By definition, P_n converges weakly to P as $n \to \infty$ ($P_n \xrightarrow[n\to\infty]{\mathrm{w}} P$) if, for every real continuous bounded function g in $\mathcal{X}$,

$$\lim_{n\to\infty} \int_{\mathcal{X}} g\, \mathrm{d}P_n = \int_{\mathcal{X}} g\, \mathrm{d}P.$$

There are some equivalents of the weak convergence of probability measures in terms of some classes of sets. Recall that the set $A \in \mathcal{B}(\mathcal{X})$ is a continuity set of the measure P if $P(\partial A) = 0$, where ∂A denotes the boundary of the set A. We will use the following convenient lemma.

Lemma 1. *The following statements are equivalent:*

(i) $P_n \xrightarrow[n\to\infty]{\mathrm{w}} P$;

(ii) *For all open sets* $G \subset \mathcal{X}$,

$$\liminf_{n\to\infty} P_n(G) \geqslant P(G);$$

(iii) *For all continuity sets* A *of* P,

$$\lim_{n\to\infty} P_n(A) = P(A).$$

For the proof, see, for example [38], Theorem 2.1.

Historically, the first applications of probabilistic methods in the theory of the function $\zeta(s)$ were described by H. Borhr and B. Jessen. Let J denote the Jordan measure on $\mathbb{R}$, R the rectangle on the complex plane with edges parallel to the axes, and

$$\mathcal{A} = \left\{ s \in \mathbb{C} : \sigma > \frac{1}{2} \right\} \setminus \bigcup_{\beta_j + i\gamma_j} \left\{ s = \sigma + i\gamma_j : \frac{1}{2} < \sigma \leqslant \beta_j \right\},$$

where $\beta_j + i\gamma_j$ runs over all zeros of $\zeta(s)$ in the strip $1/2 < \sigma < 1$. Then, they proved [39,40] that the limit

$$\lim_{T\to\infty} \frac{1}{T} \mathrm{J}\{\tau \in [0,T] : \sigma + it \in \mathcal{A},\ \log\zeta(s+it) \in \mathrm{R}\}$$

exists and depends only on σ and R.

Later, for the description of the chaotic behavior of $\zeta(s)$ by limit theorems, a more convenient method involving the weak convergence of probability measures began to be cultivated.

Denote by $\mathcal{H}(D)$ the space of analytic functions on the strip D equipped with the topology of uniform convergence on compact sets. A probabilistic approach to the proof of Theorem 3 is based on the weak convergence of the probability measure

$$P_T(A) \stackrel{\text{def}}{=} \frac{1}{T}\text{meas}\{\tau \in [0,T] : \zeta(s+it_\tau) \in A\}, \quad A \in \mathcal{B}(\mathcal{H}(D)),$$

as $T \to \infty$. For a limit theorem for P_T, a certain topological structure is needed. Let

$$\mathbb{T} = \prod_{p\in\mathbb{P}} \{s \in \mathbb{C} : |s| = 1\}.$$

The infinite-dimensional torus $\mathbb{T}$ consists of all functions $\omega : \mathbb{P} \to \{s \in \mathbb{C} : |s| = 1\}$, and, with the operation of pairwise multiplication and product topology, it is a compact topological group. Therefore, on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$, the probability Haar measure m_H exists, and we have the probability space $(\mathbb{T}, \mathcal{B}(\mathbb{T}), m_H)$. Denote by $\omega = (\omega(p) : p \in \mathbb{P})$ elements of $\mathbb{T}$, and, on the probability space $(\mathbb{T}, \mathcal{B}(\mathbb{T}), m_H)$, define the $\mathcal{H}(D)$-valued random element $\zeta(s,\omega)$ by the product

$$\zeta(s,\omega) = \prod_{p\in\mathbb{P}} \left(1 - \frac{\omega(p)}{p^s}\right)^{-1}.$$

Notice that the above infinite product, for almost all $\omega \in \mathbb{T}$, is uniformly convergent on compact subsets of the strip D; see Theorem 5.1.7 of [23]. Let P_ζ be the distribution of the element $\zeta(s,\omega)$; thus,

$$P_\zeta(A) = m_H\{\omega \in \mathbb{T} : \zeta(s,\omega) \in A\}, \quad A \in \mathcal{B}(\mathcal{H}(D)).$$

The probabilistic behavior of the function $\zeta(s)$ is described by the following lemma.

Lemma 2. *We have*

$$P_T \xrightarrow[T\to\infty]{\text{w}} P_\zeta.$$

Proof. For $A \in \mathcal{B}(\mathcal{H}^r(D))$, define

$$P_{T,r}(A) = \frac{1}{T}\text{meas}\{\tau \in [0,T] : (L(s+ia_1t_\tau,\chi_1), \dots, L(s+ia_rt_\tau,\chi_r)) \in A\}$$

and

$$P_{L_1,\dots,L_r}(A) = m_H^r\{\omega \in \mathbb{T}^r : (L(s,\omega,\chi_1), \dots, L(s,\omega,\chi_r)) \in A\},$$

where

$$L(s,\omega,\chi_j) = \prod_{p\in\mathbb{P}} \left(1 - \frac{\omega(p)\chi_j(p)}{p^s}\right)^{-1}, \quad j = 1, \dots, r,$$

with $\omega \in \mathbb{T}$. Here, m_H^r is the Haar measure on $(\mathbb{T}^r, \mathcal{B}(\mathbb{T}^r))$. Then, in [34], under the hypothesis that $a_1, \dots, a_r$ are algebraic numbers that are linearly independent over $\mathbb{Q}$, the relation

$$P_{T,r} \xrightarrow[T\to\infty]{\text{w}} P_{L_1,\dots,L_r}$$

has been obtained. From this, with $r = 1$, $a_1 = 1$, and $\chi_1(m) \equiv 1$, the lemma follows. □

We note that Theorem 3 can be proven directly by a similar method to that of the proofs of Propositions 1 and 2, with the application of the above-mentioned properties of the function t_τ.

The next ingredient for the proof of Theorems 4 and 5 is the support of the limit measure P_ζ in Lemma 2. Recall that the support of P_ζ is a minimal closed set $S \in \mathcal{H}(D)$

such that $P_\zeta(S) = 1$. The set S consists of all elements $g \in \mathcal{H}(D)$, for which, for every open neighborhood G, the inequality $P(G) > 0$ is satisfied.

Define
$$S = \{g \in \mathcal{H}(D) : g(s) \neq 0 \text{ on } D, \text{ or } g(s) \equiv 0\}.$$

Lemma 3. *The support of the measure P_ζ is the set S.*

The proof of the lemma is given in [21,23].

Since, by Lemma 2, the asymptotic behavior of $\zeta(s)$ is described by the measure P_ζ, and, in view of Lemma 3, the support of P_ζ consists of non-vanishing on D functions, we intuitively feel that $\zeta(s) \neq 0$ in D. This suggests Theorems 4 and 5.

3. Proofs of Theorems 4 and 5

We will introduce, in the space $\mathcal{H}(D)$, the metric that induces the topology of uniform convergence on compact sets. Let $\{K_j : j \in \mathbb{N}\} \subset D$ be a set of compact embedded sets such that
$$D = \bigcup_{j=1}^{\infty} K_j,$$
and every compact set $K \subset D$ lies in some set K_j. It is well known—see [41]—that such a sequence always exists. In our case, we can take, for example, closed rectangles. Now, for $g_1, g_2 \in \mathcal{H}(D)$, set
$$\rho(g_1, g_2) = \sum_{j=1}^{\infty} 2^{-j} \frac{\sup_{s \in K_j} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_j} |g_1(s) - g_2(s)|}.$$

Then, ρ is the desired metric in $\mathcal{H}(D)$.

Proof of Theorem 4. Obviously, the necessity follows from the first part of Theorem 3. If the RH is true, then, for every $K \in \mathcal{K}$, the function $\zeta(s)$ lies in the set $H_0(K)$. Therefore, by Theorem 3, for every $K \in \mathcal{K}$ and $\varepsilon > 0$,
$$\liminf_{T \to \infty} \frac{1}{T} \text{meas}\left\{\tau \in [0, T] : \sup_{s \in K} |\zeta(s) - \zeta(s + i t_\tau)| < \varepsilon\right\} > 0. \tag{3}$$

The latter inequality holds easily also without using Theorem 3. In fact, for $K \in \mathcal{K}$ and $\delta > 0$, let
$$G_{\delta,K} = \left\{g \in \mathcal{H}(D) : \sup_{s \in K} |g(s) - \zeta(s)| < \delta\right\}.$$

Since K is a compact set, $G_{\delta,K}$ is an open neighborhood of $\zeta(s)$. If the RH is true, then $\zeta(s) \in S$. Hence, in view of Lemma 3, the set $G_{\delta,K}$ is an open neighborhood of the element $\zeta(s)$ in the support of the measure P_ζ; thus,
$$P_\zeta(G_{\delta,K}) > 0.$$

This, together with Lemmas 1 and 2, shows that
$$\liminf_{T \to \infty} P_T(G_{\delta,K}) \geqslant P_\zeta(G_{\delta,K}) > 0,$$
and the definitions of P_T and $G_{\delta,K}$ prove inequality (3).

Sufficiency. Suppose that (3) is true. We will show that the RH is true as well.

On the contrary, suppose that the RH is not true. Then, $\zeta(s)$ has zeros in the strip D; therefore, $\zeta(s) \notin S$. Hence, by Lemma 3, $\zeta(s)$ is not an element of the support of the measure P_ζ. By a support property, there is an open neighborhood G of $\zeta(s)$ such that $P_\zeta(G) = 0$. Then, there is an open set $G_\delta \stackrel{\text{def}}{=} \{g \in \mathcal{H}(D) : \rho(g, \zeta) < \delta\}$, $\delta > 0$, lying in the set G. Our purpose is to prove that there exists a set $K \in \mathcal{K}$ and $\varepsilon > 0$ such that $G_{\varepsilon,K}$ lies in G_δ. Let K_{j_0} be a set from the definition of the metric ρ such that

$$\sum_{j>j_0} 2^{-j} < \frac{\delta}{2}. \tag{4}$$

By the definition of the sequence $\{K_j\}$, we have $K_{j_0} \supset K_j$ for $j = 1, \dots, j_0$. Therefore, for $g \in G_{\delta/2,K_{j_0}}$, by (4),

$$\rho(g, \zeta) = \left(\sum_{j=1}^{j_0} + \sum_{j>j_0} \right) \frac{1}{2^j} \frac{\sup_{s \in K_j} |g(s) - \zeta(s)|}{1 + \sup_{s \in K_j} |g(s) - \zeta(s)|} < \frac{\delta}{2} \sum_{j=1}^{j_0} \frac{1}{2^j} + \sum_{j>j_0} \frac{1}{2^j} < \delta.$$

This shows that $G_{\varepsilon,K_{j_0}}$ lies in the set G_δ for all $0 < \varepsilon < \delta/2$. Therefore, $G_{\varepsilon,K_{j_0}} \subset G$ for the latter values of ε. In consequence, for $0 < \varepsilon < \delta/2$, we have

$$P_\zeta(G_{\varepsilon,K_{l_0}}) = 0. \tag{5}$$

The boundary $\partial G_{\varepsilon,K_{j_0}}$ lies in the set

$$\left\{ g \in \mathcal{H}(D) : \sup_{s \in K_{j_0}} |g(s) - \zeta(s)| = \varepsilon \right\}.$$

Therefore, $\partial G_{\varepsilon_1,K_{j_0}} \cap \partial G_{\varepsilon_2,K_{j_0}} = \emptyset$ for different positive ε_1 and ε_2. From this, it follows that the set $G_{\varepsilon,K_{j_0}}$ is a continuity set of the measure P_ζ for all but at most countably many $\varepsilon > 0$. Hence, there is $0 < \widehat{\varepsilon} < \delta/2$ such that the set $G_{\widehat{\varepsilon},K_{j_0}}$ is a continuity set of P_ζ and satisfies (5). Therefore, by Lemmas 1 and 2,

$$\lim_{T \to \infty} P_T(G_{\widehat{\varepsilon},K_{j_0}}) = P_\zeta(G_{\widehat{\varepsilon},K_{j_0}}) = 0.$$

Hence,

$$\liminf_{T \to \infty} P_T(G_{\widehat{\varepsilon},K_{j_0}}) = 0,$$

and this contradicts inequality (3). Therefore, the RH is true. □

Proof of Theorem 5. *Necessity.* Suppose that the RH is true. Then, $\zeta(s) \in H_0(K)$ for every $K \in \mathcal{K}$. Therefore, the second part of Theorem 3 implies that the limit

$$\lim_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s) - \zeta(s + it_\tau)| < \varepsilon \right\} \tag{6}$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

This can be also proven directly. Using the notation of the proof of Theorem 4, we have

$$P_\zeta(G_{\varepsilon,K}) > 0. \tag{7}$$

Moreover, as in the proof of Theorem 4, we obtain that the set $G_{\varepsilon,K}$ is a continuity set of the measure P_ζ, for all but at most countably many $\varepsilon > 0$. Therefore, by Lemmas 1 and 2, we find by (7) that the limit (6) exists and is positive for all but at most countably many $\varepsilon > 0$.

Sufficiency. Suppose that the limit (6) exists and is positive for all but at most countably many $\varepsilon > 0$. We have to prove that the RH is true.

On the contrary, suppose that the RH is not true. As in the proof of Theorem 4, we obtain that there exists $\delta > 0$ and a compact set $K_\delta \subset D$ such that the limit (6) exists and is zero for all $0 < \varepsilon < \delta/2$. However, this contradicts the hypothesis that the limit (6) is positive for all but at most countably many $\varepsilon > 0$, and this contradiction proves the RH. The theorem is proven. □

4. Conclusions

The famous Riemann hypothesis (RH) asserts that the function

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}, \quad \sigma > 1,$$

and given by analytic continuation elsewhere, has zeros different from $s = -2m$, $m \in \mathbb{N}$, lying only on the line $\sigma = 1/2$. There are many equivalents of the RH. In the paper, we propose equivalents of the RH stated in terms of the self-approximation of $\zeta(s)$ by shifts $\zeta(s + it_\tau)$, where t_τ is the solution of the equation

$$\theta(t) = (\tau - 1)\pi, \quad \tau \geqslant 0,$$

and $\theta(t)$ denotes the increment in the argument of the function $\pi^{-s/2}\Gamma(s/2)$ along the segment connecting the points $s = 1/2$ and $s = 1/2 + it$. Let $\mathcal{K}$ be the class of compact sets of the strip $\{s \in \mathbb{C} : 1/2 < \sigma < 1\}$ with connected complements. Then, the RH is true if and only if the inequality

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas}\left\{\tau \in [0, T] : \sup_{s\in K} |\zeta(s) - \zeta(s + it_\tau)| < \varepsilon\right\} > 0$$

holds for every $K \in \mathcal{K}$ and $\varepsilon > 0$, or, for every $K \in \mathcal{K}$, the limit

$$\lim_{T\to\infty} \frac{1}{T} \text{meas}\left\{\tau \in [0, T] : \sup_{s\in K} |\zeta(s) - \zeta(s + it_\tau)| < \varepsilon\right\}$$

exists and is positive for all but at most countably many $\varepsilon > 0$.

The history of mathematics shows that the function $\zeta(s)$ has various connections with physics and other natural sciences. At present, we do not see any connection between the obtained criteria for the RH and physical phenomena, but this may be possible in the future. We believe that the proof (or disproof) of the RH could have a certain influence on investigations of some physical processes. On the other hand, it is impossible to prove the RH with even very large numerical calculations.

For the function $\zeta(s)$, discrete universality theorems on the application of analytic functions by shifts $\zeta(s + i\varphi(k))$, $k \in \mathbb{N}$, with certain functions are also known. Theorems 4 and 5 can be stated for Gram points. Moreover, the use of limit and universality theorems in short intervals is possible.

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